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Two-point functions of Coulomb gauge Yang-Mills theory

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The functional approach to Coulomb gauge Yang-Mills theory is considered within the standard, second order, formalism. The Dyson-Schwinger equations and Slavnov-Taylor identities concerning the two-point functions are derived explicitly and one-loop perturbative results are presented.

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I. INTRODUCTION

Coulomb gauge Yang-Mills theory (and by extension quantum chromodynamics) is a fascinating, yet frustrating endeavor. On the one hand, Coulomb gauge offers great potential for understanding such issues as confinement [1,2]; on the other, the intrinsic noncovariance of the formalism makes any perturbative calculation formidably complicated. Many approaches to solving (or providing reliable approximations to solving) the problems in Coulomb gauge have been forwarded. Recent among these are the Hamiltonian approach of Ref. [3], based on the original work of Christ and Lee [4]. A lattice version of the Coulomb gauge action also exists [5], which has led to numerical studies, for example, Refs. [6]. Functional methods based on the Lagrangian formalism have also been considered, especially within the first order (phase space) formalism [1,7] and most recently, one-loop perturbative results for both the ultraviolet divergent and finite parts of the various two-point functions have been obtained [8]. Similar results were previously obtained for the gluon propagator functions under a different formalism (using the chromoelectric field directly as a degree of freedom and without ghosts) and using different methods to evaluate the integrals [9].

In this paper, we consider the (standard, second order) functional approach to Coulomb gauge Yang-Mills theory. We derive the Dyson-Schwinger equations and Slavnov-Taylor identities for the two-point functions that arise in the construction and using the techniques of [8] we present results for the one-loop perturbative dressing functions.

The paper is organized as follows. In the next section, the functional formalism used is described. Section III concerns the decomposition of the functions used. The (nonperturbative) Dyson-Schwinger equations and Slavnov-Taylor identities relating the various Green's functions are derived in Sec. IV. In Sec. V, the one-loop perturbative results are obtained. Finally, there is a summary and outlook.

II. FUNCTIONAL FORMALISM

Let us begin by considering Coulomb gauge Yang-Mills theory. We use the framework of functional methods to derive the basic equations that will later give rise to the Dyson-Schwinger equations, Slavnov-Taylor identities, Feynman rules, etc. Throughout this work, we will use the notation and conventions established in [7,8]. We work in Minkowski space (until the perturbative integrals are to be explicitly evaluated) with metric $g_{\mu\nu} = \mathrm{diag}(1, -\vec{1})$. Greek letters $(\mu, \nu, \ldots)$ denote Lorentz indices, roman subscripts $(i, j, \ldots)$ denote spatial indices and superscripts $(a, b, \ldots)$ denote color indices. We will sometimes also write configuration space coordinates $(x, y, \ldots)$ as subscripts where no confusion arises.

The Yang-Mills action is defined as

$$\mathcal{S}_{\mathrm{YM}} = \int d^4x\left[-\frac{1}{4}F_{\mu\nu}^aF^{a\mu\nu}\right] \tag{2.1}$$

where the (antisymmetric) field strength tensor F is given in terms of the gauge field A_μ^a:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^bA_\nu^c. \tag{2.2}$$

In the above, the f^{abc} are the structure constants of the $SU(N_c)$ group whose generators obey $[T^a, T^b] = \imath f^{abc}T^c$. The Yang-Mills action is invariant under a local $SU(N_c)$ gauge transform characterized by the parameter θ_x^a:

$$U_x = \exp\{-\imath\theta_x^aT^a\}. \tag{2.3}$$

The field strength tensor can be expressed in terms of the chromoelectric and chromomagnetic fields ($\sigma = A^0$)

$$\begin{aligned}\vec{E}^a &= -\partial^0\vec{A}^a - \vec{\nabla}\sigma^a + gf^{abc}\vec{A}^b\sigma^c,\\ B_i^a &= \epsilon_{ijk}[\nabla_jA_k^a - \tfrac{1}{2}gf^{abc}A_j^bA_k^c]\end{aligned} \tag{2.4}$$

such that $\mathcal{S}_{\mathrm{YM}} = \int(E^2 - B^2)/2$. The electric and magnetic terms in the action do not mix under the gauge transform which for the gauge fields is written

$$A_\mu \to A'_\mu = U_xA_\mu U_x^\dagger - \frac{\imath}{g}(\partial_\mu U_x)U_x^\dagger. \tag{2.5}$$

Given an infinitesimal transform $U_x = 1 - \imath\theta_x^aT^a$, the variation of the gauge field is

$$\delta A_\mu^a = -\frac{1}{g}\hat{D}_\mu^{ac}\theta^c \tag{2.6}$$

where the covariant derivative in the adjoint representation is given by

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$$\hat{D}^{ac}_{\mu} = \delta^{ac}\partial_{\mu} + gf^{abc}A^{b}_{\mu}. \tag{2.7}$$

Consider the functional integral

$$Z = \int \mathcal{D}\Phi \exp\{\imath\mathcal{S}_{\rm YM}\} \tag{2.8}$$

where Φ denotes the collection of all fields. Since the action is invariant under gauge transformations, Z is divergent by virtue of the integration over the gauge group. To overcome this problem we use the Faddeev-Popov technique and introduce a gauge-fixing term along with an associated ghost term [10]. Using a Lagrange multiplier field to implement the gauge-fixing, in Coulomb gauge ($\vec{\nabla}\cdot\vec{A} = 0$) we can then write

$$Z = \int \mathcal{D}\Phi \exp\{\imath\mathcal{S}_{\rm YM} + \imath\mathcal{S}_{fp}\},$$
$$\mathcal{S}_{fp} = \int d^4x[-\lambda^a\vec{\nabla}\cdot\vec{A}^a - \bar{c}^a\vec{\nabla}\cdot\vec{D}^{ab}c^b]. \tag{2.9}$$

The new term in the action is invariant under the standard BRS transform whereby the infinitesimal gauge parameter θ^a is factorized into two Grassmann-valued components $\theta^a = c^a\delta\lambda$ where $\delta\lambda$ is the infinitesimal variation (not to be confused with the colored Lagrange multiplier field λ^a). The BRS transform of the new fields reads

$$\delta\bar{c}^a = \frac{1}{g}\lambda^a\delta\lambda \qquad \delta c^a = -\tfrac{1}{2}f^{abc}c^bc^c\delta\lambda \qquad \delta\lambda^a = 0. \tag{2.10}$$

It is at this point that this work diverges from Ref. [7] in that we remain here within the standard (second order) formalism. By including source terms to Z, we construct the generating functional, $Z[J]$:

$$Z[J] = \int \mathcal{D}\Phi \exp\{\imath\mathcal{S}_{\rm YM} + \imath\mathcal{S}_{fp} + \imath\mathcal{S}_s\} \tag{2.11}$$

where

$$\mathcal{S}_s = \int d^4x[\rho^a\sigma^a + \vec{J}^a\cdot\vec{A}^a + \bar{c}^a\eta^a + \bar{\eta}^ac^a + \xi^a\lambda^a]. \tag{2.12}$$

It is convenient to introduce a compact notation for the sources and fields and we denote a generic field Φ_α with source J_α such that the index α stands for all attributes of the field in question (including its type) such that we can write

$$\mathcal{S}_s = J_\alpha\Phi_\alpha \tag{2.13}$$

where summation over all discrete indices and integration over all continuous arguments is implicitly understood. Expanding the various terms we have explicitly

$$\mathcal{S}_{\rm YM} = \int d^4x\Big\{-\frac{1}{2}A_i^f[\delta_{ij}\partial_0^2 - \delta_{ij}\nabla^2 + \nabla_i\nabla_j]A_j^f - A_i^f\partial_0\nabla_i\sigma^f - \frac{1}{2}\sigma^f\nabla^2\sigma^f$$
$$+ gf^{fbc}[-(\partial_0A_i^f)A_i^b\sigma^c - (\nabla_i\sigma^f)A_i^b\sigma^c + (\nabla_jA_k^f)A_j^bA_k^c] + g^2f^{fbc}f^{fde}\Big[\frac{1}{2}A_i^b\sigma^cA_i^d\sigma^e - \frac{1}{4}A_i^bA_j^cA_i^dA_j^e\Big]\Big\}. \tag{2.14}$$

The field equations of motion are derived from the observation that the integral of a total derivative vanishes, up to boundary terms. The boundary terms vanish, although this is not trivial in the light of the Gribov problem [2] (the reader is directed to Ref. [7] and references therein for a discussion of this topic). Writing $\mathcal{S} = \mathcal{S}_{\rm YM} + \mathcal{S}_{fp}$, we have that

$$0 = \int \mathcal{D}\Phi \frac{\delta}{\delta\imath\Phi_\alpha}\exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\}. \tag{2.15}$$

The explicit form of the field equations of motion is given in the appendix.

