Bosonized supersymmetry from the Majorana-Dirac-Staunton theory and massive higher-spin fields

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We propose a (3 + 1)D linear set of covariant vector equations, which unify the spin-0 "new Dirac equation" with its spin-1/2 counterpart, proposed by Staunton. Our equations describe a spin-(0, 1/2) supermultiplet with different numbers of degrees of freedom in the bosonic and fermionic sectors. The translation-invariant spin degrees of freedom are carried by two copies of the Heisenberg algebra. This allows us to realize space-time supersymmetry in a bosonized form. The grading structure is provided by an internal reflection operator. Then the construction is generalized by means of the Majorana equation to a supersymmetric theory of massive higher-spin particles. The resulting theory is characterized by a nonlinear symmetry superalgebra, that, in the large-spin limit, reduces to the super-Poincaré algebra with or without tensorial central charge.

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I. INTRODUCTION

In 1932, Ettore Majorana [1,2] proposed a Lorentz invariant linear differential equation, associated with infinite-dimensional unitary representations of the Lorentz group. The subsequent development of the concept of the infinite-component fields [3–6] culminated in the construction of the dual resonance models, and lead eventually to superstring theory [7–12].

The Majorana equation has massive, massless, and tachyonic solutions (see Refs. [13,14] for recent reviews). In the massive case, the equation describes two series of positive-energy particles with arbitrary integer or half-integer spin. The equation does not fix the mass, however, rather provides a spin-dependent, Regge-like mass spectrum, (3.10) below.[1] The simultaneous presence of integer and half-integer spins suggests, together with the positivity of the energy, that some kind of *supersymmetry* could be involved in the Majorana construction [13,16].

In 1971, Dirac [17] put forward a linear spinor set of equations, from which the Majorana and Klein-Gordon equations follow as integrability (consistency) conditions. This "new Dirac equation" describes massive particles with zero spin.

A couple of years later, Staunton [18] proposed, instead of the spinorial Dirac approach, a vector equation, which involves a new parameter, κ. Staunton's new system is only consistent for $\kappa = 1/2$ or 1. For $\kappa = 1/2$, his equation coincides with one of the consistency relations implied by the equations of Dirac; it describes hence a spin 0 massive particle. The second value, $\kappa = 1$, yields a spin-1/2 particle of nonzero mass. The Staunton equations imply, once again, the Klein-Gordon and Majorana equations as integrability conditions. With some abuse of language, the $\kappa = 1/2$ (i.e. spin-0) equation of Staunton will be referred to as "the new Dirac system" (to which it is equivalent), and the $\kappa = 1$ (i.e. spin-1/2) equation will be referred to as "the Staunton system."

The Dirac and Staunton solutions both have positive energy. Their masses can also be derived from the Majorana spectrum [(3.10) below] with appropriate mass parameters, out of which those solutions which carry the lowest possible spin, namely, spin-0 and spin-1/2, respectively, were selected.

In this paper we show first that the Dirac and Staunton equations can be merged into a single supersymmetric system. Then, with the help of the modified Majorana equation, we generalize the construction to a supersymmetric theory of massive higher-spin particles. It is worth stressing that supersymmetrization is achieved here without enlarging the system by adding new degrees of freedom, as it is done usually. The necessary degrees of freedom have already been present in the Dirac-Staunton and Majorana frameworks. The underlying space is in fact decomposed into two subspaces and the Dirac and respectively Staunton equations merely select one sector and kill the other. Our unified system simply activates them simultaneously.

The two subspaces of the Majorana equation are initially unrelated. A smart choice of the mass parameter, however, creates a supersymmetry between the two sectors. This is similar to what happens for a planar anisotropic oscillator, for which rational tuning of the frequency ratio generates a (nonlinear) symmetry [19].

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[1]For a discussion of the spin-statistics relation for the Majorana field, see Refs. [6,15].

Since we only use bosonic variables, what we get here is *bosonized supersymmetry.* The unification of the even and odd spin representations and their supersymmetry relies on using a nonlocal operator, namely, the *reflection operator.*

Examples in which supersymmetry is realized within a purely bosonic system were presented recently in $D < 4$ dimensions [20–22]. For all these systems, the role of the grading operator is played by the nonlocal reflection operator.

The present paper extends these results to $(3 + 1)$ dimensions.

In the theories of Majorana, Dirac, and Staunton the field equations involve, in their internal structure, two copies of the Heisenberg algebra, associated with an internal planar harmonic oscillator, $[q_i, \eta_i] = i\delta_{ij}$, $i = 1, 2$, as well as ten quadratic products built out of these generators. Six quadratic combinations span the Lorentz algebra. The remaining four form a Lorentz vector. These ten generators span, together, the anti-de Sitter (AdS) $so(2, 3)$ algebra (analogously as Dirac matrices and their commutators do). The Heisenberg algebra generators q_i, η_i, $i = 1, 2$, can be united into a four-component operator, say L_a. The latter transforms covariantly (namely as a spinor) under the action of $so(2, 3)$ and provides us with a (bosonized) representation of the superalgebra $osp(1|4)$.

Then we can build the reflection operator $\mathcal{R} = (-1)^{(N_1+N_2)}$, where N_1 and N_2 are the number operators of the Heisenberg algebras. $\mathcal{R}$ commutes with the $so(2, 3)$ subalgebra, anticommutes with the supercharge L_a, and has eigenvalues ± 1. It provides us therefore with the grading operator of $osp(1|4)$. The operator $\mathcal{R}$ can be identified with a certain class of finite $SO(2, 3)$ transformations, namely, with internal reflection $q_i \mapsto -q_i$, or alternatively, a nonlocal, finite rotation (by π) in the 2D plane, spanned by the q_i.

Technically, the unification of the Dirac and Staunton equations boils down to first promoting Staunton's parameter κ into an operator by inserting the reflection operator, $\mathcal{R}$, Eq. (4.3) below, and then putting $\hat{\kappa}$ into Staunton's general equation. On the ± 1 eigenspaces of $\mathcal{R}$, $\hat{\kappa}$ takes precisely the correct "Dirac" and "Staunton" values, $\kappa = 1/2$ and 1, respectively. The restriction of our new equation reproduces, therefore, the spinless Dirac and the spin-$1/2$ Staunton equations, projected into the corresponding eigenspaces of $\mathcal{R}$.

The two (namely spin-0 and spin-$1/2$) sectors can be related by a Hermitian supercharge operator Q_a, which carries a spin-$1/2$ representation of the Lorentz group. As a result, we get a nonlinear extension of the usual super-Poincaré algebra by non-Abelian tensor conserved charges, which appear in the anticommutator of the supercharge.

Then we construct a generalized Majorana equation that provides us with a supersymmetric system of fields with spins $(j, j + 1/2)$. In the generic case of integer or half-integer j, such a system is described by a $(3 + 1)$D bosonized supersymmetry whose form has been slightly modified when compared to the simplest, $j = 0$, case.

The generalization is achieved in a way similar to the one we followed for the Dirac-Staunton theory: we modify the mass parameter in the original Majorana equation by introducing into it the operator $\mathcal{R}$ in a way that guarantees that the spin-j and $j + 1/2$ states have equal masses. Then requiring that a supercharge should exist and act as a symmetry implies the Klein-Gordon equation as a consistency condition. As a result, we obtain a bosonized supersymmetric theory of massive higher-spin particles, characterized by a nonlinear superalgebraic structure. In the large-spin limit the nonlinearity disappears, and the usual super-Poincaré algebra with or without tensorial central charge [23–27] is recovered.

The paper is organized as follows. In Sec. II, we construct, starting with two Heisenberg algebras, an infinite-dimensional, unitary representation of the $osp(1|4)$ superalgebra and the reflection operator $\mathcal{R}$. In Sec. III, we give a brief review of the theories of Majorana, of Dirac, and of Staunton. The supersymmetric theory for the spin-$(0, \frac{1}{2})$ supermultiplet is developed in Sec. IV, where the supersymmetric field equation and the corresponding superalgebra are constructed.

These results are extended to an arbitrary-spin supermultiplet by means of a generalized Majorana equation in Sec. V.

Section VI includes comments and concluding remarks.

