Black holes in the tensor-vector-scalar theory of gravity and their thermodynamics

Eva Sagi* and Jacob D. Bekenstein†

Racah Institute of Physics, Hebrew University of Jerusalem, Jerusalem 91904, Israel

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Tensor-vector-scalar (TeVeS) theory, a relativistic theory of gravity, was designed to provide a basis for the modified Newtonian dynamics. Since TeVeS differs from general relativity (e.g., it has two metrics, an Einstein metric and a physical metric), black hole solutions of it would be valuable for a number of endeavors ranging from astrophysical modeling to investigations into the interrelation between gravity and thermodynamics. Giannios has recently found a TeVeS analogue of the Schwarzschild black hole solution. We proceed further with the program by analytically solving the TeVeS equations for a static spherically symmetric and asymptotically flat system of electromagnetic and gravity fields. We show that one solution is provided by the Reissner-Nordström metric as physical metric, the TeVeS vector field pointing in the time direction, and a TeVeS scalar field positive everywhere (the last feature protects from superluminal propagation of disturbances in the fields). We work out black hole thermodynamics in TeVeS using the physical metric; black hole entropy, temperature, and electric potential turn out to be identical to those in general relativity. We find it inconsistent to base thermodynamics on the Einstein metric. In light of this, we reconsider the Dubovsky-Sibiryakov scenario for violating the second law of thermodynamics in theories with Lorentz symmetry violation.

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I. INTRODUCTION

There are significant discrepancies between the visible masses of galaxies and clusters of galaxies and their masses as inferred from Newtonian dynamics. In particular, the accelerations of stars and gas in the outskirts of galaxies or those of galaxies in clusters are much too large, and the disk rotation curves of spiral galaxies, which are naively expected to drop as $r^{-1/2}$ away from galaxy centers, tend to remain flat to the last optically or radio measured point. These discrepancies are also manifest in the gravitational lensing by galaxies and clusters. It is commonly assumed that these problems stem from the existence in the said systems of large amounts of "dark matter." For instance, galaxies are assumed to be enshrouded in roundish halos of dark matter that dominate the gravitational fields far from the galaxy centers.

But the putative dark matter has yet to be identified or detected directly. Furthermore, dark matter models of galaxies require much fine-tuning of the dark halo parameters to fit the data, and there are some sharp problems outstanding such as the observationally inferred absence of cusps in the dark matter density at galaxy centers, cusps which are predicted by dark matter cosmogony. Thus many have wondered if dark matter is the whole story. An alternative, if less orthodox, approach is formalized in the modified Newtonian dynamics (MOND) paradigm [1], which proposes that Newtonian gravity progressively fails as accelerations drop below a characteristic scale $a_0 \simeq 10^{-10}$ m/s^2, which is typical of galaxy outskirts. MOND assumes that, for accelerations of order a_0 or well below it, the Newtonian relation $\boldsymbol{a} = -\boldsymbol{\nabla}\Phi_N$ is replaced by

$$\tilde{\mu}(|\boldsymbol{a}|/a_0)\boldsymbol{a} = -\boldsymbol{\nabla}\Phi_N, \tag{1}$$

where the function $\tilde{\mu}(x)$ smoothly interpolates between $\tilde{\mu}(x) = x$ at $x \ll 1$ and the Newtonian expectation $\tilde{\mu}(x) = 1$ at $x \gg 1$. This relation, with a suitable standard choice of $\mu(x)$ in the intermediate range, has proved successful not only in rationalizing the asymptotical flatness of galaxy rotation curves where acceleration scales are much below a_0, but also in explaining detailed shapes of rotation curves in the inner parts in terms of the directly seen mass, and in giving a precise account of the observed Tully-Fisher law which correlates luminosity of a disk galaxy with its asymptotic rotational velocity [2].

The pristine MOND paradigm does not fulfill the usual conservation laws, does not make it clear if the departure from Newtonian physics is in the gravity or in the inertia side of the equation $\boldsymbol{F} = m\boldsymbol{a}$, and does not teach us how to handle gravitational lensing or cosmology in the weak acceleration regimes. All these things are done by tensor-vector-scalar (TeVeS) theory [3], a covariant field theory of gravity which has MOND as its low velocity, weak accelerations limit, while its nonrelativistic strong acceleration limit is Newtonian and its relativistic limit is general relativity (GR). TeVeS sports two metrics, the "physical" metric on which all matter fields propagate, and the Einstein metric which interacts with the additional fields in the theory: a timelike dynamical vector field u, a scalar field ϕ, and a nondynamical auxiliary scalar field σ. The theory also involves a free function $\mathcal{F}$, a length scale ℓ, and two positive dimensionless constants k and K.

TeVeS is an attempt to recast MOND into a full physical theory in which the latter's novel behavior is due to the gravitational field. Some checks of its consistency and

*eva.sagi@mail.huji.ac.il
†bekenste@vms.huji.ac.il

comparisons with hard facts have been made. Thus Bekenstein showed that TeVeS's weak acceleration limit reproduces MOND, and that it also has a Newtonian limit, and calculated its parametrized post-Newtonian (PPN) coefficients β and γ, which agree with the results of solar system tests [3–5]. Skordis *et al.* [6] and Dodelson and Liguori [7] studied the evolution of homogenous and isotropic model universes in TeVeS, and showed that it may reproduce key features of the power spectra of the cosmic microwave background and the galaxy distribution. TeVeS has also been tested against a multitude of data on gravitational lensing (for some references, see Ref. [2]). All the above refer principally to situations where the gravitational potential is small on scale c^2. Since neutron stars and black holes exist in nature one must also understand strong gravity systems in the TeVeS framework.

A beginning in the investigation of the strong gravity regime of TeVeS has been made by Giannios [5]. For vacuum spherically symmetric and static situations, he showed that, under a simplifying limit (which we shall detail below), the Schwarzschild metric *qua* physical metric and a particular scalar field distribution together constitute a black hole solution of TeVeS's. This motivates us to look in this paper at more complicated cases, such as that of the charged nonrotating black hole in TeVeS. We find that the Reissner-Nordström (RN) metric as physical metric and the usual electric field together with a special configuration of TeVeS's scalar field constitute a black hole solution in TeVeS. Using this solution we investigate the thermodynamics of spherical black holes in TeVeS.