In addition to the field equations of motion, there exist identities derived by considering the BRS invariance of the action (these eventually form the Slavnov-Taylor identities). The BRS transform is continuous and we can regard it as a change of variables in the functional integral. Given that the Jacobian of such a change of variables is trivial and that the action is invariant, we have that

$$0 = \int \mathcal{D}\Phi \frac{\delta}{\delta\imath\delta\lambda}\exp\{\imath\mathcal{S} + \imath\mathcal{S}_s + \imath\delta\mathcal{S}_s\}_{\delta\lambda=0}$$
$$= \int \mathcal{D}\Phi \exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\}\int d^4x\Big[\frac{1}{g}\rho^a\partial_0c^a + f^{abc}\rho^a\sigma^bc^c - \frac{1}{g}J_i^a\nabla_ic^a + f^{abc}J_i^aA_i^bc^c + \frac{1}{g}\lambda^a\eta^a + \frac{1}{2}f^{abc}\bar{\eta}^ac^bc^c\Big]. \tag{2.16}$$

So far, the generating functional, $Z[J]$, generates all Green's functions, connected and disconnected. The generating functional of connected Green's functions is $W[J]$ where

$$Z[J] = e^{W[J]}. \tag{2.17}$$

We define the classical fields to be

$$\Phi_\alpha = \frac{1}{Z}\int \mathcal{D}\Phi\Phi_\alpha \exp\imath\mathcal{S} = \frac{1}{Z}\frac{\delta Z}{\delta\imath J_\alpha}. \tag{2.18}$$

The generating functional of proper Green's functions is the effective action, Γ, which is a function of the classical fields and is defined through a Legendre transform of W:

$$\Gamma[\Phi] = W[J] - \imath J_\alpha \Phi_\alpha. \tag{2.19}$$

We introduce a bracket notation for derivatives of W with respect to sources and of Γ with respect to classical fields (no confusion arises since the two sets of derivatives are never mixed):

$$\langle \imath J_\alpha \rangle = \frac{\delta W}{\delta \imath J_\alpha}, \qquad \langle \imath \Phi_\alpha \rangle = \frac{\delta \Gamma}{\delta \imath \Phi_\alpha}. \tag{2.20}$$

It is now possible to present the field equations of motion in terms of proper functions (the Dyson-Schwinger equations are functional derivatives of these equations). Using the results listed in the appendix we have:

$$\begin{aligned}\langle \imath A^a_{ix}\rangle =& -[\delta_{ij}\partial^2_{0x} - \delta_{ij}\nabla^2_x + \nabla_{ix}\nabla_{jx}]A^a_{jx} - \partial_{0x}\nabla_{ix}\sigma^a_x + \nabla_{ix}\lambda^a_x + gf^{abc}\int d^4y d^4z \partial_{0x}\delta(y-x)\delta(z-x)[\langle \imath J^b_{iy}\imath\rho^c_z\rangle + A^b_{iy}\sigma^c_z]\\
&- gf^{fac}\int d^4y d^4z \delta(z-x)\nabla_{ix}\delta(y-x)[\langle \imath\rho^f_y\imath\rho^c_z\rangle + \langle \imath\bar\eta^c_z\imath\eta^f_y\rangle + \sigma^f_y\sigma^c_z + \bar c^f_y c^c_z]\\
&+ gf^{abc}\int d^4y d^4z[\delta_{ij}\delta(z-x)\nabla_{kx}\delta(y-x) + \delta_{jk}\delta(y-x)\nabla_{ix}\delta(z-x) - \delta_{ki}\nabla_{jx}\delta(y-x)\delta(z-x)]\\
&\times[\langle \imath J^b_{jy}\imath J^c_{kz}\rangle + A^b_{jy}A^c_{kz}] + g^2 f^{fac}f^{fde}[\langle \imath\rho^c_x\imath J^d_{ix}\imath\rho^e_x\rangle + \sigma^c_x\langle \imath J^d_{ix}\imath\rho^e_x\rangle + \sigma^e_x\langle\imath\rho^c_x\imath J^d_{ix}\rangle + A^d_{ix}\langle\imath\rho^c_x\imath\rho^e_x\rangle + \sigma^c_x A^d_{ix}\sigma^e_x]\\
&- \frac{1}{4}g^2 f^{fbc}f^{fde}\delta_{jk}\delta_{il}[\delta^{gc}\delta^{eh}(\delta^{ab}\delta^{di} + \delta^{ad}\delta^{bi}) + \delta^{bg}\delta^{dh}(\delta^{ac}\delta^{ei} + \delta^{ae}\delta^{ci})]\\
&\times[\langle \imath J^g_{jx}\imath J^h_{kx}\imath J^i_{lx}\rangle + A^g_{jx}\langle \imath J^h_{kx}\imath J^i_{lx}\rangle + A^i_{lx}\langle \imath J^g_{jx}\imath J^h_{kx}\rangle + A^h_{kx}\langle \imath J^g_{jx}\imath J^i_{lx}\rangle + A^g_{jx}A^h_{kx}A^i_{lx}],\end{aligned} \tag{2.21}$$

$$\begin{aligned}\langle \imath\sigma^a_x\rangle =& -\partial_{0x}\nabla_{ix}A^a_{ix} - \nabla^2_x\sigma^a_x - gf^{fba}\int d^4y d^4z\delta(z-x)\partial_{0x}\delta(y-x)[\langle \imath J^f_{iy}\imath J^b_{iz}\rangle + A^f_{iy}A^b_{iz}]\\
&+ gf^{abc}\int d^4y d^4z[\nabla_{ix}\delta(y-x)\delta(z-x) + \delta(y-x)\nabla_{ix}\delta(z-x)][\langle \imath J^b_{iy}\imath\rho^c_z\rangle + A^b_{iy}\sigma^c_z]\\
&+ g^2 f^{fba}f^{fde}[\langle \imath J^b_{ix}\imath J^d_{ix}\imath\rho^e_x\rangle + A^b_{ix}\langle \imath J^d_{ix}\imath\rho^e_x\rangle + \sigma^e_x\langle \imath J^b_{ix}\imath J^d_{ix}\rangle + A^d_{ix}\langle \imath J^b_{ix}\imath\rho^e_x\rangle + A^b_{ix}A^d_{ix}\sigma^c_x],\end{aligned} \tag{2.22}$$

$$\langle \imath\lambda^a_x\rangle = -\nabla_{ix}A^a_{ix}, \tag{2.23}$$

$$\langle \imath\bar c^a_x\rangle = -\nabla^2_x c^a_x + gf^{abc}\int d^4y d^4z\nabla_{ix}\delta(y-x)\delta(z-x) \times[\langle \imath J^b_{iy}\imath\bar\eta^c_z\rangle + A^b_{iy}c^c_z]. \tag{2.24}$$

