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Generic features of Einstein-Aether black holes

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We reconsider spherically symmetric black hole solutions in Einstein-Aether theory with the condition that this theory has identical parametrized post-Newtonian parameters as those for general relativity, which is the main difference from the previous research. In contrast with previous study, we allow superluminal propagation of a spin-0 Aether-gravity wave mode. As a result, we obtain black holes having a spin-0 "horizon" inside an event horizon. We allow a singularity at a spin-0 horizon since it is concealed by the event horizon. If we allow such a configuration, the kinetic term of the Aether field can be large enough for black holes to be significantly different from Schwarzschild black holes with respect to Arnowitt-Deser-Misner mass, innermost stable circular orbit, Hawking temperature, and so on. We also discuss whether or not the above features can be seen in more generic vector-tensor theories.

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I. INTRODUCTION

Identifying the contents of dark energy and dark matter is one of the most important subjects in cosmology. It is frequently argued that gravitational theories are an alternative to dark energy and dark matter. Recently, tensor-vector-scalar (TeVeS) theories have attracted much attention since they do not only explain galaxy rotation curves but also satisfy many constraints from solar experiments [1]. Although a deficiency in explaining the mismatch between luminous and dynamical masses in clusters of galaxies by TeVeS has been pointed out [2], resolution of this problem by considering a generalized vector-tensor theory has also been reported [3]. Moreover, these vector fields might explain an accelerated expansion of the universe [3,4] and might be important in inflationary scenarios [5,6]. The origin of such a vector field is argued in [7].

However, it is nontrivial whether or not these theories satisfy the constraints by strong gravity tests. Notice the result for scalar-tensor theories where compact objects have strong deviations from those in general relativity (GR) even in the cases that satisfy weak field tests [8]. To study vector fields in a general form is difficult. For example, results in TeVeS are still limited to special cases such as [9]. Thus, as a first step, it is important to investigate a simplified model which is tractable and instructive for general cases. One such useful model would be Einstein-Aether (EA) theory [10], where all parametrized post-Newtonian (PPN) parameters [11] can be the same as those in GR [12]. EA theory is a vector-tensor theory, and TeVeS can be written as a vector-tensor theory which is the extension of EA theory [13]. In EA theory, strong gravitational cases including black holes have been analyzed to some extent [14–18].

Nevertheless, the analysis of black holes has been limited to the case in which the event horizon coincides with the spin-0 horizon [15], and this case does not necessarily satisfy weak fields tests. Thus, it is interesting to ask whether or not significant differences from the Schwarzschild black hole appear when weak fields tests are satisfied. For this reason, we argue black holes with the case in which the EA theory has identical PPN parameters as in GR.

This paper is organized as follows. In Sec. II, we explain EA theory and summarize constraints located by previous research. In Sec. III, we mention our method of analyzing black holes. In Sec. IV, we show the results and compare them with the Schwarzschild black hole. In Sec. V, consequences and future subjects are discussed. In the Appendix, we summarize basic equations. We use units in which $c = 1$ and follow the sign conventions of Misner, Thorne, and Wheeler [19], e.g., $(-,+,+,+)$ for metrics.

II. EINSTEIN-AETHER THEORY

A. Action and basic equations

We consider the following action [17]:

$$I = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\mathcal{L}, \tag{2.1}$$

$$\mathcal{L} = R - K^{ab}{}_{cd}\nabla_a u^c \nabla_b u^d + \lambda(u^2+1), \tag{2.2}$$

$$K^{ab}{}_{cd} := c_1 g^{ab}g_{cd} + c_2\delta^a_c\delta^b_d + c_3\delta^a_d\delta^b_c - c_4 u^a u^b g_{cd}, \tag{2.3}$$

where u^a is a vector field and $u^2 := u^a u_a$. c_i $(i = 1, 2, 3, 4)$ are theoretical parameters in EA theory. λ is a Lagrange multiplier ensuring the vector field u^a to be unit timelike vector everywhere.

Varying this action with respect to λ and u^a, we have

$$u^2 + 1 = 0, \tag{2.4}$$

$$c_4 \dot{u}^m \nabla_a u_m + \nabla_m J^m{}_a + \lambda u_a = 0, \tag{2.5}$$

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where

$$J^a{}_m := K^{ab}{}_{mn}\nabla_b u^n, \tag{2.6}$$

$$\dot{u}^b := u^a\nabla_a u^b. \tag{2.7}$$

Multiplying Eq. (2.5) by u_a, we have

$$\lambda = c_4\dot{u}^2 + u^a\nabla_m J^m{}_a. \tag{2.8}$$

Varying the action with respect to the metric, we have

$$\begin{aligned}G_{ab} &= \nabla_m[J^m{}_{(a}u_{b)} - J_{(a}{}^m u_{b)} + J_{(ab)}u^m] \\ &\quad + c_1(\nabla_a u_m \nabla_b u^m - \nabla_m u_a \nabla^m u_b) + c_4\dot{u}_a\dot{u}_b \\ &\quad + \lambda u_a u_b - \frac{1}{2}g_{ab}\mathcal{L}_{\rm u},\end{aligned} \tag{2.9}$$

where

$$\mathcal{L}_{\rm u} := K^{ab}{}_{cd}\nabla_a u^c \nabla_b u^d. \tag{2.10}$$

B. Present constraints in EA theory

If we assume the weak field and slow-motion limits in EA theory [12], we have to take Newton's gravitational constant as

$$G_{\rm N} = \left(1 - \frac{c_1 + c_4}{2}\right)^{-1} G, \tag{2.11}$$

to reproduce Newtonian gravity correctly. For all the PPN parameters to coincide with those in GR, we have

$$c_2 = \frac{-2c_1^2 - c_1c_3 + c_3^2}{3c_1}, \qquad c_4 = -\frac{c_3^2}{c_1}. \tag{2.12}$$

If we assume Friedmann-Robertson-Walker space-time and the Aether is aligned with a cosmological rest frame, the cosmological gravitational constant is given by [20]

$$G_{\rm cosmo} = G\left(1 + \frac{c_+ + 3c_2}{2}\right)^{-1}, \tag{2.13}$$

where $c_+ := c_1 + c_3$. Using primordial ^{4}He abundance, we have

$$|G_{\rm cosmo}/G_N - 1| < 1/8. \tag{2.14}$$

From the maximum mass of neutron stars $\sim 2M_\odot$ [21,22], we have $c_1 + c_4 \leq 0.5 \sim 1.6$, depending on equation of state [18].