II. MAJORANA REPRESENTATION AND $osp(1|4)$

The Majorana representation of the Lorentz group is an infinite-dimensional representation in which the Casimir operators,

$$C_1 = S^{\mu\nu}S_{\mu\nu} \quad \text{and} \quad C_2 = \epsilon^{\mu\nu\lambda\rho}S_{\mu\nu}S_{\lambda\rho}, \tag{2.1}$$

take the fixed values

$$C_1 = -\tfrac{3}{2}, \qquad C_2 = 0. \tag{2.2}$$

This representation can be realized in terms of two copies of Heisenberg algebras.

The Majorana representation can be embedded into a larger supersymmetric structure, namely, into $osp(1|4)$. Let us indeed consider the two-dimensional Heisenberg algebra generated by the operators q_i and η_j,

$$[q_i, \eta_j] = i\delta_{ij}.$$

We assume the coordinates q_i are rescaled by a length parameter l so that the generators q_i and η_i are dimensionless. The four-component operator

$$(L_a) = (q_1, q_2, \eta_1, \eta_2), \qquad a = 1, 2, 3, 4, \tag{2.3}$$

satisfies the relation

$$[L_a, L_b] = iC_{ab}, \qquad C_{ab} = \begin{pmatrix} 0 & I_{2\times 2} \\ -I_{2\times 2} & 0 \end{pmatrix}. \tag{2.4}$$

The antisymmetric matrix C_{ab} here can be viewed as a metric tensor in the spinor indices, see below. Defining $C^{ab} = C_{ab}$, $C_{ac}C^{bc} = \delta_a^b$, we rise and lower indices as $L^a = L_b C^{ba}$ and $L_a = C_{ab}L^b$.

Ten independent tensor products $L_a L_b$ can be constructed and combined as

$$S_{\mu\nu} = \frac{i}{2} L^a (\gamma_{\mu\nu})_a{}^b L_b, \qquad \Gamma_\mu = \frac{1}{4} L^a (\gamma_\mu)_a{}^b L_b, \qquad \mu = 0, 1, 2, 3. \tag{2.5}$$

Here the $(\gamma^\mu)_a{}^b$ are the Dirac matrices in the Majorana representation,

$$(\gamma^0)_a{}^b = \begin{pmatrix} 0 & \sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}, \qquad (\gamma^1)_a{}^b = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix},$$
$$(\gamma^2)_a{}^b = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \qquad (\gamma^3)_a{}^b = \begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix},$$

and $\gamma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ are pure imaginary matrices. Dirac matrices satisfy, with the space-time metric $\mathrm{diag}(\eta^{\mu\nu}) = (- + + +)$, the relation $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + 2i\gamma^{\mu\nu}$.

The quadratic operators (2.5) and the L_a generate the $osp(1|4)$ superalgebra,

$$[S_{\mu\nu}, S_{\lambda\rho}] = i(\eta_{\mu\lambda}S_{\nu\rho} + \eta_{\nu\rho}S_{\mu\lambda} - \eta_{\mu\rho}S_{\nu\lambda} - \eta_{\nu\lambda}S_{\mu\rho}), \tag{2.6}$$

$$[S_{\mu\nu}, \Gamma_\lambda] = i(\eta_{\mu\lambda}\Gamma_\nu - \eta_{\nu\lambda}\Gamma_\mu), \qquad [\Gamma_\mu, \Gamma_\nu] = -iS_{\mu\nu}, \tag{2.7}$$

$$[S_{\mu\nu}, L_a] = -(\gamma_{\mu\nu})_a{}^b L_b, \qquad [\Gamma_\mu, L_a] = \frac{i}{2}(\gamma_\mu)_a{}^b L_b, \tag{2.8}$$

$$\{L_a, L_b\} = -2(iS_{\mu\nu}\gamma^{\mu\nu} - \Gamma_\mu\gamma^\mu)_{ab}, \tag{2.9}$$

where $(\gamma^\mu)_{ab} = C_{bc}(\gamma^\mu)_a{}^c$ and $(\gamma^{\mu\nu})_{ab} = C_{bc}(\gamma^{\mu\nu})_a{}^c$ are symmetric matrices.

The usual creation and annihilation operators are obtained from the linear combinations $a_i^\pm = \frac{1}{\sqrt{2}}(q_i \mp i\eta_i)$, $[a_i^-, a_j^+] = \delta_{ij}$, $i, j = 1, 2$. So, the $osp(1|4)$ generators act irreducibly on the tensor product of the two Fock spaces,

$$\mathcal{O} = \{|n_1, n_2\rangle = |n_1\rangle|n_2\rangle, \qquad n_1, n_2 = 0, 1, 2, \ldots\}, \tag{2.10}$$

upon which the annihilation and creation operators act as

$$a_1^+|n_1, n_2\rangle = \sqrt{n_1+1}|n_1+1, n_2\rangle,$$
$$a_2^+|n_1, n_2\rangle = \sqrt{n_2+1}|n_1, n_2+1\rangle, \tag{2.11}$$

$$a_1^-|n_1, n_2\rangle = \sqrt{n_1}|n_1-1, n_2\rangle,$$
$$a_2^-|n_1, n_2\rangle = \sqrt{n_2}|n_1, n_2-1\rangle. \tag{2.12}$$

Here

$$N_1|n_1, n_2\rangle = n_1|n_1, n_2\rangle, \qquad N_2|n_1, n_2\rangle = n_2|n_1, n_2\rangle \tag{2.13}$$

are the number operators $N_1 = a_1^+a_1^-$ and $N_2 = a_2^+a_2^-$, respectively.

The $so(2, 3)$ subalgebra (2.6) and (2.7) acts, instead, reducibly over the whole space $\mathcal{O}$. Its irreducible representations are spanned by the subspaces

$$\mathcal{O}_+ = |++\rangle \bigoplus |--\rangle \quad \text{and}$$
$$\mathcal{O}_- = |+-\rangle \bigoplus |-+\rangle, \tag{2.14}$$

where we defined

$$|\pm\pm\rangle = \{|n_1, n_2\rangle_{\pm\pm} = |n_1\rangle_\pm|n_2\rangle_\pm, \quad n_1, n_2 = 0, 1, 2, \ldots\}, \tag{2.15}$$

$$|\pm\mp\rangle = \{|n_1, n_2\rangle_{\pm\mp} = |n_1\rangle_\pm|n_2\rangle_\mp, \quad n_1, n_2 = 0, 1, 2, \ldots\}, \tag{2.16}$$

$$|n\rangle_+ = |2n\rangle, \qquad |n\rangle_- = |2n+1\rangle. \tag{2.17}$$

In this representation the Casimir operators (2.1) take the same values (2.2) in both subspaces $\mathcal{O}_+$ and $\mathcal{O}_-$. Moreover, the square of the vector operator Γ_μ is Lorentz invariant and is also fixed here,

$$\Gamma^\mu\Gamma_\mu = \tfrac{1}{2}. \tag{2.18}$$

We also have the identities

$$\Gamma^\mu S_{\mu\nu} = S_{\nu\mu}\Gamma^\mu = -\frac{3i}{2}\Gamma_\nu, \qquad \epsilon^{\mu\nu\lambda\rho}S_{\nu\lambda}\Gamma_\rho = 0. \tag{2.19}$$

The operators $R_i = (-1)^{N_i} = \cos(\pi N_i)$, $i = 1, 2$ are defined in terms of the number operators (2.13). Acting on $\mathcal{O}$, they produce

$$R_1|n_1, n_2\rangle = (-1)^{n_1}|n_1, n_2\rangle,$$
$$R_2|n_1, n_2\rangle = (-1)^{n_2}|n_1, n_2\rangle.$$

Then we introduce the *total reflection operator*

$$\mathcal{R} = R_1R_2 = (-1)^{N_1+N_2}. \tag{2.20}$$

In accordance with (2.14) and (2.20),

$$\mathcal{R}\mathcal{O}_\pm = \pm\mathcal{O}_\pm, \qquad \mathcal{R}^2 = 1. \tag{2.21}$$

$\mathcal{R}$ plays the role of the grading operator in the $osp(1|4)$ superalgebra (2.6), (2.7), (2.8), and (2.9): the relation

$\{\mathcal{R}, a_i^{\pm}\} = 0$ implies

$$[\mathcal{R}, S_{\mu\nu}] = 0, \qquad [\mathcal{R}, \Gamma_\mu] = 0, \qquad \{\mathcal{R}, L_a\} = 0. \tag{2.22}$$