It is also useful to express the λ equation of motion in terms of connected functions:

$$\xi^a_x = \nabla_{ix}\langle \imath J^a_{ix}\rangle. \tag{2.25}$$

The identity stemming from the BRS invariance is also best expressed in terms of both connected and proper functions and reads:

$$\begin{aligned}0 = \int d^4x\Big\{&\frac{1}{g}\eta^a_x\langle \imath\xi^a_x\rangle + \frac{1}{g}\rho^a_x\partial_{0x}\langle \imath\bar\eta^a_x\rangle\\
&+ f^{abc}\rho^a_x[\langle \imath\rho^b_x\imath\bar\eta^c_x\rangle + \langle \imath\rho^b_x\rangle\langle \imath\bar\eta^c_x\rangle] - \frac{1}{g}\left[\frac{\nabla_{ix}}{(-\nabla^2_x)}J^a_{ix}\right]\eta^a_x\\
&+ f^{abc}J^a_{ix}t_{ij}(x)[\langle \imath J^b_{jx}\imath\bar\eta^c_x\rangle + \langle \imath J^b_{jx}\rangle\langle \imath\bar\eta^c_x\rangle]\\
&+ \frac{1}{2}f^{abc}\bar\eta^a_x[\langle \imath\bar\eta^b_x\imath\bar\eta^c_x\rangle + \langle \imath\bar\eta^b_x\rangle\langle \imath\bar\eta^c_x\rangle]\Big\},\end{aligned} \tag{2.26}$$

$$\begin{aligned}0 = \int d^4x\Big\{&-\frac{1}{g}\langle \imath\bar c^a_x\rangle\lambda^a_x - \frac{1}{g}\langle \imath\sigma^a_x\rangle\partial_{0x}c^a_x\\
&- f^{abc}\langle \imath\sigma^a_x\rangle[\langle \imath\rho^b_x\imath\bar\eta^c_x\rangle + \sigma^b_x c^c_x] - \frac{1}{g}\left[\frac{\nabla_{ix}}{(-\nabla^2_x)}\langle \imath A^a_{ix}\rangle\right]\langle \imath\bar c^a_x\rangle\\
&- f^{abc}\langle \imath A^a_{ix}\rangle t_{ij}(x)[\langle \imath J^b_{jx}\imath\bar\eta^c_x\rangle + A^b_{jx}c^c_x]\\
&+ \frac{1}{2}f^{abc}\langle \imath c^a_x\rangle[\langle \imath\bar\eta^b_x\imath\bar\eta^c_x\rangle + c^b_x c^c_x]\Big\},\end{aligned} \tag{2.27}$$

where we have used the common trick of using the ghost equation of motion in order to reexpress one of the interaction terms transversely, with the transverse projector in configuration space being $t_{ij}(x) = \delta_{ij} + \nabla_{ix}\nabla_{jx}/(-\nabla^2_x)$. This manipulation will be useful when we consider the Slavnov-Taylor identities for the two-point functions later on.

At this stage it is useful to explore some consequences of the above equations that lead to exact statements about the Green's functions. Introducing our conventions and notation for the Fourier transform, we have for a general two-point function (connected or proper) which obeys translational invariance:

$$\begin{aligned}\langle \imath J_\alpha(y)\imath J_\beta(x)\rangle &= \langle \imath J_\alpha(y-x)\imath J_\beta(0)\rangle \\ &= \int đk W_{\alpha\beta}(k)e^{-\imath k\cdot(y-x)}, \\ \langle \imath\Phi_\alpha(y)\imath\Phi_\beta(x)\rangle &= \langle \imath\Phi_\alpha(y-x)\imath\Phi_\beta(0)\rangle \\ &= \int đk \Gamma_{\alpha\beta}(k)e^{-\imath k\cdot(y-x)},\end{aligned} \tag{2.28}$$

where $đk = d^4k/(2\pi)^4$. Starting with Eq. (2.23), we have that the only nonzero functional derivative is

$$\langle \imath A^b_{jy}\imath\lambda^a_x\rangle = \imath\delta^{ba}\nabla_{jx}\delta(y-x) = \delta^{ba}\int đk k_j e^{-\imath k\cdot(y-x)} \tag{2.29}$$

and all other proper Green's functions involving derivatives with respect to the λ-field vanish (even in the presence of sources). In terms of connected Green's functions, Eq. (2.23) becomes Eq. (2.25) and the only nonzero functional derivative is

$$\nabla_{ix}\langle \imath\xi^b_y\imath J^a_{ix}\rangle = -\imath\delta^{ba}\delta(y-x). \tag{2.30}$$

Because Eq. (2.25) involves the contraction of a vector quantity, the information is less restricted than previously. However, we can write down the following (true once sources have been set to zero such that the tensor structure is determined):

$$\begin{gathered}\langle \imath J^b_{jy}\imath J^a_{ix}\rangle = \int đk W^{ba}_{AA}(k)t_{ij}(\vec{k})e^{-\imath k\cdot(y-x)}, \\ \langle \imath\xi^b_y\imath J^a_{ix}\rangle = \delta^{ba}\int đk\frac{k_i}{\vec{k}^2}e^{-\imath k\cdot(y-x)}, \qquad \langle \imath\rho^b_y\imath J^a_{ix}\rangle = 0,\end{gathered} \tag{2.31}$$

where $t_{ji}(\vec{k}) = \delta_{ji} - k_jk_i/\vec{k}^2$ is the transverse projector in momentum space. These relations encode the transverse nature of the vector gluon field. Turning to Eq. (2.26), we recognize that if we functionally differentiate with respect to $\imath\eta^d_y$, again with respect to $\imath\xi^e_z$ and set sources to zero, we get that

$$\langle \imath\xi^e_z\imath\xi^d_y\rangle = 0. \tag{2.32}$$

In effect, the auxiliary Lagrange multiplier field λ drops out of the formalism to be replaced by the transversality conditions, as it is supposed to.

III. FEYNMAN RULES AND DECOMPOSITIONS

Let us now discuss the Feynman rules and general decompositions of Green's functions that will be relevant to this work. The Feynman rules for the propagators can be derived from the field equations of motion (written in the appendix) by neglecting the interaction terms and functionally differentiating. Denoting the tree-level quantities with a superscript (0), the corresponding equations read:

$$\begin{aligned}J^a_{ix} &= [\delta_{ij}\partial^2_{0x} - \delta_{ij}\nabla^2_x + \nabla_{ix}\nabla_{jx}]\langle \imath J^a_{jx}\rangle^{(0)} + \partial_{0x}\nabla_{ix}\langle \imath\rho^a_x\rangle^{(0)} \\ &\quad - \nabla_{ix}\langle \imath\xi^a_x\rangle^{(0)}, \\ \rho^a_x &= \partial_{0x}\nabla_{ix}\langle \imath J^a_{ix}\rangle^{(0)} + \nabla^2_x\langle \imath\rho^a_x\rangle^{(0)}, \\ \eta^a_x &= \nabla^2_x\langle \imath\bar{\eta}^a_x\rangle^{(0)}.\end{aligned} \tag{3.1}$$

The tree-level ghost propagator is then

$$\langle \imath\bar{\eta}^a_x\imath\eta^b_y\rangle^{(0)} = -\imath\delta^{ab}\int đk\frac{1}{\vec{k}^2}e^{-\imath k\cdot(y-x)} \tag{3.2}$$

and we identify the momentum space propagator as

$$W^{(0)ab}_c(k) = -\delta^{ab}\frac{\imath}{\vec{k}^2}. \tag{3.3}$$

The rest of the propagators follow a similar pattern and their momentum space forms (without the common color factor δ^{ab}) are given in Table I. Note that it is understood that the denominator factors involving both temporal and spatial components implicitly carry the relevant Feynman prescription, i.e.,

$$\frac{1}{(k_0^2-\vec{k}^2)} \to \frac{1}{(k_0^2-\vec{k}^2+\imath 0_+)}, \tag{3.4}$$

such that the integration over the temporal component can be analytically continued to Euclidean space. It is also useful to repeat this analysis for the proper two-point functions and using the tree-level components of Eqs. (2.21), (2.22), and (2.24) we have

$$\begin{aligned}\langle \imath A^a_{ix}\rangle^{(0)} &= -[\delta_{ij}\partial^2_{0x} - \delta_{ij}\nabla^2_x + \nabla_{ix}\nabla_{jx}]A^a_{jx} - \partial_{0x}\nabla_{ix}\sigma^a_x \\ &\quad + \nabla_{ix}\lambda^a_x, \\ \langle \imath\sigma^a_x\rangle^{(0)} &= -\partial_{0x}\nabla_{ix}A^a_{ix} - \nabla^2_x\sigma^a_x, \\ \langle \imath\bar{c}^a_x\rangle^{(0)} &= -\nabla^2_x c^a_x.\end{aligned} \tag{3.5}$$

The ghost proper two-point function in momentum space is

$$\Gamma^{(0)ab}_c(k) = \delta^{ab}\imath\vec{k}^2 \tag{3.6}$$

TABLE I. Tree-level propagators [top] and two-point proper functions [bottom] (without color factors) in momentum space. Underlined entries denote exact results.