In [23], the sound modes are analyzed by expanding the metric and the Aether around the Minkowski metric. As in the case in GR, we have two spin-2 modes. As peculiar to EA theory, there are three wave modes. Two correspond to a transverse spin-1 mode, and one corresponds to a longitudinal spin-0 mode. The squared speeds of them are summarized as

$$(s_0)^2 = \frac{c_{13}}{3(c_1 - c_3)(1 - c_{13})}, \tag{2.15}$$

$$(s_1)^2 = \frac{c_1(2c_1 - c_1^2 + c_3^2)}{2(1 - c_{13})c_{13}(c_1 - c_3)}, \tag{2.16}$$

$$(s_2)^2 = \frac{1}{1 - c_{13}}, \tag{2.17}$$

where we eliminate c_2 and c_4 with Eq. (2.12).

For these sound velocities to be equal to or larger than the photon velocity, or, to ensure stability against linear perturbation in Minkowski (or Friedmann-Robertson-Walker) background and linearized energy positivity, we have [5,23–25]

$$0 < c_+ < 1, \qquad 0 < c_- := c_1 - c_3 < \frac{c_+}{3(1 - c_+)}. \tag{2.18}$$

Radiation damping was also analyzed in [26,27], which almost restricts c_+ as a function of c_- based on the observation of, say, B1913 + 16 [28].

III. ANALYSIS IN A SINGLE-NULL COORDINATE SYSTEM

Our purpose in investigating black holes in EA theory is not to give a further restriction but to understand generic features of vector-tensor theories under the condition that weak gravity tests are satisfied. This is the main difference from the previous research [15], which investigates black holes with the parameters [29]

$$c_2 = \frac{-(c_1 + c_3)^2(c_1 - c_4) - 2(c_3 + c_4)}{(c_1 - c_4)(3c_3 + 4c_4 - c_1) + 2} \tag{3.1}$$

and $c_3 = -c_4$, or $c_3 = -2c_4 + c_1$, or $c_3 = -c_1$. In these parameters, qualitative differences from Schwarzschild black holes have been shown. It is nontrivial whether or not this is true even for the case which satisfies weak gravity tests.

From this point of view, we take the following strategy. (i) We assume (2.12) since the constraints by the solar experiments are severe. (ii) We assume (2.18). Otherwise, a naked singularity appears outside the event horizon in general. As for other constraints, notice that (2.14) is satisfied if (2.12) is satisfied. Constraints from neutron stars and from radiation damping are related to strong gravity tests at least partially. For the above reasons, we do not impose these constraints. Thus, we have two theoretical parameters (c_+, c_-) with the condition (2.18).

We write a static and spherically symmetric line element in a single-null coordinate system as,

$$ds^2 = -N(r)dv^2 + 2B(r)dvdr + r^2d\Omega^2. \tag{3.2}$$

In this coordinate, the vector field takes the form of

$$u = a(r)\partial_v + b(r)\partial_r. \tag{3.3}$$

$b(r) \neq 0$ means that the Aether is not aligned with the timelike Killing field, which is inevitable because of the event horizon. From Eq. (2.4),

$$-Na^2 + 2Bab = -1. \tag{3.4}$$

We can eliminate λ with Eq. (2.8). Then, from the Einstein and Aether equations, we obtain basic equations, which can be written schematically as

$$N' = f_1(B, N, a, a'), \tag{3.5}$$

$$B' = f_2(B, N, a, a'), \tag{3.6}$$

$$a'' = f_3(B, N, a, a'), \tag{3.7}$$

where the prime denotes the derivative with respect to r. Here, we have eliminated b with Eq. (3.4). The explicit form is summarized in the Appendix.

The boundary condition at the horizon $r_{\rm h}$ is $N(r_{\rm h}) = 0$. We set $B(r_{\rm h}) = 1$. We can also set $r_{\rm h} = 1$ since there is no scale in the present theory. In this sense, it is assumed that the area coordinate r is normalized by the horizon radius below.

If we use a rescaling freedom of v as $dv' = B(\infty)dv$, the asymptotic form of the metric is written as

$$ds^2 = -\frac{N(\infty)}{B(\infty)^2}dv'^2 + 2dv'dr + r^2 d\Omega^2. \tag{3.8}$$

Thus, the boundary condition at spatial infinity for the asymptotic flatness is

$$N(\infty) = B(\infty)^2. \tag{3.9}$$

We should require

$$b(\infty) = 0, \tag{3.10}$$

for the Aether to be aligned with the timelike Killing field. Then, by Eq. (3.4), we have

$$a(\infty) = \frac{1}{B(\infty)}. \tag{3.11}$$

We can determine the pair of $a_{\rm h} := a(r_{\rm h})$ and $a'_{\rm h} := a'(r_{\rm h})$ as shooting parameters, one of which is fixed by (3.11). Thus, there remains one freedom. Fixing this freedom is done as follows.

Even in the spherically symmetric case, there is a spin-0 mode. Then, we can define the effective metric for a spin-0 mode as

$$g^{(0)}_{ab} = g_{ab} - [(s_0)^2 - 1]u_a u_b. \tag{3.12}$$

We call the horizon associated with this metric as the spin-0 horizon. The freedom mentioned above is fixed by the requirement that the regularity at the spin-0 horizon which is inside the event horizon.

However, since the asymptotic observer is insensitive to the regularity at the spin-0 horizon, we permit the singularity at the spin-0 horizon. For this reason, we leave one freedom. In concrete terms, we obtain $a_{\rm h}$ iteratively for some $a'_{\rm h}$, which is regarded as a free parameter. We use the Bulirsch-Stoer method in our numerics [30].

