

S matrix on the Moyal plane: Locality versus Lorentz invarianceA. P. Balachandran,^{1,*} A. Pinzul,^{2,†} B. A. Qureshi,^{1,‡} and S. Vaidya^{3,§}¹*Department of Physics, Syracuse University, Syracuse New York, 13244-1130, USA*²*Instituto de Física, Universidade de São Paulo, C. P. 66318, São Paulo, SP, 05315-970, Brazil*³*Centre for High Energy Physics Indian Institute of Science, Bangalore, 560012, India*

(Received 17 October 2007; published 22 January 2008)

Twisted quantum field theories on the Groenewold-Moyal plane are known to be nonlocal. Despite this nonlocality, it is possible to define a generalized notion of causality. We show that interacting quantum field theories that involve only couplings between matter fields, or between matter fields and minimally coupled $U(1)$ gauge fields are causal in this sense. On the other hand, interactions between matter fields and non-Abelian gauge fields violate this generalized causality. We derive the modified Feynman rules emergent from these features. They imply that interactions of matter with non-Abelian gauge fields are not Lorentz- and CPT -invariant.

DOI: [10.1103/PhysRevD.77.025020](https://doi.org/10.1103/PhysRevD.77.025020)

PACS numbers: 11.10.Nx, 11.30.Cp, 11.10.Lm

I. INTRODUCTION

Quantum field theories on the Groenewold-Moyal (GM) plane can be made Poincaré covariant, provided their statistics are twisted along with the coproduct on the Poincaré group [1,2]. It is also possible to write interacting quantum field theories including gauge theories, and discuss scattering amplitudes. Such models are unitary as long as the interaction Hamiltonian is Hermitian.

However, twisted quantum fields are also nonlocal [2]. Naively, this might suggest that the scattering matrix for these theories cannot be Lorentz-invariant. In this article, we will show that for a large class of noncommutative field theories, the S -matrix is indeed Lorentz-invariant because of the presence of a weakened form of locality. (The connection between locality and Lorentz-invariance of the S -matrix for noncommutative theories has also been noticed by [3]) We will also show that noncommutative non-Abelian gauge theories with matter field interactions violate even this weakened notion of locality, as a result of which the S -matrix in these theories is not Lorentz invariant (They also violate CPT [4]).

It is not difficult to understand the origin of such non-invariance. The density H_I of the interaction Hamiltonian is not a local field when $\theta^{\mu\nu} \neq 0$ in the sense that

$$[H_I(x), H_I(y)] \neq 0, \quad x \sim y \quad (1.1)$$

where $x \sim y$ means that x and y are spacelike separated. But S involves time-ordered products of H_I and the equality sign in (1.1) is used to prove its Lorentz invariance already when $\theta^{\mu\nu} = 0$. This condition on H_I , known as Bogoliubov causality [5], has been reviewed and refined by Weinberg [6,7]. For $\theta^{\mu\nu} \neq 0$, a certain generalization of this condition is sufficient for Lorentz invariance. It is fulfilled in the absence of non-Abelian gauge fields, but

is violated in the presence of the latter if nonsinglet matter fields are also present. The nonperturbative Lehmann-Symanzik-Zimmermann formalism [7] also leads to the time-ordered product of relatively nonlocal fields and is not compatible with Lorentz invariance for $\theta^{\mu\nu} \neq 0$ and matter-non-Abelian gauge field interactions. Such a breakdown of Lorentz invariance is very controlled and may provide unique signals for noncommutative spacetimes, a point which requires further study.

In Sec. II, we show that these noncommutative theories without gauge interactions obey a weaker form of the condition (1.1). Consequently, the S -matrix of such theories is Lorentz-invariant. In Sec. III, we remark that this feature is maintained in the presence of just abelian gauge fields. Next we discuss noncommutative non-Abelian gauge theories with nonsinglet matter fields, and show that we lose even this generalized notion of locality. As a result, the Lorentz invariance of the S -matrix is lost at the quantum level.

As an application of these ideas, we will derive the Feynman rules for noncommutative QCD (as a specific example) and identify specific diagrams that violate Lorentz invariance in Secs. III and IV. The Pauli principle is not violated by the S -matrix for scattering of particles of definite momenta, as we also discuss.

The phenomenology of such Lorentz and CPT violations remains to be studied.

II. LOCALITY AND LORENTZ INVARIANCE

For the purposes of our discussion, locality (causality) will have the meaning it takes in standard local quantum field theories. Thus if $\rho(\xi)$ is an observable local field ρ like the electric charge density localized at a spacetime point ξ , and x and y are spacelike separated points ($x \sim y$), then causality (locality) states that

$$[\rho(x), \rho(y)] = 0. \quad (2.1)$$

It means that $\rho(x)$ and $\rho(y)$ are simultaneously measurable.

*bal@phy.syr.edu

†apinzul@fma.if.usp.br

‡bqureshi@phy.syr.edu

§vaidya@cts.iisc.ernet.in

Causal set theory (see for example [8] for a recent review) uses a sense of causality which differs from (2.1). There is also a criticism of the conceptual foundations of (2.1) by Sorkin [9].

Let H_I be the interaction Hamiltonian density in the interaction representation. The interaction representation S -matrix is

$$S = T \exp\left(-i \int d^N x H_I(x)\right). \quad (2.2)$$

For commutative spacetimes, Bogoliubov and Shirkov [5] long ago deduced from causality and relativistic invariance that H_I is a local field:

$$[H_I(x), H_I(y)] = 0, \quad x \sim y. \quad (2.3)$$

Later Weinberg [6,7] discussed the fundamental significance of (2.3) for these spacetimes: if (2.3) fails, then S is not relativistically invariant.

