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Evolution of cosmological gravitational waves in $f(R)$ gravity

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We give a rigorous and mathematically clear presentation of the covariant and gauge-invariant theory of gravitational waves in a perturbed Friedmann-Lemaître-Robertson-Walker universe for fourth order gravity, where the matter is described by a perfect fluid with a barotropic equation of state. As an example of a consistent analysis of tensor perturbations in fourth order gravity, we apply the formalism to a simple background solution of R^n gravity. We obtain the exact solutions of the perturbation equations for scales much bigger than and smaller than the Hubble radius. It is shown that the evolution of tensor modes is highly sensitive to the choice of n and an interesting new feature arises. During the radiation dominated era, their exists a growing tensor perturbation for nearly all choices of n. This occurs even when the background model is undergoing accelerated expansion as opposed to the case of general relativity. Consequently, cosmological gravitational wave modes can in principle provide a strong constraint on the theory of gravity independent of other cosmological data sets.

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I. INTRODUCTION

In the near future, gravitational waves (GW) will become a very important source of data in cosmology. Cosmological GW are produced at very early times in the evolution of the universe and almost immediately decouple from the cosmic fluid. Consequently, they carry information about the conditions that existed at this time, thus providing a way of constraining models of inflation [1].

Even if GW are decoupled from the cosmic fluid, their presence still influences some features of the observable universe. In particular, a GW background will produce a signature that can be found in the anisotropies [2] and polarization [3] of the cosmic microwaves background (CMB). This, together with the remarkable improvements in the sensitivity of CMB measurements, opens the possibility of obtaining important information about GW in an indirect way.

In the past few years, the idea of a geometrical origin for dark energy (DE) i.e. the connection between DE and a nonstandard behavior of gravitation on cosmological scales, has attracted a considerable amount of interest.

Higher order gravity, and, in particular, fourth order gravity, has been widely studied in the case of the Friedmann-Lemaître-Robertson-Walker (FLRW) metric using a number of different techniques (see for example [4–10]). Recently a general approach was developed to analyze the phase space of the fourth order cosmologies [11–13], providing for the first time a way of obtaining exact solutions together with their stability and a general idea of the qualitative behavior of these cosmological models.

The phase space analysis shows that for FLRW models there exist classes of fourth order theories in which the cosmology evolves naturally towards an accelerated expansion phase which can be associated with a DE-like era. Although this feature is particularly attractive, a problem connected with the use of these theories is that there is too much freedom in the form of the theory itself. Consequently, it is crucial to investigate these models in some detail in order to devise observational constraints which are able to eliminate this degeneracy.

A key step in this process is the development of a full theory of cosmological perturbations. A detailed analysis of the evolution of the scalar perturbations on large scales has recently been given in [14]. Here we will focus on the evolution of the tensor perturbations, which are related to GWs. This is motivated by the well-known fact [15] that the features of GWs in general relativity (GR) are rather special and therefore the detection of any deviation from this behavior would be a genuine proof of the breakdown of standard GR.

The aim of this paper is to present a general framework within which to consistently analyze tensor perturbations of FLRW models in fourth order gravity (see [16–19] for other recent contributions to this area). As an explicit example we apply our approach to the case of R^n gravity. We investigate the possible constraints one can place on such a model through future observations of gravitational waves independently of existing cosmological data.

In order to achieve this goal, a perturbation formalism needs to be chosen that is best suited for this task. One

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possible choice is the Bardeen metric based approach [20–22] which guarantees the gauge invariance of the results. However, this approach has the drawback of introducing variables which only have a clear physical meaning in certain gauges [23]. Although this is not a big problem in the context of GR, this is not necessarily true in the case of higher order gravity and consequently can lead to a misinterpretation of the results.

In what follows we will use, instead, the covariant and gauge-invariant approach developed for GR in [23–28] which has the advantage of using perturbation variables with a clear geometrical and physical interpretation. We take advantage of the fact that in this approach the non-Einstein part of the gravitational interaction can be considered as an effective fluid (the *curvature fluid*) coupled with standard matter. This specific recasting of the field equations makes the development of cosmological perturbation theory even more transparent.

The main results of the paper are as follows. (1) We find that the evolution of tensor modes is extremely sensitive to the choice of $f(R)$ theory. (2) In the specific case of R^n gravity, the tensor modes are in general weaker due to a higher expansion rate in the background. (3) During the radiation dominated era, their exists a growing tensor mode for nearly all interesting values of n.

The paper is organized as follows. In Sec. II we give a brief review of the 1 + 3 gauge invariant covariant approach in a general setting. In Sec. III we present the equations necessary for the study of linear tensor perturbations for a general imperfect fluid. In Sec. IV we investigate how these equations are modified when considering fourth order gravity. In Sec. V we adapt these equations for the specific case of R^n gravity and study tensor perturbations both in vacuum and in the presences of dust/radiation fluid. Finally, we present our discussions and conclusions in Sec. VI.

II. THE 1 + 3 COVARIANT APPROACH TO COSMOLOGY

The starting point (and the corner stone) of our analysis is the 1 + 3 covariant approach to cosmology [29]. This approach consists of deriving a set of first order differential equations and constraints for some suitable, geometrically well-defined quantities (the 1 + 3 *equations*) that are completely equivalent to the Einstein field equations. This has the advantage of simplifying the analysis of general spacetimes which can be foliated as a set of three-dimensional (spacelike) surfaces. In the following we give a very brief introduction to the parts of this formalism used in this paper.

A. Preliminaries

We will adopt natural units ($\hbar = c = k_B = 8\pi G = 1$) throughout this paper, Latin indices run from 0 to 3. The symbol ∇ represents the usual covariant derivative and ∂ corresponds to partial differentiation. We use the $-, +, +, +$ signature and the Riemann tensor is defined by

$$R^a{}_{bcd} = W^a{}_{bd,c} - W^a{}_{bc,d} + W^e{}_{bd} W^a{}_{ce} - W^f{}_{bc} W^a{}_{df}, \tag{1}$$

where the $W^a{}_{bd}$ is the usual Christoffel symbol (i.e. symmetric in the lower indices), defined by

$$W^a{}_{bd} = \tfrac{1}{2} g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}). \tag{2}$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd} R_{cadb}. \tag{3}$$

Finally the Einstein-Hilbert action in the presence of matter is defined as

$$\mathcal{A} = \int dx^4 \sqrt{-g}\left[\frac{1}{2} R + \mathcal{L}_m\right]. \tag{4}$$

B. Kinematics

In order to derive the 1 + 3 equations, we have to choose a set of observers, i.e. a 4-velocity field u^a. This choice depends strictly on the theory of gravity that we are treating. In this section we give the set of equations for a general velocity field. In later sections we will discuss how this situation is modified in the case of $f(R)$ gravity.

Given the velocity u^a, we can define the projection tensor into the tangent 3-spaces orthogonal to the flow vector:

$$h_{ab} \equiv g_{ab} + u_a u_b \Rightarrow h^a{}_b h^b{}_c = h^a{}_c, \qquad h_{ab} u^b = 0, \tag{5}$$

and the kinematical quantities can be obtained by splitting the covariant derivative of u_a into its irreducible parts:

$$\begin{aligned} \nabla_b u_a &= \tilde{\nabla}_b u_a - A_a u_b, \\ \tilde{\nabla}_b u_a &= \tfrac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}, \end{aligned} \tag{6}$$

where $\tilde{\nabla}_a$ is the spatially totally projected covariant derivative operator orthogonal to u^a, $A_a = \dot{u}_a$ is the acceleration ($A_b u^b = 0$), Θ is the expansion parameter, σ_{ab} the shear ($\sigma_{ab} = \sigma_{(ab)}$, $\sigma^a{}_a = \sigma_{ab} u^b = 0$), and ω_{ab} is the vorticity ($\omega_{ab} = \omega_{[ab]}$, $\omega_{ab} u^b = 0$). Following the standard convention we will indicate the symmetrization over two indices of a tensor with round brackets and the antisymmetrization with square ones.