IV. PROPERTIES OF SOLUTIONS

A. Mass and Hawking temperature of EA black hole

We show several asymptotically flat solutions in Figs. 1(a)–1(c) for $c_+ = 0.4$ and $c_- = 0.1$. In the figures, we have selected five solutions. The differences of these solutions are the changing boundary value $a'_{\rm h}$, ranging from -1 to 1, as denoted in the figures. Figure 1(a) shows that we can determine an $a_{\rm h}$ that satisfies the asymptotic condition (3.11) for various values of $a'_{\rm h}$. We also show $B(r)$ in Fig. 1(b). Since $B(r) = \text{const} = 1$ for a Schwarzschild black hole, it indicates that there are differences in physical quantities from those for Schwarzschild black holes. Figure 1(c) shows a "mass" function. In AE theory, it is important to distinguish different notions of mass. If we define the mass function $m(r)$ by

$$m(r) := \frac{r}{2G}\left(1 - \frac{N}{B^2}\right), \tag{4.1}$$

we can interpret $m(\infty)$ as ADM mass $M_{\rm ADM}$. As we can see, $m(r)$ monotonically decreases. Our calculation suggests that this is generic. This is not surprising since energy conditions are not necessarily satisfied in EA theory [25].

Since Fig. 1 shows that the deviation from the Schwarzschild black hole is largest for the smallest value of $a'_{\rm h}$, it is natural to ask whether or not there is a lower limit $a'_{\rm h,crit}$ below which there is no regular solution. We show the relation $a'_{\rm h}$ and $M_{\rm ADM}$ for various values of c_+ and c_- in Fig. 2(a). Typically, $M_{\rm ADM}$ is smaller than that of a Schwarzschild black hole by about 10%, which is consistent with the result in [15]. Here, we obtain $a_{\rm h}$ iteratively to satisfy Eq. (3.9) for each $a'_{\rm h}$. For $a'_{\rm h} < a'_{\rm h,crit}$, we cannot find an appropriate value of $a_{\rm h}$. $a'_{\rm h,crit}$ depends on c_+ and c_-. As $a'_{\rm h}$ approaches $a'_{\rm h,crit}$, $dM_{\rm ADM}/da'_{\rm h}$ tends to diverge. Since we obtain solutions numerically, it is nontrivial whether $M_{\rm ADM}$ is bounded or not from below. In particular, it is important to reveal the positivity of $M_{\rm ADM}$. However, since the energy conditions are not guaranteed [25], we cannot prove it at present.

For $M_{\rm ADM}$, the difference caused by the change of c_- is not clear. We can define total energy $M_{\rm tot}$ by $G_{\rm N}M_{\rm tot} = GM_{\rm ADM}$ since the gravitational constant we feel is different from that in GR as seen in Eq. (2.11). We also exhibit the relation $a'_{\rm h} - M_{\rm tot}$ in Fig. 2(b). This figure shows the differences caused by the change of c_-. $M_{\rm tot}$ decreases as c_- increases as similar to c_+.

If we contemplate these diagrams from a different viewpoint, we notice that the horizon radius of black holes in

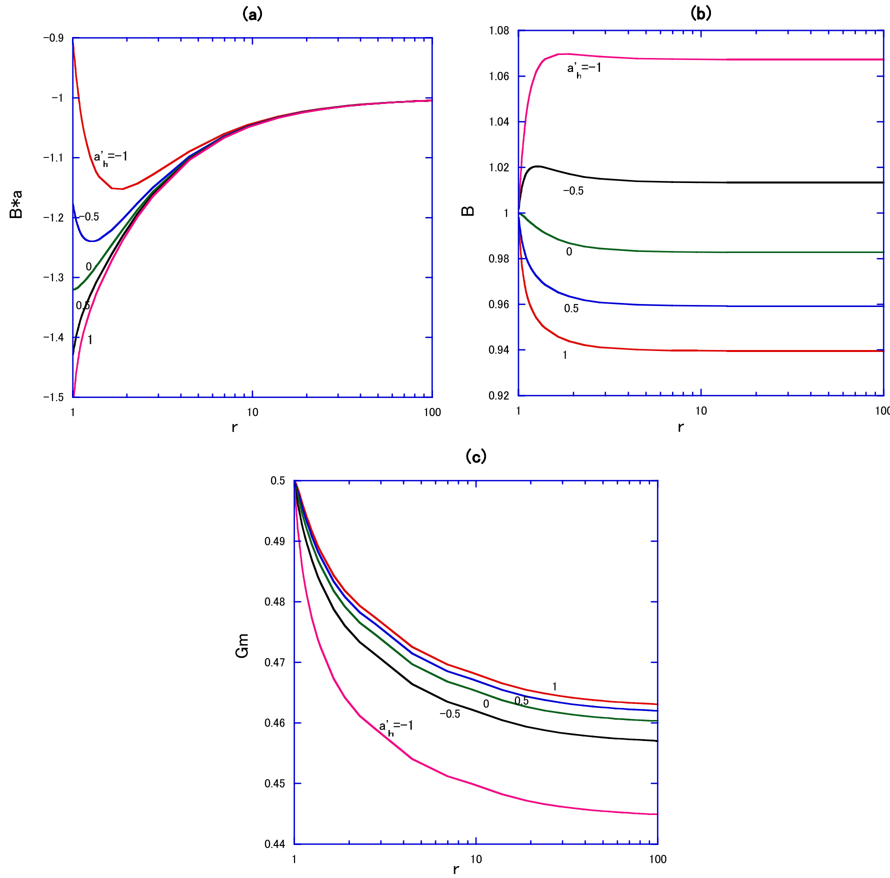

FIG. 1 (color online). Field configurations for $c_+ = 0.4$ and $c_- = 0.1$. Denoted numbers in each figure, ranging from -1 to 1, represent the values of $a'_{\rm h}$. We normalize the quantities Gm and r by the horizon radius $r_{\rm h}$. The solution with the smallest $a'_{\rm h}$ has largest deviation from a Schwarzschild black hole.