In these previous discussions, where $\theta^{\mu\nu} = 0$, H_I and their products were taken to transform in the standard way under Lorentz transformations Λ :

$$U(\Lambda)H_I(x) = H_I(\Lambda^{-1}x)U(\Lambda), \quad (2.4)$$

$$U(\Lambda)H_I(x)H_I(y) = H_I(\Lambda^{-1}x)H_I(\Lambda^{-1}y)U(\Lambda), \quad \text{etc.} \quad (2.5)$$

For $\theta^{\mu\nu} \neq 0$, the Lorentz transformation condition on H_I reduces to (2.4) in the first order term of (2.2), as our previous work shows [2], and as we explain later in this section.

However, we must use the twisted coproduct to transform tensor products of H_I . For this twisted coproduct as well, causality or rather a certain simple generalization of it, is essentially adequate to guarantee the Lorentz invariance of the S -matrix. The generalization allows for causality, but allows also for weaker possibilities. It is only ‘‘essentially’’ adequate: as Weinberg has shown [6], for a Lorentz-invariant S -matrix, there are also conditions on singularities supported at $x = y$ in the product $H_I(x)H_I(y)$.

Let us show these results.

(i) *Lorentz transformation law for the S -matrix*

The second order term in (2.2) is the leading term influenced by time-ordering. It is

$$S^{(2)} = \frac{(-i)^2}{2!} \int d^N x d^N y T(H_I(x)H_I(y)), \quad (2.6)$$

$$T(H_I(x)H_I(y)) = \theta(x_0 - y_0)H_I(x)H_I(y) + (x \leftrightarrow y). \quad (2.7)$$

Thus $S^{(2)}$ is the sum of two terms $S_1^{(2)}$ and $S_2^{(2)}$ corresponding to terms in (2.7):

$$S^{(2)} = S_1^{(2)} + S_2^{(2)}. \quad (2.8)$$

In terms of the Fourier transforms \tilde{H}_I of H_I ,

$$\tilde{H}_I(p) = \int \frac{d^N x}{(2\pi)^N} e^{ip \cdot x} H_I(x), \quad (2.9)$$

$S_1^{(2)}$ has the expression

$$S_1^{(2)} = -\frac{1}{2} \int \frac{d^N x}{(2\pi)^N} \frac{d^N y}{(2\pi)^N} \theta(x^0 - y^0) \times \int d^N k_1 d^N k_2 \tilde{H}_I(k_1) \tilde{H}_I(k_2) e_{k_1}(x) e_{k_2}(y), \quad (2.10)$$

where $e_k(x) := e^{-ik \cdot x}$.

Elsewhere [2], we worked out the twisted transformation of $e_{k_1} \otimes e_{k_2}$ under $U(\Lambda)$:

$$U(\Lambda)e_{k_1} \otimes e_{k_2} = e_{\Lambda k_1} \otimes e_{\Lambda k_2} e^{i/2 k_1 \cdot \delta_\Lambda \theta \cdot k_2} U(\Lambda_2), \quad (2.11)$$

$$\Lambda_2 = e^{-1/2(\Lambda k_1 + \Lambda k_2)_\mu \theta^{\mu\nu} \partial_\nu} \Lambda e^{1/2(k_1 + k_2)_\mu \theta^{\mu\nu} \partial_\nu}, \quad (2.12)$$

$$\delta_\Lambda \theta \equiv \Lambda^{-1} \theta \Lambda - \theta, \quad k_1 \cdot \delta_\Lambda \cdot k_2 \equiv k_{1\mu} (\delta_\Lambda \theta)^{\mu\nu} k_{2\nu}. \quad (2.13)$$

We can hence write

$$U(\Lambda)S_1^{(2)} = -\frac{1}{2} \int d^N k_1 d^N k_2 \tilde{H}_I(k_1) \tilde{H}_I(k_2) \int \frac{d^N x}{(2\pi)^N} \frac{d^N y}{(2\pi)^N} \times \theta(x^0 - y^0) e^{-(i/2)(\partial/\partial x^\mu)[\Lambda \delta_\Lambda \theta \Lambda^{-1}]^{\mu\nu}(\partial/\partial y^\nu)} \times \left(e_{\Lambda k_1}(x) e_{\Lambda k_2}(y) \left[e^{-(i/2)(\bar{\partial}/\partial x^\mu + (\bar{\partial}/\partial y^\mu))\theta^{\mu\nu} \partial_\nu} \times U(\Lambda) e^{(i/2)(\Lambda^{-1})^\sigma_\mu ((\bar{\partial}/\partial x^\sigma) + (\bar{\partial}/\partial y^\sigma))\theta^{\mu\nu} \partial_\nu} \right] \right) \quad (2.14)$$

where the derivatives do not act on $\theta(x^0 - y^0)$.

Now we note certain simple, but important facts:

(i) Since

$$\theta(x^0 - y^0) \left(\frac{\bar{\partial}}{\partial x} + \frac{\bar{\partial}}{\partial y} \right)_\mu = 0, \quad (2.15)$$

we can let $(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})_{\nu, \mu}$ to act on $\theta(x^0 - y^0)$ as well.

(ii) The expression $\frac{\partial}{\partial x^\mu} (\Lambda \delta_\Lambda \theta \Lambda^{-1})^{\mu\nu} \frac{\partial}{\partial y^\nu}$ gives zero when applied to $\theta(x^0 - y^0)$ because of the antisymmetry of $(\Lambda \delta_\Lambda \theta \Lambda^{-1})$:

$$\frac{\partial}{\partial x^\mu} (\Lambda \delta_\Lambda \theta \Lambda^{-1})_{\mu\nu} \frac{\partial}{\partial y^\nu} \theta(x^0 - y^0) = \left(\frac{\partial}{\partial x^0} (\Lambda \delta_\Lambda \theta \Lambda^{-1})_{0i} \frac{\partial}{\partial y^i} + \frac{\partial}{\partial x^i} (\Lambda \delta_\Lambda \theta \Lambda^{-1})_{i0} \frac{\partial}{\partial y^0} \right) \times \theta(x^0 - y^0) = 0. \quad (2.16)$$