We notice that, on account of the identity $(-1)^{2N_2} = 1$ and the explicit form of the AdS generators (see the appendix), the reflection operator can be identified with two specific finite transformations,

$$\mathcal{R} = -\exp(i2\pi\Gamma_0) = \exp(i2\pi S^{12}), \tag{2.23}$$

i.e., an AdS 2π rotation in the subspace of two timelike coordinates, and a 2π space rotation, respectively. Since any unitary $SO(2,3)$ transformation U commutes with $\mathcal{R}$, we have, more generally, $\mathcal{R} = -\exp(i2\pi\tilde{\Gamma}_0) = \exp(i2\pi\tilde{S}^{12})$, where $\tilde{\Gamma}_0 = U\Gamma_0 U^\dagger$, $\tilde{S}^{12} = US^{12}U^\dagger$. In any case, the reflection operator, being a π-rotation in the 2D plane spanned by the coordinates q_i, is nonlocal. In the corresponding Schrödinger representation

$$\mathcal{R}\,\psi(\vec{q}) = \psi(-\vec{q}). \tag{2.24}$$

The eigenfunctions of $\mathcal{R}$ are therefore either even or odd,

$$\mathcal{R}\,\psi_\pm(\vec{q}) = \pm\psi_\pm(\vec{q}), \qquad \psi_\pm(\vec{q}) = \tfrac{1}{2}(\psi(\vec{q}) \pm \psi(-\vec{q})). \tag{2.25}$$

III. THE RELATIVISTIC WAVE EQUATIONS OF MAJORANA, DIRAC, AND STAUNTON

In this section we briefly review the Majorana equation [1], together with the related systems of spinor and vector equations proposed by Dirac [17], and by Staunton [18].

A. The Majorana equation

The Majorana equation [1] is a Lorentz invariant equation based on the unitary infinite-dimensional (reducible) representation of the AdS algebra described above,

$$(P^\mu\Gamma_\mu - M)|\Psi(x)\rangle = 0. \tag{3.1}$$

Here the x^μ are the space-time coordinates, $P_\mu = -i\partial/\partial x^\mu$, and $S_{\mu\nu} = i[\Gamma_\mu, \Gamma_\nu]$ is the translation-invariant part of the Lorentz generators

$$\mathcal{J}_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + S_{\mu\nu}. \tag{3.2}$$

Since the AdS algebra acts irreducibly only in the subspaces $\mathcal{O}_+$ and $\mathcal{O}_-$ of the internal Fock space, $|\Psi(x)\rangle = |\Psi_+(x)\rangle + |\Psi_-(x)\rangle$ is an infinite-component field expanded in these subspaces,

$$|\Psi_\pm(x)\rangle = \sum_{\mathcal{O}_\pm} \psi^{\pm}_{n_1,n_2}(x)|n_1, n_2\rangle, \tag{3.3}$$

where the "±" label indicates that the field has been expanded over the ± eigenspaces of $\mathcal{R}$,

$$\mathcal{R}\,|\Psi_\pm(x)\rangle = \pm|\Psi_\pm(x)\rangle, \qquad |\Psi_\pm(x)\rangle \equiv \Pi_\pm|\Psi(x)\rangle. \tag{3.4}$$

Here we have introduced the projectors

$$\Pi_+ = \tfrac{1}{2}(1+\mathcal{R}), \qquad \Pi_- = \tfrac{1}{2}(1-\mathcal{R}),$$
$$\Pi_+ + \Pi_- = 1, \qquad (\Pi_\pm)^2 = \Pi_\pm, \qquad \Pi_+\Pi_- = 0. \tag{3.5}$$

Note that

$$[\Pi_\pm, S_{\mu\nu}] = [\Pi_\pm, \Gamma_\mu] = 0, \qquad \Pi_\pm L_a = L_a\Pi_\mp. \tag{3.6}$$

The square of the spin vector, built out of the space part of $S_{\mu\nu}$, $S_i = \frac{1}{2}\epsilon_{ijk}S_{jk}$, is

$$S_i S_i = \hat{J}(\hat{J}+1), \tag{3.7}$$

where

$$\hat{J} = \frac{N_1 + N_2}{2}. \tag{3.8}$$

$\hat{J}$ is related to the AdS operator Γ_0 by

$$\Gamma_0 = \hat{J} + \tfrac{1}{2}. \tag{3.9}$$

The Majorana equation (3.1) has massive, massless, and tachyonic solutions. Below we restrict our analysis to the massive sector. Passing to the rest frame, we put $P^\mu = (m_J, 0, 0, 0)$ in (3.1). Then using (3.9), we obtain the celebrated J-dependent mass spectrum,

$$m_J = \frac{M}{(J+\frac{1}{2})}, \tag{3.10}$$

where J, the spin, is the eigenvalue of $\hat{J}$ acting over the physical subspace.

The Majorana equation admits two independent sets of solutions, composed of integer and of half-integer spins, respectively. These values correspond precisely to the eigen-subspaces $\mathcal{O}_+$ and $\mathcal{O}_-$ of the reflection operator, see (2.21). This follows from

$$\mathcal{R} = (-1)^{2\hat{J}}, \tag{3.11}$$

inferred from (2.20) and (3.8).

The solutions (3.3) of the Majorana equation are superpositions of those solutions which carry spin $J_\pm$ and mass $m_{J_\pm}$,

$$|\Psi_\pm(x)\rangle = \sum_{J_\pm}|\Psi_{J_\pm}(x)\rangle, \qquad J_+ = 0, 1, 2, \ldots,$$
$$J_- = 1/2,\ 3/2,\ 5/2, \ldots$$

Consistently, the square of the Pauli-Lubanski vector $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}S_{\nu\lambda}P_\rho$,

$$W^\mu W_\mu = -\tfrac{1}{2}S^{\mu\nu}S_{\mu\nu}P^2 + S_{\mu\nu}S^{\mu\lambda}P^\nu P_\lambda = \tfrac{1}{4}P^2 + (P\Gamma)^2, \tag{3.12}$$

takes, when restricted to these states, the (on-shell) value

$$W^\mu W_\mu|\Psi_J(x)\rangle = m_J^2 J(J+1)|\Psi_J(x)\rangle. \tag{3.13}$$

In this way, in the massive sector $P^2 < 0$ the Majorana equation (3.1) describes an infinite sum of irreducible representations of the Poincaré group of arbitrary spin J and of mass m_J related via Eq. (3.10). At the same time we can expand

$$|\Psi_{J_+}(x)\rangle = \sum_{n_1,n_2=0}^{\infty}(\psi_{++}^{n_1n_2}(x)|n_1,n_2\rangle_{++} + \psi_{--}^{n_1n_2}(x)|n_1,n_2\rangle_{--}), \quad \text{in } \mathcal{O}_+, \tag{3.14}$$

$$|\Psi_{J_-}(x)\rangle = \sum_{n_1,n_2=0}^{\infty}(\psi_{+-}^{n_1n_2}(x)|n_1,n_2\rangle_{+-} + \psi_{-+}^{n_1n_2}(x)|n_1,n_2\rangle_{-+}), \quad \text{in } \mathcal{O}_-, \tag{3.15}$$

In the rest-frame,

$$|\Psi_{J_+}^{(0)}(x)\rangle = \left(\sum_{n=0}^{J_+}\psi_{++}^n|J_+ - n, n\rangle_{++} + \sum_{n=1}^{J_+}\psi_{--}^n|J_+ - n, n-1\rangle_{--}\right)\exp(-itm_{J_+}), \tag{3.16}$$

$$|\Psi_{J_-}^{(0)}(x)\rangle = \sum_{n=0}^{J_- -(1/2)}\left(\psi_{+-}^n\left|J_- - \frac{1}{2} - n, n\right\rangle_{+-} + \psi_{-+}^n\left|J_- - \frac{1}{2} - n, n\right\rangle_{-+}\right)\exp(-itm_{J_-}), \tag{3.17}$$

where the $\psi_{\pm\pm}^n$, $\psi_{\pm\mp}^n$ are arbitrary constants and $t = x^0$. These expansions correspond to the superposition of the $2J+1$ possible polarization states (with $J = J_+$ or $J = J_-$). Every state is an eigenvector of the operator $S_z = S_{12} = \frac{1}{2}(N_1 - N_2)$, which is the projection of the spin on the z-axis; it has eigenvalues $\{-J, -J+1, \ldots, J-1, J\}$. (3.14) and (3.15) can be obtained by a suitable Lorentz transformation of (3.16) and (3.17). Hence, only $2J+1$ components are independent in (3.14) and (3.15).