EA theory is larger than that of a Schwarzschild black hole for fixed $GM_{\rm tot}$ (or $GM_{\rm ADM}$). Therefore, one might think that black holes in EA theory have larger entropy. However, since we have the Lorentz violating field, it is nontrivial to establish black hole thermodynamics [31,32]. Thus, the comparison of the black hole entropy, which is

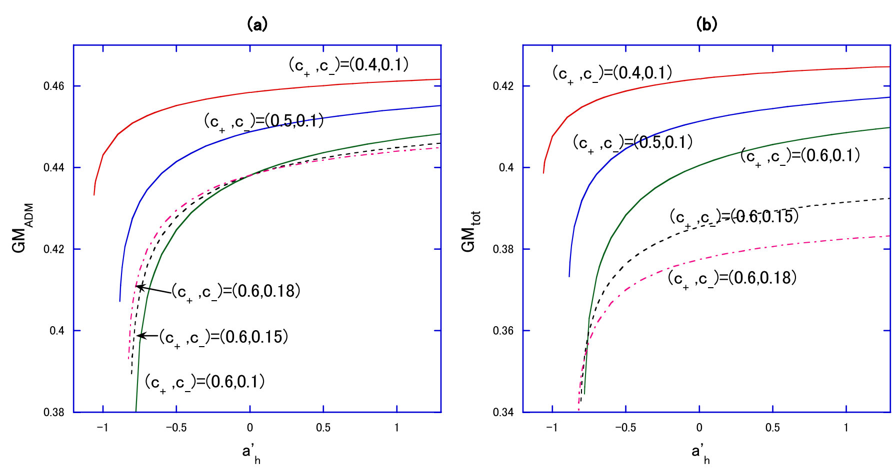

FIG. 2 (color online). (a) $a'_{\rm h}$ vs $GM_{\rm ADM}$ and (b) $a'_{\rm h}$ vs $GM_{\rm tot}$ for several sets of c_+ and c_-. Physical quantities are normalized by the horizon radius $r_{\rm h}$. Notice that there is a lower limit $a'_{\rm h,crit}$ below which there is no regular solution. Near $a'_{\rm h,crit}$, $GM_{\rm ADM}$, and $GM_{\rm tot}$ depend on $a'_{\rm h}$ remarkably.

crucial to discuss the stability of black holes, belongs among our future tasks.

It is also important to reveal what happens at the critical point, $a'_{\rm h} = a'_{\rm h,crit}$. The key point is the factor $(-1 + Na^2)$ in the denominator in (A4). For $a'_{\rm h} = a'_{\rm h,crit}$, $(-1 + Na^2)$ becomes zero at finite r. Thus, solutions disappear. We also show the $a'_{\rm h}$ dependence of Hawking temperature $T_{\rm H}$ for $c_+ = 0.6$ and $c_- = 0.1$ in Fig. 3. From this diagram, it is supposed that $T_{\rm H}$ diverges for the solution at $a'_{\rm h} = a'_{\rm h,crit}$. Thus, It is intriguing to consider an evaporation process of such black holes.

B. ISCO of EA black hole

We shall turn to more realistic problems. We consider the possibility of distinguishing black holes in EA theory from Schwarzschild black hole by observation. In Ref. [18], the innermost stable circular orbit (ISCO) for neutron stars in EA theory was analyzed. The result is that the deviation from the Schwarzschild black hole is at most several percent. But this is not necessarily the case in the present situation, as shown below. The differences occur since we have the freedom parametrized by $a'_{\rm h}$ and the Aether is not static. These facts will be important if we consider observations such as Constellation-X [33], which tracks the motion of individual elements near black holes.

From an equation for timelike geodesics for a unit mass particle, we have effective potential V as

$$V(r) = \frac{N}{B^2}\left(\frac{L^2}{r^2} + 1\right), \tag{4.2}$$

where L is the angular momentum normalized by the horizon radius.

We show the typical configurations of V in Fig. 4 for EA theory (with $c_+ = 0.6$, $c_- = 0.1$, and $a'_{\rm h} = 0.78 \simeq a'_{\rm h,crit}$) and for GR (Schwarzschild black hole), where the angular momentum of the test particle L is fixed as $L = 1.5$. We find that a potential minimum exists even for $L = 1.5$ in EA theory.

We show the dependence of $r_{\rm ISCO}$ (normalized by $r_{\rm h}$) on $a'_{\rm h}$ in Fig. 5(a). Notice that $r_{\rm ISCO} = 3$ for the Schwarzschild black hole. Therefore, the difference is nearly 10% for $a'_{\rm h} \simeq a'_{\rm h,crit}$. It is also impressive to write the ISCO normalized by $GM_{\rm tot}$ (or $GM_{\rm ADM}$), which is shown in Fig. 5(b). In this case, we can find the difference from the Schwarzschild black hole ($r_{\rm ISCO}/GM_{\rm ADM} = 6$) is more than 20%.

Finally, let us comment on the parameter region of (c_+, c_-) in which black hole solutions exist. We obtained solutions even for $c_+, c_- > 1$, which seems to conflict with the previous results [15]. However, since we do not assume regularity at the spin-0 horizon against the case in [15], it is not inconsistent. The qualitative properties are the same as in other parameter regions, although quantitative differences from Schwarzschild black holes become larger for large (c_+, c_-) as we expect from Fig. 2. These features are the same as in [15] where the consistency with the weak gravity tests are not necessarily imposed.

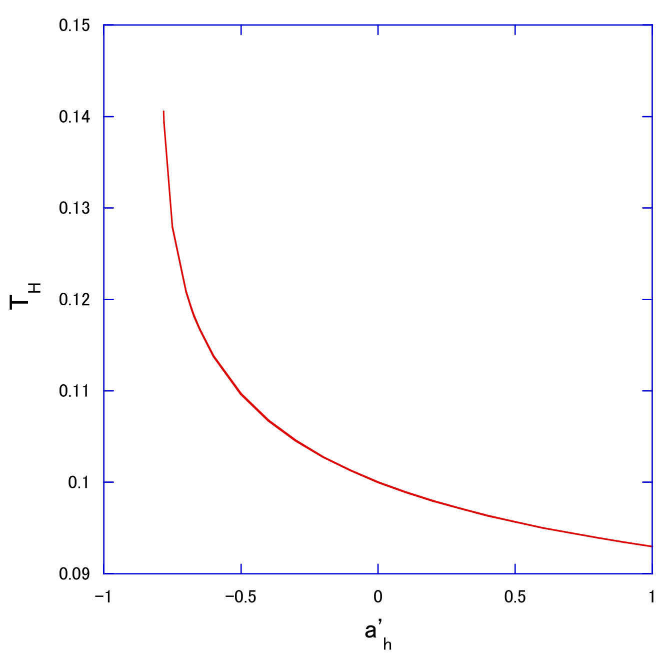

FIG. 3 (color online). Hawking temperature $T_{\rm H}$ (normalized by $r_{\rm h}$) for $c_+ = 0.6$ and $c_- = 0.1$ suggesting that $T_{\rm H}$ diverges for the solution at $a'_{\rm h,crit}$.

