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Hawking radiation and covariant anomalies

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Generalizing the method of Wilczek and collaborators we provide a derivation of Hawking radiation from charged black holes using only covariant gauge and gravitational anomalies. The reliability and universality of the anomaly cancellation approach to Hawking radiation is also discussed.

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I. INTRODUCTION

Hawking radiation is an important quantum effect in black hole physics. Specifically, it arises in the background spacetime with event horizons. The radiation has a spectrum with Planck distribution giving the black holes one of its thermodynamic properties that make it consistent with the rest of physics. Hawking's original result [1] has since been rederived in different ways thereby reinforcing the conclusion to a certain extent. However, the fact that no one derivation is truly clinching has led to open problems leading to alternative approaches with fresh insights.

An anomaly in quantum field theory is a breakdown of some classical symmetry due to the process of quantization (for reviews, see [2–4]). Specifically, for instance, a gauge anomaly is an anomaly in gauge symmetry, taking the form of nonconservation of the gauge current. Such anomalies characterize a theoretical inconsistency, leading to problems with the probabilistic interpretation of quantum mechanics. The cancellation of gauge anomalies gives strong constraints on model building. Likewise, a gravitational anomaly [5,6] is an anomaly in general covariance, taking the form of nonconservation of the energy-momentum tensor. There are other types of anomalies but here we shall be concerned with only gauge and gravitational anomalies. The simplest case for these anomalies which is also relevant for the present analysis, occurs for $1 + 1$ dimensional chiral fields.

Long back Christensen and Fulling [7] reproduced Hawking's result by exploiting the trace anomaly in the energy-momentum tensor of quantum fields in a Schwarzschild black hole background. The use of anomalies, though in a different form, has been powerfully resurrected recently by Robinson and Wilczek [8]. They observed that effective field theories become two dimensional and chiral near the event horizon of a Schwarzschild black hole. This leads to a two dimensional gravitational anomaly. The existence of energy flux of Hawking's radiation is necessary to cancel this anomaly. The method of [8] was soon extended to charged black holes [9] by using the gauge anomaly in addition to the gravitational anomaly. Further advances and applications of this approach may be found in a host of papers [10–25], including a recent review [26].

The approach of [8,9] is based on the fact that a two dimensional chiral (gauge and/or gravity) theory is anomalous. Such theories admit two types of anomalous currents and energy-momentum tensors; the consistent and the covariant [2–4]. The covariant divergence of these currents and energy-momentum tensors yields either the consistent or the covariant form of the gauge and gravitational anomaly, respectively [2–6,27,28]. The consistent current and anomaly satisfy the Wess-Zumino consistency condition but do not transform covariantly under a gauge transformation. Expressions for the covariant current and anomaly, on the contrary, transform covariantly under gauge transformation but do not satisfy the Wess-Zumino condition. Similar conclusions also hold for the gravitational case, except that currents are now replaced by energy-momentum tensors and gauge transformations by general coordinate transformations. In [8,9] the charge and the energy-momentum flux of the Hawking radiation is obtained by a cancellation of the consistent anomaly. However the boundary condition necessary to fix the parameters are obtained from a vanishing of the covariant current at the event horizon.

In this paper we generalize the method of [8,9] by presenting a unified description totally in terms of covariant expressions. This discussion is specifically done for Hawking radiation from charged black holes. The charge flux is determined by a cancellation of the covariant gauge anomaly while the energy-momentum flux is fixed by cancellation of the covariant gravitational anomaly. These are the only inputs. Also, we show that the analysis of [8,9] is resilient and the results are unaffected by taking more general expressions for the consistent anomaly which occur due to peculiarities of two dimensional spacetime.

