Robustness of the vacuum wave function and other matters for Yang-Mills theory

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In the first part of this paper, we present a set of simple arguments to show that the two-dimensional gauge anomaly and the (2 + 1)-dimensional Lorentz symmetry determine the leading Gaussian term in the vacuum wave function of (2 + 1)-dimensional Yang-Mills theory. This is to highlight the robustness of the wave function and its relative insensitivity to the choice of regularizations. We then comment on the correspondence with the explicit calculations done in earlier papers. We also make some comments on the nature of the gauge-invariant configuration space for Euclidean three-dimensional gauge fields (relevant to (3 + 1)-dimensional Yang-Mills theory).

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I. INTRODUCTION

There has recently been a revival of interest in the Hamiltonian approach to Yang-Mills theories in 2 + 1 and in 3 + 1 dimensions. This is partly because of earlier work where it was noticed that in a Hamiltonian approach in 2 + 1 dimensions, one could utilize some of the niceties of two-dimensional gauge theories [1–3]. In particular, one could choose the $A_0 = 0$ gauge and for the remaining two spatial components a matrix parametrization of the form $A = \frac{1}{2}(A_1 + iA_2) = -\partial M M^{-1}$, where M is a complex matrix, could be used. On the matrix M, gauge transformations act homogeneously by left-multiplication and hence the reduction to the gauge-invariant set of variables is more easily accomplished. This led to the computation of the volume element for the gauge-invariant configuration space, the reduction of the Hamiltonian (to gauge-invariant variables), and the computation of the vacuum wave function. The expectation value of the Wilson loop could be calculated and gave a value for string tension in good agreement with lattice simulations.

There have been more recent attempts to extend this analysis to obtain estimates of glueball masses [4]. There have also been attempts to extend the discussion of the gauge-invariant configuration space to 3 + 1 dimensions, where results have been more limited [5,6]. It is also worth mentioning that there have been a number of other analyses which are similar in spirit, i.e., within the general framework of the Hamiltonian approach to Yang-Mills theory, but different in details [7].

The calculations presented in [1–3] are simplified by the parametrization we used and known results for two-dimensional gauge fields. Nevertheless, they are still quite involved. In particular, we need to have proper regularization for all the terms in the Hamiltonian, the wave function, etc. While this was sorted out in detailed calculations, the reason why each component-result in the chain of argument should be true was not always transparent. Can we understand the essential elements of these results based on simple invariance arguments so that sensitivity to regularization is clearly eliminated? The following comments will address this question. We will present arguments to show that the leading Gaussian term in the wave function as calculated in [2,3] is obtained from the two-dimensional gauge anomaly and (2 + 1)-dimensional Lorentz invariance. Detailed properties of regularization are not needed. We will then comment on the points of correspondence between these arguments and the detailed calculations of the earlier papers. In the last section, we present some considerations on the gauge-invariant configuration space of three-dimensional Euclidean gauge fields which is relevant for a Hamiltonian analysis of (3 + 1)-dimensional gauge theories.

II. ROBUSTNESS OF THE WAVE FUNCTION

We will start with a sequence of arguments which will show that the leading terms in the wave function have a certain degree of robustness. For this we will use the two-dimensional anomaly calculation combined with (2 + 1)-dimensional Lorentz (Galilean) invariance and, to some extent, the perturbative limit.

A. The volume element for gauge-invariant configurations

We start with the calculation of the volume element on the gauge-invariant configuration space. Once we have chosen the gauge condition $A_0 = 0$, the spatial components of the gauge potential may be parametrized as

$$A = -\partial M M^{-1}, \qquad \bar{A} = M^{\dagger -1} \bar{\partial} M^{\dagger}. \tag{1}$$

Here M is a complex matrix which is an element of the complexification of the gauge group. Thus, for the group

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$SU(N)$ which we shall consider here, $M \in SL(N, \mathbb{C})$. The gauge-invariant Hermitian matrix $H = M^\dagger M$ will describe the physical (gauge-invariant) degrees of freedom. It may be considered as parametrizing $SL(N, \mathbb{C})/SU(N)$. (A basis for the Lie agebra of $SU(N)$, in the fundamental representation, will be taken as the set of $N \times N$ traceless Hermitian matrices t^a, $a = 1, 2, \cdots, N^2 - 1$, with $[t^a, t^b] = i f^{abc} t^c$ and $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$.)

Denoting the space of gauge potentials $\{A, \bar{A}\}$ as $\mathcal{A}$ and the set of all gauge transformations as $\mathcal{G}_*$, we are interested in the volume element of the gauge-invariant configuration space $\mathcal{A}/\mathcal{G}_*$. The parametrization (1) leads to

$$d\mu(\mathcal{A}/\mathcal{G}_*) = \det(-D\bar{D}) d\mu(H), \tag{2}$$

where $d\mu(H)$ is the Haar measure on the coset space $SL(N, \mathbb{C})/SU(N)$. The determinant in this equation can be calculated by evaluating its variation. Defining $\Gamma = \log \det(-D\bar{D})$, we can write

$$\frac{\delta\Gamma}{\delta\bar{A}^a(\vec{x})} = -i\mathrm{Tr}[\bar{D}^{-1}(\vec{x}, \vec{y})T^a]_{\vec{y}\to\vec{x}}. \tag{3}$$