In Sec. II we recapitulate the fundamentals of TeVeS, while in Sec. III we describe Giannios' results for the nonrotating vacuum black hole. In Sec. IV we go on to solve the TeVeS equations for the case of a charged nonrotating black hole, obtaining a physical metric which coincides with the RN metric of GR. Section V presents a resolution of the problem pointed out by Giannios: the uncharged black hole solution he found seems to permit superluminal propagation near the black hole horizon. Next, in Sec. VI we examine how the familiar concepts of black hole thermodynamics apply to our black hole solutions, and check their consistency using several prescriptions. We calculate the relevant thermodynamic quantities using the physical metric, and show that the Einstein metric is inappropriate for discussing thermodynamics. In this light we discuss anew the potential thermodynamic inconsistency described by Dubovsky and Sibiryakov (DS) for theories with broken Lorentz symmetry [8].

II. THE TEVES EQUATIONS

The acronym TeVeS refers to the tensor-vector-scalar content of the theory. The tensor part pertains to the two metrics, $g_{\mu\nu}$, dubbed the Einstein metric, on which the vector and the scalar fields propagate, and the physical metric $\tilde{g}_{\mu\nu}$, on which matter, electromagnetic fields, etc. propagate. The physical metric is obtained from the Einstein metric through the following relation:

$$\tilde{g}_{\alpha\beta} = e^{-2\phi} g_{\alpha\beta} - 2u_\alpha u_\beta \sinh(2\phi). \tag{2}$$

Thus one passes from the space of $g_{\alpha\beta}$ to that of $\tilde{g}_{\alpha\beta}$ by stretching spacetime along the vector u by a factor $e^{2\phi}$, and shrinking it by the same factor orthogonally to that vector. This prescription retains MOND phenomenology, while augmenting the gravitational lensing by clusters and galaxies to fit observations.

The dynamics of the metrics and the fields are derivable from an action principle. The action in TeVeS is the sum of four terms. The first two are the familiar Hilbert-Einstein action and the matter action for field variables collectively denoted f:

$$S_g = \frac{1}{16\pi G}\int g^{\alpha\beta}R_{\alpha\beta}\sqrt{-g}d^4x, \tag{3}$$

$$S_m = \int \mathcal{L}(\tilde{g}_{\mu\nu}, f^\alpha, f^\alpha_{;\mu}, \cdots)\sqrt{-\tilde{g}}d^4x. \tag{4}$$

Next comes the vector field's action (K is a dimensionless positive coupling constant)

$$S_v = -\frac{K}{32\pi G}\int\Big[(g^{\alpha\beta}g^{\mu\nu}u_{[\alpha,\mu]}u_{[\beta,\nu]}) - \frac{2\lambda}{K}(g^{\mu\nu}u_\mu u_\nu + 1)\Big]\sqrt{-g}d^4x, \tag{5}$$

which includes a constraint that forces the vector field to be timelike (and unit normalized); λ is the corresponding Lagrange multiplier. The presence of a nonzero u^α establishes a preferred Lorentz frame, thus breaking Lorentz symmetry. Finally, we have the scalar's action (k is a dimensionless positive parameter while ℓ is a constant with the dimensions of length, and $\mathcal{F}$ is a dimensionless free function)

$$S_s = -\frac{1}{2k^2\ell^2 G}\int \mathcal{F}(k\ell^2 h^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta})\sqrt{-g}d^4x. \tag{6}$$

Above, $h^{\alpha\beta} \equiv g^{\alpha\beta} - u^\alpha u^\beta$ with $u^\alpha \equiv g^{\alpha\beta}u_\beta$. The scalar action is written here differently than in Ref. [3]; we have eliminated the nondynamical field σ and redefined the function $\mathcal{F}$. The new form makes it easier to understand the strong acceleration limit of the theory, which is especially relevant to the present work.

Variation of the total action with respect to $g^{\alpha\beta}$ yields the TeVeS Einstein equations for $g_{\alpha\beta}$;

$$G_{\alpha\beta} = 8\pi G(\tilde{T}_{\alpha\beta} + (1 - e^{-4\phi})u^\mu \tilde{T}_{\mu(\alpha}u_{\beta)} + \tau_{\alpha\beta}) + \theta_{\alpha\beta}. \tag{7}$$

The sources here are the usual matter energy-momentum tensor $\tilde{T}_{\alpha\beta}$, the variational derivative of S_m with respect to $\tilde{g}^{\alpha\beta}$, as well as the energy-momentum tensors for the scalar and vector fields:

$$\tau_{\alpha\beta} \equiv \frac{\mu}{kG}(\phi_{,\alpha}\phi_{,\beta} - u^{\mu}\phi_{,\mu}u_{(\alpha}\phi_{,\beta)}) - \frac{\mathcal{F}g_{\alpha\beta}}{2k^2\ell^2 G}, \quad (8)$$

$$\theta_{\alpha\beta} \equiv K(g^{\mu\nu}u_{[\mu,\alpha]}u_{[\nu,\beta]} - \tfrac{1}{4}g^{\sigma\tau}g^{\mu\nu}u_{[\sigma,\mu]}u_{[\tau,\nu]}g_{\alpha\beta}) - \lambda u_{\alpha}u_{\beta}, \quad (9)$$

with

$$\mu(x) \equiv \mathcal{F}'(x). \quad (10)$$

Each choice of $\mathcal{F}$ defines a separate TeVeS theory, and $\mu(x)$ is similar in nature to the function $\tilde{\mu}$ in MOND. In particular, $\mu(x) \simeq 1$ corresponds to high acceleration, i.e., to the Newtonian limit.

The equations of motion for the vector and scalar fields are, respectively,

$$[\mu(kl^2h^{\gamma\delta}\phi_{,\gamma}\phi_{,\delta})h^{\alpha\beta}\phi_{,\alpha}]_{;\beta} = kG[g^{\alpha\beta} + (1 + e^{-4\phi})u^{\alpha}u^{\beta}]\tilde{T}_{\alpha\beta}, \quad (11)$$

$$u^{[\alpha;\beta]}{}_{;\beta} + \lambda u^{\alpha} + \frac{8\pi}{k}\mu u^{\beta}\phi_{,\beta}g^{\alpha\gamma}\phi_{,\gamma} = 8\pi G(1 - e^{-4\phi})g^{\alpha\mu}u^{\beta}\tilde{T}_{\mu\beta}. \quad (12)$$

Additionally, there is the normalization condition on the vector field:

$$u^{\alpha}u_{\alpha} = g_{\alpha\beta}u^{\alpha}u^{\beta} = -1. \quad (13)$$

The Lagrange multiplier λ can be calculated from the vector equation.