Hence it too can be permitted to act on $\theta(x^0 - y^0)$. Each term in the expression

$$\hat{f}(x, y) = \frac{\partial}{\partial x^\mu} (\Lambda \delta_\Lambda \theta \Lambda^{-1})^{\mu\nu} \frac{\partial}{\partial y^\nu} \quad (2.17)$$

contains at least one spatial derivative. In particular only the following terms have time derivatives:

$$\frac{\partial}{\partial x^0}(\Lambda \delta_\lambda \theta \Lambda^{-1})^{0i} \frac{\partial}{\partial y^i} + \frac{\partial}{\partial x^i}(\Lambda \delta_\lambda \theta \Lambda^{-1})^{i0} \frac{\partial}{\partial y^0}.$$

Thus suppose we encounter a term like the following:

$$\chi(x, y) = \theta(x^0 - y^0) \left[\hat{f}(x, y) \alpha_1(x) \alpha_2(y) \right] \quad (2.18)$$

where $\hat{f}(x, y)$ acts only on α_i . Then it is a total spatial divergence:

$$\begin{aligned} \chi(x, y) &= \frac{\partial}{\partial x^i} \left((\Lambda \delta_\lambda \theta \Lambda^{-1})^{ij} \frac{\partial}{\partial y^j} [\theta(x^0 - y^0) \alpha_1(x) \alpha_2(y)] \right) \\ &+ \frac{\partial}{\partial y^i} \left((\Lambda \delta_\lambda \theta \Lambda^{-1})^{0i} \theta(x^0 - y^0) \frac{\partial \alpha_1(x)}{\partial x^0} \alpha_2(y) \right) \\ &+ \frac{\partial}{\partial x^i} (\Lambda \delta_\lambda \theta \Lambda^{-1})^{i0} \theta(x^0 - y^0) \left[\alpha_1(x) \frac{\partial \alpha_2(y)}{\partial y^0} \right]. \end{aligned} \quad (2.19)$$

Here time derivatives do not act on $\theta(x^0 - y^0)$.

It follows that

$$\int d^4x d^4y \chi(x, y) = 0 \quad (2.20)$$

and hence that

$$\begin{aligned} &\int d^4x d^4y \theta(x^0 - y^0) \left[e^{-(i/2)\hat{f}(x,y)} \alpha_1(x) \alpha_2(y) \right] \\ &= \int d^4x d^4y \theta(x^0 - y^0) \alpha_1(x) \alpha_2(y). \end{aligned} \quad (2.21)$$

This identity incidentally easily generalizes to the following sort of identity as well:

$$\begin{aligned} &\int \prod_{i=1}^N d^4x_i T e^{-(i/2)\{\hat{f}(x_1, x_2) + \hat{f}(x_2, x_3) + \dots + \hat{f}(x_{n-1}, x_n)\}} \alpha_1(x_1) \\ &\quad \times \alpha_2(x_2) \cdots \alpha_N(x_N) \\ &= \int \prod_{i=1}^N d^4x_i T [\alpha_1(x_1) \alpha_2(x_2) \cdots \alpha_N(x_N)]. \end{aligned} \quad (2.22)$$

Here in the left-hand side, the \hat{f} 's do not act on the step functions in time-ordering.

We can hence write

$$\begin{aligned} U(\Lambda) S_1^{(2)} &= -\frac{1}{2} \int d^N k_1 d^N k_2 \tilde{H}_I(k_1) \tilde{H}_I(k_2) \int \frac{d^N x}{(2\pi)^N} \frac{d^N y}{(2\pi)^N} \\ &\quad \times [\theta(x^0 - y^0) e_{\Lambda k_1}(x) e_{\Lambda k_2}(y)] \\ &\quad \times e^{-(i/2)(\bar{\partial}/\partial x^\mu + \bar{\partial}/\partial y^\mu) \theta^{\mu\nu} \partial_\nu} \\ &\quad \times U(\Lambda) e^{(i/2)(\Lambda^{-1})^\sigma{}_\mu (\bar{\partial}/\partial x^\sigma + \bar{\partial}/\partial y^\sigma) \theta^{\mu\nu} \partial_\nu}. \end{aligned} \quad (2.23)$$

We now expand the exponentials, integrate term by term and throw away surface terms. A similar calculation can be

done for $U(\Lambda) S_2^{(2)}$ as well. We thus finally find,

$$\begin{aligned} U(\Lambda) S^{(2)} U(\Lambda)^{-1} &= -\frac{1}{2} \int d^N x d^N y T(H_I(\Lambda^{-1}x) H_I(\Lambda^{-1}y)) \\ &= -\frac{1}{2} \int d^N x d^N y \{ \theta((\Lambda x)^0 - (\Lambda y)^0) \\ &\quad \times H_I(x) H_I(y) + x \leftrightarrow y \} \end{aligned} \quad (2.24)$$

just as for $\theta^{\mu\nu} = 0$.

As such a calculation extends to all orders in H_I , we have

$$U(\Lambda) S U(\Lambda)^{-1} = T \exp\left(-i \int d^N x H_I(\Lambda^{-1}x)\right). \quad (2.25)$$

If x and y are timelike separated, then time-ordering is invariant under $\Lambda \in \mathcal{L}_+^1$ (and parity): $\theta((\Lambda x)^0 - (\Lambda y)^0) = \theta(x^0 - y^0)$. But that is not the case if x and y are spacelike separated, $x \sim y$. In a causal theory, the result

$$[H_I(x), H_I(y)] = 0, \quad x \sim y \quad (2.26)$$

holds and helps restore Lorentz invariance of S despite time-ordering. Weinberg [6,7] can be consulted for a detailed proof.

The condition (2.26) is only a sufficient condition for Lorentz invariance, it is not necessary as well. We shall see below that nongauge noncommutative theories fulfill a weaker form of (2.26) and are still Lorentz-invariant.

(ii) *Nongauge noncommutative theories*

The qft's on the GM plane are not local. This is the case even without gauge fields. Still in the absence of gauge fields, we showed elsewhere [10] that the S -operator has no θ -dependence. Hence it is Lorentz-invariant if its associated $\theta^{\mu\nu} = 0$ theory is.