Here $(T^a)_{mn} = -i f^a_{mn}$ are the generators of the Lie algebra in the adjoint representation. The coincident-point limit of the Green's function $\bar{D}^{-1}(\vec{x}, \vec{y})$ is singular and needs regularization. Since the volume element $d\mu(\mathcal{A}/\mathcal{G}_*)$ must be gauge invariant, we choose a gauge-invariant regularization. For any gauge-invariant regularization, this leads to

$$\mathrm{Tr}[\bar{D}^{-1}_{\mathrm{reg}}(\vec{x}, \vec{y})T^a]_{\vec{y}\to\vec{x}} = \frac{2c_A}{\pi} \mathrm{Tr}[(A(\vec{x}) - M^{\dagger -1}(\vec{x})\partial M^\dagger(\vec{x}))t^a], \tag{4}$$

where c_A is the quadratic Casimir invariant for the adjoint representation defined by $f^{amn} f^{bmn} = c_A \delta^{ab}$. Using this result in (3), and with a similar result for the variation of Γ with respect to A^a, and integrating, we get, up to an additive constant, $\Gamma = 2c_A \mathcal{S}_{\mathrm{wzw}}(H)$, where $\mathcal{S}_{\mathrm{wzw}}(H)$ is the Wess-Zumino-Witten (WZW) action for the Hermitian matrix field H,

$$\mathcal{S}_{\mathrm{wzw}}(H) = \frac{1}{2\pi}\int \mathrm{Tr}(\partial H \bar{\partial} H^{-1}) + \frac{i}{12\pi} \times \int \epsilon^{\mu\nu\alpha}\mathrm{Tr}(H^{-1}\partial_\mu H H^{-1}\partial_\nu H H^{-1}\partial_\alpha H). \tag{5}$$

For the volume element (2), we then have, up to a multiplicative constant,

$$d\mu(\mathcal{A}/\mathcal{G}_*) = d\mu(H)\exp(2c_A \mathcal{S}_{wzw}(H)). \tag{6}$$

The calculation in (4) is essentially the calculation of the gauge anomaly in two dimensions and, therefore, the result (6) is quite robust; different regulators will lead to the same result so long as gauge invariance is preserved.

B. The action of T on J^a

This result is closely related to another, namely, the action of the kinetic energy operator on the current

$$J^a = \frac{c_A}{\pi}(\partial H H^{-1})^a. \tag{7}$$

This is the current for the WZW action in (5). The current J^a is the gauge-invariant variable in terms of which all observables can be constructed. For the action of T, we find

$$TJ^a(\vec{x}) = -\frac{e^2}{2}\int d^2y \frac{\delta^2 J^a(\vec{x})}{\delta\bar{A}^b(\vec{y})\delta A^b(\vec{y})} = \frac{e^2 c_A}{2\pi} M^{\dagger am}\mathrm{Tr}[T^m \bar{D}^{-1}(\vec{y}, \vec{x})]_{\vec{y}\to\vec{x}} = mJ^a(\vec{x}), \tag{8}$$

where $m = e^2 c_A/2\pi$. Notice that the basic calculation involved is the same as in (4); therefore, this result also follows from the two-dimensional gauge anomaly.

There should be no surprise that the two results (6) and (8) are related. As argued in [2], the self-adjointness of the kinetic energy operator T relates it to the gauge-invariant volume element.

C. Identifying the vacuum wave function

Consider now the vacuum wave function which we may write as $\Psi_0 = e^P$ where P is a functional of the current J and its derivatives. We write $P = -\beta V + \cdots$, where V is the potential energy $\int B^2/2e^2$, or $(\pi/mc_A)\int \bar{\partial}J^a\bar{\partial}J^a$ in terms of the current. (These have to be understood with proper regularization; we will not need the explicit form of the regularization for the argument we present. It is discussed in the next section.) The action of the kinetic energy operator on V, considered as a functional of J, leads to an equation of the form

$$[T, V] = aV + \frac{4\pi}{c_A}\int (\mathcal{D}\bar{\partial}J)^a \frac{\delta}{\delta J^a}, \tag{9}$$

where

$$\mathcal{D}_{x\,ab} = \frac{c_A}{\pi}\partial_x \delta_{ab} + i f_{abc} J_c(\vec{x}). \tag{10}$$

Notice that, on dimensional grounds, $\int(\delta^2 V/\delta\bar{A}\delta A)$ should be proportional to V. This is the reason for postulating the first term on the right-hand side in (9). The computation of the coefficient a has to be done with proper regularization. However, the second term does not involve the intricacies of regularization, it follows directly from the variation of $\int B^2$ with respect to A.

Using (9), we find for the action of the Hamiltonian on $\Psi_0 \approx e^{-\beta V}$,

$$\mathcal{H}\Psi_0 = (T + V)\Psi_0 = e^P(V - \beta a V + \cdots), \tag{11}$$

where the omitted terms involve derivatives (or momenta k) due to the second set of terms in (9). In an expansion in powers of k/e^2, these are negligible. Thus, to lowest order in k/e^2, we must cancel the V-dependent terms to get a solution to the vacuum wave function. This requires $\beta = 1/a$. The vacuum wave function, to this order, is thus

$$\Psi_0 \approx \exp(-V/a). \tag{12}$$

We now go back to the result (8). This states that, in the extreme strong coupling limit where we neglect V entirely, J^a is an eigenstate of T with eigenvalue m. Notice that we can write this state as $J^a\Psi_0$ since $\Psi_0 \approx 1$ in the extreme strong coupling limit. We can see that, once we include the modification to Ψ_0 due to V, this is the corrected eigenstate of the Hamiltonian to first order in V and in k/e^2. In fact, we find

$$\begin{aligned}(T+V)J^a\Psi_0 &= e^P(T+V-\beta[T,V]+\cdots)J^a\\ &= \left(m+\frac{k^2}{a}+\cdots\right)J^a e^P\\ &\quad + e^P J^a(V-\beta a V+\cdots)\\ &= \left(m+\frac{k^2}{a}+\cdots\right)J^a\Psi_0.\end{aligned} \tag{13}$$

We see that we have, indeed, found the corrected eigenstate to first order in the $1/e^2$ expansion; the eigenvalue is $m + k^2/a$. This eigenvalue must have the form $m + k^2/2m$ for this to become the standard relativistic formula for the energy, to this order. This identifies a as $2m$. Going back to (9), we can now write

$$[T,V] = 2mV + \frac{4\pi}{c_A}\int(\mathcal{D}\bar{\partial}J)^a\frac{\delta}{\delta J^a}. \tag{14}$$

Notice that we have only assumed a to be nonzero. Its actual value is then fixed by Lorentz invariance and the action of T on J^a. Since the latter is given by the anomaly, and hence is quite robust, we see that (14) is unambiguously obtained. The vacuum wave function to this order of calculation is thus $\Psi_0 \approx \exp(-V/2m)$. (In (13), we have only used the first correction to m in a k/m-expansion. As shown elsewhere [3], there is a set of terms which add up to give the full relativistic expression for the energy.)