III. NEUTRAL SPHERICAL BLACK HOLES

In his work on black holes, Giannios [5] worked in the limit $\mu \to 1$, which also entails $\mathcal{F}(x) = x$. Since we shall later work in the same limit, here we shall justify it in more detail than he did. Near the horizon of a black hole of mass m, the Newtonian acceleration amounts to $10^{23}(M_{\odot}/m)a_0$. Thus, even for the most massive black holes suspected ($10^{10}m_{\odot}$), the accelerations are strong on scale a_0 out to at least a million times the gravitational radius, i.e. well into the asymptotically flat region which determines the metric properties. This means MOND effects are suppressed while the full complexity of the TeVeS equations is still evident. In the said limit, and under the assumption that the vector field points in the time direction (which has support in the more general context of static solutions [3]), Giannios obtained an exact spherically symmetric analytical solution to the TeVeS equations, for metric, scalar, and vector fields.

The Einstein metric is taken in isotropic coordinates, $x^0 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \varphi$,

$$ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta} = -e^{\nu}dt^2 + e^{\zeta}(dr^2 + r^2d\Omega^2), \quad (14)$$

where henceforth $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\theta^2$. Since ν, ζ are functions of r only, and the vector field points in the time direction, its r dependence is fully determined by the normalization condition Eq. (13) and the requirement that u^{α} be future pointing:

$$u^{\alpha} = (e^{-\nu/2}, 0, 0, 0). \quad (15)$$

Then the relation between the physical and field metrics reduces to

$$\tilde{g}_{tt} = e^{2\phi}g_{tt}, \quad (16)$$

$$\tilde{g}_{ii} = e^{-2\phi}g_{ii}. \quad (17)$$

Giannios first solved the TeVeS equations assuming that $K = 0$, thus decoupling the vector field from the theory, and then performed a transformation involving K, which recovered the more general solution. For $K = 0$ the Einstein's tt, rr, and $\theta\theta$ equations are, respectively,

$$\frac{2\zeta'}{r} + \frac{(\zeta')^2}{4} + \zeta'' = -\frac{4\pi(\phi')^2}{k}, \quad (18)$$

$$\frac{\zeta' + \nu'}{r} + \frac{(\zeta')^2}{4} + \frac{\zeta'\nu'}{2} = \frac{4\pi(\phi')^2}{k}, \quad (19)$$

$$\frac{\nu' + \zeta'}{2r} + \frac{(\nu')^2 + 2\zeta'' + 2\nu''}{4} = -\frac{4\pi(\phi')^2}{k}, \quad (20)$$

and the scalar equation takes the form

$$\phi'' + \frac{\phi'(r(\nu' + \zeta') + 4)}{2r} = 0. \quad (21)$$

Since there are only three unknown functions, $\nu(r)$, $\zeta(r)$, and $\phi(r)$, one of the four equations is obviously superfluous.

Combining the rr and $\theta\theta$ Einstein equations gives the simple differential equation

$$2(\nu + \zeta)'' + \frac{6(\nu + \zeta)'}{r} + ((\nu + \zeta)')^2 = 0. \quad (22)$$

This has the solution

$$\nu + \zeta = 2\ln\left(\frac{r^2 - r_c^2}{r^2}\right), \quad (23)$$

where the additive integration constant has been set to zero in order to have an asymptotically flat spacetime, namely, ν, $\zeta \to 0$ when $r \to \infty$.

The second integration constant, r_c, can be evaluated by expanding $\nu + \zeta$ above in $1/r$ and comparing with the $1/r$ expansions (with $K = 0$) of the metric coefficients of the exterior solution for a spherical mass [3,5],

$$e^{\nu} = 1 - \frac{r_g}{r} + \frac{1}{2}\frac{r_g^2}{r^2} + \cdots, \quad (24)$$

$$e^{\zeta} = 1 + \frac{r_g}{r} + \frac{1}{16}\left[6 - \frac{2k}{\pi}\left(\frac{Gm_s}{r_g}\right)^2\right]\frac{r_g^2}{r^2} + \cdots. \quad (25)$$

Here m_s is a mass scale [3] defined by the expansion

$$\phi(r) = \phi_c - \frac{kGm_s}{4\pi r} + \cdots \tag{26}$$

for the solution of Eq. (21). For a ball of nonrelativistic fluid, m_s is very close to the Newtonian mass, and r_g is a scale of length that can be linked to the object's mass [3]. However, the relation between Gm_s and r_g depends on the system under consideration, and is different for stars and black holes. At any rate, for $K = 0$, r_c is found to be

$$r_c = \frac{r_g}{4}\sqrt{1 + \frac{k}{\pi}\left(\frac{Gm_s}{r_g}\right)^2}. \tag{27}$$

Making the educated *guess* that

$$\zeta' = \frac{4r_c^2}{r(r^2 - r_c^2)} - \frac{r_g}{r^2 - r_c^2}, \tag{28}$$

Giannios determines ν to be

$$\nu = \frac{r_g}{2r_c}\ln\left(\frac{r - r_c}{r + r_c}\right). \tag{29}$$

The correctness of Eqs. (28) and (29) can be checked by substituting them into the sum of Eqs. (18) and (19), or the difference of Eqs. (18) and (20); both combinations are independent of the equation pair already used. The determination of the Einstein metric for $K = 0$ is completed by the trivial integration of Eq. (28). Finally, the scalar field is found now by integrating Eq. (21) and fixing the two integration constants just as in Eq. (26),

$$\phi(r) = \phi_c + \frac{kGm_s}{8\pi r_c}\ln\left(\frac{r - r_c}{r + r_c}\right). \tag{30}$$

Going on to the more general case $K \neq 0$, Giannios finds that just replacing Eq. (27) by

$$r_c = \frac{r_g}{4}\sqrt{1 + \frac{k}{\pi}\left(\frac{Gm_s}{r_g}\right)^2 - \frac{K}{2}} \tag{31}$$

in the above solutions for ν, ζ, and ϕ will produce an exact solution of the TeVeS equations for $K \neq 0$ [equations which are the $Q = 0$ case of Eqs. (49)–(52) below].