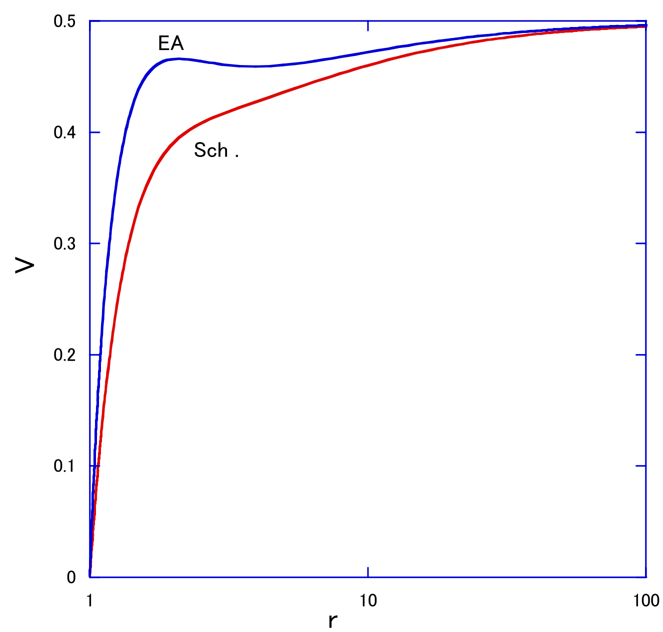

FIG. 4 (color online). The potential V for EA theory ($c_+ = 0.6$, $c_- = 0.1$, and $a'_{\rm h} = 0.78 \simeq a'_{\rm h,crit}$) and for a Schwarzschild black hole where the angular momentum of the test particle L (normalized by $r_{\rm h}$) is fixed by $L = 1.5$. There is a potential minimum in EA theory while there is none for a Schwarzschild black hole.

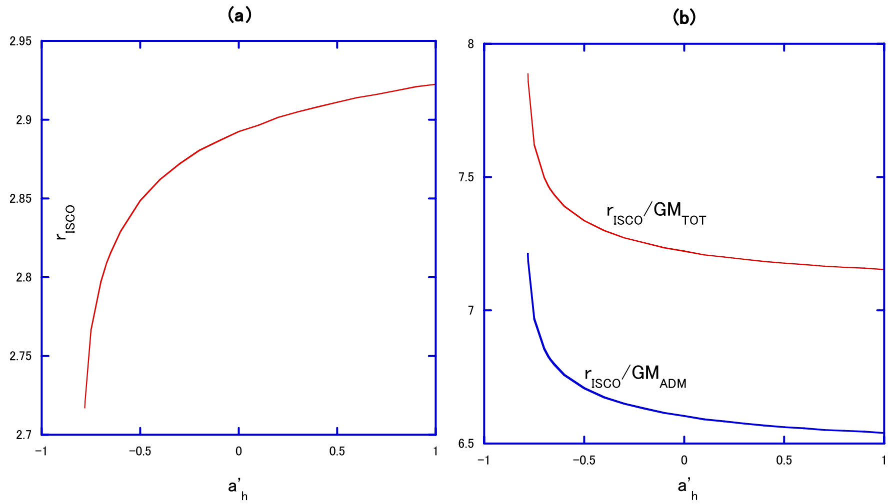

FIG. 5 (color online). The $a_{\rm h}'$ dependence of the innermost stable circular orbit (ISCO) for EA theory with $c_+ = 0.6$ and $c_- = 0.1$. (a) ISCO normalized by $r_{\rm h}$. (b) ISCO normalized by $GM_{\rm tot}$ and $GM_{\rm ADM}$.

V. CONCLUSION AND DISCUSSION

We have reanalyzed black hole solutions in EA theory while assuming that all the PPN parameters are the same as those for GR, resulting in two theoretical parameters c_+ and c_-. This is a main difference from the previous study [15]. As another difference, we do not assume regularity at the spin-0 horizon since this is inside the event horizon. Interestingly, we find $a_{\rm h,crit}'$ below which there is no regular black hole solution. Near $a_{\rm h,crit}'$, the deviation of black hole mass $M_{\rm tot}$ (or $M_{\rm ADM}$) and ISCO $r_{\rm ISCO}$ from those for the Schwarzschild black hole becomes large.

These results are instructive for other cases. If we consider the case with rotation, freedom of the vector field is added to (3.3). Then, it also contributes the kinetic term of the vector field, enhancing the differences from the vacuum solution. This would also be true in other vector-tensor theories. For this reason, it is important to consider rotational black holes in vector-tensor theories, if we are to constrain them.

Although we have revealed many properties of EA black holes, some important problems remain to be investigated. One is the positivity of the energy, which is necessary for the stability of the system. Related to this, to establish the black hole thermodynamics is also important. As a consistency check, we should also perform the linear perturbation for the black holes [34].

The other is whether or not regular spin-0 horizon happens as a result of gravitational collapse. In [17], it is shown that regular spin-0 horizon happens if we consider a gravitational collapse of a massless scalar field. Thus, it is important to investigate this feature in a general case. It is also interesting to investigate the critical behavior of such a system [35]. Of course, these are not problems particular only to EA theory but also issue confronting in more generic vector-tensor theories. Thus, it is desirable to investigate them in a unified way.