II. GENERAL DISCUSSION ON COVARIANT AND CONSISTENT ANOMALIES

Here we briefly summarize some results on anomalies highlighting the peculiarities of two dimensional spacetime. First, the consistent gauge anomaly as taken in [8,9] is considered,

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$$\nabla_\mu J^\mu = \pm \frac{e^2}{4\pi} \bar{\epsilon}^{\rho\sigma} \partial_\rho A_\sigma = \pm \frac{e^2}{4\pi\sqrt{-g}} \epsilon^{\rho\sigma} \partial_\rho A_\sigma \quad (1)$$

where $+(-)$ corresponds to left(right)-handed fields, respectively. Here $g_{\mu\nu}$ is the two dimensional $(r - t)$ part of the complete Reissner-Nordstrom metric given by [8,9]

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)} dr^2 - r^2 d\Omega^2_{(d-2)}. \quad (2)$$

so that $-g = -\det g_{\mu\nu} = 1$ and $d\Omega^2_{(d-2)}$ is the line element on the $(d - 2)$ sphere. The gauge potential is defined as $A = -\frac{Q}{r} dt$.

Now a word regarding our conventions. As is evident from (1) the antisymmetric tensor $\bar{\epsilon}^{\rho\sigma}$ differs from its numerical counterpart $\epsilon^{\rho\sigma}$ ($\epsilon^{01} = -\epsilon_{01} = 1$) by the factor $\sqrt{-g}$. Since here $\sqrt{-g} = 1$, the two get identified. Henceforth we shall always use $\epsilon^{\rho\sigma}$, omitting the $\sqrt{-g}$ factor.

The current J^μ in (1) is called the consistent current and satisfies the Wess-Zumino consistency condition. Effectively this means that the following integrability condition holds [2,27];

$$\frac{\delta J^\mu(x)}{\delta A_\nu(y)} = \frac{\delta J^\nu(y)}{\delta A_\mu(x)}. \quad (3)$$

The covariant divergence of the consistent current yields the consistent anomaly. The structure appearing in (1) is the minimal form, since only odd parity terms occur. However it is possible that normal parity terms appear in (1). Indeed, as we now argue, such a term is a natural consequence of two dimensional properties.

To fix our notions, consider the interaction Lagrangian for a chiral field ψ in the presence of an external gauge potential A^μ in 1 + 1 dimensions,

$$\mathcal{L}_I = \bar{\psi}\left(\frac{1 \pm \gamma_5}{2}\right)\gamma_\mu A^\mu \psi. \quad (4)$$

Using the property of two dimensional γ-matrices,

$$\gamma_5 \gamma^\mu = -\epsilon^{\mu\nu}\gamma_\nu, \quad (5)$$

it is found that A_μ couples as a chiral combination $(g^{\mu\nu} \pm \epsilon^{\mu\nu})A_\nu$. Note that the usual flat space identity (5) holds due to the specific structure of the two dimensional metric. Hence the expression for the anomaly in (1) generalizes to

$$\nabla_\mu \bar{J}^\mu = \partial_\mu \bar{J}^\mu = \pm \frac{e^2}{4\pi} \partial_\alpha[(\epsilon^{\alpha\beta} \pm g^{\alpha\beta})A_\beta]. \quad (6)$$

This is a nonminimal form for the consistent anomaly dictated by the symmetry of the Lagrangian, and has appeared earlier in the literature [6]. It is clear that if J^μ is a consistent current then $\bar{J}^\mu$, which is given by

$$\bar{J}^\mu = J^\mu + \frac{e^2}{4\pi} A^\mu \quad (7)$$

is also a consistent current since the extra piece satisfies the integrability condition (3).

It is possible to modify the new consistent current (7), by adding a local counterterm, such that it becomes covariant,

$$\tilde{J}^\mu = \bar{J}^\mu \mp \frac{e^2}{4\pi} A_\alpha(\epsilon^{\alpha\mu} \pm g^{\alpha\mu}). \quad (8)$$

The current $\tilde{J}^\mu$ yields the gauge covariant anomaly,

$$\nabla_\mu \tilde{J}^\mu = \pm \frac{e^2}{4\pi} \epsilon^{\alpha\beta} F_{\alpha\beta}. \quad (9)$$

Note that the covariant current (8) does not satisfy the Wess-Zumino consistency condition since the counterterm violates the integrability condition (3). Moreover the gauge covariant anomaly (9) has a unique form dictated by the gauge transformation properties. This is contrary to the consistent anomaly which may have a minimal (1) or nonminimal (6) structure.