The physical metric now follows from Eqs. (16) and (17):

$$\tilde{g}_{tt} = -\left(\frac{r - r_c}{r + r_c}\right)^a, \tag{32}$$

$$\tilde{g}_{rr} = \frac{(r^2 - r_c^2)^2}{r^4}\left(\frac{r - r_c}{r + r_c}\right)^{-a} \tag{33}$$

with $a \equiv \frac{r_g}{2r_c} + \frac{kGm_s}{4\pi r_c}$. In order for this result to represent a black hole, the candidate event horizon $r = r_c$ must have bounded surface area, and must not be a singular surface. The surface area is proportional to $\tilde{g}_{rr}(r_c)$, which has a factor $(r - r_c)^{2-a}$; for this to be bounded requires $a \leq 2$. The Ricci scalar of the above metric is

$$R = \frac{2(a^2 - 4)r_c^2 r^4 (r - r_c)^{a-4}}{(r + r_c)^{a+4}}. \tag{34}$$

We notice that R will blow up as $r \rightarrow r_c$ unless $a = 2$ or $a > 4$. Thus, only the value $a = 2$ describes a black hole. The definition of a then gives another relation between r_c and r_g,

$$r_c = \frac{r_g}{4} + \frac{kGm_s}{8\pi}, \tag{35}$$

and the physical metric takes the final form

$$\tilde{g}_{tt} = -\left(\frac{r - r_c}{r + r_c}\right)^2, \tag{36}$$

$$\tilde{g}_{rr} = \left(\frac{r + r_c}{r}\right)^4, \tag{37}$$

which we recognize as the Schwarzschild metric in isotropic coordinates.

Unlike GR's Schwarzschild black hole, the TeVeS neutral spherical black hole is "dressed" with a scalar field ϕ, a solution of Eq. (30). This field does not induce a singularity at the horizon because of the particular structure of the TeVeS equations. However, the logarithmic divergence of ϕ at the horizon was a cause of concern to Giannios. It was earlier shown [3] that the absence of superluminal propagation of the various TeVeS fields is guaranteed only when $\phi \geq 0$. But here ϕ diverges logarithmically at $r = r_c$, and becomes already negative sufficiently close to r_c even if $\phi_c > 0$. We will show in the next section how this apparent problem is solved.

IV. CHARGED SPHERICAL BLACK HOLES

The next natural step is to look for an electrovacuum static spherically symmetric solution to the TeVeS equations, the analog of the RN solution of GR. We again take $\mu \approx 1$. Again we assume that the vector field points in the time direction, and that both the physical and the Einstein metrics are spherically symmetric. These are essential simplifying assumptions which enable us to find a specific solution to the TeVeS field equations. Other solutions may exist for which the vector field is endowed with a radial component. However, to judge from the neutral case, as analyzed by Giannios [5], the PPN parameter β of such a solution with very low charge would be in contradiction with recent observations [9] in the solar system. It would be odd if the PPN structure of a black hole's far field were that different from the sun's. By contrast, still in the neutral case, a TeVeS solution with the vector field pointing in the cosmological time direction yields PPN parameters identical to those of GR [5].

We continue to work in isotropic coordinates, for which the transition between physical and Einstein metrics is simplest: as seen earlier, in view of the normalization condition (13), the transformation (2) is equivalent to

$$\tilde{g}_{\alpha\beta} = \begin{cases} e^{-2\phi} g_{ii} & i = r, \theta, \phi, \\ e^{2\phi} g_{tt}. \end{cases} \tag{38}$$

The Einstein metric again takes the form (14), and the physical metric will have a similar form, namely,

$$d\tilde{s}^2 = \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta = -e^{\tilde{\nu}} dt^2 + e^{\tilde{\zeta}}(dr^2 + r^2 d\Omega^2), \tag{39}$$

with the following relation among $\tilde{\nu}$, $\tilde{\zeta}$, ν, and ζ:

$$\zeta(r) = \tilde{\zeta}(r) + 2\phi(r), \tag{40}$$

$$\nu(r) = \tilde{\nu}(r) - 2\phi(r). \tag{41}$$

Now, the cosmological value of ϕ should be nonzero in our evolving universe: $\phi(r \to \infty) = \phi_c$. Thus the requirement that the Einstein metric be asymptotically Minkowski (both ζ and ν vanish as $r \to \infty$), needed to maintain consistency with previous work [3], introduces a factor $\pm 2\phi_c$ in the physical metric coefficients,

$$\tilde{\zeta}(r \to \infty) = -2\phi_c, \tag{42}$$

$$\tilde{\nu}(r \to \infty) = 2\phi_c. \tag{43}$$

This is equivalent to a rescaling of the coordinates which depends on a cosmological epoch, and will have to be taken into account when considering physical quantities in the framework of TeVeS.

The energy-momentum tensor no longer vanishes; it is given by

$$\tilde{T}_{\alpha\beta} = \frac{1}{4\pi}\left(\tilde{F}_{\alpha\rho}\tilde{F}^{\rho}_{\beta} - \frac{1}{4}\tilde{g}_{\alpha\beta}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma}\right), \tag{44}$$

with $\tilde{F}_{\alpha\beta}$, the electromagnetic field tensor (not its dual), obtained by solving Maxwell's equations in vacuum written wholly with the metric $\tilde{g}_{\alpha\beta}$, namely,

$$\tilde{\nabla}_\beta \tilde{F}^{\alpha\beta} = (-\tilde{g})^{-1/2}\partial_\beta[(-\tilde{g})^{1/2}\tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu}\tilde{F}_{\mu\nu}] = 0. \tag{45}$$

In the isotropic metric Eq. (39), and with the assumption of spherical symmetry and the absence of magnetic monopoles, the only nonvanishing component of the electromagnetic field tensor is

$$\tilde{F}_{rt} = \frac{Q}{r^2} e^{(1/2)(\tilde{\nu}(r) - \tilde{\zeta}(r))} = \frac{Q}{r^2} e^{(1/2)(\nu(r) - \zeta(r)) + 2\phi(r)}. \tag{46}$$

The constant of integration Q will be shown in Sec. IV to coincide with the physical electric charge of the black hole.