Starting with this formula for the vacuum wave function, in Ref. [3], we obtained a series for P, in powers of k/m. The leading terms, with two powers of the current J, were summed up to give

$$\Psi_0 \approx \exp\left[-\frac{2\pi^2}{e^2 {c_A}^2}\int\bar{\partial}J_a\left[\frac{1}{(m+\sqrt{m^2-\nabla^2})}\right]\bar{\partial}J_a + \mathcal{O}(J^3)\right]. \tag{15}$$

So far, we have basically argued for the robustness of the leading term of this expression where we neglect the momenta or ∇^2. (It is worth noting that this is also the form which gives the fully relativistic formula $\sqrt{k^2+m^2}$ for the action of $T + V$ on $J^a\Psi_0$.)

D. Another argument for the form of Ψ_0

There is another check of this formula that we can do, starting from (6). Using the formula for the gauge-invariant volume element, we can write for the inner product of the wave functions,

$$\begin{aligned}\langle 1|2\rangle &= \int d\mu(\mathcal{A}/\mathcal{G}_*)\Psi_1^*\Psi_2\\ &= \int d\mu(H)e^{2c_A\mathcal{S}_{\rm wzw}(H)}\Psi_1^*\Psi_2.\end{aligned} \tag{16}$$

As we have argued elsewhere [1,2], the WZW action in the exponent for the volume element is related to a mass gap. This is seen explicitly by writing $\Psi = \exp[-c_A\mathcal{S}_{\rm wzw}(H)]\Phi$. The inner product then simplifies as

$$\langle 1|2\rangle = \int d\mu(H)\Phi_1^*\Phi_2. \tag{17}$$

The Hamiltonian acting on Φ's is given by $\mathcal{H}_\Phi = e^{c_A\mathcal{S}_{\rm wzw}}\mathcal{H}e^{-c_A\mathcal{S}_{\rm wzw}}$. For the argument we are going to present, it is sufficient to consider the small φ-expansion where $H = \exp(t^a\varphi^a) \approx 1 + t^a\varphi^a$. In this case

$$\begin{aligned}c_A\mathcal{S}_{\rm wzw} &\approx -\frac{c_A}{4\pi}\int\partial\varphi^a\bar{\partial}\varphi^a+\cdots,\\ \mathcal{H}_\Phi &\approx \frac{1}{2}\int\left[-\frac{\delta}{\delta\phi^a\delta\phi^a}+\phi^a(m^2-\nabla^2)\phi^a\right]+\cdots,\end{aligned} \tag{18}$$

where $\phi^a = \sqrt{c_A(-\nabla^2)/8\pi m}\,\varphi^a$. We see that the leading term in $\mathcal{H}_\Phi$ corresponds to a free field of mass m (actually $\dim G$ fields, counting the multiplicity due to the index a.) To arrive at this result we have used the fact that

$$T \approx m\left[\int\varphi^a\frac{\delta}{\delta\varphi^a} - \frac{4\pi}{c_A}\int\frac{\delta}{\delta\varphi^a(x)}\left(\frac{1}{-\nabla^2}\right)_{x,y}\times\frac{\delta}{\delta\varphi^a(y)}+\cdots\right]. \tag{19}$$

The first term in this expression follows from (8). The second term does not involve the intricacies of regularization; it is just the rewriting of $-\delta^2/\delta A^2$ to the perturbative linear order in φ. (If we write $A \approx -\partial\theta$, φ is given as $\varphi = \theta + \bar{\theta}$, and we get the second term on the right-hand side of (19) when $\delta/\delta A\delta/\delta\bar{A}$ acts on functionals of φ.) Thus, to the order we have calculated, (19) also follows from the gauge anomaly calculation.

Since (18) is the Hamiltonian for free fields, the vacuum wave function is trivially constructed as

$$\Phi_0 \approx \exp\left[-\frac{1}{2}\int\phi^a\sqrt{m^2-\nabla^2}\phi^a\right]. \tag{20}$$

Going back to Ψ_0, we find

$$\Psi_0 = e^{-c_A S_{\rm wzw}} \Phi_0 \approx \exp\left(\frac{c_A}{4\pi}\int \partial\varphi^a \bar{\partial}\varphi^a + \cdots\right) \exp\left[-\frac{c_A}{16\pi m} \times \int (-\nabla^2\varphi)^a \sqrt{m^2-\nabla^2}\varphi^a + \cdots\right] \approx \exp\left[-\frac{c_A}{\pi m}\int (\bar{\partial}\partial\varphi^a)\left[\frac{1}{m+\sqrt{m^2-\nabla^2}}\right](\bar{\partial}\partial\varphi^a) + \cdots\right]. \tag{21}$$

The basic argument can now be formulated as follows. Let us say we start with the Yang-Mills theory in 2 + 1 dimensions. Then the inner product is given by (16); further Ψ_0 should be a functional of J. So far we do not need to make any small φ-approximations. Now we can say that, whatever Ψ_0 is, it should agree with (21) in the small φ-limit. The only functional of J which has this property is (15). (It is easily checked that (15) agrees with (21) in the small φ-limit, using $J = (c_A/\pi)\partial H H^{-1} \approx (c_A/\pi)\partial\varphi$.) Thus, we see that, in short, the volume element and the perturbative small φ-limit restrict Ψ_0 to the form (15). The formula for the measure, which is determined by the anomaly, and the form of T in (19), which is also determined by the anomaly, are the key ingredients for this argument.