Now we will concentrate our attention on the gravity sector. If we omit the ingoing modes the energy-momentum tensor near the horizon will not conserve, while there is no difficulty in the region outside the horizon. The analysis [8,9] for obtaining the flow of energy-momentum tensor was done by using the minimal form of the consistent $d = 2$ anomaly [3,5,6,28], for right-handed fields,

$$\nabla_\mu T^\mu_\nu = \frac{1}{96\pi} \epsilon^{\beta\delta} \partial_\delta \partial_\alpha \Gamma^\alpha_{\nu\beta}, \quad (10)$$

Here we consider the general form for $d = 2$ consistent gravitational anomaly. It is worthwhile to point out that the consistent gravitational anomaly and the consistent gauge anomaly are analogous satisfying similar consistency conditions. This is easily observed here by comparing (10) with (1) where the affine connection plays the role of the gauge potential. We therefore omit the details and write the generalized anomaly by an inspection of (6) on how to include the normal parity term. The result is

$$\nabla_\mu \bar{T}^\mu_\nu = \frac{1}{96\pi} \partial_\delta \partial_\alpha[(\epsilon^{\beta\delta} + g^{\beta\delta})\Gamma^\alpha_{\nu\beta}] = \mathcal{A}_\nu. \quad (11)$$

The covariant energy-momentum tensor, on the other hand, has the divergence anomaly,

$$\nabla_\mu \tilde{T}^\mu_\nu = \frac{1}{96\pi} \epsilon_{\nu\mu} \partial^\mu R = \tilde{\mathcal{A}}_\nu. \quad (12)$$

This is called the covariant anomaly as distinct from the consistent anomaly (10).

III. COVARIANT GAUGE ANOMALY AND CHARGE FLUX

The current is conserved outside the horizon so that $\nabla_\mu \tilde{J}^\mu_{(o)} = \partial_\mu \tilde{J}^\mu_{(o)} = \partial_r \tilde{J}^r_{(o)} = 0$. Near the horizon there are only outgoing (right-handed) fields and the current becomes (covariantly) anomalous (9),

$$\partial_r \tilde{J}^r_{(H)} = \frac{e^2}{2\pi} F_{rt} = \frac{e^2}{2\pi} \partial_r A_t. \qquad (13)$$

The solution in the different regions is given by

$$\tilde{J}^{\,r}_{(o)} = c_o, \qquad (14)$$

$$\tilde{J}^{\,r}_{(H)} = c_H + \frac{e^2}{2\pi}[A_t(r) - A_t(r_+)], \qquad (15)$$

where c_o and c_H are integration constants.

The current is now written as a sum of two contributions from the two regions, $\tilde{J}^\mu = \tilde{J}^\mu_{(o)}\Theta(r - r_+ - \epsilon) + \tilde{J}^\mu_{(H)}H$, where $H = 1 - \Theta(r - r_+ - \epsilon)$. Then by using the conservation equations, the Ward identity becomes

$$\begin{aligned}\partial_\mu \tilde{J}^\mu &= \partial_r \tilde{J}^r \\ &= \partial_r\left(\frac{e^2}{2\pi}A_t H\right) + \delta(r - r_+ - \epsilon) \\ &\quad \times\left(\tilde{J}^r_{(o)} - \tilde{J}^r_{(H)} + \frac{e^2}{2\pi}A_t\right). \end{aligned} \qquad (16)$$