Because both $\mathcal{O}_+$ and $\mathcal{O}_-$ carry irreducible representations of the Lorentz group, the solutions of integer and half-integer spin are, in principle, independent. We can make an important observation, however. The direct sum $\mathcal{O}_+ \bigoplus \mathcal{O}_-$ spans an irreducible representation of the $osp(1|4)$ superalgebra, where the spinor supercharge operator, L_a, interchanges the subspaces: L_a: $\mathcal{O}_+ \leftrightarrow \mathcal{O}_-$.

So we have the possibility to construct a (super)symmetry based on the L_a operators, which connect the solutions of the Majorana equation that live in the even ($\mathcal{O}_+$) and the odd ($\mathcal{O}_-$) sectors, respectively.

B. The "new Dirac equation"

The "new Dirac equation" (NDE) proposed by Dirac [17] four decades after Majorana's work, reads, in our conventions,[2]

$$D_a|\Psi(x)\rangle = 0, \quad \text{where } D_a = (-iP^\mu\gamma_\mu + m)_a{}^b L_b. \tag{3.18}$$

The formal similarity with the usual spin-1/2 Dirac equation is merely superficial: $|\Psi(x)\rangle$ here is an infinite-component field (due to its expansion in Fock space), and has no spinor index a.

Contracting D_a operator in (3.18) with $L^b(\lambda)_b{}^a$, where $(\lambda)_a{}^b$ is an arbitrary 4×4 matrix, we obtain 15 independent consistency equations (for $\lambda = \gamma_0\gamma_1\gamma_2\gamma_3$ contraction gives the identity $0 = 0$) that can be organized as follows:

$$(P^\mu\Gamma_\mu - \tfrac{1}{2}m)|\Psi(x)\rangle = 0, \tag{3.19}$$

$$(m\Gamma_\mu + \tfrac{1}{2}P_\mu - iS_{\mu\nu}P^\nu)|\Psi(x)\rangle = 0, \tag{3.20}$$

$$(\Gamma_\mu P_\nu - \Gamma_\nu P_\mu + imS_{\mu\nu})|\Psi(x)\rangle = 0, \tag{3.21}$$

$$W^\mu|\Psi(x)\rangle = 0, \tag{3.22}$$

where W^μ is the Pauli-Lubanski vector. The Klein-Gordon equation appears as a consistency condition, requiring the commutator to vanish,

$$[D_a, D_b]|\Psi(x)\rangle = iC_{ab}(P^2 + m^2)|\Psi(x)\rangle = 0. \tag{3.23}$$

The Klein-Gordon equation (3.23) selects, out of all solutions of Majorana equation (3.19), the one with the lowest possible spin, $J = 0$, as seen from the mass formula (3.10) with $M = \frac{m}{2}$. The NDE (3.18) describes therefore a spinless massive particle of positive energy. The solution of the Dirac equation in the internal space-coordinate representation (i.e. $\psi_+(x,q) = \langle q|\Psi_{J_+=0}(x)\rangle$) is

$$\psi_+(x,q) = A\exp\left\{\frac{-m(q_1^2+q_2^2) - ip^2(q_1^2-q_2^2) + i2p^3q_1q_2}{2(p^0-p^1)}\right\} \times \exp(ix^\mu p_\mu), \tag{3.24}$$

where A is an arbitrary constant. Since this is an even function under internal reflection, we have

$$\mathcal{R}\,\psi_+(x,q) = \psi_+(x,q).$$

In the rest frame the solution reduces to the ground state of

[2] The correspondence of Dirac's notations [17] with ours is $q_a = L_a$, $\alpha^0 = (\gamma^0)^{ab}$, $\alpha^1 = (\gamma^2)^{ab}$, $\alpha^2 = (\gamma^3)^{ab}$, and $\alpha^3 = (\gamma^1)^{ab}$.

a planar harmonic oscillator,

$$\psi_+^{(0)}(t,q) = A\exp\left\{\frac{-(q_1^2+q_2^2)}{2}\right\}\exp(-itm) = A\langle q|00\rangle\exp(-itm). \quad (3.25)$$

C. The Staunton equation

In 1974 Staunton [18] observed that the Majorana and Klein-Gordon equations can both be obtained directly from the consistency condition (3.20), instead of the original Dirac equation, (3.18). Then, Staunton's idea was to modify (3.20) by putting an arbitrary coefficient, κ, in front of P_μ. In his analysis, Staunton arrived at the conclusion that the modified equation is consistent with the Poincaré representation for only two values of this parameter, namely, for $\kappa = \frac{1}{2}$ and $\kappa = 1$. We express this as

$$D_\mu^{(\kappa)}|\Psi(x)\rangle = 0, \qquad D_\mu^{(\kappa)} = m\Gamma_\mu + \kappa P_\mu - iS_{\mu\nu}P^\nu. \quad (3.26)$$

As consistency conditions, (3.26) implies the Klein-Gordon and Majorana equations,

$$(P^2+m^2)|\Psi(x)\rangle = 0 \quad \text{and} \quad (P^\mu\Gamma_\mu - m\kappa)|\Psi(x)\rangle = 0. \quad (3.27)$$

We require that the commutator annihilates the physical states,

$$[D_\mu, D_\nu]|\Psi(x)\rangle = [-i(P^2+m^2)S_{\mu\nu} - P_\mu D_\nu + P_\nu D_\mu] \times |\Psi(x)\rangle = 0.$$

Then, contracting this equation with $S^{\mu\nu}$ and taking into account the first relation from (2.2), we find that (3.26) implies the Klein-Gordon equation.

The Majorana equation appears in turn upon contracting (3.26) with P^μ and using the Klein-Gordon equation.

From (3.12) and (3.27) we get, for $\kappa = \frac{1}{2}$, $W^\mu W_\mu = 0$. The spin is hence zero, and (3.26) is equivalent to the original Dirac equation (3.18). For $\kappa = 1$ we have, instead,

$$W^\mu W_\mu = m^2\,\tfrac{1}{2}(1+\tfrac{1}{2}), \quad (3.28)$$

so that (3.26) describes a spin-1/2 particle.

For $\kappa = 1$, the general solution of (3.26) can be expressed, in internal coordinate space, in terms of the solution (3.24) of the Dirac system,

$$\Psi_-(x,q) = (Bq_1 + Cq_2)\Psi_+(x,q), \quad (3.29)$$

where B, C are arbitrary constants. Note that $\Psi_-(x,q)$ is an odd function of q_i,

$$\mathcal{R}\,\Psi_-(x,q) = -\Psi_-(x,q).$$

In the rest frame, (3.29) reduces to the first exited state of a planar harmonic oscillator,

$$\Psi_-^{(0)}(t,q) = (Bq_1 + Cq_2)\exp\left\{\frac{-(q_1^2+q_2^2)}{2}\right\}\exp(-itm) = (B\langle q|10\rangle + C\langle q|01\rangle)\exp(-itm). \quad (3.30)$$

In the Fock space, the rest-frame solutions (3.25) and (3.30) take the form

$$|\Psi_+^{(0)}(x)\rangle = \exp(-itm)\psi_{++}^0|00\rangle_{++}, \quad \text{spin } 0,$$
$$|\Psi_-^{(0)}(x)\rangle = \exp(-itm)(\psi_{+-}^0|00\rangle_{+-} + \psi_{-+}^0|00\rangle_{-+}), \quad \text{spin } \tfrac{1}{2}, \quad (3.31)$$

(see (3.16) for $J_+ = 0$, and (3.17) for $J_- = \frac{1}{2}$). After an arbitrary Lorentz transformation, all states in the corresponding Fock subspace $\mathcal{O}_+$ or $\mathcal{O}_-$ can be occupied, cf. (3.14) and (3.15). All coefficients will be linear combinations of the only independent coefficient ψ_{++}^{00} (spin-0 case), or of ψ_{+-}^{00} and ψ_{-+}^{00} (spin-1/2 case). Note here that

$$S_z|00\rangle_{++} = 0, \qquad S_z|00\rangle_{+-} = -\tfrac{1}{2}|00\rangle_{+-}, \qquad S_z|00\rangle_{-+} = \tfrac{1}{2}|00\rangle_{-+}.$$