E. How does this apply to the string tension?

The vacuum expectation value of any operator $\mathcal{O}$ is given by

$$\langle \mathcal{O}\rangle = \int d\mu(\mathcal{A}/\mathcal{G}_*)\Psi_0^*\Psi_0\mathcal{O} = \int d\mu(\mathcal{A}/\mathcal{G}_*)e^{-S}\mathcal{O}, \tag{22}$$

where S is defined by $\Psi_0^*\Psi_0 = e^{-S}$. The expectation value is, thus, the functional average in a two-dimensional gauge theory with the action S. Based on arguments given above, for modes of low momentum, the wave function for the vacuum can be taken as

$$\Psi_0 \approx \exp\left[-\frac{\pi}{2m^2 c_A}\int \bar{\partial}J^a\bar{\partial}J^a\right] = \exp\left[-\frac{1}{8g^2}\int F^a_{ij}F^a_{ij}\right], \tag{23}$$

where $g^2 = me^2$, so that $S \approx S^{(2)}_{\rm YM}$, where $S^{(2)}_{\rm YM}$ is the two-dimensional Yang-Mills action with coupling constant g^2. The expectation value of the Wilson loop operator (in the representation R) then obeys an area law given by

$$\langle W_R(C,A)\rangle = \int d\mu(\mathcal{A}/\mathcal{G}_*)e^{-S}W_R(C,A) \approx \int d\mu(\mathcal{A}/\mathcal{G}_*)e^{-S^{(2)}_{\rm YM}}W_R(C,A) \sim \exp[-\sigma_R \mathcal{A}(C)], \tag{24}$$

TABLE I. Comparison of $\sqrt{\sigma}/e^2$ as predicted by (25) (upper entry) and lattice estimates (lower entry, in red) from [8,9]. k is the rank of the representation.

Group	Representations $k=1$ Fund.	$k=2$ antisym.	$k=3$ antisym.	$k=2$ sym.	$k=3$ sym.	$k=3$ mixed
$SU(2)$	0.345					
	0.335					
$SU(3)$	0.564					
	0.553					
$SU(4)$	0.772	0.891		1.196		
	0.759	0.883		1.110		
$SU(5)$	0.977					
	0.966					
$SU(6)$	1.180	1.493	1.583	1.784	2.318	1.985
	1.167	1.484	1.569	1.727	2.251	1.921
$SU(N) N \to \infty$	0.1995 N					
	0.1976 N					

where $\mathcal{A}(C)$ is the area of the loop C and the string tension σ_R is given by

$$\sigma_R = e^4 \frac{c_A c_R}{4\pi}, \tag{25}$$

where c_R is the Casimir invariants of the R representation. As mentioned elsewhere, and as shown in Table I, this formula is in good agreement with the lattice estimates [8], the difference being less than 3% for all cases, and less than 0.88% as $N \to \infty$, even though the deviations are still statistically significant [9].

We have argued that the leading term of the vacuum wave function (15), and hence the leading term in S (which is quadratic in the currents), is quite robust. Therefore, if there are any corrections to the string tension, they should arise, not from modification of the wave function, but due to the approximation of S by $S^{(2)}_{\rm YM}$ in the evaluation of the expectation value (24). Thus corrections to σ should be due to terms in S which are higher than quadratic in the J's.

On general grounds, we should expect some corrections to the formula for the string tension. It has been argued that the ratios of string tensions should deviate from the ratios of Casimir invariants on the basis of the $1/N$-expansion [10]. Also, for Wilson loops in the adjoint representation (or other representations which are invariant under the center of the group), we should expect screening rather than confinement or area law. We have presented reasons to show how screening and the corresponding string-breaking effect can arise from a judicious resummation of the higher order corrections which can lead to the formation of color-singlet bound states of a "gluon" with the external charge whose world line trajectory is represented by (part of) the Wilson loop. An estimate of the string-breaking energy along these lines gives a result within 8.8% of the lattice estimates [11].

III. CORRESPONDENCE WITH EXPLICIT CALCULATIONS

A. How do we regularize the Hamiltonian?

We now turn to the question: How are the results given so far explicitly realized when we solve the Schrödinger equation after regularization of the Hamiltonian? This was done in some detail in [2], so the following comments are more in the nature of clarifying remarks. The Hamiltonian consists of the kinetic term T, which is a functional differential operator, and V, the potential energy. Since Lorentz transformations can mix the two, there has to be a concordance between the regularization of these two terms to ensure that the full theory has Lorentz symmetry.

In the regularized expression for any quantity in field theory, one can have terms which are suppressed by powers of k/M where k is a typical momentum and M is the regulator mass. The details of such terms differ from regulator to regulator and constitute regularization ambiguities. These regularization-dependent terms are, of course, negligible if we consider processes of momenta $k \ll M$. In other words, once we introduce a regulator, we must apply the results only to processes with $k \ll M$. This is well-known lore in field theory, but is worth emphasizing in the context of regularization of terms in the Hamiltonian. Now, of the two terms in the Hamiltonian, the kinetic energy requires more care regarding regularization, so we consider it first. As a regularized expression, we may take the kinetic energy operator as

$$T_{(\epsilon)} = \frac{e^2}{2}\int_{u,v} \Pi_{rs}(\vec{u},\vec{v})\bar{p}_r(\vec{u})p_s(\vec{v}),$$
$$\Pi_{rs}(\vec{u},\vec{v}) = \int_x \bar{\mathcal{G}}_{ar}(\vec{x},\vec{u})K_{ab}(\vec{x})\mathcal{G}_{bs}(\vec{x},\vec{v}), \tag{26}$$

where $K_{ab} = 2\,\mathrm{Tr}(t_a H t_b H^{-1})$ is the adjoint representative of H. The functions $\bar{\mathcal{G}}_{ma}(\vec{x},\vec{y})$, $\mathcal{G}_{ma}(\vec{x},\vec{y})$ are given by

$$\bar{\mathcal{G}}_{ma}(\vec{x},\vec{y}) = \frac{1}{\pi(x-y)}[\delta_{ma} - e^{-|\vec{x}-\vec{y}|^2/\epsilon}(K(x,\bar{y})K^{-1}(y,\bar{y}))_{ma}],$$
$$\mathcal{G}_{ma}(\vec{x},\vec{y}) = \frac{1}{\pi(\bar{x}-\bar{y})}[\delta_{ma} - e^{-|\vec{x}-\vec{y}|^2/\epsilon}(K^{-1}(y,\bar{x})K(y,\bar{y}))_{ma}]. \tag{27}$$

These are the regularized versions of the corresponding Green's functions

$$\bar{G}(\vec{x},\vec{y}) = \frac{1}{\pi(x-y)}, \qquad G(\vec{x},\vec{y}) = \frac{1}{\pi(\bar{x}-\bar{y})}. \tag{28}$$

The parameter $\sqrt{\epsilon}$ acts as a short-distance cutoff; it is the regularization parameter, taken to be arbitrarily small compared to other distance scales in the theory. In the naive $\epsilon \to 0$ limit, we find

$$T_{(\epsilon)}]_{(\epsilon\to 0)} = -\frac{e^2}{2}\int \frac{\delta^2}{\delta A^a \delta \bar{A}^a} \tag{29}$$

so that (26) can indeed be interpreted as the regularized version of the kinetic energy.