$W^{(0)}$	A_j	σ	λ
A_i	$t_{ij}(\vec{k})\frac{\imath}{(k_0^2-\vec{k}^2)}$	$\underline{0}$	$\underline{\frac{(-k_i)}{\vec{k}^2}}$
σ	$\underline{0}$	$\frac{\imath}{\vec{k}^2}$	$\frac{(-k^0)}{\vec{k}^2}$
λ	$\underline{\frac{k_j}{\vec{k}^2}}$	$\frac{k^0}{\vec{k}^2}$	$\underline{0}$

$\Gamma^{(0)}$	A_j	σ	λ
A_i	$-\imath k_0^2\delta_{ij} + \imath\vec{k}^2 t_{ij}(\vec{k})$	$\imath k^0 k_i$	$\underline{k_i}$
σ	$\imath k^0 k_j$	$-\imath\vec{k}^2$	$\underline{0}$
λ	$\underline{-k_j}$	$\underline{0}$	$\underline{0}$

and the rest are presented (without color factors) in Table I. It is immediately apparent that the gluon polarization is *not* transverse in contrast to Landau gauge.

The tree-level vertices are determined by taking the various interaction terms of Eqs. (2.21), (2.22), (2.23), and (2.24) and functionally differentiating. Since, in this study, we are interested only in the eventual one-loop perturbative results we omit the tree-level four-point functions (Γ_{4A} and $\Gamma_{AA\sigma\sigma}$). Defining all momenta as incoming, we have:

$$\begin{aligned}\Gamma^{(0)abc}_{\sigma AAjk}(p_a, p_b, p_c) &= \imath g f^{abc}\delta_{jk}(p_b^0 - p_c^0),\\ \Gamma^{(0)abc}_{\sigma A\sigma j}(p_a, p_b, p_c) &= -\imath g f^{abc}(p_a - p_c)_j,\\ \Gamma^{(0)abc}_{3Aijk}(p_a, p_b, p_c) &= -\imath g f^{abc}[\delta_{ij}(p_a - p_b)_k\\ &\quad + \delta_{jk}(p_b - p_c)_i + \delta_{ki}(p_c - p_a)_j],\\ \Gamma^{(0)abc}_{\bar{c}cAi}(p_{\bar{c}}, p_c, p_A) &= -\imath g f^{abc} p_{\bar{c}i}.\end{aligned} \tag{3.7}$$

In addition to the tree-level expressions for the various two-point functions (connected and proper) it is necessary to consider their general nonperturbative structures. These structures are determined by considering the properties of the fields under the discrete transforms of time-reversal and parity (the noncovariant analogue of Lorentz invariance arguments for covariant gauges). Using the same techniques as in Ref. [7] we can easily write down the results in momentum space. For the ghost, we have

$$W_c^{ab}(k) = -\delta^{ab}\frac{\imath}{\vec{k}^2}D_c(\vec{k}^2), \qquad \Gamma_c^{ab}(k) = \delta^{ab}\imath\vec{k}^2\Gamma_c(\vec{k}^2) \tag{3.8}$$

and the rest are presented in Table II. With the exception of the ghost, all dressing functions are scalar functions of *two* independent variables, k_0^2 and $\vec{k}^2$. The ghost dressing functions are functions of $\vec{k}^2$ only for exactly the same reasons as in the first order formalism [7]. At tree-level, all dressing functions are unity.

The dressing functions for the propagators and two-point proper functions are related via the Legendre transform. The connection follows from

$$\begin{aligned}\frac{\delta \imath J_\beta}{\delta \imath J_\alpha} = \delta_{\alpha\beta} = -\imath\frac{\delta}{\delta \imath J_\alpha}\langle \imath\Phi_\beta\rangle = \frac{\delta\Phi_\gamma}{\delta \imath J_\alpha}\langle \imath\Phi_\gamma \imath\Phi_\beta\rangle\\ = \langle \imath J_\alpha \imath J_\gamma\rangle\langle \imath\Phi_\gamma \imath\Phi_\beta\rangle.\end{aligned} \tag{3.9}$$

TABLE II. General form of propagators [top] and two-point proper functions [bottom] (without color factors) in momentum space. All dressing functions are functions of k_0^2 and $\vec{k}^2$.

W	A_j	σ	λ
A_i	$t_{ij}(\vec{k})\frac{\imath}{(k_0^2-\vec{k}^2)}D_{AA}$	0	$\frac{(-k_i)}{\vec{k}^2}$
σ	0	$\frac{\imath}{\vec{k}^2}D_{\sigma\sigma}$	$\frac{(-k^0)}{\vec{k}^2}D_{\sigma\lambda}$
λ	$\frac{k_j}{\vec{k}^2}$	$\frac{k^0}{\vec{k}^2}D_{\sigma\lambda}$	0

Γ	A_j	σ	λ
A_i	$-\imath(k_0^2-\vec{k}^2)t_{ij}(\vec{k})\Gamma_{AA} - \imath k_0^2\frac{k_ik_j}{\vec{k}^2}\bar{\Gamma}_{AA}$	$\imath k^0 k_i\Gamma_{A\sigma}$	k_i
σ	$\imath k^0 k_j \Gamma_{A\sigma}$	$-\imath\vec{k}^2\Gamma_{\sigma\sigma}$	0
λ	$-k_j$	0	0

(Recall here that there is an implicit summation over all discrete indices and integration over continuous variables labeled by γ.) Considering all the possibilities in turn, we find that

$$\begin{aligned}D_{AA} = \Gamma_{AA}^{-1}, \qquad D_{\sigma\sigma} = \Gamma_{\sigma\sigma}^{-1}, \qquad D_c = \Gamma_c^{-1},\\ D_{\sigma\lambda} = \Gamma_{A\sigma}\Gamma_{\sigma\sigma}^{-1} = \bar{\Gamma}_{AA}\Gamma_{A\sigma}^{-1}.\end{aligned} \tag{3.10}$$

Actually, while we have included $D_{\sigma\lambda}$ up to this point, since there is no vertex involving the λ-field this propagator will not directly play any role in the formalism. However, indirectly it does turn out to have a meaning as will be shown in the next section.

IV. DYSON–SCHWINGER EQUATIONS AND SLAVNOV-TAYLOR IDENTITIES

With the observation that

$$\frac{\delta}{\delta \imath\Phi_\beta}\langle \imath J_\gamma \imath J_\alpha\rangle = -\langle \imath J_\gamma \imath J_\varepsilon\rangle\langle \imath\Phi_\varepsilon \imath\Phi_\beta \imath\Phi_\delta\rangle\langle \imath J_\delta \imath J_\alpha\rangle \tag{4.1}$$

[stemming from the Legendre transform and following from Eq. (3.9)], the derivation of the Dyson-Schwinger equations becomes relatively straightforward. Starting with Eq. (2.21), omitting the terms that will not contribute at one-loop perturbatively and recognizing the tree-level vertices in configuration space, we have that

$$\begin{aligned}\langle \imath A_{ix}^a\rangle &= \imath[\delta_{ij}\partial_{0x}^2 - \delta_{ij}\nabla_x^2 + \nabla_{ix}\nabla_{jx}]\imath A_{jx}^a + \imath\partial_{0x}\nabla_{ix}\imath\sigma_x^a - \imath\nabla_{ix}\imath\lambda_x^a - \int d^4y d^4z \Gamma^{(0)cab}_{\sigma AAij}(z,x,y)[\langle \imath J_{jy}^b \imath\rho_z^c\rangle - \imath A_{jy}^b \imath\sigma_z^c]\\ &\quad - \int d^4y d^4z \frac{1}{2!}\Gamma^{(0)cab}_{\sigma A\sigma i}(z,x,y)[\langle \imath\rho_y^b \imath\rho_z^c\rangle - \imath\sigma_y^b \imath\sigma_z^c] - \int d^4y d^4z \frac{1}{2!}\Gamma^{(0)abc}_{3Aijk}(x,y,z)[\langle \imath J_{jy}^b \imath J_{kz}^c\rangle - \imath A_{jy}^b \imath A_{kz}^c]\\ &\quad + \int d^4y d^4z \Gamma^{(0)bca}_{\bar{c}cAi}(y,z,x)[\langle \imath\bar{\eta}_z^c \eta_y^b\rangle + \imath c_z^c \imath c_y^b] + \ldots\end{aligned} \tag{4.2}$$