Since we assumed the vector field to point in the time direction, then, as in the vacuum case, the normalization condition (13) determines its functional dependence:

$$u^\alpha = (e^{-\nu/2}, 0, 0, 0). \tag{47}$$

It follows that the spatial components of the vector equation (12) are identically satisfied, while its temporal component serves to determine the Lagrange multiplier λ to be substituted in the Einstein equations:

$$\lambda = -\frac{K(r\nu'\zeta' + 2r\nu'' + 4\nu')}{4re^{\zeta}} + \frac{GQ^2 e^{2\phi}(e^{4\phi} - 1)}{r^4 e^{2\zeta}}. \tag{48}$$

We now turn to the Einstein equations (7), and the scalar equation (11). Upon substitution of the Lagrange multiplier (48) and the electromagnetic field tensor (46), the tt, rr, and $\theta\theta$ equations become, respectively,

$$\frac{2\zeta'}{r} + \frac{(\zeta')^2}{4} + \zeta'' + \frac{K(8\nu' + 2r\nu'\zeta' + r(\nu')^2 + 4r\nu'')}{8r} = -\frac{4\pi(\phi')^2}{k} - \frac{e^{-\zeta+2\phi}GQ^2}{r^4}, \tag{49}$$

$$\frac{\zeta' + \nu'}{r} + \frac{(\zeta')^2}{4} + \frac{\zeta'\nu'}{2} + \frac{K(\nu')^2}{8} = \frac{4\pi(\phi')^2}{k} - \frac{e^{-\zeta+2\phi}GQ^2}{r^4}, \tag{50}$$

$$\frac{(\nu' + \zeta')}{2r} + \frac{((\nu')^2 + 2\zeta'' + 2\nu'')}{4} - \frac{K(\nu')^2}{8} = -\frac{4\pi(\phi')^2}{k} + \frac{e^{-\zeta+2\phi}GQ^2}{r^4}. \tag{51}$$

The scalar equation is

$$\phi'' + \frac{(r(\nu' + \zeta') + 4)\phi'}{2r} = \frac{e^{-\zeta+2\phi}kGQ^2}{4\pi r^4}. \tag{52}$$

These are four equations for three unknowns, $\zeta(r)$, $\nu(r)$, and $\phi(r)$, so one of the equations is actually redundant. We shall use two combinations of the three Einstein equations plus the scalar equation.

By adding the rr and $\theta\theta$ equations, we again obtain, as in the vacuum case, Eq. (22). This time we write the solution

$$\zeta + \nu = 2\ln\left(\frac{r^2 - r_h^2}{r^2}\right). \tag{53}$$

Here one integration constant has been set so as to have an asymptotically flat spacetime, namely, ν, $\zeta \to 0$ when $r \to \infty$. The other constant, r_h, will be set by the boundary conditions on the horizon.

The remaining equations for ν, ζ, and ϕ are not immediately solvable. To make progress we shall *assume* that the *physical* metric $\tilde{g}_{\alpha\beta}$ is of RN form, solve for the scalar field in this framework, and check that all TeVeS equations are satisfied. This will give us a pair of charged black hole solutions of TeVeS; existence of other solutions is yet to be excluded.

In Schwarzschild coordinates $x^0 = t$, $x^1 = R$, $x^2 = \theta$, and $x^3 = \varphi$, the RN metric may be written as

$$ds^2 = -(1 - R_+/R)(1 - R_-/R)dt^2 + \frac{dR^2}{(1 - R_+/R)(1 - R_-/R)} + R^2 d\Omega^2 \quad (54)$$

where R_+ and R_- are the coordinates of the outer and inner horizons, respectively. We may transform the metric to isotropic form by going over to a new radial coordinate r defined implicitly by

$$R(r) = r + (R_+ - R_-)^2/16r + (R_+ + R_-)/2. \quad (55)$$

This gives

$$ds^2 = -\frac{(4r - (R_+ - R_-))^2(4r + (R_+ - R_-))^2}{(16r^2 + 8r(R_+ + R_-) + (R_+ - R_-)^2)^2} dt^2 + \frac{(16r^2 + 8r(R_+ + R_-) + (R_+ - R_-)^2)^2}{256r^4} \times (dr^2 + r^2 d\Omega^2).$$

We recall that in GR R_+ and R_- satisfy the relations $R_+ + R_- = 2G_N m$ and $R_+ R_- = G_N q^2$, where we write the gravitational constant as G_N to distinguish it from plain G, the coupling constant in TeVeS. Here we shall assume that the *physical* metric of TeVeS has the above form, while leaving the parameters R_+ and R_- to be determined later. However, the proposed metric is asymptotically Minkowskian, while as previously mentioned, we require rather that the Einstein metric be asymptotically Minkowskian. This means that, in the generic physical metric Eq. (39), we must set

$$\tilde{\nu}(r) = \ln\frac{(4r - (R_+ - R_-))^2(4r + (R_+ - R_-))^2}{(16r^2 + 8r(R_+ + R_-) + (R_+ - R_-)^2)^2} + 2\phi_c, \quad (56)$$

$$\tilde{\zeta}(r) = \ln\frac{(16r^2 + 8r(R_+ + R_-) + (R_+ - R_-)^2)^2}{256r^4} - 2\phi_c. \quad (57)$$

To simplify these, note that by Eqs. (40) and (41) we have $\zeta + \nu = \tilde{\zeta} + \tilde{\nu}$, whereupon, in view of Eq. (53),

$$\frac{(r^2 - r_h^2)^2}{r^4} = \frac{(4r - (R_+ - R_-))^2(4r + (R_+ - R_-))^2}{256r^4}. \quad (58)$$

We may thus relate R_+ and R_- to the integration constant r_h appearing in (53):

$$r_h = \tfrac{1}{4}(R_+ - R_-). \quad (59)$$

Since $R(r = r_h) = R_+$, $r = r_h$ is the outer black hole horizon in isotropic coordinates. In terms of r_h and $M \equiv (R_+ + R_-)/2$ the physical metric coefficients are

$$e^{\tilde{\zeta}} = \frac{(r^2 + r_h^2 + Mr)^2}{r^4} e^{-2\phi_c}, \quad (60)$$

$$e^{\tilde{\nu}} = \frac{(r^2 - r_h^2)^2}{(r^2 + r_h^2 + Mr)^2} e^{2\phi_c}. \quad (61)$$

It is useful at this point to trade the charge Q for a dimensionless positive parameter α defined by

$$GQ^2 = \alpha^2 R_+ R_- = \alpha^2(M^2 - 4r_h^2). \quad (62)$$

This replaces the GR relation $R_+ R_- = G_N q^2$. The value of α will be determined by the Einstein equations (49) and (50).