This result comes about as follows.

Let us consider a spin zero field Φ for simplicity as in [10]. For Φ , the annihilation operators for momentum p will be denoted by a_p . Then using eq. (7.11) of [11], we get $a_p e_p = c_p e_p e^{(1/2)\bar{\partial}_\mu \theta^{\mu\nu} P_\nu}$ (where c_p is the annihilation operator for $\theta^{\mu\nu} = 0$ and P_ν is the Fock space momentum operator) so that

$$\Phi(x) = \Phi^{(0)}(x) e^{(1/2)(\bar{\partial}/\partial x^\mu) \theta^{\mu\nu} P_\nu} \quad (2.27)$$

where $\Phi^{(0)}(x)$ is made of c_p 's and c_p^\dagger 's.

We must take $*$ -products of e_p 's when evaluating products of Φ 's at the same point since $e_p \in \mathcal{A}_\theta(\mathbb{R}^N)$. It becomes the ordinary product when we substitute (2.27) as proved in [10] and we get for the $*$ -product of n Φ 's,

$$\Phi(x) * \Phi(x) \cdots * \Phi(x) = (\Phi^{(0)}(x))^n e^{(1/2)(\bar{\partial}/\partial x^\mu) \theta^{\mu\nu} P_\nu} \quad (2.28)$$

$((\Phi^{(0)}(x))^n$ involves only commutative products of functions.)

Thus in the absence of gauge fields,

$$H_I(x) = H_I^{(0)}(x)e^{(1/2)(\bar{\partial}/\partial x^\mu)\theta^{\mu\nu}P_\nu}, \quad (2.29)$$

$H_I^{(0)}(x)$ being the interaction density for $\theta^{\mu\nu} = 0$.

Notice that

$$\int d^4x H_I(x) = \int d^4x H_I^{(0)}(x), \quad (2.30)$$

because the exponential factor in (2.29) becomes 1 on integration over x . Also, the Lorentz transformation prop-

erties of H_I can be obtained by transforming the operators $H_I^{(0)}$ and P_ν in the standard way [2,10,12]. Hence the Lorentz transformation property of the left-hand side of (2.29) can be obtained assuming (2.4).

Since

$$[P_\nu, H_I(y)] = -i \frac{\partial}{\partial y^\nu} H_I(y), \quad (2.31)$$

(2.29) gives for example

$$H_I(x)H_I(y) = H_I^{(0)}(x)e^{(1/2)(\bar{\partial}/\partial x^\mu)\theta^{\mu\nu}(-i)(\partial/\partial y_\nu)} H_I^{(0)}(y)e^{(1/2)((\bar{\partial}/\partial x^\mu)+(\bar{\partial}/\partial y^\mu))\theta^{\mu\nu}P_\nu}. \quad (2.32)$$

Hence,

$$T(H_I(x_1)H_I(x_2)\cdots H_I(x_k)) = T(H_I^{(0)}(x_1)e^{(1/2)(\bar{\partial}/\partial x_1^\mu)\theta^{\mu\nu}P_\nu^{(1)}} H_I^{(0)}(x_2)e^{(1/2)(\bar{\partial}/\partial x_2^\mu)\theta^{\mu\nu}P_\nu^{(2)}} \cdots H_I^{(0)}(x_k)e^{(1/2)(\bar{\partial}/\partial x_1^\mu)+\cdots+(\bar{\partial}/\partial x_k^\mu)\theta^{\mu\nu}P_\nu}), \quad (2.33)$$

$$P_\mu^{(j)} = -i \left(\frac{\partial}{\partial x_{j+1}^\mu} + \cdots \frac{\partial}{\partial x_k^\mu} \right), \quad j \leq k-1 \quad (2.34)$$

where the derivatives in (2.33) do not act on the step-functions in the definition of the time-ordered product. But we can let them act on the step functions as well in view of the discussion from (2.15) to (2.22). [We must adapt it only slightly to reach this conclusion.] Then integrating over x_i 's and discarding surface terms as in (2.22), we find that S is independent of $\theta^{\mu\nu}$.

This is a fundamental result of [10] in proving the absence of UV-IR mixing in nongauge noncommutative theories.

In the same way, we can show that $U(\Lambda)SU(\Lambda^{-1})$ given in (2.25) is independent of $\theta^{\mu\nu}$:

$$U(\Lambda)SU(\Lambda^{-1}) = T \exp \left(-i \int d^N x H_I(\Lambda^{-1}x) \right) := S^{(0)}. \quad (2.35)$$

Thus if the $\theta^{\mu\nu} = 0$ theory has a causal interaction Hamiltonian density $H_I^{(0)}$ and the operator product $H_I^{(0)}(x)H_I^{(0)}(y)$ is not too singular at $x = y$ so that $S^{(0)}$ is Lorentz invariant, then S is also Lorentz invariant.

(iii) *Generalized causality*

We see from (2.35) that the following generalized causality condition holds in nongauge theories for any $\theta^{\mu\nu}$: for some choice of the constant λ , the operator

$$H_I^{(\lambda)}(x) = H_I(x)e^{-(1+\lambda/2)(\bar{\partial}/\partial x^\mu)\theta^{\mu\nu}P_\nu} \quad (2.36)$$

is local:

$$[H_I^{(\lambda)}(x), H_I^{(\lambda)}(y)] = 0, \quad x \sim y. \quad (2.37)$$

This is our generalized causality relation. Our arguments show that if

$$S = T e^{-i \int d^N x dH_I(x)} \quad (2.38)$$

and

$$S^{(\lambda)} = T e^{-i \int d^N x dH_I^{(\lambda)}(x)}, \quad (2.39)$$

then

$$S = S^{(\lambda)}. \quad (2.40)$$

Weinberg's arguments show that $S^{(0)}$ is Lorentz-invariant if (2.26) holds unless singularities at coincident points (mentioned before) spoil it. Therefore $S^{(\lambda)}$ will also be Lorentz-invariant if (2.37) holds and singularities at coincident points do not spoil it.