Taking the functional derivative with respect to $\imath A_{lw}^f$, using Eq. (4.1), setting sources to zero and Fourier transforming to momentum space (each step is straightforward so we omit the details for clarity) we get the Dyson-Schwinger equation for the gluon polarization:

$$\Gamma^{af}_{AAil}(k) = \delta^{af}[-\imath(k_0^2-\vec{k}^2)\delta_{il}-\imath k_ik_l] + \int \text{đ}\omega \Gamma^{(0)cab}_{\sigma AAij}(\omega-k,k,-\omega)W^{bd}_{AAjm}(\omega)\Gamma^{edf}_{\sigma AAml}(k-\omega,\omega,-k)W^{ec}_{\sigma\sigma}(\omega-k)$$
$$+\frac{1}{2!}\int \text{đ}\omega\Gamma^{(0)cab}_{\sigma A\sigma i}(\omega-k,k,-\omega)W^{bd}_{\sigma\sigma}(\omega)\Gamma^{dfe}_{\sigma A\sigma l}(\omega,-k,k-\omega)W^{ec}_{\sigma\sigma}(\omega-k)$$
$$+\frac{1}{2!}\int \text{đ}\omega\Gamma^{(0)abc}_{3Aijk}(k,-\omega,\omega-k)W^{bd}_{AAjm}(\omega)\Gamma^{dfe}_{3Amln}(\omega,-k,k-\omega)W^{ec}_{AAnk}(\omega-k)$$
$$-\int \text{đ}\omega\Gamma^{(0)bca}_{\bar{c}cAi}(\omega-k,-\omega,k)W^{cd}_{c}(\omega)\Gamma^{def}_{\bar{c}cAl}(\omega,k-\omega,-k)W^{eb}_{c}(\omega-k)+\ldots \quad (4.3)$$

Turning now to Eq. (2.22), we have

$$\langle \imath\sigma^a_x\rangle = \imath\partial_{0x}\nabla_{ix}\imath A^a_{ix} + \imath\nabla^2_x\imath\sigma^a_x - \int d^4yd^4z\frac{1}{2!}\Gamma^{(0)abc}_{\sigma AAjk}(x,y,z)[\langle \imath J^b_{jy}\imath J^c_{kz}\rangle - \imath A^b_{jy}\imath A^c_{kz}]$$
$$-\int d^4yd^4z\Gamma^{(0)abc}_{\sigma A\sigma j}(x,y,z)[\langle\imath J^b_{jy}\imath\rho^c_z\rangle - \imath A^b_{jy}\imath\sigma^c_z]+\ldots \quad (4.4)$$

where again, terms that do not contribute at the one-loop perturbative level are omitted. There are two functional derivatives of interest, those with respect to $\imath\sigma^f_w$ and $\imath A^f_{lw}$, which give rise to the following two Dyson-Schwinger equations:

$$\Gamma^{af}_{\sigma\sigma}(k) = \delta^{af}(-\imath\vec{k}^2) + \frac{1}{2!}\int \text{đ}\omega\Gamma^{(0)abc}_{\sigma AAjk}(k,-\omega,\omega-k)W^{bd}_{AAjm}(\omega)\Gamma^{fde}_{\sigma AAmn}(-k,\omega,k-\omega)W^{ec}_{AAnk}(\omega-k)$$
$$+\int \text{đ}\omega\Gamma^{(0)abc}_{\sigma A\sigma j}(k,-\omega,\omega-k)W^{bd}_{AAjm}(\omega)\Gamma^{fde}_{\sigma A\sigma m}(-k,\omega,k-\omega)W^{ec}_{\sigma\sigma}(\omega-k)+\ldots \quad (4.5)$$

$$\Gamma^{af}_{\sigma Al}(k) = \delta^{af}\imath k_0k_l + \frac{1}{2!}\int \text{đ}\omega\Gamma^{(0)abc}_{\sigma AAjk}(k,-\omega,\omega-k)W^{bd}_{AAjm}(\omega)\Gamma^{dfe}_{3Amln}(\omega,-k,k-\omega)W^{ec}_{AAnk}(\omega-k)$$
$$+\int \text{đ}\omega\Gamma^{(0)abc}_{\sigma A\sigma j}(k,-\omega,\omega-k)W^{bd}_{AAjm}(\omega)\Gamma^{edf}_{\sigma AAml}(k-\omega,\omega,-k)W^{ec}_{\sigma\sigma}(\omega-k)+\ldots \quad (4.6)$$

Next we consider the ghost equation, Eq. (2.24), which can be written

$$\langle \imath\bar{c}^a_x\rangle = \imath\nabla^2_x\imath c^a_x + \int d^4yd^4z\Gamma^{(0)abc}_{\bar{c}cAi}(x,y,z)$$
$$\times[\langle\imath J^c_{iz}\imath\bar{\eta}^b_y\rangle - \imath A^c_{iz}\imath c^b_y]. \quad (4.7)$$

The ghost Dyson-Schwinger equation is subsequently

$$\Gamma^{af}_{c}(k) = \delta^{af}\imath\vec{k}^2 + \int \text{đ}\omega\Gamma^{(0)abc}_{\bar{c}cAi}(k,-\omega,\omega-k)W^{bd}_{c}(\omega)$$
$$\times\Gamma^{dfe}_{\bar{c}cAj}(\omega,-k,k-\omega)W^{ec}_{AAji}(\omega-k). \quad (4.8)$$

In addition to the Dyson-Schwinger equations, the Green's functions are constrained by Slavnov-Taylor identities. These are the functional derivatives of Eq. (2.27). Since Eq. (2.27) is Grassmann-valued, we must first functionally differentiate with respect to $\imath c^d_y$. We are not interested (here) in further ghost correlations, so we can then set ghost sources to zero. Also, there is no further information to be gained by considering the Lagrange multiplier field λ^a, and we set its source to zero also. Equation (2.27) then becomes

$$\frac{\imath}{g}\partial_{0y}\langle\imath\sigma^d_y\rangle - f^{abd}\langle\imath\sigma^a_y\rangle\imath\sigma^b_y - f^{abd}\imath A^b_{jy}t_{ji}(y)\langle\imath A^a_{iy}\rangle$$
$$=\int d^4x\bigg\{-f^{abc}\langle\imath\sigma^a_x\rangle\frac{\delta}{\delta\imath c^d_y}\langle\imath\rho^b_x\imath\bar{\eta}^c_x\rangle$$
$$+\frac{1}{g}\bigg[\frac{\nabla_{ix}}{(-\nabla^2_x)}\langle\imath A^a_{ix}\rangle\bigg]\langle\imath\bar{c}^a_x\imath c^d_y\rangle$$
$$-f^{abc}\langle\imath A^a_{ix}\rangle t_{ij}(x)\frac{\delta}{\delta\imath c^d_y}\langle\imath J^b_{jx}\imath\bar{\eta}^c_x\rangle\bigg\}. \quad (4.9)$$

Taking the functional derivatives of this with respect to $\imath\sigma^e_z$ or $\imath A^e_{kz}$ and setting all remaining sources to zero gives rise to the following two equations:

$$\frac{\imath}{g}\partial_{0y}\langle\imath\sigma^e_z\imath\sigma^d_y\rangle = \int d^4x\bigg\{\frac{1}{g}\bigg[\frac{\nabla_{ix}}{(-\nabla^2_x)}\langle\imath\sigma^e_z\imath A^a_{ix}\rangle\bigg]\langle\imath\bar{c}^a_x\imath c^d_y\rangle$$
$$-f^{abc}\langle\imath\sigma^e_z\imath\sigma^a_x\rangle\frac{\delta}{\delta\imath c^d_y}\langle\imath\rho^b_x\imath\bar{\eta}^c_x\rangle$$
$$-f^{abc}\langle\imath\sigma^e_z\imath A^a_{ix}\rangle t_{ij}(x)\frac{\delta}{\delta\imath c^d_y}\langle\imath J^b_{jx}\imath\bar{\eta}^c_x\rangle\bigg\}, \quad (4.10)$$

$$\frac{\imath}{g}\partial_{0y}\langle \imath A_{kz}^{e}\imath\sigma_y^d\rangle = \int d^4x\Big\{\frac{1}{g}\Big[\frac{\nabla_{ix}}{(-\nabla_x^2)}\langle \imath A_{kz}^e \imath A_{ix}^a\rangle\Big]\langle \imath\bar{c}_x^a\imath c_y^d\rangle - f^{abc}\langle \imath A_{kz}^e\imath\sigma_x^a\rangle\frac{\delta}{\delta\imath c_y^d}\langle \imath\rho_x^b\imath\bar{\eta}_x^c\rangle - f^{abc}\langle \imath A_{kz}^e\imath A_{ix}^a\rangle t_{ij}(x)\frac{\delta}{\delta\imath c_y^d}\langle \imath J_{jx}^b\imath\bar{\eta}_x^c\rangle\Big\}. \tag{4.11}$$