One can now consider the action of this operator on functionals $\Psi(\lambda')$, which is some product of fields and their derivatives with an average separation of points between fields being $\sqrt{\lambda'}$. When $T_{(\epsilon)}$ acts on this, it can generate terms which diverge as $\epsilon \to 0$, terms which are finite as $\epsilon \to 0$, and terms which vanish as $\epsilon \to 0$. The first type of terms would indicate that we must do an additional subtraction to define a "renormalized" kinetic energy operator. The second set of terms corresponds to physically meaningful results. The last set of terms represents regularization ambiguities. They vanish when ϵ goes to zero, but they may be in the form of powers of ϵ/λ'. If we take λ' comparable to ϵ, the results can be ambiguous. (For example, a different regularization may give different results for these terms.) The correct procedure is to keep ϵ much smaller than λ'; the regularization in (26) and (27) only applies with this caveat.

The regularized expression for the potential energy can be taken as

$$V_{(\lambda')} = \frac{\pi}{mc_A}\left[\int_{x,y}\sigma(\vec{x},\vec{y};\lambda')\bar{\partial}J_a(\vec{x}) \times (K(x,\bar{y})K^{-1}(y,\bar{y}))_{ab}\bar{\partial}J_b(\vec{y}) - \frac{c_A\,\mathrm{dim}G}{\pi^2\lambda'^2}\right],$$
$$\sigma(\vec{x},\vec{y};\lambda') = \frac{1}{\pi\lambda'}\exp[-|\vec{x}-\vec{y}|^2/\lambda']. \tag{30}$$

In using this expression for solving the Schrödinger equation, we will encounter terms like $[T_{(\epsilon)}, V_{(\lambda')}]$, in other words, the action of T on V. From what was stated earlier, for consistency, we must keep λ' much larger than ϵ. Explicit calculation then shows that

$$T_{(\epsilon)}V_{(\lambda')} = 2m[1 + \tfrac{1}{2}\log(\lambda'/2\epsilon)]V_{(\lambda')} + \cdots, \tag{31}$$

where the omitted terms correspond to powers of ϵ or λ'. This equation shows that we have a potential log-divergence. In addition to the regularization, we must define a renormalized $T_{(\lambda)}$ as

$$T_{(\lambda)} = T_{(\epsilon)} + \frac{e^2}{2}\log(2\epsilon/\lambda)\mathcal{Q},$$
$$\mathcal{Q} = \epsilon\int \sigma(\vec{u},\vec{v};\epsilon)K_{rs}(u,\bar{v})(\bar{p}_r(\vec{u}) - i\bar{\partial}J_r(\vec{u}))p_s(\vec{v}). \tag{32}$$

$T_{(\lambda)}$ corresponds to a subtraction scale of λ. Since we are interested in the "local" operator T, eventually we must take λ to be very small compared to the distance scales in the theory, i.e., $\lambda \ll e^{-4}$. Using $T_{(\lambda)}$ we find

$$T_{(\lambda)}V_{(\lambda')} = 2m[1 + \tfrac{1}{2}\log(\lambda'/\lambda)]V_{(\lambda')} + \cdots \tag{33}$$

B. Lorentz transformation once more

Consider now an infinitesimal Lorentz transformation corresponding to velocity v_i. For the electric and magnetic fields we have

$$\delta E_i \approx -\epsilon_{ij} v_j B, \qquad \delta B \approx \epsilon_{ij} v_i E_j. \tag{34}$$

For simplicity, consider a transformation along the x-axis, so that $v_2 = 0$. The transformation of the Hamiltonian is now given as

$$\delta \mathcal{H} = \delta T_{(\lambda)} + \delta V_{(\lambda')} = v_1 \int (BE_2)_{(\lambda)} + v_1 \int (BE_2)_{(\lambda')}. \tag{35}$$

The two terms on the right-hand side must combine to produce twice the momentum density $P_1 \sim \int BE_2$. Now, for $\int (BE_2)_{(\lambda')}$, there are no modes of momenta larger than $1/\sqrt{\lambda'}$, on average. For this to combine with the first term, we must therefore conclude that the smallest value for λ must be λ'. The consistent regularization, keeping as many modes as possible for both terms would be to have $\lambda = \lambda'$, with $e^2 \ll 1/\sqrt{\lambda}$. Thus $\mathcal{H} = T_{(\lambda)} + V_{(\lambda)}$, and, going back to (33), we get

$$T_{(\lambda)} V_{(\lambda)} = 2m V_{(\lambda)}. \tag{36}$$

This result holds when λ is taken to be very, very small, $\lambda \to 0$, keeping $\epsilon \ll \lambda \ll e^{-4}$. This is effectively the result (14) and the construction of the wave function then follows the arguments given after that equation.

Even though the Lorentz transformation properties were not explicitly used in [2], the regularization and detailed calculations presented there followed the same general approach and gave the result (36). It is also worth mentioning that there are regularizations in the literature which do not lead to (36), or (14), and which, from our arguments, do not respect the Lorentz symmetry [4]. (Mansfield in [7] also presents another regularization, and also raises the question of Lorentz invariance.)