III. GAUGE THEORIES WITH MATTER FIELDS

Suppose we have a charged scalar field Φ ,

$$\Phi(x) = \int d\mu(p)(a_p e^{-ip \cdot x} + b^\dagger(p) e^{ip \cdot x}) \quad (3.1)$$

that obeys twisted statistics. Then Φ can be written in terms of the corresponding commutative counterpart $\Phi^{(0)}$ using (2.27), where

$$P_\mu = \int d\mu(q) q_\mu [a^\dagger(q)a(q) + b^\dagger(q)b(q)] \\ = \text{the total momentum operator.} \quad (3.2)$$

As we discussed in Section 7 of [11], we require that the definition of the covariant derivative D_μ of the field Φ preserves statistics, transforms covariantly under Poincaré transformations and has the commutator $[D_\mu, D_\nu]$ given by the curvature $F_{\mu\nu}^c$ of the commutative gauge fields. This immediately tells us that D_μ is of the form

$$D_\mu \Phi = (D_\mu^{(0)} \Phi^{(0)}) e^{(1/2) \bar{\partial} \theta^{\mu\nu} P_\nu} \quad (3.3)$$

where

$$D_\mu^{(0)} = \partial_\mu + A_\mu^{(0)} \quad (3.4)$$

and $A_\mu^{(0)}$ is the commutative gauge field. This choice satisfies all our requirements of a covariant derivative. It also obeys gauge invariance at the quantum level [11]. Any gauge group can be treated in this approach, unlike some other approaches.

Note that since the gauge symmetry generators are the same as those for $\theta^{\mu\nu} = 0$, the $(F_{\mu\nu}^{(0)})^2$ term of the gauge field “kinetic energy term” also transforms correctly.

Similar arguments can be made about the transformation properties under the Poincaré group.

The interaction Hamiltonian splits into two parts:

$$H_\theta^I = \int d^3x [\mathcal{H}_\theta^{MG} + \mathcal{H}_\theta^G], \quad (3.5)$$

MG = matter-gauge,

G = pure gauge field

$$\mathcal{H}_\theta^{MG} = \mathcal{H}_0^{MG} e^{1/2 \bar{\partial} \theta^{\mu\nu} P_\nu}, \quad (3.6)$$

$$\mathcal{H}_\theta^G = \mathcal{H}_0^G. \quad (3.7)$$

We include matter-gauge field and pure matter field couplings in \mathcal{H}_θ^{MG} , while \mathcal{H}_θ^G contains only gauge field terms.

For QED, $\mathcal{H}_\theta^G = 0$ and the S -operator of the theory is the same as in the commutative case:

$$S_\theta^{\text{QED}} = S_0^{\text{QED}}. \quad (3.8)$$

[However, in [13], we developed another approach to gauge theories where (3.8) is not true.]

For the standard model (SM), $\mathcal{H}_\theta^G = \mathcal{H}_0^G \neq 0$. As this term has no statistics twist,

$$S_\theta^{\text{SM}} \neq S_0^{\text{SM}} \quad (3.9)$$

because of the cross-terms in the S -matrix between \mathcal{H}_θ^{MG} and \mathcal{H}_θ^G . In particular, this inequality happens in QCD. Processes like $qg \rightarrow qg$ via a gluon exchange interaction actually also violate Lorentz invariance, as we explain below.

The generalized causality condition (2.37) is not fulfilled in non-Abelian gauge theories with matter-gauge field couplings. It is enough to show this in QCD as we now will.

We have, as in (2.27),

$$\Psi(x) = \Psi^{(0)}(x) e^{(1/2) \bar{\partial} \theta^{\mu\nu} P_\nu}. \quad (3.10)$$

P_ν is the *total* momentum operator of the quark and gluon fields as in (2.27). That is so for the following reason. Under covariant transport, Ψ and $D_\mu \Psi$ must have similar

braiding properties. In particular since

$$\begin{aligned} \Psi(x) \Psi(y) &= e^{-(i/2) (\bar{\partial} / \partial x^\mu) \otimes \theta^{\mu\nu} (\bar{\partial} / \partial y^\nu)} (\Psi^{(0)}(x) \Psi^{(0)}(y)) \\ &\times e^{(1/2) \bar{\partial}_\mu \theta^{\mu\nu} P_\nu}, \end{aligned} \quad (3.11)$$

we need

$$\begin{aligned} D_\mu \Psi(x) D_\nu \Psi(y) &= e^{-(i/2) (\bar{\partial} / \partial x^\mu) \otimes \theta^{\mu\nu} (\bar{\partial} / \partial y^\nu)} (D_\mu^{(0)} \Psi^{(0)}(x)) \\ &\times D_\nu^{(0)} \Psi^{(0)}(y) e^{1/2 \bar{\partial}_\mu \theta^{\mu\nu} P_\nu}. \end{aligned} \quad (3.12)$$

So this requires

$$[P_\mu, D_\lambda \Psi] = -i \partial_\mu D_\lambda \Psi. \quad (3.13)$$

As D_λ involves the gluon field, P_μ must contain its momentum too. It follows that

$$H_\theta^I = \frac{e}{2} (\bar{\Psi}^{(0)} \gamma \cdot A \Psi^{(0)}) e^{(1/2) \bar{\partial}_\mu \theta^{\mu\nu} P_\nu} + H_\theta^G \quad (3.14)$$

where $H_\theta^G = H_0^G$ contains three- and four-gluon terms and gluon fields are free.

As

$$[P_\mu, H_\theta^G] = -i \partial_\mu H_\theta^G, \quad (3.15)$$

it is clear that

$$[H_\theta^I(x), H_\theta^I(y)] \neq 0 \quad \text{if } x \sim y, \quad (3.16)$$

the nonvanishing term coming from

$$[H_\theta^{MG}(x), H_\theta^G(y)] + x \leftrightarrow y. \quad (3.17)$$

Thus H_θ^I is not local. It does not fulfill our generalized locality condition as well. Thus in the next subsection, we explicitly show that diagrams involving $H_\theta^{MG} H_\theta^G$ lead to violations of Lorentz invariance in scattering. This proves that H_θ^I does not fulfill our generalized causality.