IV. OUR UNIFIED SUPERSYMMETRIC THEORY

We have seen that the Dirac and Staunton equations extract, via the Klein-Gordon equation, the lowest spin states, namely $J = 0$ and $J = \frac{1}{2}$, respectively, from the Majorana spectrum (3.1). Now we show how these two cases can be merged into a single supersymmetric one. We posit the equation

$$(D_\mu^{1/2}\Pi_+ + D_\mu^1\Pi_-)|\Psi(x)\rangle = 0, \quad (4.1)$$

where the D_μ^κ's are the operators in the Staunton equation (3.26), and the $\Pi_\pm$ are the projectors (3.5). Then (4.1) becomes

$$\mathcal{D}_\mu|\Psi(x)\rangle = 0, \qquad \mathcal{D}_\mu = m\Gamma_\mu + \hat{\kappa}P_\mu - iS_{\mu\nu}P^\nu. \quad (4.2)$$

Our Eq. (4.2) amounts hence to promoting Staunton's constant κ to an *operator*, $\hat{\kappa}$, on $\mathcal{O}$,

$$\hat{\kappa} = \tfrac{1}{4}(3 - \mathcal{R}), \quad (4.3)$$

whose eigenvalues are precisely those appropriate for the new Dirac and Staunton equations,

$$\hat{\kappa}|\Psi_+(x)\rangle = \tfrac{1}{2}|\Psi_+(x)\rangle, \qquad \hat{\kappa}|\Psi_-(x)\rangle = |\Psi_-(x)\rangle. \quad (4.4)$$

Let us note, however, that the similarity of (4.2) with Staunton's equation (3.26) is deceiving, in that the operator $\hat{\kappa}$ is nonlocal in the internal translation-invariant variables q_i, $i = 1, 2$. Moreover, since the general solution of (4.2) is an arbitrary combination of (3.24) and (3.29), our equation activates simultaneously the spin-0 and spin-1/2 fields. Projecting $\Pi_+\mathcal{D}_\mu|\Psi(x)\rangle = D_\mu^{1/2}|\Psi_+(x)\rangle$, we get the spin-0 Dirac system reduced onto $\mathcal{O}_+$, and for

$\Pi_-\mathcal{D}_\mu|\Psi(x)\rangle = D^1_\mu|\Psi_-(x)\rangle$ we get the spin-1/2 Staunton system reduced onto $\mathcal{O}_-$. In this way, our new equation describes a spin-$(0,\frac{1}{2})$ supermultiplet. Then consistency of our new Eq. (4.2) implies, once again, a Klein-Gordon and a Majorana equation

$$(P^2 + m^2)|\Psi(x)\rangle = 0 \tag{4.5}$$

and

$$(P^\mu\Gamma_\mu - m\hat{\kappa})|\Psi(x)\rangle = 0, \tag{4.6}$$

respectively. The first one fixes the mass, *in both sectors*, to be m. The mass term in the Majorana equation is now an operator $M = m\hat{\kappa}$, which takes different values in the even and odd subspaces of the Hilbert space,

$$\begin{aligned} \Pi_+(P^\mu\Gamma_\mu - m\hat{\kappa})|\Psi(x)\rangle &= (P^\mu\Gamma_\mu - \tfrac{1}{2}m)|\Psi_+(x)\rangle, \qquad \text{integer spin,} \\ \Pi_-(P^\mu\Gamma_\mu - m\hat{\kappa})|\Psi(x)\rangle &= (P^\mu\Gamma_\mu - m)|\Psi_-(x)\rangle, \qquad \text{half-integer spin.} \end{aligned} \tag{4.7}$$

The Klein-Gordon equation (4.5) implies that the spin of every solution in (4.7) is necessarily the lowest possible one, namely, zero for $|\Psi_+(x)\rangle$ and $\frac{1}{2}$ for $|\Psi_-(x)\rangle$. This is consistent with the mass formula (3.10), yielding the *same* mass for the fields $|\Psi_+(x)\rangle$ and $|\Psi_-(x)\rangle$,

$$\frac{m/2}{0+1/2} = \frac{m}{1/2+1/2} = m,$$

cf. also (4.5). Because of our specific representation (2.5) of the $SO(2,3)$ group, we have the relations (2.19), from which we obtain the identities

$$W^\mu\mathcal{D}_\mu \equiv 0, \qquad \mathcal{P}^\mu\mathcal{D}_\mu \equiv 0, \tag{4.8}$$

where W^μ is the Pauli-Lubanski vector, and

$$\mathcal{P}_\mu = \tfrac{1}{2}P_\mu + (3\hat{\kappa} - 1)(P\Gamma)\Gamma_\mu + i\hat{\kappa}S_{\mu\nu}P^\nu. \tag{4.9}$$

Relations (4.8) indicate that only two components of $\mathcal{D}_\mu$ yield independent equations. Four components are necessary to assure the covariance of the equations.

Now we identify the supercharge operator. Let us consider the Hermitian 4-component spinor operator

$$Q_a = \frac{1}{\sqrt{m}}(-i\mathcal{R}P^\mu\gamma_\mu + m)_a{}^bL_b, \qquad a = 1,\ldots,4, \tag{4.10}$$

where the L_a are those internal $osp(1|4)$ generators in (2.3). This is an observable operator with respect to our equations,

$$\begin{aligned} [\mathcal{D}_\mu, Q_a] = &-\frac{i}{2m}\mathcal{R}(P^2 + m^2)(\gamma_\mu)_a{}^bL_b \\ &+ \left(i\gamma_\mu - \frac{1}{m}P_\mu\right)_a^{\ b}D_b\Pi_+ \approx 0, \end{aligned} \tag{4.11}$$

and consequently, also with respect to the Klein-Gordon and the Majorana equations. In (4.11) D_b is the Dirac operator from (3.18); here and in what follows $\approx$ denotes equality on the surface defined by the corresponding field equations.

This operator transforms the spin-0 particle into the spin-1/2 particle, and vice versa. To show this, let $|\Psi_\pm(x)\rangle = \Pi_\pm|\Psi(x)\rangle$ be solutions of (4.2). Then, due to Eqs. (3.6) and (4.11), we have $Q_a\Pi_\pm = \Pi_\mp Q_a$, and

$$Q_a|\Psi_\mp(x)\rangle \approx |\Psi_\pm(x)\rangle.$$

It is illustrative to verify this in the rest frame, where

$$Q_a^{(0)}|\Psi_\pm^{(0)}(x)\rangle = \sqrt{2m}(a_1^\pm, a_2^\pm, \pm ia_1^\pm, \pm ia_2^\pm)|\Psi_\pm^{(0)}(x)\rangle. \tag{4.12}$$

With (3.31), Eq. (4.12) yields

$$Q_a^{(0)}|\Psi_+^{(0)}(x)\rangle \approx |\Psi_-^{(0)}(x)\rangle, \qquad Q_a^{(0)}|\Psi_-^{(0)}(x)\rangle \approx |\Psi_+^{(0)}(x)\rangle.$$

The Q_a operator satisfies nonlinear anticommutation relations,

$$\begin{aligned} \{Q_a, Q_b\} = &(-3P^\mu + Z^\mu)(\gamma_\mu)_{ab} - 4imZ^{\mu\nu}(\gamma_{\mu\nu})_{ab} \\ &+ \frac{2}{m}(P^2 + m^2)\Big(iS_{\mu\nu}\gamma^{\mu\nu} + 4i\frac{1}{P^2}P_\mu S_{\nu\lambda}P^\lambda\gamma^{\mu\nu} \\ &+ \Gamma_\mu\gamma^\mu\Big)_{ab} - \frac{4}{m}(P\Gamma - m\hat{\kappa})(\gamma_\mu P^\mu)_{ab}, \end{aligned} \tag{4.13}$$

where

$$Z^\mu = -\mathcal{R}P^\mu, \qquad Z^{\mu\nu} = \pi^{\mu\rho}\pi^{\nu\lambda}S_{\rho\lambda}, \tag{4.14}$$

and

$$\pi_{\mu\nu} = \eta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} \approx \eta_{\mu\nu} + \frac{P_\mu P_\nu}{m^2}. \tag{4.15}$$

We note that $Z_{\mu\nu}$ is a covariant expression of the spin operator S_{ij}. (In the rest frame, the projectors reduce to $\pi_{\mu\nu} = (0, \delta_{ij})$ so that $Z_{\mu\nu}$ reduces to S_{ij}.) Note also that on shell, the first term in the anticommutator (4.13), $(-3P^\mu + Z^\mu)\gamma_\mu = -(3+\mathcal{R})(P\gamma)$, is positive definite.