We will close this section with a few more comments. As noted after Eq. (15), the form of the wave function given there is what leads to the relativistic formula for the energy for $J^a \Psi_0$. We might ask whether it is important to have Lorentz symmetry for a state like $J^a \Psi_0$, since it may not be a physical state in the full theory. It is worth emphasizing that the argument for (15), or the basic argument leading to (21), relies only on the energy spectrum being consistent with Lorentz invariance in the small φ-limit, with $J^a \approx (c_A/\pi)\partial \varphi^a$. In this case, we are in the regime of perturbation theory. Indeed, the gluon propagator, in a resummed perturbation theory, can be constructed in terms of the propagator for φ^a. Therefore, any result which is not consistent with Lorentz invariance would lead to trouble already at this level. We should generally expect that the gluon propagator, in a resummation of perturbation theory, should have poles consistent with relativistic symmetry. The general argument presented is thus independent of the question whether $J^a \Psi_0$ is an acceptable physical state in the full theory.

Also, regarding wave functions of the type discussed in [4], the following comment might be useful. In computing expectation values using our wave function, we end up with the averages in a two-dimensional field theory with action S defined by $\Psi_0^* \Psi_0 = e^{-S}$, as in (22). We can envisage doing this integral by first integrating over a set of high momentum modes, obtaining an effective action $S_{\rm eff}$ and then completing the integration over the remaining modes (indicated by $d\tilde{\mu}$) at the second stage,

$$\langle \mathcal{O} \rangle = \int d\mu(\mathcal{A}/\mathcal{G}_*) e^{-S} \mathcal{O} = \int d\tilde{\mu}(\mathcal{A}/\mathcal{G}_*) e^{-S_{\rm eff}} \mathcal{O}. \tag{37}$$

In this case, the calculation is equivalent to using a wave function $\Psi_0 \sim e^{-(1/2)S_{\rm eff}}$ for the low momentum modes. (And it will work only for suitable low momentum observables $\mathcal{O}$.) There is no reason why this intermediate step should be consistent with Lorentz symmetry, since cutoffs are imposed on the spatial momenta only, although the final result, after completing all integrations, should be. This may be one way of understanding the usefulness of wave functions like the one proposed in [4].

IV. THE CONFIGURATION SPACE FOR THREE-DIMENSIONAL GAUGE FIELDS: GENERAL COMMENTS

We now turn to some general properties of the gauge-invariant configuration space for Euclidean gauge fields in three spatial dimensions. This would be appropriate for a Hamiltonian analysis for (3 + 1)-dimensional gauge theories in the $A_0 = 0$ gauge, or for a covariant path integral calculation for the (Wick-rotated version of) (2 + 1)-dimensional Yang-Mills theory.

A. Is the volume of the configuration space finite?

For two-dimensional gauge fields, the total volume of the configuration space is

$$\int d\mu(\mathcal{C}) = \int d\mu(H) e^{2c_A S_{\rm wzw}(H)} < \infty. \tag{38}$$

This is the partition function of the Hermitian WZW model and is finite with some regularization (to a finite number of modes). The contrast to be emphasized here is with the Abelian theory for which $c_A = 0$ and the integral diverges for each mode. This result is important for two reasons. First of all, it is possible to find configurations which are separated by an infinite distance on the configuration space $\mathcal{C}$. The finiteness of $\int d\mu(\mathcal{C})$ shows that these have zero transverse measure, i.e., zero volume in the directions transverse to the line connecting the two configurations. Such far-separated configurations are therefore not important to the question of the spectrum of the Laplacian (i.e., the kinetic energy operator) on $\mathcal{C}$. Second, in continuation of this reasoning, we see that $\mathcal{S}_{\rm wzw}(H)$ provides a cutoff for low momentum modes. This property is crucial for the existence of a mass gap.

One can now ask the question whether similar properties are obtained for the three-dimensional gauge fields. There have been a number of attempts at calculations of the volume element for the (3 + 1)-dimensional theory [5,6]. These have generally been in special parametrizations for the fields. However, here, we shall consider some general properties. The naive volume element $[dA]/vol(\mathcal{G}_*)$ is difficult to analyze, so it is useful to define it as the limit of a "regularized" version as

$$d\mu(\mathcal{C})_{3d} = \frac{[dA]}{\mathrm{vol}(\mathcal{G}_*)}\exp\left(-\frac{1}{4\mu}\int F^2\right)\Big]_{\mu\to\infty}, \qquad (39)$$

where μ has the dimensions of mass. The right-hand side is the functional measure for the Euclidean (Wick-rotated) version of (2 + 1)-dimensional Yang-Mills theory with a coupling constant $e^2 = \mu$. Therefore we can evaluate various quantities by the Hamiltonian techniques we have developed for the (2 + 1)-dimensional theory. In particular, the total volume is given by the Euclidean version of the vacuum-to-vacuum transition amplitude,

$$\begin{aligned}\int d\mu(\mathcal{C})_{3d} &= \int \frac{[dA]}{\mathrm{vol}(\mathcal{G}_*)}\exp\left(-\frac{1}{4\mu}\int F^2\right)\Big]_{\mu\to\infty}\\ &= \langle 0|e^{-\beta\mathcal{H}}|0\rangle]_{\beta,\mu\to\infty}\\ &= \int d\mu(\mathcal{C})_{2d}\Psi_0^*\Psi_0]_{\mu\to\infty}. \end{aligned} \qquad (40)$$

As $\beta \to \infty$, only the ground state survives in the expectation value; this gives the last equality. Ψ_0 is the ground state wave function for $e^2 = \mu$. We need the large e^2 (or μ) limits of Ψ_0 which is known from (23). Thus

$$\begin{aligned}\int d\mu(\mathcal{C})_{3d} &= \int d\mu(\mathcal{C})_{2d}\exp\left(-\frac{1}{4e_{2d}^2}\int F^2\right)\\ &= 2-\text{dim. Yang-Mills partition function for } e_{2d}^2\\ &= \frac{\mu^2 c_A}{2\pi}\\ &= \text{WZW partition function as } \mu\to\infty < \infty. \end{aligned} \qquad (41)$$

This leads to the (somewhat surprising) conclusion that the total volume of the configuration space is finite, even in three dimensions.