Feynman rules and examples

Let $\Psi(x)$ be the noncommutative quantum field representing the quark. Using (2.27), it can be written in terms of the field $\Psi^{(0)}(x)$ (with the $\theta^{\mu\nu} = 0$ creation-annihilation operators) as

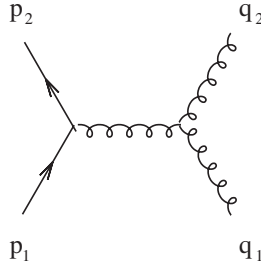
$$\Psi(x) = \Psi^{(0)}(x) e^{(1/2) \bar{\partial} \theta^{\mu\nu} P_\nu}. \quad (3.18)$$

P_ν is the *total* momentum operator of the quark and gluon fields as emphasized above.

Diagrams involving $H_\theta^{MG} H_\theta^G$ lead to violations of Lorentz invariance in scattering, as we will show below.

The discussion generalizes to the θ -deformed standard model (SM_θ) or any such θ -deformed theory.

In the expansion of S , terms involving just H_θ^{MG} or just $H_\theta^G \equiv H_0^G$ are independent of θ . The dependence on θ comes from terms which involve product H_θ^{MG} with H_θ^G . The simplest such term is

FIG. 1. A Feynman diagram with a nontrivial θ -dependence.

$$S^{(2)} = \frac{(-i)^2}{2!} \int d^4x_1 d^4x_2 T(H_\theta^{MG}(x_1)H_0^G(x_2)). \quad (3.19)$$

It contributes to quark-gluon (qg) scattering at the tree level, as shown in Fig. 1.

We now simplify $S^{(2)}$. Such simplifications generalize to arbitrary terms in S as we later indicate.

(i) *Simplifications for Fig. 1*

(a) The first simplification comes from integrating over d^3x_1 and throwing away surface terms from spatial derivatives in $\bar{\partial}_\mu \theta^{\mu\nu} P_\nu$. This lets us replace $H_\theta^{MG}(x_1)$ by

$$\hat{H}_\theta^{MG}(x_1) = H_0^{MG}(x_1) e^{(1/2)(\bar{\partial}/\partial x_{10})\theta^{0i}P_i}. \quad (3.20)$$

(b) We have for $i = 1, 2, 3$,

$$\left[P_i, \int d^3x_2 H_0^G(x_2) \right] = 0. \quad (3.21)$$

Hence we can move P_i to the right extreme:

$$S^{(2)} = -\frac{1}{2} \int d^4x_1 d^4x_2 T \times \left(H_0^{MG}(x_1) H_0^G(x_2) e^{(1/2)(\bar{\partial}/\partial x_{10})\theta^{0i}P_i} \right) \quad (3.22)$$

where $\frac{\bar{\partial}}{\partial x_{10}}$ does *not* act on the step functions in time defining T .

From (3.22) we see that P_i can be replaced by the total incident momentum $P_{inc,i} = (p_1 + q_1)_i$ when considering the process in Fig. 1:

$$S^{(2)} = -\frac{1}{2} \int d^4x_1 d^4x_2 \left\{ \theta(x_{10} - x_{20}) \times \left(H_0^{MG}(x_1) e^{1/2(\bar{\partial}/\partial x_{10})\bar{\theta}^0 \cdot \vec{P}_{inc}} H_0^G(x_2) \right) + \theta(x_{20} - x_{10}) \times \left(H_0^G(x_2) H_0^{MG}(x_1) e^{(1/2)(\bar{\partial}/\partial x_{10})\bar{\theta}^0 \cdot \vec{P}_{inc}} \right) \right\}, \quad (3.23)$$

$$\bar{\theta}^0 \cdot \vec{P}_{inc} = \theta^{0i} P_{inc,i}. \quad (3.24)$$

Now

$$H_0^{MG}(x_1) e^{(1/2)(\bar{\partial}/\partial x_{10})\bar{\theta}^0 \cdot \vec{P}_{inc}} = H_0^{MG}\left(\vec{x}_1, x_{10} + \frac{1}{2}\bar{\theta}^0 \cdot \vec{P}_{inc}\right). \quad (3.25)$$

The θ -deformation thus twists the fields at the $q - g - g$ vertex.

(c) By a change of variables, we can shift the deformation to the $g - g - g$ vertex instead:

$$S^{(2)} = -\frac{1}{2} \int d^4x_1 d^4x_2 \left\{ \theta\left(x_{10} - x_{20} - \frac{1}{2}\bar{\theta}^0 \cdot \vec{P}_{inc}\right) \times \left(H_0^{MG}(x_1) H_0^G(x_2) \right) + \theta\left(x_{20} + \frac{1}{2}\bar{\theta}^0 \cdot \vec{P}_{inc} - x_{10}\right) \left(H_0^G(x_2) H_0^{MG}(x_1) \right) \right\} = -\frac{1}{2} \int d^4x_1 d^4x_2 T \left(H_0^{MG}(x_1) H_0^G\left(\vec{x}_2, x_{20} - \frac{1}{2}\bar{\theta}^0 \cdot \vec{P}_{inc}\right) \right). \quad (3.26)$$

The ability to shift the twist between a quark-quark-gluon and a 3- or 4-gluon vertex connected to it in this manner is often useful. It is thus sufficient (see also below) to give the twisted gluon propagator to calculate Feynman diagrams.