In the u^a frame, the *Weyl or conformal curvature tensor* C_{abcd} can be split into its electric (E_{ab}) and magnetic (H_{ab}) components, respectively:

$$\begin{aligned} E_{ab} = C_{acbd} u^c u^d \Rightarrow E^a{}_a = 0, \qquad E_{ab} = E_{(ab)}, \\ E_{ab} u^b = 0, \end{aligned} \tag{7}$$

$$H_{ab} = \tfrac{1}{2}\eta_{ade}C^{de}{}_{bc}u^c \Rightarrow H^a{}_a = 0, \qquad H_{ab} = H_{(ab)}, \qquad H_{ab}u^b = 0. \tag{8}$$

In what follows we will use orthogonal projections of vectors and the orthogonally projected symmetric trace-free part of tensors. They are defined as follows:

$$v^{\langle a\rangle} = h^a{}_b v^b, \qquad X^{\langle ab\rangle} = [h^{(a}{}_c h^{b)}{}_d - h^{ab}h_{cd}]X^{cd}. \tag{9}$$

Angle brackets may also be used to denote orthogonal projections of covariant time derivatives along u^a:

$$\dot{v}^{\langle a\rangle} = h^a{}_b \dot{v}^b, \qquad \dot{X}^{\langle ab\rangle} = [h^{(a}{}_c h^{b)}{}_d - \tfrac{1}{3}h^{ab}h_{cd}]\dot{X}^{cd}. \tag{10}$$

C. Energy-momentum tensors

The choice of frame, i.e., choice of velocity field u^a and therefore the projection tensor h_{ab}, allows one to obtain an irreducible decomposition of a generic energy-momentum tensor (EMT), $T^{\rm tot}_{ab}$. The following unbarred quantities have been derived from the total EMT, quantities relating to the effective fluids will be denoted with sub/superscripts in order to help to avoid confusion in later sections and to generalize to a multifluid system:

$$T^{\rm tot}_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \tag{11}$$

where μ is the total energy density and p is the total isotropic pressure of the fluid, q_a represents the total energy flux, π_{ab} is the total anisotropic pressure. Additionally, we have the following constraints:

$$q_a u^a = 0, \qquad \pi^a{}_a = 0, \qquad \pi_{ab} = \pi_{(ab)}, \qquad \pi_{ab}u^b = 0.$$

The various components of the total energy-momentum tensor can be isolated in the following way:

$$\mu = T^{\rm tot}_{ab}u^a u^b, \tag{12}$$

$$p = \tfrac{1}{3}T^{\rm tot}_{ab}h^{ab}, \tag{13}$$

$$q_a = -T^{\rm tot}_{cd}u^c h^d{}_a, \tag{14}$$

$$\pi_{ab} = T^{\rm tot}_{\langle ab\rangle}. \tag{15}$$

In a general fluid the pressure, energy density, and entropy are related to each other by an equation of state $p = p(\mu, s)$. A fluid is considered *perfect* if q_a and π_{ab} vanish, and *barotropic* if the entropy is a constant, i.e. the equation of state reduces to $p = p(\mu)$.

D. Propagation and constraint equations

Writing the Ricci and the Bianchi identities in terms of the 1 + 3 variables defined above, we obtain a set of evolution equations (here the "curl" is defined as $({\rm curl}X)^{ab} = \eta^{cd\langle a}\tilde{\nabla}_c X^{b\rangle}{}_d$):

$$\dot{\Theta} - \tilde{\nabla}_a\dot{u}^a = -\tfrac{1}{3}\Theta^2 + (\dot{u}_a\dot{u}^a) - 2\sigma^2 + 2\omega^2 - \tfrac{1}{2}(\mu + 3p), \tag{16}$$

$$\dot{\omega}^{\langle a\rangle} - \tfrac{1}{2}\eta^{abc}\tilde{\nabla}_b\dot{u}_c = -\tfrac{2}{3}\Theta\omega^a + \sigma^a{}_b\omega^b, \tag{17}$$

$$\dot{\sigma}^{\langle ab\rangle} - \tilde{\nabla}^{\langle a}\dot{u}^{b\rangle} = -\tfrac{2}{3}\Theta\sigma^{ab} + \dot{u}^{\langle a}\dot{u}^{b\rangle} - \sigma^{\langle a}{}_c\sigma^{b\rangle c} - \omega^{\langle a}\omega^{b\rangle} - (E^{ab} - \tfrac{1}{2}\pi^{ab}), \tag{18}$$

$$\begin{aligned}(\dot{E}^{\langle ab\rangle} + \tfrac{1}{2}\dot{\pi}^{\langle ab\rangle}) - ({\rm curl}\,H)^{ab} + \tfrac{1}{2}\tilde{\nabla}^{\langle a}q^{b\rangle} &= -\tfrac{1}{2}(\mu + p)\sigma^{ab} - \Theta(E^{ab} + \tfrac{1}{6}\pi^{ab}) \\ &\quad + 3\sigma^{\langle a}{}_c(E^{b\rangle c} - \tfrac{1}{6}\pi^{b\rangle c}) - \dot{u}^{\langle a}q^{b\rangle} \\ &\quad + \eta^{cd\langle a}[2\dot{u}_c H^{b\rangle}{}_d + \omega_c(E^{b\rangle}{}_d + \tfrac{1}{2}\pi^{b\rangle}{}_d)],\end{aligned} \tag{19}$$

$$\begin{aligned}\dot{H}^{\langle ab\rangle} + ({\rm curl}\,E)^{ab} - \tfrac{1}{2}({\rm curl}\pi)^{ab} &= -\Theta H^{ab} + 3\sigma^{\langle a}{}_c H^{b\rangle c} + \tfrac{3}{2}\omega^{\langle a}q^{b\rangle} \\ &\quad - \eta^{cd\langle a}[2\dot{u}_c E^{b\rangle}{}_d - \tfrac{1}{2}\sigma^{b\rangle}{}_c q_d - \omega_c H^{b\rangle}{}_d],\end{aligned} \tag{20}$$

$$\dot{\mu} + \tilde{\nabla}_a q^a = -\Theta(\mu + p) - 2(\dot{u}_a q^a) - (\sigma^a{}_b\pi^b{}_a), \tag{21}$$

$$\dot{q}^{\langle a\rangle} + \tilde{\nabla}^a p + \tilde{\nabla}_b\pi^{ab} = -\tfrac{4}{3}\Theta q^a - \sigma^a{}_b q^b - (\mu + p)\dot{u}^a - \dot{u}_b\pi^{ab} - \eta^{abc}\omega_b q_c, \tag{22}$$

and a set of constraints

$$\tilde{\nabla}_b(E^{ab} + \tfrac{1}{2}\pi^{ab}) - \tfrac{1}{3}\tilde{\nabla}^a\mu + \tfrac{1}{3}\Theta q^a - \tfrac{1}{2}\sigma^a{}_b q^b - 3\omega_b H^{ab} - \eta^{abc}[\sigma_{bd}H^d{}_c - \tfrac{3}{2}\omega_b q_c] = 0, \tag{23}$$

$$\tilde{\nabla}_b H^{ab} + (\mu + p)\omega^a + 3\omega_b(E^{ab} - \tfrac{1}{6}\pi^{ab}) + \eta^{abc}[\tfrac{1}{2}\tilde{\nabla}_b q_c + \sigma_{bd}(E^d{}_c + \tfrac{1}{2}\pi^d{}_c)] = 0, \tag{24}$$

$$\tilde{\nabla}_b\sigma^{ab} - \tfrac{2}{3}\tilde{\nabla}^a\Theta + \eta^{abc}[\tilde{\nabla}_b\omega_c + 2\dot{u}_b\omega_c] + q^a = 0, \tag{25}$$

$$\tilde{\nabla}_a\omega^a - (\dot{u}_a\omega^a) = 0, \tag{26}$$

$$H^{ab} + 2\dot{u}^{\langle a}\omega^{b\rangle} + \tilde{\nabla}^{\langle a}\omega^{b\rangle} - ({\rm curl}\,\sigma)^{ab} = 0, \tag{27}$$

that are completely equivalent to the Einstein equations. It is from these equations that we derive the general evolution equations for linear tensor perturbations.

III. TENSOR PERTURBATION EQUATIONS

A. The background

The equations presented in the previous section hold in any spacetime we may wish to analyze. However, in what follows we will focus on the class of spacetimes that can be thought of as describing an "almost" FLRW model, motivated by the fact that current observations suggest that the universe appears to deviate only slightly from homogeneity and isotropy. We can define a FLRW spacetime in terms of the variables above. Homogeneity and isotropy imply

$$\sigma = \omega = 0, \qquad \tilde{\nabla}_a f = 0, \tag{28}$$

where f is any scalar quantity; in particular

$$\tilde{\nabla}_a \mu = \tilde{\nabla}_a p = 0 \Rightarrow \dot{u}_a = 0. \tag{29}$$

It follows that the governing equations for this background are

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \tfrac{1}{2}(\mu + 3p) = 0, \tag{30}$$

$$\tilde{R} = 2[-\tfrac{1}{3}\Theta^2 + \mu], \tag{31}$$

$$\dot{\mu} + \Theta(\mu + p) = 0. \tag{32}$$

Now in order to describe small deviations from a FLRW spacetime, we simply take all the quantities that are zero in the background as being first order, and retain in the equations (Eq. (16)–(27)) only the terms that are linear in these quantities, i.e. we drop all second order terms. This procedure corresponds to the *linearization* in the 1 + 3 covariant approach and it greatly simplifies the system of equations. In particular, the scalar, vector, and tensor parts of the perturbations are decoupled, so that we are able to treat them separately. In what follows we will focus only on the tensor perturbations.