The only indeterminate function remaining now is the scalar field. The scalar equation (52) can be rewritten in terms of the new parameters M, r_h, and α, as

$$\phi'' + \frac{2r\phi'}{r^2 - r_h^2} - \frac{k\alpha^2(M^2 - 4r_h^2)e^{2\phi_c}}{4\pi(r^2 + r_h^2 + Mr)^2} = 0. \quad (63)$$

Its general solution is

$$\phi = \phi_c + \frac{ke^{2\phi_c}\alpha^2}{4\pi}[(1 + C)\ln(r + r_h) + (1 - C)\ln(r - r_h) - \ln(r^2 + r_h^2 + Mr)], \quad (64)$$

with ϕ_c and C integration constants, the first already familiar. Since we guessed the form of the metric, we need to verify that the Einstein equations are satisfied. From the requirement that Eq. (50) be satisfied, we obtain values for α and C:

$$\alpha^2 = \frac{4\pi(2 - K)e^{-2\phi_c}}{k(2 - K) + 8\pi}, \quad (65)$$

$$C_\pm = \pm\frac{\sqrt{2k^2(2 - K) + 8\pi kK}}{(2 - K)k}. \quad (66)$$

Equation (49) is then satisfied identically. Since we have already used the sum of Eqs. (50) and (51) to get the solution (53), we see that all TeVeS equations are satisfied. Thus the RN metric from GR with a suitable choice of parameters is the physical metric of TeVeS spherical charged black holes.

We shall soon see that a physically acceptable solution can be had only for $K < 2$. For such solutions the sign of the quantity under the square root in Eq. (66) is positive. The two TeVeS solutions (corresponding to the two signs of C) are most clearly presented in terms of the coefficients $\delta_\pm = (k/4\pi)\alpha^2(1 + C_\pm)e^{2\phi_c}$, or

$$\delta_\pm = \frac{(2 - K)k \pm \sqrt{2k^2(2 - K) + 8\pi kK}}{(2 - K)k + 8\pi}. \quad (67)$$

In view of Eq. (47) we finally obtain the solutions

$$d\tilde{s}^2 = -\frac{(r^2 - r_h^2)^2}{(r^2 + r_h^2 + Mr)^2} e^{2\phi_c} dt^2 + \frac{(r^2 + r_h^2 + Mr)^2}{r^4} e^{-2\phi_c}(dr^2 + r^2 d\Omega^2), \quad (68)$$

$$\phi(r) = \phi_c + \delta_\pm \ln(r + r_h) + \delta_\mp \ln(r - r_h) - \tfrac{1}{2}(\delta_+ + \delta_-)\ln(r^2 + r_h^2 + Mr), \tag{69}$$

$$u^\alpha = \left(\frac{(r - r_h)^{\delta_\mp - 1}(r + r_h)^{\delta_\pm - 1}}{(r^2 + r_h^2 + Mr)^{(\delta_+ + \delta_- - 2)/2}}, 0, 0, 0\right). \tag{70}$$

V. RESOLVING THE SUPERLUMINAL PARADOX

We have found two black hole solutions for each value of R_h and M. The requirement that superluminal propagation be excluded selects one of them as physically viable. As mentioned in Sec. III, in a region where $\phi < 0$, superluminal propagation of the TeVeS fields is not ruled out. This acausal behavior would be unacceptable. Now, since $\ln(r - r_h)$ in Eq. (69) is arbitrarily large and negative near enough to the horizon, its coefficient must be negative in order that ϕ has a chance to be nonnegative everywhere. It is easy to see that for $K > 0$, $k > 0$, δ_+ is always positive. Thus the solution, Eqs. (68)–(70), with lower signs is immediately excluded on grounds that it permits superluminal propagation. But is the second solution viable in this sense?

Focusing on the solution with upper signs, we must now exclude the parameter range $K \geq 2 + 8\pi/k$; the equality here corresponds to unbounded δ_- and ϕ, while the inequality leads to $\delta_- > 0$ and superluminal propagation. The range $2 \leq K < 2 + 8\pi/k$, although palatable in this sense, gives $\alpha^2 \leq 0$. We shall show in Sec. VI that this is unphysical. For $0 < K < 2$ we have $\delta_- < 0$, while $\alpha^2 > 0$. Thus a physically viable black hole solution of TeVeS can exist only for $0 < K < 2$ (we continue to assume that $k > 0$). It is the solution with the lower indices in Eqs. (68)–(70).

Close enough to the horizon, ϕ of this solution is necessarily positive because of the $\delta_- \ln(r - r_h)$ which is arbitrarily large. Additionally, the asymptotic value of ϕ is ϕ_c, the cosmological value of the scalar, which may be assumed to be positive [3]. Hence the question of whether $\phi(r)$ is positive in the intermediate region hinges on whether it has a negative minimum outside the horizon, or not.

To find out, we look at its derivative,

$$\phi'(r) = \frac{(M + 2r_h)(r + r_h)^2\delta_- + (M - 2r_h)(r - r_h)^2\delta_+}{2(r^2 - r_h^2)(r^2 + r_h^2 + Mr)}. \tag{71}$$

The numerator here is quadratic in r and thus has two roots. Now in the case $K < 2$ we have $\delta_- < 0$, but $\delta_+ + \delta_- > 0$. Then, because $M > 2r_h$ [see Eq. (59) and the following discussion], both roots are real. Furthermore, if

$$M < 2r_h \frac{\delta_+ - \delta_-}{\delta_+ + \delta_-} = 2r_h \frac{\sqrt{2k^2(2 - K) + 8\pi Kk}}{(2 - K)k}, \tag{72}$$

both roots are at $r < 0$, so for $r > r_h$ the field $\phi(r)$ has no minimum and must be everywhere positive.

Focus now on the case

$$M > 2r_h \frac{\sqrt{2k^2(2 - K) + 8\pi Kk}}{(2 - K)k}. \tag{73}$$

Now ϕ does have a minimum outside the horizon. In Fig. 1 we plot $\phi - \phi_c$ for several values of M/r_h. We see that, unless M/r_h is very large, the dip below the axis (which grows roughly as $\ln M/r_h$) is modest compared to unity. Hence, a modest positive ϕ_c (which is expected from cosmological models [3]) will be enough to keep $\phi(r)$ positive throughout the black hole exterior, except for black holes with exponentially large values of M/r_h for which a region of negative ϕ will occur near the horizon.