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APPENDIX: BASIC EQUATIONS FOR EINSTEIN-AETHER SYSTEM

The equation for N is

$$\sum_{i=0}^{6} H_i a^i = 0, \tag{A1}$$

where

$$\begin{aligned}
H_6 &= c_+[(3c_- - c_+)N^2 + 2(3c_- + c_+)rNN' \\
&\quad + (3c_- - c_+)r^2N'^2, \\
H_5 &= 2c_+rN[(3c_- + c_+)N + (3c_- - c_+)rN']a', \\
H_4 &= -12(c_- + c_+)B(r)^2 + 2[-3c_-(-2 + c_+) \\
&\quad + c_+(6 + c_+)]N - 2[3c_-(-2 + c_+) \\
&\quad + (-6 + c_+)c_+]rN' + (3c_- - c_+)c_+r^2N^2a'^2, \\
H_3 &= -2c_+(3c_- + c_+)r^2N'a', \\
H_2 &= -c_+a^2[-3c_- + c_+ + 2(3c_- + c_+)r^2Na'^2], \\
H_1 &= -2c_+(3c_- + c_+)ra', \\
H_0 &= (3c_- - c_+)c_+r^2a'^2.
\end{aligned}$$

(A1) is the quadratic equation for N' (notice H_6). If we solve (A1) about N', we obtain the equation which satisfies asymptotically flatness as

$$N' = \frac{\sum_{i=0}^{3} h_i a^i + 2\sqrt{3}\sqrt{X}}{(3c_- - c_+)c_+ r a^3}, \tag{A2}$$

where

$$\begin{aligned}
&h_3 = -c_+(3c_- + c_+)N, \qquad h_2 = c_+(c_+ - 3c_-)rNa', \\
&h_1 = 3c_-(c_+ - 2) + (c_+ - 6)c_+, \qquad h_0 = c_+(3c_- + c_+)ra', \\
&X = [-3c_-^2(c_+ - 1) - (c_+ - 3)c_+^2 + c_-c_+(6 - 4c_+ + c_+^2)]a^2 + c_+[(3c_-^2 + 2c_-c_+ - c_+^2)B^2 \\
&\quad + 2c_+(c_- + c_+ - c_-c_+)N]a^4 + c_-c_+^3N^2a^6 + c_+[3c_-^2(c_+ - 1) + c_-(c_+ - 4)c_+ - c_+^2]raa' \\
&\quad - c_+[3c_-^2(c_+ - 1) + c_-(c_+ - 2)c_+ + c_+^2]rNa^3a' + c_-c_+^3r^2a'^2.
\end{aligned}$$

Notice the denominator in (A2). For $c_+ = 3c_-$, we should use (A1).

The equation for B is

$$B' = \frac{B\sum_{i=0}^{8} g_i a^i}{Y}, \tag{A3}$$

where

$$\begin{aligned}
g_8 &= -c_+[3c_-^2(c_+ - 1) + c_-(c_+ - 2)c_+ + c_+^2]N[(3c_- - c_+)N^2 + 2(3c_- + c_+)rNN' + (3c_- - c_+)r^2N'^2], \\
g_7 &= 2c_+[3c_-^2(c_+ - 1) + c_-(c_+ - 2)c_+ + c_+^2]rN^2[(3c_- + c_+)N + (3c_- - c_+)rN']a', \\
g_6 &= -\{12[3c_-^3(c_+ - 1) + c_-(c_+ - 1)c_+^2 + c_+^3 + c_-^2c_+(4c_+ - 5)]B^2N + [-c_+^3(12 + c_+) + c_-c_+^2(12 - 19c_+ + 3c_+^2) \\
&\quad + 9c_-^3(4 - 9c_+ + 5c_+^2) + 3c_-^2c_+(20 - 33c_+ + 8c_+^2)]N^2 + 4[-3c_+^3 + c_-c_+^2(3 - 9c_+ - 2c_+^2) \\
&\quad + 9c_-^3(1 - 3c_+ + 2c_+^2) + 3c_-^2c_+(5 - 12c_+ + 4c_+^2)]rNN' + c_+[27c_-^3(c_+ - 1) - 21c_-^2c_+ + c_+^3 \\
&\quad + c_-c_+^2(7 + 5c_+)]r^2N'^2 + c_+[-9c_-^3(c_+ - 1) + 3c_-^2c_+ + c_-(-5 + c_+)c_+^2 + c_+^3]r^2N^3a'^2\}, \\
g_5 &= -2c_+rN\{[27c_-^3(c_+ - 1) - c_+^3 - c_-c_+^2(13 + 5c_+) + 3c_-^2c_+(-13 + 6c_+)]N + 2[18c_-^3(c_+ - 1) \\
&\quad + 3c_-^2(-5 + c_+)c_+ + c_+^3 + c_-c_+^2(4 + 3c_+)]rN'\}a', \\
g_4 &= 12[9c_-^3(c_+ - 1) - c_+^3 + c_-c_+^2(-11 + 3c_+) + c_-^2c_+(-19 + 12c_+)]B^2 + [c_+^3(12 + c_+) \\
&\quad + 9c_-^3(12 - 19c_+ + 7c_+^2) + c_-c_+^2(132 - 65c_+ + 9c_+^2) + 3c_-^2c_+(76 - 79c_+ + 16c_+^2)]N + 2[-(c_+ - 6)c_+^3 \\
&\quad + c_-c_+^2(66 - 31c_+ - 5c_+^2) + 27c_-^3(2 - 3c_+ + c_+^2) + 3c_-^2c_+(38 - 37c_+ + 6c_+^2)]rN' - c_+[45c_-^3(c_+ - 1) \\
&\quad + 3c_+^3 + 3c_-^2c_+(-13 + 4c_+) + c_-c_+^2(9 + 7c_+)]r^2N^2a'^2, \\
g_3 &= -2c_+r\{[9c_-^3(c_+ - 1) + c_+^3 + c_-c_+^2(1 + c_+) + 3c_-^2c_+(-3 + 2c_+)]N + [-27c_-^3(c_+ - 1) + c_+^3 \\
&\quad + 3c_-^2c_+(5 + 2c_+) + c_-c_+^2(-11 + 5c_+)]rN'\}a', \\
g_2 &= -c_+\{27c_-^3(c_+ - 1) + c_+^3 + c_-c_+^2(-17 + 5c_+) + 3c_-^2c_+(-15 + 8c_+) + [-63c_-^3(c_+ - 1) + 3c_+^3 \\
&\quad + 3c_-^2c_+(11 + 4c_+) + c_-c_+^2(-27 + 11c_+)]r^2Na'^2\}, \\
g_1 &= -2c_+[-27c_-^3(c_+ - 1) + 3c_-^2(13 - 6c_+)c_+ + c_+^3 + c_-c_+^2(13 + 5c_+)]ra', \\
g_0 &= -c_+[27c_-^3(c_+ - 1) - 21c_-^2c_+ + c_+^3 + c_-c_+^2(7 + 5c_+)]r^2a'^2, \\
Y &= 12[c_-^2(c_+ - 1) + c_-(c_+ - 2)c_+ - c_+^2]ra^2\{2[-3c_-(c_+ - 1) + c_+]Na^2 + [3c_-(c_+ - 1) + c_+](N^2a^4 + 1)\}.
\end{aligned}$$