B. The general linear tensor perturbation equations

The 1 + 3 covariant description of gravitational waves in the context of cosmology has been considered by [28]. The linearized gravitational waves are described by the transverse and trace-free degrees of freedom once scalars have been switched off. Therefore, focusing only on tensor perturbations, the necessary evolution equations are

$$\dot{\sigma}_{ab} + \tfrac{2}{3}\Theta\sigma_{ab} + E_{ab} - \tfrac{1}{2}\pi_{ab} = 0, \tag{33}$$

$$\dot{H}_{ab} + H_{ab}\Theta + (\mathrm{curl}\, E)_{ab} - \tfrac{1}{2}(\mathrm{curl}\, \pi)_{ab} = 0, \tag{34}$$

$$\dot{E}_{ab} + E_{ab}\Theta - (\mathrm{curl}\, H)_{ab} + \tfrac{1}{2}(\mu + p)\sigma_{ab} + \tfrac{1}{6}\Theta\pi_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} = 0, \tag{35}$$

together with the conditions

$$\tilde{\nabla}_b H^{ab} = 0, \qquad \tilde{\nabla}_b E^{ab} = 0, \qquad H_{ab} = (\mathrm{curl}\, \sigma)_{ab}. \tag{36}$$

Note that, since the linear tensor perturbations are frame invariant, the structure of the equations does not depend on the choice of 4-velocity, u_a. In the following, however, we shall choose the frame associated with standard matter ($u_a = u_a^m$). The motivation for such a choice is the fact that real observers are attached to galaxies and these galaxies follow the standard matter geodesics. Taking the time derivative of the above equations we obtain

$$\ddot{\sigma}_{ab} - \tilde{\nabla}^2\sigma + \tfrac{5}{3}\Theta\dot{\sigma}_{ab} + (\tfrac{1}{9}\Theta^2 + \tfrac{1}{6}\mu - \tfrac{3}{2}p)\sigma_{ab} = \dot{\pi}_{ab} + \tfrac{2}{3}\Theta\pi_{ab}, \tag{37}$$

$$\ddot{H}_{ab} - \tilde{\nabla}^2 H_{ab} + \tfrac{7}{3}\Theta\dot{H}_{ab} + \tfrac{2}{3}(\Theta^2 - 3p)H_{ab} = (\mathrm{curl}\, \dot{\pi})_{ab} + \tfrac{2}{3}\Theta(\mathrm{curl}\, \pi)_{ab}, \tag{38}$$

$$\ddot{E}_{ab} - \tilde{\nabla}^2 E_{ab} + \tfrac{7}{3}\Theta\dot{E}_{ab} + \tfrac{2}{3}(\Theta^2 - 3p)E_{ab} - \tfrac{1}{6}\Theta(\mu + p)(1 + 3c_s^2)\sigma_{ab} = -[\tfrac{1}{2}\ddot{\pi}_{ab} - \tfrac{1}{2}\tilde{\nabla}^2\pi_{ab} + \tfrac{5}{6}\Theta\dot{\pi}_{ab} + \tfrac{1}{3}(\Theta^2 - \mu)\pi_{ab}], \tag{39}$$

where $c_s^2 = \dot{p}/\dot{\mu}$ and we have used the Raychaudhuri equation [Eq. (16)], the energy conservation equation [Eq. (32)], and the commutator identity

$$(\mathrm{curl}\, \dot{X})_{ab} = (\mathrm{curl}\, X)^{\cdot}_{ab} + \tfrac{1}{3}(\mathrm{curl}\, X)\Theta. \tag{40}$$

These equations generalize the tensor perturbation equations for an imperfect fluid that were derived in [30]. Once the form of the anisotropic pressure has been determined in Eqs. (37)–(39), the equations can be solved to give the evolution of tensor perturbations. As already noted in [28], the presence of a term that contains the shear in Eq. (39) makes this equation effectively third order, so that it is not possible to write down a closed wave equation for E_{ab}. If $\pi_{ab} = 0$, it is easy to show that for consistency, the solution for this field must also satisfy a wave equation because the shear is a solution of a wave equation and Eq. (33) holds. This will also be the case here because in our case $\pi_{ab} \propto \sigma_{ab}$ and so the anisotropic pressure will also satisfy a wave equation.

Following standard harmonic analysis, Eqs. (37) and (38) may be reduced to ordinary differential equations. It is standard [23] to use trace-free symmetric tensor eigenfunctions of the spatial the Laplace-Beltrami operator defined by

$$\tilde{\nabla}^2 Q_{ab} = -\frac{k^2}{a^2} Q_{ab}, \tag{41}$$

where $k = 2\pi a/\lambda$ is the wave number and $\dot{Q}_{ab} = 0$. Developing σ_{ab} and H_{ab} in terms of the Q_{ab}, Eqs. (37) and (38) reduce to

$$\ddot{\sigma}^{(k)} + \frac{5}{3}\Theta\dot{\sigma}^{(k)} + \left(\frac{1}{9}\Theta^2 + \frac{1}{6}\mu - \frac{3}{2}p - \frac{k^2}{a^2}\right)\sigma_{ab}^{(k)} = \dot{\pi}^{(k)} + \frac{2}{3}\Theta\pi^{(k)}, \tag{42}$$

$$\ddot{H}^{(k)} + \frac{7}{3}\Theta\dot{H}^{(k)} + \frac{2}{3}\left(\Theta^2 - 3\mu - \frac{k^2}{a^2}\right)H^{(k)} = (\mathrm{curl}\,\dot{\pi})^{(k)} + \frac{2}{3}\Theta(\mathrm{curl}\,\pi)^{(k)}, \tag{43}$$

and Eq. (33) reads

$$E^{(k)} = -\dot{\sigma}^{(k)} - \tfrac{2}{3}\Theta\sigma^{(k)} + \tfrac{1}{2}\pi^{(k)}. \tag{44}$$

IV. GENERAL EQUATIONS FOR FOURTH ORDER GRAVITY

The classical action for a fourth order theory of gravity is given by

$$\mathcal{A} = \int d^4x\sqrt{-g}[\Lambda + c_0R + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + \mathcal{L}_m], \tag{45}$$

where we have used the Gauss Bonnet theorem [31] and $\mathcal{L}_m$ represents the matter contribution. In situations where the metric has a high degree of symmetry, this action can be further simplified. In particular, in the homogeneous and isotropic case the action for a general fourth order theory of gravity takes the form

$$\mathcal{A} = \int dx^4\sqrt{-g}[f(R) + \mathcal{L}_m], \tag{46}$$

where $\mathcal{L}_m$ represents the matter contribution. Such modifications to the linear Einstein-Hilbert action can typically arise in effective actions derived from higher-dimensional theories of gravity [19]. Varying the action with respect to the metric gives the gravitational field equations:

$$f'R_{ab} - \tfrac{1}{2}g_{ab}f = (g^c{}_a g^d{}_b - g_{ab}g^{cd})S_{cd} + T^m_{ab}, \tag{47}$$

where $f = f(R)$, $f' = f'(R) \equiv \partial f(R)/\partial R$, $T^m_{\mu\nu} = \frac{2}{\sqrt{-g}} \times \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}}$ represents the stress energy tensor of standard matter, and $S_{ab} = \nabla_a\nabla_b f'(R)$. The trace of Eq. (47) gives

$$f'R - 2f = -3S + T^m, \tag{48}$$

where $S = g^{ab}S_{ab}$. The various components of S_{ab} can be decomposed as

$$\begin{aligned} S_{ab} &= f''[\tilde{\nabla}_a\tilde{\nabla}_b R - \tilde{\nabla}_a\dot{R}u_b - u_a u^c\nabla_c(\tilde{\nabla}_b R) + \ddot{R}u_a u_b \\ &\quad - \dot{R}(\tilde{\nabla}_a u_b - u_a\dot{u}_b)] + f'''[\tilde{\nabla}_a R\tilde{\nabla}_b R \\ &\quad - \dot{R}(\tilde{\nabla}_b R u_a + \tilde{\nabla}_a R u_b) + \dot{R}^2 u_a u_b], \\ S &= f''(\tilde{\nabla}^c\tilde{\nabla}_c R + \dot{u}^c\tilde{\nabla}_c R - \ddot{R} - \Theta\dot{R}) \\ &\quad + f'''(\tilde{\nabla}^c R\tilde{\nabla}_c R - \dot{R}^2). \end{aligned} \tag{49}$$

These equations reduce to the standard Einstein field equations when $f(R) = R$. It is crucial for our purposes to be able to write Eq. (47) in the form

$$G_{ab} = T^{\mathrm{tot}}_{ab} = \tilde{T}^m_{ab} + T^R_{ab}, \tag{50}$$

where $\tilde{T}^m_{ab} = \frac{T^m_{ab}}{f'}$ and T^R_{ab} is defined as

$$T^R_{ab} = \frac{1}{f'}\left[\frac{1}{2}(f - f'R)g_{ab} + (g^c{}_a g^d{}_b - g_{ab}g^{cd})S_{cd}\right]. \tag{51}$$

The right-hand side of Eq. (50) represents two effective "fluids": the *curvature fluid* (associated with T^R_{ab}) and the *effective matter fluid* (associated with $\tilde{T}^m_{ab}$). This step is important because it allows us to treat fourth order gravity as standard Einstein gravity in the presence of two "effective" fluids. This means that once the effective thermodynamics of these fluids has been studied, we can apply the covariant gauge-invariant approach in the standard way.

The conservation properties of these effective fluids are given by the Bianchi identities $T^{\mathrm{tot};b}_{ab}$. When applied to the total stress energy tensor, these identities reveal that if standard matter is conserved, the total fluid is also conserved even though the curvature fluid may in general possess off-diagonal terms [11,32,33]. In other words, no matter how complicated the effective stress energy tensor T^{tot}_{ab} is, it will always be divergence free if $T^{m;b}_{ab} = 0$. When applied to the single effective tensors, the Bianchi identities read

$$\tilde{T}^{m;b}_{ab} = \frac{T^{m;b}_{ab}}{f'} - \frac{f''}{f'^2}T^m_{ab}R^{;b}, \tag{52}$$

$$T^{R;b}_{ab} = \frac{f''}{f'^2}\tilde{T}^m_{ab}R^{;b}, \tag{53}$$

with the last expression being a consequence of total energy-momentum conservation. It follows that the individual effective fluids are not conserved but exchange energy and momentum.