Now, using Eq. (4.1), we have that

$$f^{abc}\frac{\delta}{\delta\imath c_y^d}\langle \imath\rho_x^b\imath\bar{\eta}_x^c\rangle = -f^{abc}\langle \imath\bar{\eta}_x^c\imath\eta_\alpha\rangle\langle \imath\bar{c}_\alpha\imath c_y^d\imath\Phi_\gamma\rangle\langle \imath J_\gamma\imath\rho_x^b\rangle = \delta^{ad}\tilde{\Sigma}_{\sigma;\bar{c}c}(x,y). \tag{4.12}$$

Taking the Fourier transform

$$\tilde{\Sigma}_{\sigma;\bar{c}c}(x,y) = \int đk\tilde{\Sigma}_{\sigma;\bar{c}c}(k)e^{-\imath k\cdot(x-y)} \tag{4.13}$$

we get that

$$\tilde{\Sigma}_{\sigma;\bar{c}c}(k) = N_c\int đ\omega W_c(k-\omega)\Gamma_{\bar{c}c\gamma}(k-\omega,-k,\omega)\times W_{\gamma\sigma}(\omega). \tag{4.14}$$

Since the ghost Green's functions are independent of the ghost line's energy scale [7], after ω_0 has been integrated out, there is no external energy scale and

$$\tilde{\Sigma}_{\sigma;\bar{c}c}(k) = \tilde{\Sigma}_{\sigma;\bar{c}c}(\vec{k}). \tag{4.15}$$

However, under time-reversal the σ-field changes sign (such that the action remains invariant) which in momentum space means that under the transform $k_0 \to -k_0$, $\tilde{\Sigma}_{\sigma;\bar{c}c}(k)$ must change sign and so, given Eq. (4.15) we have the result that

$$\tilde{\Sigma}_{\sigma;\bar{c}c}(k) = 0. \tag{4.16}$$

In the case of the term

$$\delta^{af}\tilde{\Sigma}_{Aj;\bar{c}c}(x,y) = f^{abc}\frac{\delta}{\delta\imath c_y^d}\langle \imath J_{jx}^b\imath\bar{\eta}_x^c\rangle \tag{4.17}$$

we can see automatically that in momentum space, $\tilde{\Sigma}_{Aj;\bar{c}c}(k) \sim k_j$ and that the transverse projector that acts on it in Eqs. (4.10) and (4.11) will kill the term. We thus have

$$\frac{\imath}{g}\partial_{0y}\langle \imath\sigma_z^e\imath\sigma_y^d\rangle = \int d^4x\Big\{\frac{1}{g}\Big[\frac{\nabla_{ix}}{(-\nabla_x^2)}\langle \imath\sigma_z^e\imath A_{ix}^a\rangle\Big]\langle \imath\bar{c}_x^a\imath c_y^d\rangle\Big\}, \tag{4.18}$$

$$\frac{\imath}{g}\partial_{0y}\langle \imath A_{kz}^e\imath\sigma_y^d\rangle = \int d^4x\Big\{\frac{1}{g}\Big[\frac{\nabla_{ix}}{(-\nabla_x^2)}\langle \imath A_{kz}^e\imath A_{ix}^a\rangle\Big]\langle \imath\bar{c}_x^a\imath c_y^d\rangle\Big\}, \tag{4.19}$$

which in terms of the momentum space dressing functions gives

$$\Gamma_{\sigma\sigma}(k_0^2,\vec{k}^2) = \Gamma_{A\sigma}(k_0^2,\vec{k}^2)\Gamma_c(\vec{k}^2), \tag{4.20}$$

$$\Gamma_{A\sigma}(k_0^2,\vec{k}^2) = \bar{\Gamma}_{AA}(k_0^2,\vec{k}^2)\Gamma_c(\vec{k}^2). \tag{4.21}$$

The Slavnov-Taylor identities for the two-point functions above are rather revealing. They are the Coulomb gauge equivalent of the standard covariant gauge result that the longitudinal part of the gluon polarization remains bare [11]. We notice that they relate the temporal, longitudinal and ghost degrees of freedom in a manner reminiscent of the quartet mechanism in the Kugo-Ojima confinement criterion [12]. Also, they represent Gauß' law as applied to the Green's functions. Equation (4.9) suggests that proper functions involving the temporal σ-field can be systematically eliminated and replaced by functions involving the vector $\vec{A}$ and ghost fields although whether this is desirable remains to be seen.

We can now return to the general decompositions of the two-point functions. We see that as a consequence of either of the two Slavnov-Taylor identities above, Eqs. (4.20) or (4.21), Eq. (3.10) reduces to $D_{\sigma\lambda} = D_c$, reassuring us that at least the formalism is consistent. We also see that there are only three independent two-point dressing functions, whereas (accounting for the tensor structure of the gluon polarization) we have five Dyson-Schwinger equations. We will investigate this perturbatively in the next section.

V. ONE-LOOP PERTURBATION THEORY

Let us now consider the one-loop perturbative form of the two-point dressing functions that are derived from the Dyson-Schwinger equations. So far, all quantities are expressed in Minkowski space. The perturbative integrals must however be evaluated in Euclidean space. The analytic continuation to Euclidean space ($k_0 \to \imath k_4$) is straightforward given the Feynman prescription for denominator factors. Henceforth, all dressing functions will be written in Euclidean space and are functions of k_4^2 and $\vec{k}^2$. The Euclidean four momentum squared is $k^2 = k_4^2 + \vec{k}^2$. We write the perturbative expansion of the two-point dressing functions as follows:

$$\Gamma_{\alpha\beta} = 1 + g^2\Gamma_{\alpha\beta}^{(1)}. \tag{5.1}$$

The loop integrals will be dimensionally regularized with the (Euclidean space) integration measure

$$đ\omega = \frac{d\omega_4 d^d\vec{\omega}}{(2\pi)^{d+1}} \tag{5.2}$$

(spatial dimension $d = 3 - 2\varepsilon$). The coupling acquires a dimension:

$$g^2 \to g^2\mu^\varepsilon, \tag{5.3}$$

where μ is the square of some nonvanishing mass scale squared. This factor is included in $\Gamma^{(1)}_{\alpha\beta}$ such that the new coupling and $\Gamma^{(1)}$ are dimensionless. By inserting the appropriate tree-level factors into the Dyson-Schwinger equations, extracting the color and tensor algebra we get the following integral expressions for the various two-point proper dressing functions:

$$(d-1)\Gamma^{(1)}_{AA}(k_4^2,\vec{k}^2) = -N_c\int\frac{\mu^\varepsilon \bar{d}\omega(k_4+\omega_4)^2}{k^2\omega^2(\vec{k}-\vec{\omega})^2}t_{ij}(\vec{\omega})t_{ji}(\vec{k}) - N_c\int\frac{\mu^\varepsilon \bar{d}\omega}{k^2\vec{\omega}^2(\vec{k}-\vec{\omega})^2}\omega_i\omega_j t_{ji}(\vec{k}) - 2N_c\int\frac{\mu^\varepsilon \bar{d}\omega}{k^2\omega^2(k-\omega)^2}t_{li}(\vec{k})t_{jm}(\vec{\omega})t_{nk}(\vec{k}-\vec{\omega})[\delta_{ij}k_k-\delta_{jk}\omega_i-\delta_{ki}k_j][\delta_{ml}k_n-\delta_{ln}k_m-\delta_{nm}\omega_l], \tag{5.4}$$