The terms in the second line in (4.13) include, as commuting factors, the Klein-Gordon and the Majorana operators. Putting them to zero, the on-shell anticommutator is obtained,

$$\{Q_a, Q_b\} \approx (-3P^\mu + Z^\mu)(\gamma_\mu)_{ab} - 4imZ^{\mu\nu}(\gamma_{\mu\nu})_{ab}. \tag{4.16}$$

The translation and Lorentz generators P_μ and $\mathcal{J}_{\mu\nu}$, together with the supercharge Q_a, obey the commutation relations

$$[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\lambda\rho}] = i(\eta_{\mu\lambda}\mathcal{J}_{\nu\rho} + \eta_{\nu\rho}\mathcal{J}_{\mu\lambda} - \eta_{\mu\rho}\mathcal{J}_{\nu\lambda} - \eta_{\nu\lambda}\mathcal{J}_{\mu\rho}), \tag{4.17}$$

$$[\mathcal{J}_{\mu\nu}, P_\lambda] = i(\eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu), \qquad [P_\mu, P_\nu] = 0,$$
$$[\mathcal{J}_{\mu\nu}, Q_a] = -(\gamma_{\mu\nu})_a{}^b Q_b, \qquad [P_\mu, Q_a] = 0. \tag{4.18}$$

The operators P_μ, $\mathcal{J}_{\mu\nu}$, and Q_a, together with the operators Z_μ and $Z_{\mu\nu}$ appearing in the anticommutator of the supercharge, form the set of symmetry generators for our system,

$$[\mathcal{D}_\mu, \mathcal{A}] \approx 0, \quad \text{for } \mathcal{A} = \mathcal{J}_{\mu\nu}, P_\mu, Q_a, Z_\mu, Z_{\mu\nu}. \tag{4.19}$$

The reflection operator plays, in our superalgebraic structure, the role of the grading operator,

$$[\mathcal{R}, \mathcal{J}_{\mu\nu}] = 0, \qquad [\mathcal{R}, P_\mu] = 0, \qquad [\mathcal{R}, Z_\mu] = 0,$$
$$[\mathcal{R}, Z_{\mu\nu}] = 0, \qquad \{\mathcal{R}, Q_a\} = 0. \tag{4.20}$$

However here, unlike the 2 + 1 dimensions [22], we do not have a Lie superalgebraic structure on shell, since, although Z_μ and $Z_{\mu\nu}$ are translationally invariant vector respectively antisymmetric tensor operators, their commutators with the supercharge Q_a are nontrivial, nonlinear in symmetry generators,

$$[Z_\mu, Q_a] = 2Z_\mu Q_a,$$
$$[Z_{\mu\nu}, Q_a] = -\pi_{\mu\lambda}\pi_{\nu\rho}(\gamma^{\lambda\rho})_a{}^b Q_b. \tag{4.21}$$

Note also that here

$$[Z_\mu, Z_\nu] = 0, \quad [Z_\mu, Z_{\nu\lambda}] = 0,$$
$$[Z_{\mu\nu}, Z_{\lambda\rho}] = i(\pi_{\mu\lambda}Z_{\nu\rho} + \pi_{\nu\rho}Z_{\mu\lambda} - \pi_{\mu\rho}Z_{\nu\lambda} - \pi_{\nu\lambda}Z_{\mu\rho}), \tag{4.22}$$

to be compared with the tensorial extensions which appear in supergravity, and for superbranes [23–25].

In spite of these complications due to nonlinearity, the invariant operator playing the role of the Casimir operator is easily identified: up to the m^2 factor, it can be, namely, identified as the superspin (see below),

$$\mathcal{C} = W^\mu W_\mu - \tfrac{1}{64}\chi^\mu\chi_\mu, \qquad [\mathcal{C}, \mathcal{A}] \approx 0, \tag{4.23}$$

where $\chi_\mu = Q^a(\gamma_\mu)_a{}^b Q_b$. On shell it takes the value

$$\mathcal{C} = m^2.$$

We also note that although we have a nonlinear, W-type [19], symmetry superalgebra, the Jacobi identities are valid owing to the associativity of all involved operator products.

V. SUPERSYMMETRIC HIGHER-SPIN MAJORANA-KLEIN-GORDON SYSTEM

As we already noted, the main properties of the Majorana equation strongly suggest that some kind of supersymmetry could be involved. We have also shown that, for the lowest (namely the zero and one-half) spin states, a supersymmetric theory can indeed be constructed. It is therefore natural to ask if it is possible to extend the supersymmetry to some arbitrary-spin massive supermultiplet. *A priori*, we know that P^2 should be a Casimir operator. Requiring supersymmetry, we expect the appearance of the Klein-Gordon equation as a consistency condition. In fact, when we impose, simultaneously,

$$(P^2 + m^2)|\Psi_J(x)\rangle = 0, \qquad (P^\mu\Gamma_\mu - M_J)|\Psi_J(x)\rangle = 0, \tag{5.1}$$

where $M_J = (J + \frac{1}{2})m$, $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, we extract from the infinite spectrum of the Majorana equation an irreducible representation of the Poincaré group with the mass m, spin J, and positive energy.

Now, we extend the supersymmetry of the spin-$(0, \frac{1}{2})$ supermultiplet constructed above to an arbitrary spin-(J_+, J_-) supermultiplet such that $|J_+ - J_-| = \frac{1}{2}$. By convention, J_+ is integer and J_- is half-integer.

First, we generalize the Majorana equation (3.1) to

$$(P^\mu\Gamma_\mu - \hat{M}_S)|\Psi(x)\rangle = 0, \tag{5.2}$$

where the mass parameter has been traded for an operator, cf. (4.6),

$$\hat{M}_S = \frac{m}{2}(J_+ + J_- + 1 + (J_+ - J_-)\mathcal{R}), \tag{5.3}$$

$$J_+ = 0, 1, \ldots, \qquad J_- = \tfrac{1}{2}, \tfrac{3}{2}, \ldots, \qquad |J_+ - J_-| = \tfrac{1}{2}. \tag{5.4}$$

Projected to the even and odd Fock subspaces, this equation is equivalent to

$$\Pi_\pm(P^\mu\Gamma_\mu - \hat{M}_S)|\Psi(x)\rangle = (P^\mu\Gamma_\mu - M_{J_\pm})|\Psi_\pm(x)\rangle = 0, \tag{5.5}$$

where $|\Psi_\pm(x)\rangle = \Pi_\pm|\Psi(x)\rangle$, see (3.4). By construction, the solution of (5.2) is the sum of $|\Psi_+(x)\rangle$ and $|\Psi_-(x)\rangle$ belonging to the integer (respectively half-integer) spin subseries of solutions of the Majorana Eq. (3.1). It follows from the mass formula (3.10) that the states with spins J_+ and J_- have again equal masses, namely m. Equation (5.2) has a supermultiplet in its spectrum therefore. The equations which describe it read

$$(P^2 + m^2)|\Psi_S(x)\rangle = 0, \qquad (P^\mu\Gamma_\mu - \hat{M}_S)|\Psi_S(x)\rangle = 0. \tag{5.6}$$

Their spin content is

$$\left(\mathcal{S}-\frac{\triangle J}{2},\mathcal{S}+\frac{\triangle J}{2}\right),\qquad \mathcal{S}=\frac{J_{+}+J_{-}}{2},\qquad \triangle J=J_{-}-J_{+}=\pm\frac{1}{2}. \tag{5.7}$$

The observable Hermitian supercharge is

$$Q_a^{(\pm)}=\frac{1}{\sqrt{m}}(\mp i\mathcal{R}P^{\mu}\gamma_{\mu}+m)_a{}^bL_b,\qquad \triangle J=\pm\frac{1}{2}. \tag{5.8}$$

Its commutator with Eqs. (5.6) vanishes on shell,

$$[P^{\mu}\Gamma_{\mu}-\hat{M}_S,Q_a^{(\pm)}]=\pm\frac{\mathcal{R}L_a}{2}(P^2+m^2)\approx 0,\qquad [(P^2+m^2),Q_a^{(\pm)}]=0. \tag{5.9}$$

The first equation from (5.9) means that supersymmetry itself requires satisfying the Klein-Gordon equation. Hence, the Q_a is an (observable) odd symmetry generator for Eqs. (5.6). Being a spinor operator, it intertwines physical states of spin J_+ and of spin J_-.