It is worth noting here that, even if the energy-momentum tensor associated with the effective matter source is not conserved, standard matter still follows the usual conservation equations $T^{m;b}_{ab} = 0$. It is also important to stress that the fluids with T^R_{ab} and $\tilde{T}^m_{ab}$ defined above are *effective* and consequently can admit features that one would normally consider unphysical for a standard matter field. This means that all the thermodynamical quantities associated with the curvature defined below should be considered *effective* and not bounded by the usual constraints associated with matter fields. It is important to understand that this does not compromise any of the thermodynamical properties of standard matter represented by the Lagrangian $\mathcal{L}_m$.

In the matter frame u^m_a, the various components of the total energy-momentum tensor, Eq. (12), can be rewritten in terms of the two effective fluids:

$$\mu = \frac{\mu^m}{f'} + \mu^R, \tag{54}$$

$$p = \frac{p^m}{f'} + p^R, \tag{55}$$

$$q_a = \frac{q_a^m}{f'} + q_a^R, \tag{56}$$

$$\pi_{ab} = \frac{\pi_{ab}^m}{f'} + \pi_{ab}^R, \tag{57}$$

where we assume that standard matter is a perfect fluid, i.e. $q_a^m = 0$ and $\pi_{ab}^m = 0$. The effective thermodynamical quantities for the curvature fluid are

$$\mu^R = \frac{1}{f'}\Big[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R + f''\dot{u}_b\tilde{\nabla}R\Big], \tag{58}$$

$$p^R = \frac{1}{f'}\Big[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\Theta f''\dot{R} - \frac{2}{3}f''\tilde{\nabla}^2 R - \frac{2}{3}f'''\tilde{\nabla}^a R\tilde{\nabla}_a R - \frac{1}{3}f''\dot{u}_b\tilde{\nabla}R\Big], \tag{59}$$

$$q_a^R = -\frac{1}{f'}\Big[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \frac{1}{3}f''\tilde{\nabla}_a R\Big], \tag{60}$$

$$\pi_{ab}^R = \frac{1}{f'}[f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R - f''\sigma_{ab}\dot{R}]. \tag{61}$$

The twice contracted Bianchi identities lead to evolution equations for μ^m, μ^R, q_a^R:

$$\dot{\mu}^m = -\Theta(\mu^m + p^m), \tag{62}$$

$$\dot{\mu}^R + \tilde{\nabla}^a q_a^R + \Theta(\mu^R + p^R) + 2(\dot{u}^a q_a^R) + (\sigma^{ab}\pi_{ba}^R) = \mu^m\frac{f''\dot{R}}{f'^2}, \tag{63}$$

$$\dot{q}_{\langle a\rangle}^R + \tilde{\nabla}_a p^R + \tilde{\nabla}^b\pi_{ab}^R + \frac{4}{3}\Theta q_a^R + \sigma_a^b q_b^R + (\mu^R + p^R)\dot{u}_a + \dot{u}^b\pi_{ab}^R + \eta_a^{bc}\omega_b q_c^R = \mu^m\frac{f''\tilde{\nabla}_a R}{f'^2}, \tag{64}$$

and a relation connecting the acceleration $\dot{u}_a$ to μ^m and p^m follows from momentum conservation of standard matter:

$$\tilde{\nabla}^a p^m = -(\mu^m + p^m)\dot{u}^a. \tag{65}$$

Note that, as we have seen in the previous section the *curvature* fluid and the effective *matter* exchange energy and momentum. The decomposed interaction terms in Eqs. (63) and (64) are given by $\mu^m\frac{f''\tilde{\nabla}_a R}{f'^2}$ and $\mu^m\frac{f''\dot{R}}{f'^2}$.

It is easy to see that the *curvature* fluid is in general an imperfect fluid, i.e. has energy flux (q_a) and anisotropic pressure (π_{ab}). Since we are only interested in linear tensor perturbations, we need only be concerned with the tensor anisotropic pressure, which is proportional to the shear, σ_{ab}. We now present the second order evolution equations resulting from the standard harmonic analysis of Eqs. (42)–(44) in the case of $f(R)$ theories of gravity:

$$\ddot{\sigma}^{(k)} + \Big(\frac{5}{3}\Theta + \dot{R}\frac{f''}{f'}\Big)\dot{\sigma}^{(k)} + \Big\{\frac{1}{9}\Theta^2 + \frac{1}{f'}\Big(\frac{1}{6}\mu^m - \frac{3}{2}p^m\Big) + \frac{k^2}{a^2} - \frac{1}{2}\Theta\dot{R}\frac{f''}{f'} - \frac{5}{6}\frac{1}{f'}(f - f'R) - \dot{R}^2\Big[\frac{1}{2}\frac{f'''}{f'} + \Big(\frac{f''}{f'}\Big)^2\Big] - \frac{1}{2}\ddot{R}\frac{f''}{f'}\Big\}\sigma^{(k)} = 0, \tag{66}$$

$$\ddot{H}^{(k)} + \Big(\frac{7}{3}\Theta + \dot{R}\frac{f''}{f'}\Big)\dot{H}^{(k)} + \Big\{\frac{2}{3}\Theta^2 - \frac{2}{f'}p^m + \frac{k^2}{a^2} - \frac{1}{3}\Theta\dot{R}\frac{f''}{f'} - \frac{1}{f'}(f - f'R) - \dot{R}^2\Big[\frac{f'''}{f'} + \Big(\frac{f''}{f'}\Big)^2\Big] - \ddot{R}\frac{f''}{f'}\Big\}H^{(k)} = 0, \tag{67}$$

$$E^{(k)} = -\dot{\sigma}^{(k)} - \Big(\frac{2}{3}\Theta + \frac{1}{2}\dot{R}\frac{f''}{f'}\Big)\sigma^{(k)}. \tag{68}$$

For our purposes it will be particularly useful to consider these equations in the so-called long wavelength limit. In this limit the wave number k is considered to be so small that the wavelength $\lambda = 2\pi a/k$ associated with it is much larger than the Hubble radius. Equation (41) then implies that all the Laplacians can be neglected and the spatial dependence of the perturbation variables can be factored out.

V. TENSOR PERTURBATIONS IN R^n GRAVITY

To proceed, we must now fix our theory of gravity, i.e. we must choose the form of $f(R)$. We will consider a toy model (R^n-gravity) which is the simplest example of fourth order theory of gravity but exhibits many of the properties of such theories. In this theory $f(R) = \chi R^n$ and the action reads

$$\mathcal{A} = \int d^4x\sqrt{-g}[\chi R^n + \mathcal{L}_M], \tag{69}$$

where χ a the coupling constant with suitable dimensions and $\chi = 1$ for $n = 1$. If $R \neq 0$, the field equations for this theory read

$$G_{ab} = \chi^{-1}\frac{\tilde{T}^m_{ab}}{nR^{n-1}} + T^R_{ab}, \tag{70}$$

where

$$\tilde{T}^{\,m}_{ab} = \chi^{-1}\frac{T^m_{ab}}{nR^{n-1}}, \tag{71}$$

$$T^R_{ab} = (n-1)\bigg\{-\frac{R}{2n}g_{ab} + \bigg[\frac{R^{;cd}}{R} + (n-2)\frac{R^{;c}R^{;d}}{R^2}\bigg] \times (g_{ca}g_{db} - g_{cd}g_{ab})\bigg\}. \tag{72}$$

The FLRW dynamics of this model have been investigated in detail via a dynamical systems approach in [11], where a complete phase space analysis was performed. This work demonstrated that for specific intervals of the parameter n there exist a set of initial conditions with nonzero measure for which the cosmic histories include a transient decelerated phase (during which large-scale structure can form) which evolves towards one with accelerated expansion. These transient almost Friedmann models existed for $0.28 \lesssim n \lesssim 1.35$ in the case of a dust filled ($w = 0$) universe and for $0.31 \lesssim n \lesssim 1.29$ in the case of a radiation filled ($w = 1/3$) universe. As we will discuss in later sections, these allowed intervals of n could be reduced significantly with future observations of gravitational waves. This model was also investigated as a possible explanation for the observed flatness of the rotation curves of spiral galaxies and the observed late times acceleration of the universe [34]. The authors found a very good agreement between this model and observational data when $n = 3.5$. This is however at odds with the results of [11]. Thus, if one requires a transient decelerated phase (during which large-scale structure can form) and a solution to the dark matter and dark energy problem, the R^n model appears not to be viable. However, the aim of this paper is to show that the study of tensor perturbations can in principle provide a strong constraint on the theory of gravity independent of existing cosmological data sets and consequently this work will provide a template for a more extensive study of tensor perturbations of $f(R)$ cosmologies.

In what follows we begin by analyzing the evolution of tensor perturbations in the absence of standard matter. We then consider the case of dust/radiation dominated evolution. Although we will give the full solutions, the discussion of the physics will be restricted to the long wavelength limit.