$$\bar{\Gamma}^{(1)}_{AA}(k_4^2,\vec{k}^2) = -N_c\int\frac{\mu^\varepsilon \bar{d}\omega(k_4+\omega_4)^2}{k_4^2\vec{k}^2\omega^2(\vec{k}-\vec{\omega})^2}k_ik_jt_{ij}(\vec{\omega}) - N_c\int\frac{\mu^\varepsilon \bar{d}\omega}{k_4^2\vec{k}^2\vec{\omega}^2(\vec{k}-\vec{\omega})^2}\left[\frac{1}{2}\vec{k}\cdot(2\vec{\omega}-\vec{k})^2-\vec{k}\cdot\vec{\omega}\,\vec{k}\cdot(\vec{\omega}-\vec{k})\right] - \frac{1}{2}N_c\int\frac{\mu^\varepsilon \bar{d}\omega\vec{k}\cdot(\vec{k}-2\vec{\omega})^2}{k_4^2\vec{k}^2\omega^2(\vec{k}-\vec{\omega})^2}t_{ij}(\vec{\omega})t_{ji}(\vec{k}-\vec{\omega}), \tag{5.5}$$

$$\Gamma^{(1)}_{\sigma\sigma}(k_4^2,\vec{k}^2) = -\frac{1}{2}N_c\int\frac{\mu^\varepsilon \bar{d}\omega(k_4-2\omega_4)^2}{\vec{k}^2\omega^2(k-\omega)^2}t_{ij}(\vec{\omega})t_{ji}(\vec{k}-\vec{\omega}) - 4N_c\int\frac{\mu^\varepsilon \bar{d}\omega}{\vec{k}^2\omega^2(\vec{k}-\vec{\omega})^2}k_ik_jt_{ij}(\vec{\omega}), \tag{5.6}$$

$$\Gamma^{(1)}_{A\sigma}(k_4^2,\vec{k}^2) = \frac{1}{2}N_c\int\frac{\mu^\varepsilon \bar{d}\omega(k_4-2\omega_4)}{k_4\vec{k}^2\omega^2(k-\omega)^2}\vec{k}\cdot(\vec{k}-2\vec{\omega})t_{ij}(\vec{\omega})t_{ji}(\vec{k}-\vec{\omega}) - 2N_c\int\frac{\mu^\varepsilon \bar{d}\omega}{\vec{k}^2\omega^2(\vec{k}-\vec{\omega})^2}k_ik_jt_{ij}(\vec{\omega}), \tag{5.7}$$

$$\Gamma^{(1)}_{c}(\vec{k}^2) = -N_c\int\frac{\mu^\varepsilon \bar{d}\omega}{\vec{k}^2\omega^2(\vec{k}-\vec{\omega})^2}k_ik_jt_{ij}(\vec{\omega}). \tag{5.8}$$

At this stage, we are in a position to check the two Slavnov-Taylor identities for the two-point functions. The first of these, Eq. (4.20), reads at one-loop:

$$\Gamma^{(1)}_{\sigma\sigma} - \Gamma^{(1)}_{A\sigma} - \Gamma^{(1)}_{c} = 0. \tag{5.9}$$

Inserting the integral expressions above and eliminating overall constants, the left-hand side reads

$$\Gamma^{(1)}_{\sigma\sigma} - \Gamma^{(1)}_{A\sigma} - \Gamma^{(1)}_{c} \sim -\frac{1}{2}\int\frac{\bar{d}\omega(k_4-2\omega_4)}{k_4\vec{k}^2\omega^2(k-\omega)^2}\times k\cdot(k-2\omega)t_{ij}(\vec{\omega})t_{ji}(\vec{k}-\vec{\omega}) - \int\frac{\bar{d}\omega}{\vec{k}^2\omega^2(\vec{k}-\vec{\omega})^2}k_ik_jt_{ij}(\vec{\omega}). \tag{5.10}$$

By expanding the transverse projectors and scalar products, it is relatively trivial to show that this does indeed vanish. The second identity, Eq. (4.21), reads

$$\Gamma^{(1)}_{A\sigma} - \bar{\Gamma}^{(1)}_{AA} - \Gamma^{(1)}_{c} = 0 \tag{5.11}$$

and the left-hand side is:

$$\Gamma^{(1)}_{A\sigma} - \bar{\Gamma}^{(1)}_{AA} - \Gamma^{(1)}_{c} \sim \frac{1}{2}\int\frac{\bar{d}\omega\vec{k}\cdot(\vec{k}-2\vec{\omega})}{\omega^2(k-\omega)^2}k\cdot(k-2\omega)t_{ij}(\vec{\omega})t_{ji}(\vec{k}-\vec{\omega}) + \int\frac{\bar{d}\omega(\omega_4^2+2k_4\omega_4)}{\omega^2(\vec{k}-\vec{\omega})^2}k_ik_jt_{ij}(\vec{\omega}) + \int\frac{\bar{d}\omega}{\vec{\omega}^2(\vec{k}-\vec{\omega})^2}\left[\frac{1}{2}\vec{k}\cdot(2\vec{\omega}-\vec{k})^2-\vec{k}\cdot\vec{\omega}\,\vec{k}\cdot(\vec{\omega}-\vec{k})\right]. \tag{5.12}$$

Again, it is straightforward to show that this vanishes. Thus, we have reproduced the Slavnov-Taylor identity results that tell us that there are only three independent two-point dressing functions.

The evaluation of the integrals that give Γ_{AA}, $\Gamma_{\sigma\sigma}$, and Γ_c is far from trivial. However, using the techniques developed in [8] it is possible. For brevity, we do not go into the details here and simply quote the results. They are, as $\varepsilon\to 0$:

$$\Gamma_{AA}^{(1)}(x,y)=\frac{N_c}{(4\pi)^{2-\varepsilon}}\left\{-\left[\frac{1}{\varepsilon}-\gamma-\ln\left(\frac{x+y}{\mu}\right)\right]+\frac{64}{9}-3z+g(z)\left[\frac{1}{2z}-\frac{14}{3}+\frac{3}{2}z\right]-\frac{f(z)}{4}\left[\frac{1}{z}-1+11z-3z^2\right]\right\},$$
$$\Gamma_{\sigma\sigma}^{(1)}(x,y)=\frac{N_c}{(4\pi)^{2-\varepsilon}}\left\{-\frac{11}{3}\left[\frac{1}{\varepsilon}-\gamma-\ln\left(\frac{x+y}{\mu}\right)\right]-\frac{31}{9}+6z+g(z)(1-3z)-f(z)\left[\frac{1}{2}+2z+\frac{3}{2}z^2\right]\right\}, \quad (5.13)$$
$$\Gamma_{c}^{(1)}(y)=\frac{N_c}{(4\pi)^{2-\varepsilon}}\left\{-\frac{4}{3}\left[\frac{1}{\varepsilon}-\gamma-\ln\left(\frac{y}{\mu}\right)\right]-\frac{28}{9}+\frac{8}{3}\ln 2\right\},$$

where $x=k_4^2$, $y=\vec{k}^2$, $z=x/y$ and we define two functions:

$$f(z)=4\ln 2\frac{1}{\sqrt{z}}\arctan\sqrt{z}-\int_0^1\frac{dt}{\sqrt{t}(1+zt)}\ln(1+zt),$$
$$g(z)=2\ln 2-\ln(1+z). \quad (5.14)$$

(The integral occurring in $f(z)$ can be explicitly evaluated in terms of dilogarithms [8].) Defining a similar notation for the perturbative expansion of the propagator functions:

$$D_{\alpha\beta}=1+g^2D_{\alpha\beta}^{(1)} \quad (5.15)$$

we then have, via Eq. (3.10), the final results:

$$D_{AA}^{(1)}(x,y)=-\Gamma_{AA}^{(1)}(x,y), \qquad D_{\sigma\sigma}^{(1)}(x,y)=-\Gamma_{\sigma\sigma}^{(1)}(x,y),$$
$$D_c^{(1)}(y)=-\Gamma_c^{(1)}(y). \quad (5.16)$$