In fact, for $r_h = 0$, which by Eq. (59) means that $R_+ = R_-$, i.e., that the physical metric is extremal RN, the two solutions for ϕ are identical:

$$\phi = \phi_c - \frac{(2 - K)k}{(2 - K)k + 8\pi}\ln\left(1 + \frac{M}{r}\right). \tag{74}$$

Thus for $K < 2$ and $r_h = 0$, the variable part of ϕ is negative and can be very large for $r \ll M$. This is unacceptable as it permits superluminal propagation. We may conclude that, provided $0 < K < 2$, $k > 0$ and ϕ_c somewhat above zero, the superluminality issue raised by Giannios does not arise for the TeVeS charged black hole solution with the lower signs in Eqs. (68)–(70). The above conclusion does not apply to black holes near the extremally charged case.

What about Giannios' case $Q = 0$ for which he found conditions conducive to superluminal propagation (see end of Sec. III)? The TeVeS equations (49)–(52) are smooth with Q, so we may take the limit $Q \to 0$ of their solutions, Eqs. (68)–(70). In this limit, according to Eq. (62), $M = 2r_h$, while by Eq. (59), $r_h = R_+/4$. Thus our metric (68) reduces exactly to Giannios' equations (36) and (37) with the obvious identification $r_c = r_h$; that is, we recover the fact that the physical metric is Schwarzschild. In the same limit, our scalar field solutions (69) reduce to the pair

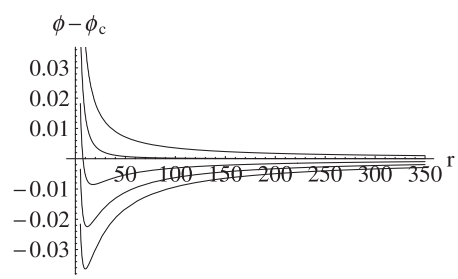

FIG. 1. Our solution for $\phi - \phi_c$ as a function of r for several values of M/r_h: the higher M/r_h, the lower the curve. Both axes are in arbitrary units.

$$\phi = \phi_c + \delta_{\mp} \ln\left(\frac{r - r_h}{r + r_h}\right), \tag{75}$$

whereas Giannios obtained only one scalar solution. We notice that the solution with upper sign has $\phi > 0$ for all $r > r_h$ provided that we stick to the parameter ranges $0 < K < 2$, $k > 0$ for which $\delta_- < 0$. The solution with the lower sign has $\phi < 0$ sufficiently near $r = r_h$; this is Giannios' solution, and it is indeed excluded because it allows superluminal propagation.

To sum up, in our study of spherical static black holes in TeVeS, we have found a viable charged black hole solution for the parameter range $0 < K < 2$, $k > 0$. The limiting case $Q \to 0$ of this is a viable neutral black hole solution. Since black holes are seen in nature with virtual certainty, the above results tell us that only the range $0 < K < 2$, $k > 0$ of TeVeS needs to be considered as physical. This range includes the values that have been explored in the confrontation of TeVeS with observations [2,3,6,7].

VI. BLACK HOLE THERMODYNAMICS

It has been clear for a long time that black holes are really thermodynamic systems characterized by temperature and entropy. Thus a discussion of black hole solutions in TeVeS would be incomplete without a survey of their thermodynamic properties. However, before we can talk about thermodynamics for the charged black hole in TeVeS, we must first identify the physical values of attributes of the black hole solution. By physical values we mean the quantities that an asymptotically Minkowski observer would measure using instruments made of matter, measurements which are thus referred to the physical metric. These values need not be identical with those of quantities naively associated with the attributes. For example, we do not know *a priori* that the masslike quantity M and the chargelike quantity Q appearing in our solution are indeed the physical mass and charge of the black hole. In fact, we shall see that M is related to the physical mass in a nontrivial fashion.

We first note that the G appearing in the TeVeS equations is not Newton's constant, but, as shown elsewhere [10], is related to it through

$$G_N = \left(\frac{(2 - K)k + 8\pi}{4\pi(2 - K)}\right)G. \tag{76}$$

It will be useful to also write the above relation in terms of the constant α defined by Eq. (65):

$$G_N = (G/\alpha^2)e^{-2\phi_c}. \tag{77}$$

Experimentally $G_N > 0$; it also seems natural that the fundamental coupling constant G be positive; hence we must require $\alpha^2 > 0$. This explains why in Sec. IV we ruled out the parameter range $2 \le K < 2 + 8\pi/k$.

Next, recall that if we use the same coordinates for the Einstein and physical metrics, the transformation (2) implies that our physical metric is not asymptotically Minkowski. Thus, asymptotically, the relation between the physical distance $\tilde{x}$ and the corresponding spatial lengthlike coordinate (denoted x) must be

$$\tilde{x} = e^{-\phi_c}x. \tag{78}$$

Focus now on M. According to Eq. (61), we may write the asymptotical expansion for $e^{\tilde{\nu}}$ as

$$e^{\tilde{\nu}} \approx \left(1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right)e^{2\phi_c}. \tag{79}$$

Thus r here is not the physical distance $\tilde{r}$, but it is related to it through Eq. (78). Rewriting $e^{\tilde{\nu}}$ in terms of the latter gives

$$e^{\tilde{\nu}} \approx \left(1 - \frac{2Me^{-\phi_c}}{\tilde{r}} + O\left(\frac{1}{r^2}\right)\right)e^{2\phi_c}. \tag{80}$$

From the customary linear approximation we see that M and physical mass m are related by

$$Me^{-\phi_c} = G_N m. \tag{81}$$

Likewise, the physical charge q can be easily identified by integrating the flux of the electromagnetic field tensor through a spherical shell at spatial infinity:

$$q = \lim_{r\to\infty} \frac{1}{4\pi} \int_{S^2} \tilde{F}^{tr} e^{\tilde{\zeta}} r^2 \sin\theta d\theta d\phi. \tag{82}$$

Use of $e^{\tilde{\zeta}}$ in forming the area element guarantees that we are calculating a physical flux: according to Eq. (40) the factor $e^{-2\phi_c}$ required by Eq. (78) is supplied by the $e^{\tilde{\zeta}}$. Substituting $\tilde{F}^{\rm tr}$ from Eq. (46) gives

$$\begin{aligned} q &= \lim_{r\to\infty} \frac{1}{4\pi} \int_{S^2} \frac{Q}{r^2} e^{-(1/2)(\tilde{\nu}+\tilde{\zeta})} r^2 \sin\theta d\theta d\phi \\ &= \lim_{r\to\infty} \frac{1}{4\pi} \int_{S^2} \frac{Qr^2}{r^2 - r_h^2} \sin\theta d\theta d\phi = Q. \end{aligned} \tag{83}$$

Thus our charged black hole is characterized by mass m and charge Q as measured by physical asymptotic observers for which the metric is $\tilde{g}_{\alpha\beta}$.