A. The vacuum case

We start by considering tensor perturbations in the absence of matter. This class of theories then admits the following exact solution:

$$a(t) = a_0 t^q, \qquad q = \frac{(1-n)(2n-1)}{n-2}, \qquad K = 0. \tag{73}$$

The expansion parameter is given by

$$\Theta(t) = \frac{3q}{t}. \tag{74}$$

For the purposes of this paper we restrict our attention to expanding models. This requires $q > 0$, which in turn restricts the parameter n. In order to have an expanding background we require $0 < n < 1/2$ and $1 < n < 2$ (we recover a static vacuum solution for $n = 1/2, 1$). We will only investigate models with values of n which satisfy the second inequality (since we wish to investigate models close to GR). The equation of state (EOS) of the total effective fluid in the background is then [4]

$$w = \frac{p}{\mu} = -\frac{1}{3}\frac{(6n^2 - 7n - 1)}{(2n-1)(n-1)}. \tag{75}$$

The EOS is singular and the poles occur at $n = 1/2$ and $n = 1$. Additionally, we have accelerated expansion, ($w < -1/3$) for $n > (1+\sqrt{3})/2 \approx 1.366$ and in the limit $n \to \infty$ we have $w \to -1$. Substituting into Eqs. (66)–(68) we obtain

$$\ddot{\sigma}^{(k)} + \frac{3(1-n)(4n-3)}{(n-2)t}\dot{\sigma}^{(k)} + \bigg[\frac{n(4n-5)(n-1)(8n-7)}{(n-2)^2t^2} + k^2t^{-2q}\bigg]\sigma^{(k)} = 0, \tag{76}$$

$$\ddot{H}^{(k)} + \frac{(1-n)(16n-11)}{(n-2)t}\dot{H}^{(k)} + \bigg[\frac{2(6n^2-8n+1)(n-1)(5n-4)}{(n-2)^2t^2} + k^2t^{-2q}\bigg]H^{(k)} = 0, \tag{77}$$

$$E^{(k)} = -\dot{\sigma}^{(k)} + \frac{(n-1)(5n-4)}{(n-2)t}\sigma^{(k)}. \tag{78}$$

In the long wavelength/superhorizon limit ($k = 0$), the above equations admit the following solutions:

$$\sigma^{(k)} = A_1 t^{[n(4n-5)]/(n-2)} + A_2 t^{[(8n-7)(n-1)]/(n-2)}, \tag{79}$$

$$H^{(k)} = A_3 t^{(6n^2-8n+1)/(n-2)} + A_4 t^{[2(5n-4)(n-1)]/(n-2)}, \tag{80}$$

$$E^{(k)} = A_1(n-2)t^{[2(n-1)(2n-1)]/(n-2)} + A_2\frac{(n-1)(5n-4)}{(n-2)}t^{(8n^2-16n+9)/(n-2)}. \quad (81)$$

However, the physical quantity of interest is the dimensionless expansion normalized shear $\Sigma = \sigma/H$:

$$\Sigma^{(k)} = \tilde{\Sigma}_1 t^{(4n^2-4n-2)/(n-2)} + \tilde{\Sigma}_2 t^{[(2n-1)(4n-5)]/(n-2)}. \quad (82)$$

In Fig. 1 we have plotted the exponents of each mode of the solutions given above as a function of n in order to better see how the large-scale behavior varies. The black (gray) line represents the growing (decaying) mode and the points represent the value of the exponents in the case of GR ($n = 1$). In the GR limit we recover a static vacuum model in the background and $\tilde{\Sigma}_i$ always grows indicating that this model is unstable with respect to tensor perturbations. In the case of larger values of n, the $\tilde{\Sigma}_1$ mode grows (decays) for $n \lesssim 1.366$ ($n \gtrsim 1.366$) and the $\tilde{\Sigma}_2$ mode grows (decays) for $n < 1.25$ ($n > 1.25$). This is consistent with the background dynamics in that all perturbation modes are decaying when we have accelerated expansion ($w < -1/3$) in the background.

For the sake of completeness we present the results of the general case ($k \neq 0$). The solutions are given in terms of Bessel functions of the first and second kind (J and Y, respectively):

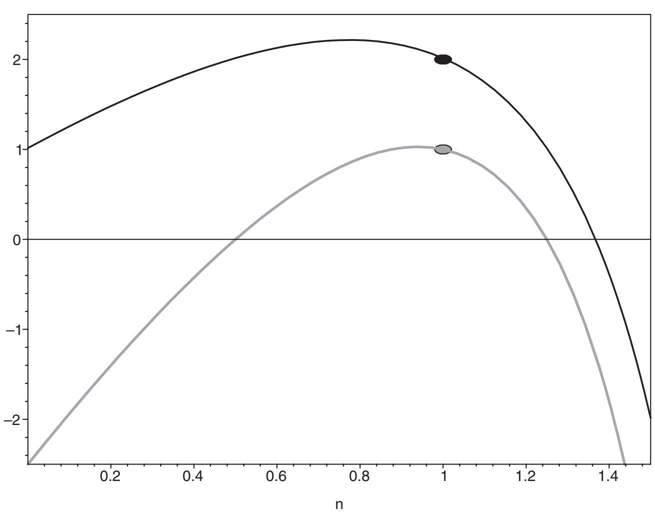

FIG. 1. The exponents of each mode of the solution [Eq. (83)] against n in the vacuum case. The black (gray) line represents the growing (decaying) mode. The points represent the value of the exponents in the case of GR ($n = 1$). As n is increased the background expansion rate increases, resulting in a weaker growth rate for tensor perturbations. When the critical value of $n \approx 1.366$ is reached, no growing modes can be supported in this model.

$$\sigma^{(k)} = t^{[(2n-1)(6n-7)]/2(n-2)}\left[A_1 J\left(s, \frac{kt^r}{r}\right) + A_2 Y\left(s, \frac{kt^r}{r}\right)\right], \quad (83)$$

$$H^{(k)} = t^{[(2n-1)(8n-9)]/2(n-2)}\left[A_3 J\left(s, \frac{kt^r}{r}\right) + A_4 Y\left(s, \frac{kt^r}{r}\right)\right], \quad (84)$$

$$E^{(k)} = A_1 t^{(12n^2-22n+11)/[2(n-2)]}\left[\frac{n^2+2n-5}{(2-n)} J\left(s, \frac{kt^r}{r}\right) + kt^r J\left(s+1, \frac{kt^r}{r}\right)\right] \quad (85)$$

$$+ A_2 t^{(12n^2-22n+11)/[2(n-2)]}\left[\frac{n^2+2n-5}{(2-n)} Y\left(s, \frac{kt^r}{r}\right) + kt^r Y\left(s+1, \frac{kt^r}{r}\right)\right], \quad (86)$$

where we have introduced the following parameters:

$$r = \frac{2n^2-2n-1}{n-2}, \qquad s = -1 + \frac{3(2n-3)}{2(n-2)r}. \quad (87)$$

The normalized shear Σ is now of the form

$$\Sigma^{(k)} = t^{[3(4n^2-6n+1)]/[2(n-2)]}\left[\tilde{\Sigma}_1 J\left(s, \frac{kt^r}{r}\right) + \tilde{\Sigma}_2 Y\left(s, \frac{kt^r}{r}\right)\right], \quad (88)$$

where both the $\tilde{\Sigma}_i$ modes grow (decay) for $n \lesssim 1.290$ ($n \gtrsim 1.290$).

B. The fluid case

We will now consider the case of tensor perturbations in the presence of matter which is described by a perfect fluid with barotropic EOS index, w_m. This class of theories then admits the following exact solution:

$$a(t) = a_0 t^{2n/[3(1+w_m)]}, \qquad K = 0. \quad (89)$$

The expansion parameter is given by

$$\Theta(t) = \frac{2n}{(1+w_m)t}. \quad (90)$$

As in the previous case we restrict our attention to expanding models. Additionally, we are mainly interested in the case where the perfect fluid describes dust ($w_m = 0$) or radiation ($w_m = 1/3$). This is due to the fact that these cases are the most relevant when considering GW detection via the CMB or direct detectors, e.g. LISA and BBO. To ensure an expanding model we now require $n > 0$, provided $w_m > -1$.