Several comments are in order here. First, the expressions for Γ_{AA} and $\bar{\Gamma}_{AA}$, Eqs. (5.4) and (5.5), respectively, contain energy divergent integrals of the form

$$\int\frac{\,{\mathchar'26\mkern-12mu d}\omega\{1,\omega_i,\omega_i\omega_j\}}{\vec{\omega}^2(\vec{k}-\vec{\omega})^2}. \quad (5.17)$$

These integrals cancel explicitly, though it should be remarked that this cancellation is more obvious in the first order formalism [8]. Second, with respect to the temporal variable x, all the results above are strictly finite for Euclidean and spacelike Minkowski momenta—any singularities occur for $z=x/y=-1$ (the light-cone) with branch cuts extending in the timelike direction. This means that the analytic continuation between Euclidean and Minkowski space can be justified. Thirdly, the coefficient of the ε-pole for $D_{\sigma\sigma}$ and the combination $D_{AA}D_c^2$ is $11N_c/3(4\pi)^2$ which is minus the value of the first coefficient of the β-function. This confirms that $g^2D_{\sigma\sigma}$ [13] and $g^2D_{AA}D_c^2$ (the Coulomb gauge analogue of the Landau gauge nonperturbative running coupling) are renormalization group invariants at this order in perturbation theory. Fourthly, the results above for D_{AA}, $D_{\sigma\sigma}$ and D_c are identical to those calculated within the first order formalism [8].

VI. SUMMARY AND OUTLOOK

The two-point functions (connected and proper) of Coulomb gauge Yang-Mills theory have been considered within the standard, second order formalism. Functional methods have been used to derive the relevant Dyson-Schwinger equations and Slavnov-Taylor identities. One-loop perturbative results have been presented and the Slavnov-Taylor identities that concern them verified.

Suffice it to say that it is tautological for the situation in Coulomb gauge to be somewhat different from covariant gauges such as Landau gauge. The proper $\vec{A}$-$\vec{A}$ two-point function is explicitly not transverse, nor does its longitudinal component remain bare beyond tree-level. This longitudinal component can however be written in terms of the temporal gluon and ghost two-point functions via the Slavnov-Taylor identities. Indeed, the Slavnov-Taylor identities show that there are only three independent two-point dressing functions: the (transverse) spatial gluon propagator dressing function (D_{AA}), the temporal gluon propagator dressing function ($D_{\sigma\sigma}$) and the ghost propagator dressing function (D_c). With the exception of the ghost dressing function, all are noncovariantly expressed in terms of two variables: k_4^2 (or k_0^2 in Minkowski space) and $\vec{k}^2$. Perturbatively it is seen that the analytic continuation between Euclidean and Minkowski space (and vice versa) is valid and that the Slavnov-Taylor identities hold.

It is worthwhile to discuss some of the differences and similarities between the (second order formalism) results presented here and the previous, first order formalism results of Ref. [8]. Recall that the first order formalism is constructed from the second order formalism by effectively replacing the terms in the action that are quadratic in the σ-field with terms linear in σ, but at the expense of introducing two new fields: the transverse $\vec{\pi}$-field and the scalar ϕ-field [1,7]. It should be emphasized that this is a technical procedure which does not alter the physical content of the theory, or the gauge. The various terms in the action correspond to the tree-level Green's functions of the theory—the fact that the two formalisms use different fields merely reflects a different decomposition of the available field degrees of freedom while the resultant physical degrees of freedom will be the same. Now, the gauge transform is the same for both formalisms (the gauge transform of the additional fields within the first order formalism is fully determined from the behavior of the $\vec{E}$-field in the second order formalism [7]) and the Slavnov-Taylor identities will have the same form and physical content, albeit that the various Green's functions are differently decomposed. Also, the form of the loop integrals in the Dyson-Schwinger equations will not change since their structure is determined by the Legendre transform. There will however be different terms in the Dyson-Schwinger equations since there is one term in each equation for each of the relevant interaction terms of the action—thus, for example, loops involving the $\Gamma_{AA\sigma\sigma}$ tree-level vertex appear in the second order formalism (although we have not explicitly included them in this study because they first contribute at two-loops perturbatively) while loops involving the $\Gamma_{\pi A\sigma}$ vertex appear in the first order formalism. The different loop integrals of the two formalisms again correspond simply to the different decompositions of the field degrees of freedom. If one were to include quarks into the theory, then since the quark propagator and vertices do not change at tree-level (and hence leading order perturbatively) between the formalisms, it would be expected that the temporal and spatial gluon propagators that contribute perturbatively at one-loop order to physical processes are identical in both formalisms. This is explicitly observed to be the case and thus, while the first and second order formalisms do rearrange the field degrees of freedom, the physical degrees of freedom turn out to be the same. In fact, this argument also applies to the ghost Green's functions—the tree-level propagator and vertex is unaltered and the one-loop (leading order) ghost propagator is identical in both formalisms.

There are many further questions to be addressed. The perturbative structure of the vertex functions, the addition of the quark sector and the construction of physical scattering matrix elements from noncovariant components are all important next steps. The issue of noncovariant renormalization prescriptions must also be understood. The connection of the functional formalism with other approaches such as the Hamiltonian formalism [3] and lattice calculations must also be established. Aside from the technical issues of noncovariance, it remains an important goal to understand the differences between Coulomb gauge and Landau gauge—the primary aim being to understand more about the physical mechanism of confinement. As we learn more about the Coulomb gauge (especially nonperturbatively), such discussions will surely enhance our knowledge of the theory of the strong interaction considerably. Clearly, there is a lot of work yet to be done.

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APPENDIX: EXPLICIT FORM OF THE FIELD EQUATIONS OF MOTION

For completeness, we write the explicit form of the various field equations of motion represented by Eq. (2.15):

$$\begin{aligned} J_{ix}^a Z[J] = \int \mathcal{D}\Phi \exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\}\Big\{&[\delta_{ij}\partial_{0x}^2 - \delta_{ij}\nabla_x^2 + \nabla_{ix}\nabla_{jx}]A_{jx}^a + \partial_{0x}\nabla_{ix}\sigma_x^a - \nabla_{ix}\lambda_x^a + gf^{fac}(\nabla_{ix}\bar{c}_x^c)c_x^c \\ &- gf^{fbc}[\delta^{af}\partial_{0x}A_{ix}^b\sigma_x^c - \delta^{ab}\sigma_x^c\nabla_{ix}\sigma_x^f + \delta^{ab}A_{jx}^c\nabla_{ix}A_{jx}^f + 2\delta^{ac}A_{jx}^b\nabla_{jx}A_{ix}^f - \delta^{af}A_{ix}^c\nabla_{jx}A_{jx}^b] \\ &- g^2 f^{fac}f^{fde}\sigma_x^c A_{ix}^d\sigma_x^e + \frac{1}{4}g^2 f^{fbc}f^{fde}[\delta^{ab}A_{jx}^c A_{ix}^d A_{jx}^e + A_{jx}^b\delta^{ac}A_{jx}^d A_{ix}^e + A_{ix}^b A_{jx}^c\delta^{ad}A_{jx}^e + A_{jx}^b A_{ix}^c A_{jx}^d\delta^{ae}]\Big\}, \end{aligned} \quad (A1)$$

$$\begin{aligned} \rho_x^a Z[J] = \int \mathcal{D}\Phi \exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\}\{&\partial_{0x}\nabla_{ix}A_{ix}^a + \nabla_x^2\sigma_x^a - g^2 f^{fba}f^{fde}A_{ix}^b A_{ix}^d\sigma_x^e \\ &- gf^{fbc}[-\delta^{ac}A_{ix}^b\partial_{0x}A_{ix}^f - \delta^{ac}A_{ix}^b\nabla_{ix}\sigma_x^f + \delta^{af}\nabla_{ix}A_{ix}^b\sigma_x^c]\}, \end{aligned} \quad (A2)$$

$$\xi_x^a Z[J] = \int \mathcal{D}\Phi \exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\}\{\nabla_{ix}A_{ix}^a\}, \quad (A3)$$

$$\eta_x^a Z[J] = \int \mathcal{D}\Phi \exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\}\{\nabla_x^2 c_x^a - gf^{abc}\nabla_{ix}A_{ix}^b c_x^c\}. \quad (A4)$$

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