On shell (given by Eqs. (5.6)), instead of (4.16), we obtain the anticommutation relation

$$\{Q_a^{(\pm)},Q_b^{(\pm)}\}\approx -2(1+2\mathcal{S})P^{\mu}(\gamma_{\mu})_{ab}\pm Z^{\mu}(\gamma_{\mu})_{ab}-4imZ^{\mu\nu}(\gamma_{\mu\nu})_{ab},\qquad \triangle J=\pm\tfrac{1}{2}, \tag{5.10}$$

where

$$\mathcal{S}=\tfrac{1}{4},\tfrac{5}{4},\tfrac{9}{4},\ldots,\quad \text{for } \triangle J=\tfrac{1}{2},\quad\text{and}\quad \mathcal{S}=\tfrac{3}{4},\tfrac{7}{4},\tfrac{11}{4},\ldots,\quad \text{for } \triangle J=-\tfrac{1}{2},$$

and Z's are the same as in (4.14).

The form of the superalgebraic structure (4.16), (4.17), (4.18), (4.20), (4.21), and (4.22), with (4.16) exchanged for (5.10), is preserved. The $Z's$ here are the conserved charges with respect to Eqs. (5.6). The invariant operator related to superspin is now

$$\mathcal{C}^{+}=W^{\mu}W_{\mu}-\tfrac{1}{64}\chi^{\mu}\chi_{\mu},\qquad \triangle J=+\tfrac{1}{2}, \tag{5.11}$$

$$\mathcal{C}^{-}=W^{\mu}W_{\mu}+\tfrac{1}{3\cdot 64}\chi^{\mu}\chi_{\mu},\qquad \triangle J=-\tfrac{1}{2}. \tag{5.12}$$

It takes the on-shell values

$$\mathcal{C}^{+}=2m^2(\mathcal{S}+\tfrac{3}{4})(\mathcal{S}+\tfrac{1}{4}),\qquad \mathcal{S}=\tfrac{1}{4},\tfrac{5}{4},\tfrac{9}{4},\ldots, \tag{5.13}$$

$$\mathcal{C}^{-}=\frac{2m^2}{3}\left(\mathcal{S}-\frac{1}{4}\right)\left(\mathcal{S}+\frac{5}{4}\right),\qquad \mathcal{S}=\frac{3}{4},\frac{7}{4},\frac{11}{4},\ldots \tag{5.14}$$

In this way, the Majorana-Klein-Gordon (5.6) system describes, universally, a massive supermultiplet of spin content $(J_+,J_-)=(\mathcal{S}-\frac{\triangle J}{2},\mathcal{S}+\frac{\triangle J}{2})$. Our previous results are plainly recovered for $\mathcal{S}=\frac{1}{4}$ and $\triangle J=\frac{1}{2}$.

A. The large superspin limit

It is interesting to study the behavior of our superalgebra for large values of the superspin. Let us first redefine the supercharges (5.8),

$$\mathcal{Q}_a^{(\pm)}=\frac{1}{\sqrt{1+2\mathcal{S}}}Q_a^{(\pm)}. \tag{5.15}$$

Off shell they satisfy the anticommutation relation

$$\begin{aligned}\{\mathcal{Q}_a^{(\pm)},\mathcal{Q}_b^{(\pm)}\}={}&-2P^{\mu}(\gamma_{\mu})_{ab}\pm\frac{1}{1+2\mathcal{S}}Z^{\mu}(\gamma_{\mu})_{ab}\\&-\frac{4im}{1+2\mathcal{S}}Z^{\mu\nu}(\gamma_{\mu\nu})_{ab}+\frac{2(P^2+m^2)}{m(1+2\mathcal{S})}\\&\times\left(iS_{\mu\nu}\gamma^{\mu\nu}+4i\frac{1}{P^2}P_{\mu}S_{\nu\lambda}P^{\lambda}\gamma^{\mu\nu}+\Gamma_{\mu}\gamma^{\mu}\right)_{ab}\\&-\frac{4}{m(1+2\mathcal{S})}(P\Gamma-\hat{M}_{\mathcal{S}})(\gamma_{\mu}P^{\mu})_{ab},\end{aligned} \tag{5.16}$$

while the commutator of Z_μ and $Z_{\mu\nu}$ with $\mathcal{Q}_a^{(\pm)}$ remains of the form (4.21). When $\mathcal{S}\to\infty$, (5.16) takes the usual form of the $N=1$ supersymmetric anticommutation relation,

$$\{\mathcal{Q}_a^{(\pm)},\mathcal{Q}_b^{(\pm)}\}=-2P^{\mu}(\gamma_{\mu})_{ab}. \tag{5.17}$$

On the other hand, defining

$$W_{\mu\nu}=-\frac{4m}{1+2\mathcal{S}}Z_{\mu\nu}$$

we get, in this limit,

$$\{\mathcal{Q}_a^{(\pm)},\mathcal{Q}_b^{(\pm)}\}=-2P^{\mu}(\gamma_{\mu})_{ab}+W^{\mu\nu}(\gamma_{\mu\nu})_{ab},\quad\text{and}\quad [W_{\mu\nu},\mathcal{Q}_a^{(\pm)}]=0. \tag{5.18}$$

Using (4.22), we have

$$[W_{\mu\nu},W_{\lambda\rho}]=\frac{4m}{1+2\mathcal{S}}(\pi_{\mu\lambda}W_{\nu\rho}+\pi_{\nu\rho}W_{\mu\lambda}-\pi_{\mu\rho}W_{\nu\lambda}-\pi_{\nu\lambda}W_{\mu\rho}),$$

and in the limit $\mathcal{S}\to\infty$ we find that the $W_{\mu\nu}$ turns into an Abelian tensorial central charge.

So, with (5.17) or (5.18), off shell we obtain the super-Poincaré algebra without or with tensorial central extension. Note that in the large-spin limit the nonlinearity in the superalgebraic structure disappears off shell.

Remember that our construction includes, from the beginning, a hidden length parameter l, used to transform the canonical operators q_i and η_i into dimensionless 2D Heisenberg generators. This parameter can be identified with the AdS radius. Then we note also here that the (un)extended Poincaré superalgebra we have gotten can be obtained as a limit of the $osp(1|4)$, when the AdS radius tends to infinity [26,27].

VI. CONCLUDING REMARKS AND OUTLOOK

We have constructed the covariant $(3+1)$D vector set of linear differential equations, which describe supermultiplets of spins-0 and $1/2$ fields. In this theory, the spin degrees of freedom are carried by an internal 2D Heisenberg algebra. Extending the construction to get a supermultiplet of spins $(j, j+1/2)$, requires, however, to use a modified, first order Majorana equation, augmented with the second order Klein-Gordon equation.

Our results here can be compared with those in the $(2+1)$D case [20,28–30], where an analogous supersymmetric construction has been carried out in [22]. It is based on the 1D deformed Heisenberg algebra with reflection [31], $[a^-, a^+] = 1 + \nu R$, $\{a^\pm, R\} = 0$, $R^2 = 1$, and involves the $osp(1|2)$ superalgebra that allowed us to describe, universally, either an anyonic supermultiplet of spins $\pm(s, s+1/2)$ or a supermultiplet, $(j, j+1/2)$, of usual fields of integer and half-integer spin. In the former case, $s = \frac{1}{4}(1+\nu) > 0$ can take arbitrary real values for the unitary infinite-dimensional representations of the deformed Heisenberg algebra, characterized by the deformation parameter values $\nu > -1$. The latter case arises when we choose finite-dimensional nonunitary representations of the algebra corresponding to the negative odd values of ν [31]. The underformed Heisenberg algebra ($\nu = 0$) gives a semionic supermultiplet $(1/4, 3/4)$ [32].