The equation for a is

$$a'' = \frac{\sum_{i=0}^{13} f_i a^i}{c_+r(-1 + Na^2)Y}, \tag{A4}$$

where

$$f_{13} = 2c_-c_+^4N^3[(3c_- - c_+)N^2 + 2(3c_- + c_+)rNN' + (3c_- - c_+)r^2N'^2],$$

$$f_{12} = 4c_-c_+^4rN^4[(3c_- + c_+)N + (3c_- - c_+)rN']a',$$

$$f_{11} = c_+^2N^2\{-24c_-c_+(c_- + c_+)B^2N + [9c_-^3(c_+ - 1) - 7c_+^3 + c_-c_+^2(17 + 7c_+) - 3c_-^2c_+(-5 + 8c_+)]N^2 + 2[9c_-^3(c_+ - 1) + c_-(13 - 5c_+)c_+^2 + 7c_+^3 - 3c_-^2c_+(1 + 4c_+)]rNN' + [9c_-^3(c_+ - 1) + 3c_-^2(5 - 12c_+)c_+ - 7c_+^3 + c_-c_+^2(17 + 3c_+)]r^2N'^2 + 2c_-(3c_- - c_+)c_+^2r^2N^3a'^2\},$$

$$f_{10} = c_+^2rN^2\{[9c_-^3(c_+ - 1) + 15c_+^3 - 3c_-^2c_+(9 + 4c_+) - c_-c_+^2(3 + 5c_+)]N^2 - 16c_+[-2c_-c_+ + c_+^2 + c_-^2(-3 + 6c_+)]rNN' + [-9c_-^3(c_+ - 1) + 3c_-^2c_+ + c_-(-5 + c_+)c_+^2 + c_+^3]r^2N'^2\}a',$$

$$f_9 = c_+N(-12[3c_-^3(c_+ - 1) + c_-(3 - 7c_+)c_+^2 + 3c_+^3 - c_-^2c_+(3 + 4c_+)]B^2N + 2[(18 - 13c_+)c_+^3 + 3c_-^2c_+(-6 + 7c_+) + c_-c_+^2(18 - 37c_+ + 3c_+^2) - 9c_-^3(2 - 5c_+ + 3c_+^2)]rNN' + 6c_+[-3c_-^3(c_+ - 1) - 7c_-c_+^2 + c_+^3 + c_-^2c_+(-5 + 9c_+)]r^2N'^2 - 3c_+[3c_-^3(c_+ - 1) + c_-(-5 + c_+)c_+^2 + 3c_+^3 + c_-^2c_+(-11 + 20c_+)]r^2N^3a'^2 - 2N^2\{2[9c_-^3(c_+ - 1)^2 - c_+^3(3 + 5c_+) + c_-c_+^2(3 + 10c_+ + 2c_+^2) - 3c_-^2c_+(-5 + c_+ + 3c_+^2)] - c_+[-9c_-^3(c_+ - 1) + 3c_-^2c_+ + c_-(-5 + c_+)c_+^2 + c_+^3]r^3N'a'^2\}),$$

$$f_8 = c_+rNa'\{12[3c_-^3(c_+ - 1) + c_-(c_+ - 1)c_+^2 + c_+^3 + c_-^2c_+(4c_+ - 5)]B^2N - 2[c_+^3(-6 + 7c_+) + c_-c_+^2(6 - 23c_+ + c_+^2) + 9c_-^3(2 - 5c_+ + 3c_+^2) + 3c_-^2c_+(10 - 25c_+ + 12c_+^2)]N^2 - 12[3c_-^3(c_+ - 1)^2 - c_+^3 + c_-c_+^2(1 + 10c_+) + c_-^2c_+(5 + 4c_+ - 13c_+^2)]rNN' + 4c_-c_+[9c_-^2(c_+ - 1) - 6c_-c_+ + c_+^2(3 + c_+)]r^2N'^2 + c_+[-9c_-^3(c_+ - 1) + 3c_-^2c_+ + c_-(-5 + c_+)c_+^2 + c_+^3]r^2N^3a'^2\},$$

$$f_7 = c_+(24[3c_-^3(c_+ - 1) + c_-(3 - 4c_+)c_+^2 + 3c_+^3 - c_-^2c_+(3 + c_+)]B^2N + 2c_+[27c_-^3(c_+ - 1) + c_+^2(-48 + 5c_+) + c_-c_+(-96 + 47c_+ + c_+^2) + 3c_-^2(-16 + 5c_+ + 4c_+^2)]rNN' + [c_+^3(12 + c_+) - c_-c_+^2(12 - 13c_+ + c_+^2) + 9c_-^3(-4 + 3c_+ + c_+^2) - 3c_-^2c_+(20 - 13c_+ + 8c_+^2)]r^2N'^2 - 4c_+[-9c_-^3(c_+ - 1) + 2c_+^3 + c_-c_+^2(29 + c_+) - 6c_-^2c_+(-6 + 7c_+)]r^2N^3a'^2 + 2N^2\{-9c_+^3(2 + c_+) + c_-c_+^2(-18 + 21c_+ + c_+^2) + 9c_-^3(2 - 5c_+ + 3c_+^2) - 3c_-^2c_+(-6 + 5c_+ + 4c_+^2) + c_+[45c_-^3(c_+ - 1) + c_+^3 + 3c_-^2c_+(-11 + 2c_+) + c_-c_+^2(13 + 5c_+)]r^3N'a'^2\}),$$