To make the current anomaly free the first term must be canceled by quantum effects of the classically insignificant ingoing modes. This is the Wess-Zumino term induced by these modes near the horizon. Effectively it implies a redefinition of the current as $\tilde{J}'^r = (\tilde{J}^r - \frac{e^2}{2\pi}A_t H)$ which is anomaly free provided the coefficient of the delta function vanishes, leading to the condition,

$$c_o = c_H - \frac{e^2}{2\pi}A_t(r_+). \qquad (17)$$

The coefficient c_H is fixed by requiring the vanishing of the covariant current at the horizon. This yields $c_H = 0$ from (15). Hence the value of the charge flux is given by

$$c_o = -\frac{e^2}{2\pi}A_t(r_+) = \frac{e^2 Q}{2\pi r_+}. \qquad (18)$$

This is precisely the current flow of the Hawking blackbody radiation with a chemical potential [9].

IV. COVARIANT GRAVITATIONAL ANOMALY AND ENERGY-MOMENTUM FLUX

In the presence of a charged field the classical energy-momentum tensor is no longer conserved but gives rise to the Lorentz force law, $\nabla_\mu \tilde{T}^\mu_\nu = F_{\mu\nu}\tilde{J}^\mu$. The corresponding anomalous Ward identity for covariantly regularized quantities is then given by,

$$\nabla_\mu \tilde{T}^\mu_\nu = F_{\mu\nu}\tilde{J}^\mu + \tilde{\mathcal{A}}_\nu, \qquad (19)$$

where $\tilde{\mathcal{A}}_\nu$ is the covariant gravitation anomaly (12). Since the current $\tilde{J}^\mu$ itself is anomalous one might envisage the possibility of an additional term in (19) proportional to the gauge anomaly. Indeed this happens in the Ward identity for consistently regularized objects [9]. Such a term is ruled out here because there is no such covariant piece with the correct dimensions, having one free index.

For the metric (2) the covariant anomaly is purely timelike ($\tilde{\mathcal{A}}_r = 0$) while,

$$\tilde{\mathcal{A}}_t = \partial_r \tilde{N}^r_t; \qquad \tilde{N}^r_t = \frac{[ff'' - \frac{(f')^2}{2}]}{96\pi}. \qquad (20)$$

Next, the Ward identity is solved for the $\nu = t$ component. In the exterior region there is no anomaly and the Ward identity reads

$$\partial_r \tilde{T}^r_{t(o)} = F_{rt}\tilde{J}^r_{(o)}. \qquad (21)$$

Using (14) this is solved as

$$\tilde{T}^{\,r}_{t(o)} = a_o + c_o A_t(r), \qquad (22)$$

where a_o is an integration constant. Near the horizon the anomalous Ward identity, obtained from (19), reads

$$\partial_r \tilde{T}^r_{t(H)} = F_{rt}\tilde{J}^r_H + \partial_r \tilde{N}^r_t, \qquad (23)$$

Using $\tilde{J}^r_{(H)}$ from (15) yields the solution

$$\tilde{T}^{\,r}_{t(H)} = a_H + \int_{r_+}^{r} dr \partial_r\left[c_o A_t + \frac{e^2}{4\pi}A_t^2 + \tilde{N}^r_t\right]. \qquad (24)$$

Writing the energy-momentum tensor as a sum of two combinations $\tilde{T}^\mu_\nu = \tilde{T}^\mu_{\nu(o)}\Theta(r - r_+ - \epsilon) + \tilde{T}^\mu_{\nu(H)}H$ we find

$$\begin{aligned}\nabla_\mu \tilde{T}^\mu_t = \partial_r \tilde{T}^r_t &= c_o \partial_r A_t(r) + \partial_r\left[\left(\frac{e^2}{4\pi}A_t^2 + \tilde{N}^r_t\right)H\right] \\ &\quad + \left(\tilde{T}^r_{t(o)} - \tilde{T}^r_{t(H)} + \frac{e^2}{4\pi}A_t^2 + \tilde{N}^r_t\right)\delta(r - r_+ - \epsilon). \end{aligned} \qquad (25)$$