1. The dust case

We now investigate the evolution of tensor perturbations in the dust dominated era. The scale factor is given by

$$a(t) = a_0 t^r, \qquad r = \frac{2n}{3}. \tag{91}$$

The EOS of the total effective fluid (dust and the effective curvature fluid) is then

$$w = -\frac{(n-1)}{n}. \tag{92}$$

The EOS is divergent for $n = 0$ and we have accelerated expansion ($w < -1/3$) when $n > 3/2$. In the limit $n \to \infty$ we have $w \to -1$. Substituting into Eqs. (66)–(68) we obtain

$$\ddot{\sigma}^{(k)} + \frac{2(2n+3)}{3t}\dot{\sigma}^{(k)} + \left[\frac{(8n-6)}{3t^2} + k^2 t^{-2r}\right]\sigma^{(k)} = 0, \tag{93}$$

$$\ddot{H}^{(k)} + \frac{2(4n+3)}{3t}\dot{H}^{(k)} + \left[\frac{2(2n^2+5n-3)}{3t^2} + k^2 t^{-2r}\right]H^{(k)} = 0, \tag{94}$$

$$E^{(k)} = -\dot{\sigma}^{(k)} - \frac{(n+3)}{3t}\sigma^{(k)}. \tag{95}$$

In the long wavelength limit ($k = 0$), the above equations admit the following solutions:

$$\sigma^{(k)} = B_1 t^{-2} + B_2 t^{(1-2r)}, \tag{96}$$

$$H^{(k)} = B_3 t^{-(r+2)} + B_4 t^{(1-2n)}, \tag{97}$$

$$E^{(k)} = -B_1\frac{(9+n)}{3}t^{-3} - B_2\frac{5n}{3}t^{-2r}, \tag{98}$$

The normalized shear is given by

$$\Sigma^{(k)} = \tilde{\Sigma}_1 t^{-1} + \tilde{\Sigma}_2 t^{2(1-r)}. \tag{99}$$

The $\tilde{\Sigma}_1$ mode is the decaying mode solution and is independent of the parameter n. This mode corresponds to the standard decaying mode found in GR. The $\tilde{\Sigma}_2$ mode grows (decays) for $n < 3/2$ ($n > 3/2$) and reduces to the GR growing mode in the limit $n \to 1$. This is consistent with the background dynamics in that all perturbation modes are decaying when we have accelerated expansion ($w < -1/3$) in the background.

In Fig. 2(a) we have plotted the exponents of each mode of the solutions given above as a function of n. The black (grey) lines represent the growing (decaying) mode and the points represent the value of the exponents in the case of GR ($n = 1$). For most of the values of n the perturbations grow slower in R^n gravity than in GR. In fact only for $n < 1$ does the $\tilde{\Sigma}_1$ mode grow with a rate faster than the usual $t^{2/3}$. In the case of GR, there is always a growing tensor perturbation mode provided the background is not undergoing accelerated expansion. In the case of R^n gravity, tensor perturbations grow at a slower rate, thus requiring a sufficiently decelerated expansion in order to support a growing mode.

Again, for the sake of completeness, in the general case ($k \neq 0$) the solutions are given in terms of Bessel functions of the first and second kind (J and Y respectively):

$$\sigma^{(k)} = t^{-(2r+1)/2}\left\{B_1 J\left[-\frac{2r-3}{2(r-1)}, \frac{kt^{(1-r)}}{(r-1)}\right) + B_2 Y\left[-\frac{2r-3}{2(r-1)}, \frac{kt^{(1-r)}}{(r-1)}\right]\right\}, \tag{100}$$

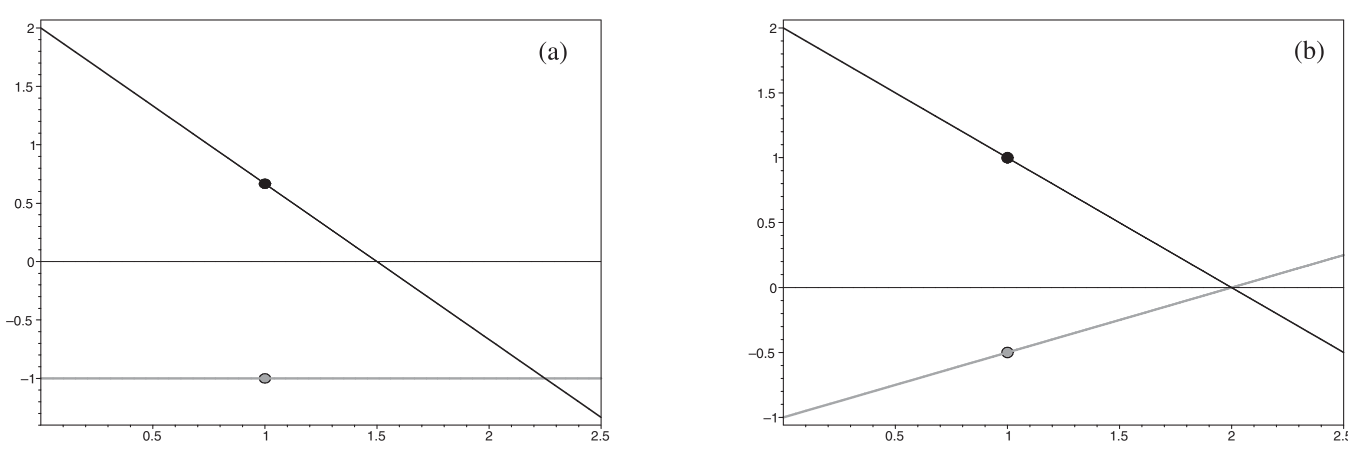

FIG. 2. The exponents of each mode of the solution for the normalized shear against n in the dust and radiation dominated eras. (a) The left panel represents the exponents of the mode in the dust dominated era. (b) The right panel represents the exponents of the mode in the radiation dominated era. The black (grey) line represents the growing (decaying) mode. The points represent the value of the exponents in the case of GR ($n = 1$).

$$H^{(k)} = t^{-(4r+1)/2}\left\{B_3 J\left[\frac{2r-3}{2(r-1)}, \frac{kt^{(1-r)}}{(r-1)}\right) + B_4 Y\left[\frac{2r-3}{2(r-1)}, \frac{kt^{(1-r)}}{(r-1)}\right]\right\}, \tag{101}$$

$$\begin{aligned} E^{(k)} = {}& B_1 t^{-(2r+3)/2}\left\{\frac{2(2r-3)}{3} J\left[\frac{2r-3}{2(r-1)}, \frac{kt^{(1-r)}}{(r-1)}\right] - kt^{-r} J\left[\frac{1}{2(r-1)}, \frac{kt^{(1-r)}}{(r-1)}\right]\right\} \\ & + B_2 t^{-(2r+3)/2}\left\{\frac{2(2r-3)}{3} Y\left[\frac{2r-3}{2(r-1)}, \frac{kt^{(1-r)}}{(r-1)}\right] - kt^{-r} Y\left[\frac{1}{2(r-1)}, \frac{kt^{(1-r)}}{(r-1)}\right]\right\}, \end{aligned} \tag{102}$$

2. The radiation case

Next, we study the evolution of tensor perturbations in the radiation dominated era. The results of this section are especially relevant if one wishes to constrain $f(R)$ models through their impact on the B-mode correlation on the CMB. The scale factor goes as

$$a(t) = a_0 t^r \qquad r = \frac{n}{2}. \tag{103}$$

The EOS of the total effective fluid (dust and the effective curvature fluid) is then

$$w = -\frac{(3n-4)}{3n}. \tag{104}$$

The EOS is divergent for $n = 0$ and we have accelerated expansion ($w > -1/3$) when $n > 2$. In the limit $n \longrightarrow \infty$ we have $w \longrightarrow -1$. Substituting into Eqs. (66)–(68) we obtain

$$\ddot{\sigma}^{(k)} + \frac{n+4}{2t}\dot{\sigma}^{(k)} + \left[\frac{(4-n)(n-1)}{2t^2} + k^2 t^{-2r}\right]\sigma^{(k)} = 0, \tag{105}$$

$$\ddot{H}^{(k)} + \frac{3n+4}{2t}\dot{H}^{(k)} - \left[\frac{n(3n-2)}{t^2} + k^2 t^{-2r}\right]H^{(k)} = 0, \tag{106}$$

$$E^{(k)} = -\dot{\sigma}^{(k)} - \frac{1}{t}\sigma^{(k)}. \tag{107}$$

In the long wavelength limit ($k = 0$), the above equations admit the following solutions:

$$\sigma^{(k)} = C_1 t^{(1-2r)} + C_2 t^{(r-2)}, \tag{108}$$

$$H^{(k)} = C_3 t^{(1-3r)} + C_4 t^{-2}, \tag{109}$$

$$E^{(k)} = C_1(2r-2)t^{-2r} + C_2(1-r)t^{(r-3)}, \tag{110}$$

The normalized shear is given by

$$\Sigma^{(k)} = \tilde{\Sigma}_1 t^{(2-2r)} + \tilde{\Sigma}_2 t^{(r-1)}. \tag{111}$$

The $\tilde{\Sigma}_1$ mode grows for $0 < n < 2$ and decays for $n > 2$. In Fig. 2(b) we have plotted the exponents of each mode of the solutions given above as a function of n. The black (grey) lines represents the growing (decaying) mode and the points represent the value of the exponents in the case of GR ($n = 1$). For $0 < n < 2$ the $\tilde{\Sigma}_1$ mode grows and the $\tilde{\Sigma}_2$ mode decays. In the range $n > 2$ the modes change behavior in that the $\tilde{\Sigma}_1$ mode decays and the $\tilde{\Sigma}_2$ mode grows. Again, for most of the values of n the perturbations grow slower in R^n gravity than in GR, and only for $0 < n < 1$ and $n > 4$ does the $\tilde{\Sigma}_1$ mode grow with a rate faster than the usual linear growth. The most interesting feature of the solutions in the radiation dominated era is the possibility of growing modes even if the universe is in a state of accelerated expansion ($n > 4$). The impact of these modes on the CMB could allow one to constrain deviations from GR. However, one should also analyze the evolution of perturbations on small scales. This analysis is beyond the scope of this paper and it is left to a future, more detailed investigation.