In $(2+1)$ dimensions spin is a pseudoscalar, and both members of the supermultiplet have, on-shell, the same number of spin degrees of freedom (namely equal to one). As a consequence, there, on shell, appears a usual Poincaré superalgebra. In the present $(3+1)$D case, the integer and half-integer spin members of a supermultiplet are described on shell by different numbers of spin components (cf. [33,34]), and on shell we have a nonlinear superalgebra, that only in the large superspin limit reduces to the Poincaré superalgebra with or without tensorial central charge.

By an appropriate generalization of the Majorana-Klein-Gordon theory presented in Sec. V, one can obtain a bosonized supersymmetric system with a more general, exotic supermultiplet that includes fields of spins shifted by $n + \frac{1}{2}$, $n = 1, 2, \ldots$ In such a generalized theory, unlike in the case $n = 0$ considered here, the supercharge will be a covariant object of spin $n + \frac{1}{2}$ of the order $2n+1$ in the space-time translation generator P_μ, and will generate some more complicated superalgebra. Such a generalization will be considered elsewhere.

Dirac observed that his new equation is inconsistent with the usual minimal U(1) gauge coupling [17,35,36], namely, that consistency requires $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0$. Hence, the electromagnetic field can only be a pure gauge. The aim of Staunton has been precisely to find an improved theory which would remove this inconsistency: coupling the particle to a gauge field is essential. However, while he shows that his theory does not produce immediately the obstruction $F_{\mu\nu} = 0$, the consistency, in fact, was not proved in [18].[3] This—fundamental—question still remains open.

Staunton's ideas have also been extended to a curved background [37]. The generalization of our theory presented here to interactions with gauge fields and gravity deserves separate study.

Our theory has a close relationship with the "supertwistor" approach that is used, in particular, for the description of higher-spin massless fields [26,27], based on the fundamental representation of $osp(1|4)$ (see also [38] for a related approach). Indeed, our Majorana spinor L_a generates, like a twistor λ_α does, the 2D Heisenberg algebra (2.4), $[L_a, L_b] = iC_{ab}$, and satisfies the relation $L^a L_a = 2i$ of the form of the helicity constraint in the twistor theory. Moreover, our vector equation (4.2) has a form (see Eq. (6.1) below) similar to the twistor relation $P_\mu = \lambda\gamma_\mu\lambda$, generating the mass zero constraint. In the supertwistor approach, in the simplest case, the Grassmann-even twistor variable λ_α is combined with the scalar Clifford algebra generator ψ, $\psi^2 = 1$, to realize the $osp(1|4)$-odd generator as a product $Q_\alpha = \psi\lambda_\alpha$. There the role of the grading operator is played by the external, Grassmann-even operator, anticommuting with ψ. In our case instead, the reflection operator $\mathcal{R}$ is identified as the grading operator.[4]

It is also instructive to compare the equations of Majorana, Dirac, and Staunton rewritten in the form

$$\frac{1}{4}P^\mu(\gamma_\mu)^{ab}L_aL_b + M = 0, \qquad \text{Majorana,}$$
$$mL^a - iP^\mu(\gamma_\mu)^{ab}L_b = 0, \qquad \text{NDE,}$$
$$\kappa P_\mu - \frac{m}{4}(\gamma_\mu)^{ab}L_aL_b - \frac{1}{2}P^\nu(\gamma_{\mu\nu})^{ab}L_aL_b = 0, \qquad \text{Staunton,} \tag{6.1}$$

[3]The commutator of the interacting Majorana and Klein-Gordon equations (58.b) and (59) from [18] produces a new, missing, nontrivial condition that includes a derivative term $\partial_\lambda F_{\mu\nu}$, and the checking process should continue.

[4]Our theory is different from the classical, related approach to higher-spin massless fields [39] based on higher-rank symmetric Lorentz tensors, see [40,41] and references therein. Here, the supersymmetric higher-spin fields are massive, and the spin degrees of freedom are hidden in the internal Fock space (cf. also [42]).

with the constraints appearing in the twistor formulation of massive spin fields, see e.g. [43–46].

So, it would be interesting to work out in more details the relation of our bosonized supersymmetry with the usual one in the supertwistor approach.

Let us note here that a kind of ‘‘generalization of global supersymmetry’’ [47] and a ‘‘bosonic counterpart of supersymmetry’’ [44] were discussed earlier in the literature in the context of the massive spin theory. The approaches of van Dam-Biedenharn, of Fedoruk-Lukierski, and our present one here share the common feature that all three theories are constructed in terms of infinite-dimensional representation of the Lorentz group, and involve internal, bosonic twistorlike variables [cf. also [48] for the massless case]. Unlike our case, the models [44,47] are characterized by an infinite number of physical states of integer and half-integer spin $J = 0, \frac{1}{2}, \ldots$, which lie either on a linear Regge trajectory $m_J^2 \propto J + \frac{1}{2}$ [47], or have a fixed mass $m^2 = \text{const}$ [44], cf. the Majorana spectrum (3.10) $m_J^2 \propto (J + \frac{1}{2})^{-2}$. The essential difference is, however, that in the approaches [44,47] the Poincaré algebra is extended by a spinorial *even* operator. The latter interchanges integer and half-integer spin physical states and satisfies *commutation* relations [cf. [48]. *Additional* Lorentz-scalar, topologically nontrivial isospin variables transmute, after quantization, the even spinorial integrals of motion into odd supercharges]. A remarkable property of the theory in [47] is that its spinorial charge involves a space-time nonlocal operator that changes not only the spin, but also the mass of the physical states consistently with the Regge character of the spectrum. In [44], P^2, instead, plays a role of the Casimir of the extended Poincaré symmetry *algebra*. In our case the bosonized supersymmetry is characterized by a nonlinear *superalgebra*, realized on a finite supermultiplet, extracted from the infinite Majorana spectrum.

In summary, we have shown that the Majorana-Dirac-Staunton theory possesses a rich structure that allowed us to construct its supersymmetric generalization without introducing any (Grassmann odd, fermionic) additional spin degrees of freedom. Our supersymmetric generalization relies on nonlocality in the internal, translation-invariant (twistorlike) bosonic variables. The superalgebraic structure we obtain admits a nontrivial internal symmetry, namely $Z_{\mu\nu}$. This nonlocality is similar to that in other bosonization constructions, where fermions are described in terms of bosonic variables [49]. Such a kind of boson-fermion relation is, in turn, rooted in the underlying nontrivial topology, see e.g. [50–55]. We hope that investigation of the field systems like those presented here could reveal further connections between supersymmetry and topology.

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APPENDIX

The generators (2.5) are, explicitly,

$$S^{01} = -\frac{i}{4}(a_1^{+2} + a_2^{+2} - a_1^{-2} - a_2^{-2}),$$
$$S^{02} = -\frac{1}{4}(a_1^{+2} - a_2^{+2} + a_1^{-2} - a_2^{-2}),$$
$$S^{03} = \frac{1}{2}(a_1^+ a_2^+ + a_1^- a_2^-),$$
$$S^{12} = \frac{1}{2}(N_1 - N_2),$$
$$S^{13} = -\frac{1}{2}(a_1^+ a_2^- + a_1^- a_2^+),$$
$$S^{23} = \frac{i}{2}(a_1^+ a_2^- - a_1^- a_2^+),$$
$$\Gamma^0 = -\frac{1}{2}(N_1 + N_2 + 1),$$
$$\Gamma^1 = \frac{1}{4}(a_1^{+2} + a_2^{+2} + a_1^{-2} + a_2^{-2}),$$
$$\Gamma^2 = -\frac{i}{4}(a_1^{+2} - a_2^{+2} - a_1^{-2} + a_2^{-2}),$$
$$\Gamma^3 = \frac{i}{2}(a_1^+ a_2^+ - a_1^- a_2^-).$$

We can check that

$$S^{\mu\nu}S_{\mu\nu} = -\frac{3}{2}, \qquad \Gamma^\mu \Gamma_\mu = \frac{1}{2},$$
$$S_i S_i = \frac{1}{2} S_{ij} S_{ij} = \frac{N_1 + N_2}{2}\left(\frac{N_1 + N_2}{2} + 1\right).$$

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