$$f_6 = c_+ra'\{-48c_-[3c_-^2(c_+ - 1) + (c_+ - 3)c_+^2 + 2c_-c_+(-3 + 2c_+)]B^2N - 4c_+[-27c_-^3(c_+ - 1) + 6c_-^2(10 - 7c_+)c_+ + 4c_+^3 + c_-c_+^2(37 + c_+)]N^2 + 8[c_+^3(3 + 2c_+) + 9c_-^3(-1 + c_+^2) + c_-c_+^2(-3 + 8c_+ + 2c_+^2) - 3c_-^2c_+(5 - 2c_+ + 3c_+^2)]rNN' - c_+[27c_-^3(c_+ - 1) - 21c_-^2c_+ + c_+^3 + c_-c_+^2(7 + 5c_+)]r^2N'^2 + 2c_+[27c_-^3(c_+ - 1) + c_+^3 + 3c_-^2c_+(-7 + 2c_+) + c_-c_+^2(7 + 3c_+)]r^2N^3a'^2\},$$

$$f_5 = -2(18[(c_+ - 4)c_+^3 + c_-^2c_+(-12 + 7c_+) - c_-c_+^2(12 - 5c_+ + c_+^2) + c_-^3(-4 + 3c_+ + c_+^2)]B^2 + [-c_+^3(-72 + 30c_+ + c_+^2) + 9c_-^3(8 - 10c_+ + c_+^2 + c_+^3) + c_-c_+^2(216 - 150c_+ + 23c_+^2 + c_+^3) + 3c_-^2c_+(72 - 70c_+ + 11c_+^2 + 2c_+^3)]rN' + c_+[(6 - 5c_+)c_+^3 + c_-c_+^2(66 - 119c_+ - 7c_+^2) + 27c_-^3(2 - 3c_+ + c_+^2) + 3c_-^2c_+(38 - 65c_+ + 40c_+^2)]r^2N^2a'^2 + N\{-2c_+^3(-36 + 9c_+ + c_+^2) + 18c_-^3(4 - 5c_+ + c_+^3) - c_-c_+^2(-216 + 126c_+ - 16c_+^2 + c_+^3) - 3c_-^2c_+(-72 + 66c_+ - 6c_+^2 + c_+^3) + c_+^2[63c_-^3(c_+ - 1) - 45c_-^2c_+ + c_+^3 + c_-c_+^2(19 + c_+)]r^3N'a'^2\}),$$

$$f_4 = -2c_+ra'\{-6[9c_-^3(c_+ - 1) - c_+^3 + c_-c_+^2(3c_+ - 11) + c_-^2c_+(-19 + 12c_+)]B^2 + [(6 - 7c_+)c_+^3 + c_-c_+^2(66 - 73c_+ - 7c_+^2) + 9c_-^3(6 - 11c_+ + 5c_+^2) + 3c_-^2c_+(38 - 55c_+ + 22c_+^2)]N + 6[-c_+^3 + c_-^2c_+(-19 + 8c_+) + c_-c_+^2(-11 + 2c_+ + c_+^2) + 3c_-^3(-3 + 2c_+ + c_+^2)]rN' + 2c_-c_+[27c_-^2(c_+ - 1) - 18c_-c_+ - (-9 + c_+)c_+^2]r^2N^2a'^2\},$$

$$
\begin{aligned}
f_3 = {} & c_+\{-36c_-^3 - 84c_-^2c_+ + 27c_-^3c_+ - 60c_-c_+^2 + 39c_-^2c_+^2 + 9c_-^3c_+^2 - 12c_+^3 + 13c_-c_+^3 + c_+^4 - c_-c_+^4 \\
& + 2[4c_+^3(3+c_+) + 18c_-^3(6 - 7c_+ + c_+^2) + c_-c_+^2(132 - 110c_+ + 3c_+^2) + 3c_-^2c_+(76 - 80c_+ + 31c_+^2)]r^2Na'^2 \\
& - 2c_+[-27c_-^3(c_+ - 1) + c_+^3 + 3c_-^2c_+(5 + 2c_+) + c_-c_+^2(-11 + 5c_+)]r^3N'a'^2\},\\
f_2 = {} & c_+ra'\{c_+^4 + c_-c_+^2(144 - 41c_+ - 7c_+^2) + 9c_-^3(16 - 19c_+ + 3c_+^2) + 3c_-^2c_+(96 - 71c_+ + 12c_+^2) \\
& - 2c_+[-45c_-^3(c_+ - 1) + c_+^3 + 3c_-^2c_+(9 + 2c_+) + c_-c_+^2(-17 + 3c_+)]r^2Na'^2\},\\
f_1 = {} & -c_+[(-36 + c_+)c_+^3 + 9c_-^3(12 - 13c_+ + c_+^2) + c_-c_+^2(36 - 35c_+ + 11c_+^2) + 3c_-^2c_+(60 - 51c_+ + 20c_+^2)]r^2a'^2,\\
f_0 = {} & -c_+^2[27c_-^3(c_+ - 1) - 21c_-^2c_+ + c_+^3 + c_-c_+^2(7 + 5c_+)]r^3a'^3.
\end{aligned}
$$

If we remove N' from (A3) and (A4), we can write them as the form in (3.6) and (3.7). Since it is too tedious, we do not perform it. From (A2) to (A4), we obtain the asymptotic form for $r \to \infty$ as

$$N(r) = B(\propto)^2 + \frac{N_1}{r} + \cdots, \tag{A5}$$

$$B(r) = B(\infty) + \frac{B_1}{r^2} + \cdots, \tag{A6}$$

$$a(r) = \frac{1}{B(\infty)} + \frac{a_1}{r} + \cdots, \tag{A7}$$

where N_1, B_1, and a_1 are constants.

We should be careful about $(-1 + Na^2)$ in the denominator in (A4) since (3.9) and (3.11) show that $(-1 + Na^2)$ asymptotically approaches zero. However, since $(-1 + Na^2) \propto 1/r$ for $r \to \infty$, this is canceled by r in the denominator in (A4).

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