B. A potential paradox and its resolution

We now consider a possible counter-argument for the finiteness of the total volume of the configuration space in three dimensions. This argument is taken/adapted from [12], where a general analysis of many properties of the configuration space is given.

The square of the Euclidean distance between the gauge orbits corresponding to the potentials A and A' can be defined as

$$L^2(A, A') = \mathrm{Inf}_g \int d^3x \mathrm{Tr}(A^g - A')^2. \qquad (42)$$

The choice of the infimum over the gauge transformations g picks the minimum distance between the orbits corresponding to A and A'. The energy functional for a configuration A is given by

$$\mathcal{E}(A) = \frac{1}{4\mu}\int d^3x F^2. \qquad (43)$$

Consider now the orbits of $A_i(x)$ and $A_i^{(s)} = sA_i(sx)$. It is easily checked that if $A_i(x)$ transforms as a connexion under gauge transformations, then so does $A^{(s)}$ (with a different gauge transformation matrix.) We find

$$L^2(A^{(s)}, 0) = \frac{1}{s}L^2(A, 0), \qquad \mathcal{E}(A^{(s)}) = s\mathcal{E}(A). \qquad (44)$$

As $s \to 0$, we scale up the distance of the configuration A from the trivial configuration $A = 0$, yet there is no cutoff imposed by $\mathcal{E}(A)$ (which scales to zero). Thus for any configuration $A_i(x)$, we can find a sequence of configurations, parametrized by s, farther and farther away with no increase in $\mathcal{E}$. (Notice that this argument will not work in two spatial dimensions.) So the question is: Since any configuration can be moved arbitrarily farther away by this scaling trick, how could one get $\int d\mu(\mathcal{C}) < \infty$?

The resolution of this paradox has to do with the dynamical generation of mass in three dimensions. As we said before, integrations done with the volume measure (39) can be viewed as the functional integration for a three-dimensional (or (2 + 1)-dimensional) Yang-Mills theory at strong coupling. In this theory there is dynamical generation of mass, so that the effective action which controls the behavior of the integral (39) has mass terms in addition to $\mathcal{E}(A)$. Therefore, we must consider not just the scaling of $\mathcal{E}(A)$, but also of the mass term which is generated when the functional integration is carried out. The mass term can be seen in the Hamiltonian approach as discussed elsewhere [1,2]. It can also be seen in a three-dimensional covariant approach by a resummation technique [13–15]. For example, we may think of doing the functional integral by progressively integrating out the higher momentum modes, obtaining a new effective action at each stage, along the lines of the Wilsonian renormalization group. To integrate out modes of momenta higher than some value M, we rewrite the 3d-action or energy functional as

$$S = \frac{1}{\mu}\left[\frac{1}{4}\int d^3x F^2 + M^2 S_m(A)\right] - \frac{M^2}{\mu}S_m(A). \qquad (45)$$

Here $S_m(A)$ is a gauge-invariant mass term for the gauge potentials, the specific form of which will be briefly discussed below. With this action, we can now consider the

Feynman diagrams generated by the bracketed set of terms. The propagators for the gauge fields are now massive and so, in integrations over the loop momenta k, the contributions of modes of $k \ll M$ are suppressed. The result will thus be the contribution of the Feynman diagrams due to modes of momenta $k \gg M$. Since S_m is gauge invariant, this gives a way of formulating the notion of the renormalization group in a gauge-invariant way. Notice that the leading mass terms cancel out at the end, so that one is left with any mass term which is dynamically generated (plus other terms with more derivatives of the fields). This procedure has been carried out to one-loop order using different types of mass terms, although the interpretation there was different. For example, it was shown in [13] that we get

$$S_{\rm eff} = \frac{1}{4\mu}\int d^3x F^2 + \lambda S_m(A), \tag{46}$$

where $\lambda \approx 1.2 M c_A/2\pi$. The volume element (39) now becomes

$$d\mu(\mathcal{C}, k \ll M)_{3d} = \frac{[dA]}{{\rm vol}(\mathcal{G}_*)} \times \exp\left(-\frac{1}{4\mu}\int F^2 - \lambda S_m(A)\right)\Big]_{\mu\to\infty}. \tag{47}$$

The remaining integration is over modes of A of momenta $k \ll M$. Returning to the scaling of the potentials, notice that the mass term scales as

$$S_m(A^{(s)}) = \frac{1}{s} S_m(A). \tag{48}$$

As $s \to 0$, we get a cutoff in the functional integral due to this mass term. This explains why it is possible to get $\int d\mu(\mathcal{C}) < \infty$.

C. The nature of the mass term

The qualitative nature of the result (46) is not sensitive to the details of the gauge-invariant mass term. However, for the sake of completeness, we give the expression for the specific mass term which was used in the calculation of (46). It is given by [16]

$$S_m(A) = \int d\Omega K(A_n, A_{\bar n}), \tag{49}$$

where n_i is a (complex) three-dimensional null vector which may be parametrized as

$$n_i = (-\cos\theta\cos\varphi - i\sin\varphi, -\cos\theta\sin\varphi + i\cos\varphi, \sin\theta). \tag{50}$$

In terms of this, $A_n = \frac{1}{2}A_i n_i$, $A_{\bar n} = \frac{1}{2}A_i \bar n_i$. Further, in (49), $d\Omega = \sin\theta d\theta d\varphi$ and denotes integration over the angles of n_i. The function $K(A_n, A_{\bar n})$ is given by

$$\begin{aligned} K(A_n, A_{\bar n}) &= -\frac{1}{\pi}\int d^2x^T\Big[\int d^2z {\rm Tr}(A_n, A_{\bar n}) + i\pi I(A_n) + i\pi I(A_{\bar n})\Big],\\ I(A_n) &= i\sum_2^\infty \frac{(-1)^m}{m}\int \frac{d^2z_1}{\pi}\cdots\frac{d^2z_n}{\pi} \times \frac{{\rm Tr}(A_n(x_1)\ldots A_n(x_m))}{\bar z_{12}\bar z_{23}\ldots \bar z_{m-1m}\bar z_{m1}}. \end{aligned} \tag{51}$$

In these expressions, $z = n\cdot\vec x$, $\bar z = \bar n\cdot\vec x$, and x^T denotes the coordinate transverse to n_i, i.e., $\vec x^T\cdot\vec n = 0$; also $z_{ij} = \bar z_i - \bar z_j$. The argument of all A's in (51) is the same for the transverse coordinate x^T. (The complex null vectors n, $\bar n$ define a choice of complex coordinates $n\cdot\vec x$, $\bar n\cdot\vec x$ at each point in space. The construction given here can thus be reinterpreted in terms of twistors for the three-dimensional space.)