In the general case ($k \neq 0$), the solutions are given in terms of Bessel functions of the first and second kind (J and Y respectively):

$$\sigma^{(k)} = t^{-(r+1)/2}\left\{C_1 J\left[\frac{3}{2}, \frac{kt^{-(1-r)}}{(r-1)}\right] + C_2 Y\left[\frac{3}{2}, \frac{kt^{(1-r)}}{(r-1)}\right]\right\}, \tag{112}$$

$$H^{(k)} = t^{-(3r+1)/2}\left\{C_3 J\left[\frac{3}{2}, \frac{kt^{(1-r)}}{(r-1)}\right] + C_4 Y\left[\frac{3}{2}, \frac{kt^{(1-r)}}{(r-1)}\right]\right\}, \tag{113}$$

$$\begin{aligned} E^{(k)} = {}& C_1 t^{-(r+3)/2}\left\{2(r-1) J\left[\frac{3}{2}, \frac{kt^{-(1-r)}}{(r-1)}\right] - kt^{-r} J\left[\frac{5}{2}, \frac{kt^{(1-r)}}{(r-1)}\right]\right\} \\ & + C_2 t^{-(r+3)/2}\left\{2(r-1) Y\left[\frac{3}{2}, \frac{kt^{(1-r)}}{(r-1)}\right] - kt^{-r} Y\left[\frac{5}{2}, \frac{kt^{(1-r)}}{(r-1)}\right]\right\}, \end{aligned} \tag{114}$$

3. The generic large-scale case

Finally, we study the evolution of large-scale ($k = 0$) tensor perturbations in the presence of a general barotropic fluid (that is we will not fix w_m except to state that $w_m > -1$). To ensure an expanding model we now require $n > 0$, provided $w_m > -1$. The scale factor goes as

$$a(t) = a_0 t^r \qquad r = \frac{2n}{3(1+w_m)}. \tag{115}$$

The EOS of the total effective fluid (radiation and the effective curvature fluid) is then

$$w = \frac{(w_m + 1 - n)}{n}. \tag{116}$$

The EOS is divergent for $n = 0$ and we have accelerated expansion ($w < -1/3$) when $n > 3(w_m + 1)/2$. In the limit $n \to \infty$ we have $w \to -1$. Substituting into Eqs. (66)–(68) we obtain

$$\ddot{\sigma}^{(k)} + \frac{10n + 6(1+w_m)(1-n)}{3(1+w_m)t}\dot{\sigma}^{(k)} + \frac{2(3+3w_m-4n)(nw_m - w_m - 1)}{3(1+w_m)^2 t^2}\sigma^{(k)} = 0, \tag{117}$$

$$\ddot{H}^{(k)} + \frac{14n + 6(1+w_m)(1-n)}{3(1+w_m)t}\dot{H}^{(k)} + \frac{16n^2 - 8n(w_m+1) + 6(n-1)(w_m+1)(w_m+1-2n)}{3(1+w_m)^2 t^2}H(k) = 0, \tag{118}$$

$$E^{(k)} = -\dot{\sigma}^{(k)} - \frac{4n + 3(1+w_m)(1-n)}{3(1+w_m)t}\sigma(k). \tag{119}$$

The solutions are then

$$\sigma^{(k)} = D_1 t^{(1-2r)} + C_2 t^{(2n-2-3r)}, \tag{120}$$

$$H^{(k)} = D_3 t^{(1-3r)} + D_4 t^{2n-2-4r}, \tag{121}$$

$$E^{(k)} = D_1(n-2)t^{-2r} + C_2(n-1-r)t^{(2n-3-3r)}, \tag{122}$$

The normalized shear is given by

$$\Sigma^{(k)} = \tilde{\Sigma}_1 t^{(2-2r)} + \tilde{\Sigma}_2 t^{(2n-1-3r)}. \tag{123}$$

The $\tilde{\Sigma}_1$ mode grows for $n < 3(w_m + 1) = 2$ and decays for $n > 3(w_m + 1) = 2$. The $\tilde{\Sigma}_2$ mode decays for the range $n < (w_m + 1) = 2w_m$ and grows for $n > (w_m + 1) = 2w_m$. In Fig. 3 we have plotted the range of parameters for which the modes grow or decay as a function of n and w_m. This divides the parameter space into four regions. In region I, $\tilde{\Sigma}_1$ decays and $\tilde{\Sigma}_2$ grows. In region II, both modes grow. In region III, $\tilde{\Sigma}_1$ grows and $\tilde{\Sigma}_2$ decays. Finally in region IV, both modes decay. The most interesting features of these solutions are those of region IV. As mentioned earlier this particular model was also investigated as a possible explanation for the observed flatness of the rotation curves of spiral galaxies and the observed late times acceleration of the universe [12]. The authors found a good agreement between this model and observational data when $n = 3.5$ in the presence of dust ($w_m = 0$). However, from our analysis we have found that such a choice of parameters ensures the absence of growing modes in the tensor perturbations. Therefore, if we wish to use this model as an explanation for dark matter, we can use gravitational wave detectors to severely constrain such theories.

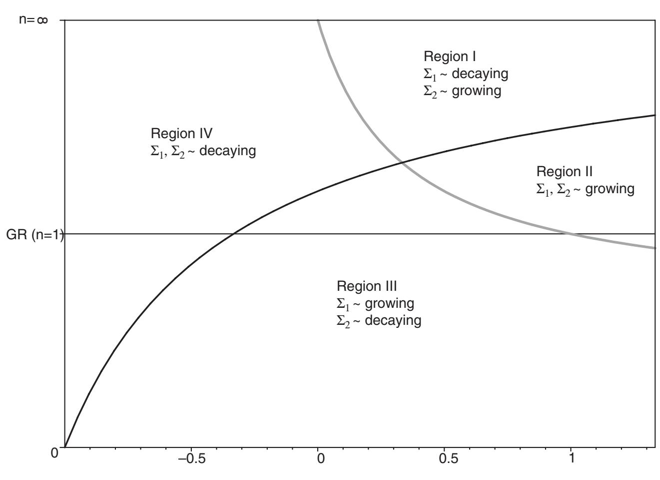

FIG. 3. The range of parameters for which the modes grow or decay as a function of n and w_m The black thick line represents the change from growth to decay for the $\tilde{\Sigma}_1$ mode. The grey thick line represents the change from growth to decay for the $\tilde{\Sigma}_2$ mode. The thin black line represents the GR case ($n = 1$). In region I, $\tilde{\Sigma}_1$ decays and $\tilde{\Sigma}_2$ grows. In region II, both modes grow. In region III, $\tilde{\Sigma}_1$ grows and $\tilde{\Sigma}_2$ decays. Finally in region IV, both modes decay.

VI. CONCLUSIONS

We have presented a mathematically well-defined method of analyzing the evolution tensor perturbations of FLRW backgrounds in fourth order gravity, providing a general template for the study of linear gravitational waves in this context. The analysis is based on two important steps. First, the recasting of the field equations for a generic fourth order theory of gravity into a form which is equivalent to GR, plus two effective fluids (the *curvature fluid* and the *effective matter fluid*). Second, using the $1 + 3$ covariant approach, it is possible to derive the general equations describing the evolution of the cosmological perturbations of these models for a FLRW background. In this paper we have only dealt with the evolution of tensor perturbations; the evolution of scalar perturbations was presented in [14] and the vector perturbations will be presented elsewhere [35]. Providing that one has a clear picture in mind of the effective nature of the fluids involved, the approach above has the advantage of making the treatment of the perturbations physically clear and mathematically rigorous.

Once the general perturbation equations were derived, we specialized them to the case of the R^n-gravity model. Using background solutions derived from an earlier dynamical systems analysis [11], we found exact solutions to the perturbation equations both in a vacuum and in the presence of matter (dust and radiation). We presented both the large-scale limit and full solutions; however, we restricted our discussions to the large-scale results. In Sec. V A we studied the evolution of tensor perturbations in vacuum. The background solution proved to be unstable under tensor perturbations in the case of GR, where the background represents a static vacuum solution. In addition, for general values of n, the rate of growth of tensor perturbations is weaker than the GR case.

In Sec. V B we studied the evolution of tensor perturbations in the presence of matter. We first considered the case of the dust dominated era. For most choices of n ($n > 1$), the perturbations were found to grow at a slower rate in R^n-gravity than in GR and no growing mode could be supported for $n > 3/2$.

Next, we studied the evolution of tensor perturbation in the radiation dominated era. Again, for most choices of n ($n > 1$) the perturbations were found to grow at a slower rate in R^n-gravity than in GR. However, it was found that their is always a growing mode present except for the special case of $n = 2$. This could have important consequences on the tensor perturbation spectrum, e.g. result in a tilt or running of the spectral index of the power spectrum. In this way the connection between the spectrum of tensor perturbations and the CMB polarization power spectrum offers an interesting independent way of testing for alternative gravity on cosmological scales.

Finally, we studied the evolution of tensor perturbation in the presence of a generic fluid ($w_m > -1$) in the large scale limit ($k = 0$). We found that there is a range of the parameters w_m and n for which no growing modes are present [$n > 3(w_m + 1)/2$ and $n < (w_m + 1)/2w_m$]. This corresponds to the choice of parameters as required to solve the dark matter problem in the work of [12]. Thus, the aforementioned theories of gravity may be constrained via an alternate method.