The first term is a classical effect coming from the Lorentz force. The second term has to be canceled by the quantum effect of the incoming modes. As before, it implies the existence of a Wess-Zumino term modifying the energy-momentum tensor as $\tilde{T}'^\mu_t = \tilde{T}^\mu_t - [(\frac{e^2}{4\pi}A_t^2 + \tilde{N}^r_t)H]$ which is anomaly free provided the coefficient of the last term vanishes. This yields the condition

$$a_o = a_H + \frac{e^2}{4\pi}A_t^2(r_+) - \tilde{N}^r_t(r_+). \qquad (26)$$

where the integration constant a_H is fixed by requiring that the covariant energy-momentum tensor vanishes at the horizon. From (24) this gives $a_H = 0$. Hence the total flux of the energy-momentum tensor is given by

$$a_o = \frac{e^2}{4\pi}A_t^2(r_+) - \tilde{N}^r_t(r_+). \qquad (27)$$

Since $f(r_+) = 0$ we find from (20) that $\tilde{N}^r_t(r_+) = -\frac{(f')^2|_{r_+}}{192\pi}$. Using the surface gravity of the black hole $\kappa = \frac{2\pi}{\beta} = \frac{(f')|_{r_+}}{2}$, the final result is expressed in terms of the inverse temperature β as

$$a_o = \frac{e^2Q^2}{4\pi r_+^2} + \frac{\pi}{12\beta^2}. \tag{28}$$

This is just the energy flux from blackbody radiation with a chemical potential [9].

V. GENERALIZED CONSISTENT ANOMALY AND FLUX

Here we show that the conclusions of [8,9] remain unaffected by taking the general form of the consistent anomaly (6) and (11). Instead of repeating their analysis we just point out the reasons for this robustness.

For static configuration and for the specific choice of the potential ($A_r = 0$), it is clear that the normal parity term in (6) vanishes. Likewise the normal parity term in the counterterm (8) also vanishes since only the $\mu = r$ component in J^μ is relevant. Hence, effectively the same structures of the consistent (gauge) anomaly and the counterterm relating the consistent and covariant currents, as used in [9], are valid. Since these were the two basic inputs the results concerning the charge flux associated with Hawking radiation remain intact.

Identical conclusions also hold for the gravitational case. Although not immediately obvious, a little algebra shows that the normal parity term in $\mathcal{A}_t$ (11) vanishes. Hence the energy-momentum flux (given by T_t^r) remains as before.

VI. DISCUSSIONS

This work was based on [9] but with a different procedure and emphasis. The flow of charge and energy momentum from charged black hole horizons were obtained by a cancellation of the covariant anomalies. Since the boundary condition involved the vanishing of the covariant current at the horizon, all calculations involved only covariant expressions. Neither the consistent anomaly nor the counterterm relating the different currents, which were essential inputs in [9], were required. Consequently our analysis was economical and, we feel, also conceptually clean. We would here like to mention that the interplay of covariant versus consistent anomalies, as occurring in [8–10], has been specifically discussed in the appendix of [15].

It should be pointed out that the flux is identified with $J^r_{(o)}$ or $T^r_{t(o)}$ which are the expressions for the currents exterior to the horizon. Here these currents are anomaly free implying that there is no difference between the covariant and consistent expressions. Actually the germ of the anomaly lies in this difference [2,27]. Hence it becomes essential, and not just desirable, to obtain the same flux in terms of the covariant expressions. In other words the Hawking flux must yield identical results whether one uses the consistent or the covariant anomalies. But the boundary condition must be covariant. This is consistent with the universality of the Hawking radiation and gives further credibility to the anomaly cancellation approach.