If we define a complex $SL(N,\mathbb{C})$-matrix L by $n\cdot A = -n\cdot\nabla L L^{-1}$, $\bar n\cdot A = L^{\dagger -1}\bar n\cdot\nabla L^\dagger$, in a way analogous to the parametrization we used for two-dimensional Euclidean fields, then this mass term can be written as

$$S_m(A) = -\int d\Omega dx^T \mathcal{S}_{wzw}(L^\dagger L). \tag{52}$$

If we expand (51) in powers of A, then the lowest order term in S_m is seen to be

$$S_m = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}A_i^a(-k)\left(\delta_{ij} - \frac{k_ik_j}{\vec k^2}\right)A_j^a(k) + \mathcal{O}(A^3). \tag{53}$$

Thus $S_m(A)$ is indeed a mass term; its gauge invariance is evident from (52).

It is worth emphasizing that, for the purpose of integrating out modes of high momenta, other mass terms, such as those given in [14,15], may also be used. Different mass terms may be viewed as different gauge-invariant completions of the basic quadratic term in (53). As pointed out in [15], generally, when these mass terms are used to calculate the corrections to the effective action, specifically the vacuum polarization, one gets terms which have a singularity at $k^2 = 0$. In the language of unitarity cuts, when continued to Minkowski signature, this may suggest that there are still massless modes. The mass term (52) does not have such threshold singularities. This may be considered a small advantage to this particular mass term, but, it should be emphasized that, for the properties of the configuration space in three Euclidean dimensions, which is what is needed for the (3 + 1)-dimensional theory, the question of continuation to Minkowski signature does not arise.

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[1] D. Karabali and V. P. Nair, Nucl. Phys. **B464**, 135 (1996); Phys. Lett. B **379**, 141 (1996); Int. J. Mod. Phys. A **12**, 1161 (1997).
[2] D. Karabali, Chanju Kim, and V. P. Nair, Nucl. Phys. **B524**, 661 (1998).
[3] D. Karabali, Chanju Kim, and V. P. Nair, Phys. Lett. B **434**, 103 (1998).
[4] R. G. Leigh, D. Minic, and A. Yelnikov, Phys. Rev. Lett. **96**, 222001 (2006); Phys. Rev. D **76**, 065018 (2007).
[5] V. P. Nair and A. Yelnikov, Nucl. Phys. **B691**, 182 (2004).
[6] L. Freidel, R. G. Leigh, and D. Minic, Phys. Lett. B **641**, 105 (2006); L. Freidel, arXiv:hep-th/0604185.
[7] There have been a number of other analytic attempts and approaches, some of them related to ours, for Yang-Mills theory in 2 + 1 dimensions. Some relevant articles are: M. B. Halpern, Phys. Rev. D **16**, 1798 (1977); **16**, 3515 (1977); **19**, 517 (1979); I. Bars and F. Green, Nucl. Phys. **B148**, 445 (1979); J. Greensite, Nucl. Phys. **B158**, 469 (1979); D. Z. Freedman and R. Khuri, Phys. Lett. A **192**, 153 (1994); M. Bauer and D. Z. Freedman, Nucl. Phys. **B450**, 209 (1995); F. A. Lunev, Phys. Lett. B **295**, 99 (1992); O. Ganor and J. Sonnenschein, Int. J. Mod. Phys. A **11**, 5701 (1996); S. R. Das and S. Wadia, Phys. Rev. D **53**, 5856 (1996); I. I. Kogan and A. Kovner, Phys. Rev. D **52**, 3719 (1995); arXiv:hep-th/0205026; P. Mansfield and D. Nolland, J. High Energy Phys. 07 (**1999**) 028; P. Mansfield, J. High Energy Phys. 04 (2004) 059; S. G. Rajeev, arXiv:hep-th/0401202; P. Orland, Phys. Rev. D **71**, 054503 (2005); **74**, 085001 (2006); **75**, 025001 (2007); Phys. Rev. D **75**, 101702 (2007).
[8] M. Teper, Phys. Rev. D **59**, 014512 (1998); B. Lucini and M. Teper, Phys. Rev. D **66**, 097502 (2002).
[9] B. Bringoltz and M. Teper, Phys. Lett. B **645**, 383 (2007).
[10] A. Armoni and M. Shifman, Nucl. Phys. **B664**, 233 (2003); Nucl. Phys. **B671**, 67 (2003).
[11] A. Agarwal, D. Karabali, and V. P. Nair, Nucl. Phys. **B790**, 216 (2008).
[12] P. Orland, arXiv:hep-th/9607134; Phys. Rev. D **70**, 045014 (2004).
[13] G. Alexanian and V. P. Nair, Phys. Lett. B **352**, 435 (1995).
[14] W. Buchmuller and O. Philipsen, Nucl. Phys. **B443**, 47 (1995); O. Philipsen, in TFT98: Proceedings of the 5th International Workshop on Thermal Field Theories and Their Applications, edited by U. Heinz, arXiv:hep-ph/9811469; F. Eberlein, Phys. Lett. B **439**, 130 (1998); Nucl. Phys. **B550**, 303 (1999); J. M. Cornwall, Phys. Rev. D **10**, 500 (1974); **26**, 1453 (1982); Phys. Rev. D **57**, 3694 (1998).
[15] R. Jackiw and S-Y. Pi, Phys. Lett. B **368**, 131 (1996); **403**, 297 (1997).
[16] V. P. Nair, Phys. Lett. B **352**, 117 (1995).