As in the case of the results found for the evolution of the scalar perturbations [15], the key question is how general these results are in terms of the form of the fourth order Lagrangian. Unfortunately this question is not easy to answer based only on the analysis presented above. The key point to consider would be the differences in the dynamics of the perturbations which, as we have seen, are very pronounced but more difficult to use because they depend largely on the features of the background. The important point, however, is that these differences do not necessarily imply a complete incompatibility with the data coming from the CMB and other observational constraints. Much more work will be needed before we can determine whether alternative gravity provides a viable alternative to standard general relativity.

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APPENDIX: COVARIANT FORMALISM VERSUS BARDEEN'S FORMALISM

As we have seen, the covariant approach is a very useful framework for studying perturbations in alternative theories of gravity. However, since most work on cosmological perturbations is usually done using the Bardeen approach [20], we will give here a brief summary of how one can relate our quantities to the standard Bardeen quantities. A detailed analysis of the connection between these formalisms is given in [23]. Here we limit ourselves to give the main results for tensor perturbations.

In Bardeen's approach to perturbations of FLRW spacetimes, the metric g_{ab} is the fundamental object, if $\bar{g}_{ab}$ is the background metric and $g_{ab} = \bar{g}_{ab} + \delta g_{ab}$ defines the metric perturbations δg_{ab} in these coordinates.

The perturbed metric can be written in the form

$$ds^2 = a^2(\eta)\{-(1 + 2A)d\eta^2 - 2B_\alpha dx^\alpha d\eta + [(1 + 2H_L)\gamma_{\alpha\beta} + 2H^T_{\alpha\beta}]dx^\alpha dx^\beta\}, \tag{A1}$$

where η is the conformal time, and the spatial coordinates are left arbitrary. This spacetime can be foliated in 3-hypersurfaces Σ characterized by constant conformal time η and metric γ_{ab}.

The quantities A and B_α are, respectively, the perturbation in the lapse function (i.e. the ratio of the proper time distance and the coordinate time one between two constant time hypersurfaces) and in the shift vector (i.e. the rate of deviation of a constant space coordinate line from the normal line to a constant time hypersurface), H_L represents the amplitude of perturbation of a unit spatial volume, and $H^T_{\alpha\beta}$ is the amplitude of anisotropic distortion of each constant time hypersurface [21].

The minimal set of perturbation variables is completed by defining the fluctuations in the energy density:

$$\mu = \bar{\mu} + \delta\mu, \qquad \delta \equiv \delta\mu/\bar{\mu}, \tag{A2}$$

and the fluid velocity:

$$u^a = \bar{u}^a + \delta u^a, \qquad \delta u^\alpha = \bar{u}^0 v^\alpha, \qquad \delta u^0 = -\bar{u}^0 A, \tag{A3}$$

together with the energy flux q_a and the anisotropic pressure π_{ab} which are GI by themselves.

These quantities are treated as 3-fields propagating on the background 3-geometry. With a suitable choice of boundary conditions [36], these quantities can be uniquely (but nonlocally) decomposed into scalars, 3-vectors, and 3-tensors:

$$B_\alpha = B_{|\alpha} + B^S_\alpha, \tag{A4}$$

$$H_{T\alpha\beta} = \nabla_{\alpha\beta} H_T + H^S_{T(\alpha|\beta)} + H^{TT}_{T\alpha\beta}, \tag{A5}$$

where the slash indicates covariant differentiation with respect to the metric γ_{ab} of Σ. In this way $\nabla_{ab} f = f_{|\beta\alpha} - \frac{1}{3}\nabla^2 f$ and $\nabla^2 f = f^{|\gamma}{}_{|\gamma}$ is the Laplacian. The superscript S on a vector means it is solenoidal ($B^{S|\alpha}_\alpha = 0$), and TT tensors are transverse ($H^{TT\beta}_{T\alpha} = 0$) and trace-free.

On the base of (A4) and (A5), it is standard to define *scalar* perturbations as those quantities which are 3-scalars, or are derived from a scalar through linear operations involving only the metric γ_{ab} and its $|$ derivative. Quantities derived from similar operations on solenoidal vectors and on TT tensors are dubbed *vector* and *tensor* perturbations. Scalar perturbations are relevant to matter clumping, i.e. correspond to density perturbations, while vector and tensor perturbations correspond to rotational perturbations and gravitational waves.

Given the homogeneity and isotropy of the background, we can separate each variable into its time and spatial dependence using the method of harmonic decomposition. In the Bardeen approach the standard harmonic decomposition is performed using the eigenfunctions of the Laplace-Beltrami operator on 3-hypersurfaces of constant curvature Σ (i.e. on the homogeneous spatial sections of FLRW universes). In particular, these harmonics are defined by

$$\nabla^2 Y^{(k)} = -k^2 Y^{(k)}, \tag{A6}$$

$$\nabla^2 Y^{(k)}_\alpha = -k^2 Y^{(k)}_\alpha, \tag{A7}$$

$$\nabla^2 Y^{(k)}_{\alpha\beta} = -k^2 Y^{(k)}_{\alpha\beta}, \tag{A8}$$

where $Y^{(k)}$, $Y^{(k)}_\alpha$, $Y^{(k)}_{\alpha\beta}$ are the scalar, vector, and tensor harmonics of order k. In this way one can decompose scalars, vectors, and tensors as

$$A = A(\eta) Y \tag{A9}$$

$$B_\alpha = B^{(0)}(\eta) Y^{(0)}_\alpha + B^{(1)}(\eta) Y^{(1)}_\alpha, \tag{A10}$$

$$H_{T\alpha\beta} = H^{(0)}_T(\eta) Y^{(0)}_{\alpha\beta} + H^{(1)}_T(\eta) Y^{(1)}_{\alpha\beta} + H^{(2)}_T(\eta) Y^{(2)}_{\alpha\beta}. \tag{A11}$$

The key property of linear perturbation theory of FLRW spacetimes, arising from the unicity of the splitting of (A4) and (A5), is that in any vector and tensor equation the scalar, vector, and tensor parts on each side are separately equal, i.e. the scalar, vector, and tensor components of the equations decouple.

All the quantities defined above can be decomposed in this way. However, before proceeding, one should note that the quantities A, B_α, H^L, $H^T_{\alpha\beta}$, δ, v^α change their values under a change of correspondence between the perturbed "world" and the unperturbed background, i.e., under a *gauge transformation*. In order to have a gauge-invariant theory, one has to look for combinations of these quantities which are gauge invariant. Bardeen constructed such GI variables to treat scalar and vector perturbations [20]. The quantities which are relevant to our analysis, $\pi_{\alpha\beta}$ and $H^{TT}_{T\alpha\beta}$ (or the harmonically decomposed object $H^{(2)}_T$), are already GI.

The variables covariantly defined in the main text are, by themselves, exact quantities (defined in any spacetime) and are GI by themselves, therefore, to first order, we can express them as linear combinations of Bardeen's GI variables. In [23] these expansions are given in full generality. Here we will limit ourselves to a few examples, giving only the tensor contributions and refer the reader to [23] for details.

The tensor part of the shear, trace-free part of the 3-Ricci tensor, the electric and magnetic parts of the Weyl tensor are given by

$$\sigma_{\alpha\beta} = a H^{(2)\prime}_T Y^{(2)}_{\alpha\beta}, \tag{A12}$$

$${}^{(3)}\mathcal{R}_{\alpha\beta} = (k^2 + 2K) H^{(2)}_T Y^{(2)}_{\alpha\beta}, \tag{A13}$$

$$E_{\alpha\beta} = -\tfrac{1}{2}[H^{(2)\prime\prime}_T - (k^2 + 2K) H^{(2)}_T] Y^{(2)}_{\alpha\beta}, \tag{A14}$$

$$H_{\alpha\beta} = a^{-2} H^{(2)\prime}_T Y^{(2)\gamma|\delta}_{(\alpha} \eta_{\beta)0\gamma\delta}, \tag{A15}$$

where the prime denotes derivative with respect to the conformal time η. The relations above can be used to give an intrinsic physical and geometrical meaning to Bardeen's variables, and also to recover his equations. For example, combining our linearized expression for the trace-free part of the 3-Ricci tensor

$${}^{(3)}\mathcal{R}_{\alpha\beta} = -\frac{\Theta}{3}(\sigma_{\alpha\beta} + \omega_{\alpha\beta}) + E_{\alpha\beta} + \frac{1}{2}\pi_{\alpha\beta}, \tag{A16}$$

with the above expressions [Eqs. (A12)–(A14)] gives Bardeen's expression for the transverse and trace-free metric perturbation evolution equation,

$$H^{(2)\prime\prime}_T + 2\frac{a'}{a} H^{(2)\prime}_T + (k^2 + 2K) H^{(2)}_T = \pi, \tag{A17}$$

where π is the harmonically decomposed anisotropic pressure. Substituting for π using Eqs. (61) and (A12), we find the general evolution equation for tensor perturbations in fourth order gravity theories to be

$$H_T^{(2)\prime\prime} + \left[2\frac{a'}{a} + \frac{\partial^2 f}{\partial R^2}\left(\frac{\partial f}{\partial R}\right)^{-1} R'\right] H_T^{(2)\prime} + (k^2 + 2K) H_T^{(2)} = 0, \tag{A18}$$

where primes denote differentiation with respect to conformal time throughout this Appendix.

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