It was shown [8,9], performing a partial wave decomposition, that physics near the horizon is described by an infinite collection of massless (1 + 1) dimensional fields, each partial wave propagating in spacetime with a metric given by the "$r - t$" sector of the complete spacetime metric (2). This simplification, which effects a dimensional reduction from d-dimensions to $d = 2$ is also exploited here. It is however noted that greybody factors have not been included. In that case dimensional reduction will not yield the real Hawking radiation for $d > 2$. For instance it is known [29] that for $d = 4$ reduction to $d = 2$ and keeping only the s-wave (i.e. $l = 0$) reduces the Hawking flux with respect to its $2d$ value.

A reason in favor of working with covariant anomalies is the fact that their functional forms are unique, being governed solely by the gauge (diffeomorphism) transformation properties. This is not so for consistent anomalies. They can and do have normal parity terms, apart from the odd parity ones. In fact, the special property (5) of two dimensions yields a natural form for this anomaly which has normal parity terms. Our observation that the results of [8,9] are still valid lend further support to this scheme of deriving Hawking radiation. The present approach can be easily extended to other (e.g. rotating) black holes with Kerr-(Newman) metric.

[1] S. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[2] H. Banerjee, R. Banerjee, and P. Mitra, Z. Phys. C **32**, 445 (1986).
[3] R. Bertlmann, *Anomalies In Quantum Field Theory* (Oxford Sciences, Oxford, 2000).
[4] K. Fujikawa and H. Suzuki, *Path Integrals and Quantum Anomalies* (Oxford Sciences, Oxford, 2004).
[5] L. Alvarez-Gaume and E. Witten, Nucl. Phys. **B234**, 269 (1984).
[6] W. A. Bardeen and B. Zumino, Nucl. Phys. **B244**, 421 (1984).
[7] S. Christensen and S. Fulling, Phys. Rev. D **15**, 2088 (1977).
[8] S. P. Robinson and F. Wilczek, Phys. Rev. Lett. **95**, 011303 (2005).
[9] S. Iso, H. Umetsu, and F. Wilczek, Phys. Rev. Lett. **96**, 151302 (2006).
[10] S. Iso, H. Umetsu, and F. Wilczek, Phys. Rev. D **74**, 044017 (2006).
[11] K. Murata and J. Soda, Phys. Rev. D **74**, 044018 (2006).

[12] E. Vagenas and S. Das, J. High Energy Phys. 10 (2006) 025.
[13] M. R. Setare, Eur. Phys. J. C **49**, 865 (2007).
[14] Z. Xu and B. Chen, Phys. Rev. D **75**, 024041 (2007).
[15] S. Iso, T. Morita, and H. Umetsu, J. High Energy Phys. 04 (2007) 068.
[16] Q. Q. Jiang and S. Q. Wu, Phys. Lett. B **647**, 200 (2007).
[17] Q. Q. Jiang, S. Q. Wu, and X. Cai, Phys. Lett. B **651**, 58 (2007).
[18] Q. Q. Jiang, S. Q. Wu, and X. Cai, Phys. Rev. D **75**, 064029 (2007).
[19] X. Kui, W. Liu, and H. Zhang, Phys. Lett. B **647**, 482 (2007).
[20] H. Shin and W. Kim, arXiv:0705.0265.
[21] J. J. Peng and S. Q. Wu, arXiv:0705.1225.
[22] Q. Q. Jiang, arXiv:0705.2068.
[23] B. Chen and W. He, arXiv:0705.2984.
[24] U. Miyamoto and K. Murata, arXiv:0705.3150.
[25] S. Iso, T. Morita, and H. Umetsu, Phys. Rev. D **76**, 064015 (2007).
[26] S. Das, S. P. Robinson, and E. C. Vagenas, arXiv: 0705.2233.
[27] H. Banerjee and R. Banerjee, Phys. Lett. B **174**, 313 (1986).
[28] R. Bertlmann and E. Kohlprath, Ann. Phys. (N.Y.) **288**, 137 (2001).
[29] R. Balbinot, A. Fabbri, P. Nicolini, V. Frolov, P. Sutton, and A. Zelnikov, Phys. Rev. D **63**, 084029 (2001).