(ii) *The twisted gluon propagator*

The twisted gluon propagator coming from (3.26) is

$$T\langle A_\mu^\alpha(x_1) A_\nu^\beta(\vec{x}_2, x_{20} - \frac{1}{2}\bar{\theta}^0 \cdot \vec{P}_{inc}) \rangle = \delta^{\alpha\beta} \eta_{\mu\nu} D_F^\theta(x_1 - x_2) \quad (3.27)$$

where in the Lorentz gauge, D_F^θ is just the twisted propagator of a massless scalar field A :

$$D_F^\theta(x) = T\langle A(x) A(\vec{0}, \frac{1}{2}\bar{\theta}^0 \cdot \vec{P}_{inc}) \rangle. \quad (3.28)$$

The Fourier expansion of A is

$$A(x) = \int \frac{d^3k}{2k_0} \left(c_k e_k(x) + c_k^\dagger e_{-k}(x) \right), \quad (3.29)$$

$$e_k(x) = e^{-ikx} = e^{-i(\vec{k} \cdot \vec{x} - k_0 x_0)}, \quad k_0 = |\vec{k}|$$

where c_k, c_k^\dagger are the $\theta^{\mu\nu} = 0$ annihilation and creation operators. Hence

$$A\left(\vec{0}, \frac{1}{2}\bar{\theta}^0 \cdot \vec{P}_{inc}\right) = \int \frac{d^3k}{2|\vec{k}|} \left(c_k e^{(i/2)|\vec{k}|\bar{\theta}^0 \cdot \vec{P}_{inc}} + c_k^\dagger e^{-(i/2)|\vec{k}|\bar{\theta}^0 \cdot \vec{P}_{inc}} \right). \quad (3.30)$$

Note that we pick up the second term here in the $\theta(x_0)$ term of the T -product, and the first term in the $\theta(-x_0)$ term, and these have opposite phases.

Now

$$D_F^0(x) = 2\pi i \int \frac{d^3k}{2|\vec{k}|} (\theta(x_0)e^{ikx} + \theta(-x_0)e^{-ikx}), \quad (3.31)$$

which comes from

$$\begin{aligned} D_F^0(x) &= \int d^4k \frac{e^{-ikx}}{k^2 - i\epsilon} \\ &= - \int d^3k e^{-i\vec{k}\cdot\vec{x}} \int \frac{dk_0}{2|\vec{k}|} e^{ik_0x_0} \\ &\quad \times \left(\frac{1}{k_0 + |\vec{k}| - i\epsilon} - \frac{1}{k_0 - |\vec{k}| + i\epsilon} \right). \end{aligned} \quad (3.32)$$

Hence

$$\begin{aligned} D_F^\theta(x) &= - \int d^3k e^{-i\vec{k}\cdot\vec{x}} \int \frac{dk_0}{2|\vec{k}|} e^{ik_0x_0} \\ &\quad \times \left(\frac{e^{-(i/2)|\vec{k}|\vec{\theta}^0\cdot\vec{P}_{\text{inc}}}}{k_0 + |\vec{k}| - i\epsilon} - \frac{e^{i/2|\vec{k}|\vec{\theta}^0\cdot\vec{P}_{\text{inc}}}}{k_0 - |\vec{k}| + i\epsilon} \right) \\ &= \int d^4k \frac{e^{-ikx}}{k^2 - i\epsilon} \left(\cos\left(\frac{|\vec{k}|\vec{\theta}^0\cdot\vec{P}_{\text{inc}}}{2}\right) \right. \\ &\quad \left. + i \frac{k_0}{|\vec{k}|} \sin\left(\frac{|\vec{k}|\vec{\theta}^0\cdot\vec{P}_{\text{inc}}}{2}\right) \right) \\ &\equiv \int d^4k e^{-ik\cdot x} \tilde{D}_F^\theta(k). \end{aligned} \quad (3.33)$$

(iii) *General rules*

In any scattering process, the twist factors $e^{(1/2)\vec{\theta}^0\cdot\vec{P}_{\text{inc}}}$ can all be replaced by $e^{(1/2)\vec{\theta}^0\cdot\vec{P}_{\text{inc}}}$ where \vec{P}_{inc} is the incident total momentum and $\vec{\theta}^0$ differentiates an appropriate time argument.

The propagator of a quark or of a gluon connecting two $q - q - g$ vertices is not changed. That is because for example

$$\begin{aligned} &\int d^4x_1 d^4x_2 \theta(x_{10} - x_{20}) H_0^{MG}(\vec{x}_1, x_{10} + \frac{1}{2}\vec{\theta}^0\cdot\vec{P}_{\text{inc}}) \\ &\quad \times H_0^{MG}(\vec{x}_2, x_{20} - \frac{1}{2}\vec{\theta}^0\cdot\vec{P}_{\text{inc}}) \\ &= \int d^4x_1 d^4x_2 \theta(x_{10} - x_{20}) H_0^{MG}(x_1) H_0^{MG}(x_2). \end{aligned} \quad (3.34)$$

In an arbitrary diagram, *a priori*, the twisted vertices are the $q - q - g$ vertices. By a change of variables, we can then shift the twist to appropriate gluon propagators. In this way, we can tell which of the gluon propagators in the diagram are twisted.

IV. LORENTZ INVARIANCE AND PAULI PRINCIPLE

(i) *Violation of Lorentz invariance*

Consider Fig. 1. It carries the propagator

$$\tilde{D}_F^\theta(k) = \frac{\cos\left(\frac{|\vec{k}|\vec{\theta}^0\cdot\vec{P}_{\text{inc}}}{2}\right) + i \frac{k_0}{|\vec{k}|} \sin\left(\frac{|\vec{k}|\vec{\theta}^0\cdot\vec{P}_{\text{inc}}}{2}\right)}{k^2 - i\epsilon}. \quad (4.1)$$

The numerator is frame-dependent. It is unity if

$$\vec{\theta}^0 \cdot \vec{P}_{\text{inc}} = 0, \quad (4.2)$$

in particular in the center-of mass system. Hence all twist effects are absent in S in any frame fulfilling (4.2). Otherwise it depends on θ . Thus as anticipated, the process violates Lorentz invariance.

The discussion of C , P , T and CPT can be found in [4].

(ii) *Pauli principle violation*

In [13], based on a different treatment of dynamics, we found Pauli principle violation in processes like electron-electron scattering. Such violation was present even for cross-sections for scattering of particles with definite momenta.

In the present approach, there is no such violation in any scattering cross-section of particles with definite momenta.

But there are expected to be signals of Pauli principle violations if initial and final particles do not have definite momenta, for example, if they are spatially localized wave packets. See for example [1, 14].

The proof is very general and very simple too: we just show below that the initial and final states of definite momenta differ from those for $\theta_{\mu\nu} = 0$ only by a phase, a result well-known. The phase disappears when we compute cross-sections, that is, in the modulus of scattering amplitudes. Hence the modulus of scattering amplitudes in the momentum basis inherits exactly the same symmetry properties from the states under particle exchange as those for $\theta_{\mu\nu} = 0$. The nontrivial dependence of S -matrix on external momenta through the term $\vec{\theta}^0\cdot\vec{P}_{\text{in}}$ does not spoil this argument because this dependence *always* involves the total momentum, which is, of course, symmetric under permutation of the individual momenta. We can even replace the actual scattering amplitudes with ones with the same symmetries under particle exchange as those for $\theta_{\mu\nu} = 0$ by setting the above-mentioned phase to 1. The result on Pauli principle follows.

The difference between arbitrary states (such as spatially localized wave packets) for $\theta_{\mu\nu} = 0$ and $\theta_{\mu\nu} \neq 0$ is not a phase [15]. Hence we cannot readily assert that the modulus of scattering amplitudes for $\theta_{\mu\nu} = 0$ and $\theta_{\mu\nu} \neq 0$ have the same symmetry under particle exchange in *any* basis.

Now for the demonstration. Consider, for example, an N -particle state of identical spin- $\frac{1}{2}$ particles. Their creation operators $a_p^{(\lambda)\dagger}$ for spin basis label λ and momentum p are related to those for $\theta_{\mu\nu} = 0$ by

$$a_p^{(\lambda)\dagger} = c_p^{(\lambda)\dagger} e^{(i/2)p\wedge P}, \quad p \wedge P := p_\mu \theta^{\mu\nu} P_\nu, \quad (4.3)$$

where P is the total momentum operator.

For the gauge field, the creation operators $\alpha_q^{(m)\dagger}$ are independent of $\theta_{\mu\nu}$.

Let us first look at a two spin- $\frac{1}{2}$ particle state:

$$a_{p_1}^{(\lambda_1)\dagger} a_{p_2}^{(\lambda_2)\dagger} |0\rangle = c_{p_1}^{(\lambda_1)\dagger} c_{p_2}^{(\lambda_2)\dagger} |0\rangle e^{(i/2)p_1 \wedge p_2}. \quad (4.4)$$

The θ -dependent term on the right side is just a phase. A similar calculation can be made for any N spin- $\frac{1}{2}$ particles and also for any state with bosons, fermions and gauge particles. Thus in the state

$$a_{p_1}^{(\lambda_1)\dagger} a_{p_2}^{(\lambda_2)\dagger} \dots a_{p_N}^{(\lambda_N)\dagger} \alpha_{q_1}^{(m_1)\dagger} \alpha_{q_2}^{(m_2)\dagger} \dots \alpha_{q_M}^{(m_M)\dagger} |0\rangle, \quad (4.5)$$

we can move all P_μ 's to the right extreme, where they contribute only a phase. For example, for $N = 2$ and $M =$

1, the above expression is

$$c_{p_1}^{(\lambda_1)\dagger} c_{p_2}^{(\lambda_2)\dagger} \alpha_{q_1}^{(m_1)\dagger} |0\rangle e^{(i/2)p_1 \wedge (p_2 + q_1)} e^{(i/2)p_2 \wedge q_1}. \quad (4.6)$$

In this way, we arrive at our conclusion about Pauli principle.

ACKNOWLEDGMENTS

It is a pleasure to thank Earnest Akofor, Sang Jo, and Anosh Joseph for discussions. Some of the results of this papers overlap with some of those of [4]. The work of A. P. B. and B. Q. is supported in part by DOE under grant number DE-FG02-85ER40231. The work of A. P. is supported by FAPESP grant number 06/56056-0.

-
- [1] A. P. Balachandran, G. Mangano, A. Pinzul, and S. Vaidya, *Int. J. Mod. Phys. A* **21**, 3111 (2006).
 - [2] A. P. Balachandran, T.R. Govindarajan, G. Mangano, A. Pinzul, B. A. Qureshi, and S. Vaidya, *Phys. Rev. D* **75**, 045009 (2007).
 - [3] J. M. Carmona, J. L. Cortes, J. Gamboa, and F. Mendez, *J. High Energy Phys.* 03 (2003) 058.
 - [4] E. Akofor, A. P. Balachandran, S. G. Jo, and A. Joseph, *J. High Energy Phys.* 08 (2007) 045.
 - [5] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, New York, 1959).
 - [6] S. Weinberg, *Phys. Rev.* **133**, B1318 (1964).
 - [7] S. Weinberg, *The Quantum Theory of Fields. Vol. 1: Foundations* (Cambridge University Press, Cambridge, 1995).
 - [8] F. Dowker, arXiv:gr-qc/0508109.
 - [9] R. D. Sorkin, arXiv:gr-qc/9302018.
 - [10] A. P. Balachandran, A. Pinzul, and B. Qureshi, *Phys. Lett. B* **634**, 434 (2006).
 - [11] A. P. Balachandran, A. Pinzul, B. Qureshi, and S. Vaidya, *Phys. Rev. D* **76**, 105025 (2007).
 - [12] A. P. Balachandran, A. Pinzul, and B. Qureshi, arXiv:0708.1779.
 - [13] A. P. Balachandran, A. Pinzul, B. A. Qureshi, and S. Vaidya, arXiv:hep-th/0608138.
 - [14] B. Chakraborty, S. Gangopadhyay, A. G. Hazra, and F. G. Scholtz, *J. Phys. A* **39**, 9557 (2006).
 - [15] A. P. Balachandran and S. G. Jo, arXiv:0704.0921.