In investigating the black hole entropy we start with the *assumption* that it is given in terms of the physical surface area of the outer horizon A by the usual formula

$$S_{\rm BH} = \frac{A}{4\hbar G_N}. \tag{84}$$

It is true that more complicated forms are known, but they usually appear in gravity theories with higher derivatives; TeVeS is free of these. The proof that our choice is correct ultimately rests on the consistency checks we present later in this section.

Obviously

$$A = 4\pi r_h^2 e^{\tilde{\zeta}(r_h)} = 4\pi(2r_h + M)^2 e^{-2\phi_c}. \tag{85}$$

From (62) we have for the outer horizon

$$r_h = \frac{1}{2}\sqrt{M^2 - GQ^2/\alpha^2}. \qquad (86)$$

Thus

$$A = 4\pi(M + \sqrt{M^2 - GQ^2/\alpha^2})^2 e^{-2\phi_c}. \qquad (87)$$

We now express A in terms of physical mass, charge, and Newton's constant using the relations (77) and (81):

$$A = 4\pi(G_N m e^{\phi_c} + \sqrt{(G_N m e^{\phi_c})^2 - G_N e^{2\phi_c} Q^2})^2 e^{-2\phi_c}, \qquad (88)$$

so that

$$S_{\rm BH} = \frac{\pi}{G_N \hbar}(G_N m + \sqrt{(G_N m)^2 - G_N Q^2})^2. \qquad (89)$$

This is identical to the familiar expression for the entropy of a RN black hole. To it corresponds the thermodynamic temperature $T_{\rm BH} = (\partial S_{\rm BH}/\partial m)_Q^{-1}$, or

$$T_{\rm BH} = \frac{\hbar}{2\pi}\frac{\sqrt{(G_N m)^2 - G_N Q^2}}{(G_N m + \sqrt{(G_N m)^2 - G_N Q^2})^2}. \qquad (90)$$

To check the consistency of our scheme, we now also calculate the temperature corresponding to our black hole solution using the Euclidean path integral approach [11]. This approach entails performing a Wick rotation of the time coordinate to obtain a Euclidean metric. The path integral for the gravitational action then becomes the partition function for a canonical ensemble. Regularity of the new coordinate system near the horizon requires the new time coordinate to be periodic, and the period is related to the black hole temperature: $T = \hbar/\text{period}$.

We first define l, the radial proper distance from the horizon using the physical metric (68):

$$dl = \frac{(r^2 + r_h^2 + Mr)}{r^2} e^{-\phi_c} dr$$
$$\Rightarrow l = (r - r_h^2/r + M\ln(r/r_h))e^{-\phi_c}. \qquad (91)$$

Consequently the physical (2-D) line element $d\tilde{\sigma}^2$ for fixed θ and ϕ following from metric (68) becomes

$$d\tilde{\sigma}^2 = -\frac{(r^2 - r_h^2)^2}{(r^2 + r_h^2 + Mr)^2} e^{2\phi_c} dt^2 + dl^2. \qquad (92)$$

Near the horizon, where $r \approx r_h$, we have

$$l \approx (2 + M/r_h)e^{-\phi_c}(r - r_h) + O((r - r_h)^2). \qquad (93)$$

Substituting this into $d\tilde{\sigma}^2$ and replacing $e^{\phi_c} dt$, the global physical time interval according to Eq. (79), by $\imath d\tau$, we obtain an expression for the Euclidean metric near the horizon,

$$d\tilde{\sigma}_E^2 = \left(\frac{2r_h e^{\phi_c}}{(2r_h + M)^2}\right)^2 l^2 (d\tau)^2 + dl^2. \qquad (94)$$

For this metric to be regular at $l = 0$ ($r = r_h$), we must regard τ as an angular variable with period

$$2\pi\frac{(2r_h + M)^2}{2r_h e^{\phi_c}}; \qquad (95)$$

the corresponding temperature is thus

$$T_{\rm BH} = \frac{\hbar r_h e^{\phi_c}}{\pi(2r_h + M)^2}. \qquad (96)$$

By means of Eqs. (77), (81), and (86), this can be reduced to precisely the form (90). Thus far, the thermodynamic description based on Eq. (84) is consistent.

Of course, our black hole solution must exhibit an electric potential. By thermodynamics we would expect that [12] $\Phi_{\rm BH} = -T_{\rm BH}(\partial S_{\rm BH}/\partial Q)_m$. This gives

$$\Phi_{\rm BH} = \frac{Q}{G_N m + \sqrt{(G_N m)^2 - G_N Q^2}}, \qquad (97)$$

which agrees with the potential of the RN black hole in GR. To verify this result we shall also calculate the electric potential by a strictly mechanical approach using the conservation of energy. We expect the increase in the black hole's energy due to the fall into it of a charged particle to equal the particle's conserved (kinetic plus electric potential) energy.

We set out from the Lagrangian for a charged particle with mass μ and charge e,

$$\mathcal{L} = e^{-\phi_c}\left(-\mu\sqrt{-\tilde{g}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}} + e\tilde{A}_\alpha\frac{dx^\alpha}{d\tau}\right), \qquad (98)$$

where $\tilde{A}_\alpha$ is the potential for $\tilde{F}_{\alpha\beta}$. The $e^{-\phi_c}$ normalization will soon be justified. The Lagrangian does not depend on t; therefore, we have the conserved canonical momentum

$$P_t = \frac{\partial\mathcal{L}}{\partial\frac{dt}{d\tau}} = \mu e^{-\phi_c}\tilde{g}_{tt}\frac{dt}{d\tau} + e\tilde{A}_t e^{-\phi_c}. \qquad (99)$$

Asymptotically $\tilde{g}_{tt} \to e^{2\phi_c}$, so that $P_t \to \mu e^{\phi_c}\frac{dt}{d\tau} + e\tilde{A}_t e^{-\phi_c}$; now, since the physical global time is given by $\tilde{t} = e^{\phi_c} t$, the first term in P_t is recognized as minus the physical value of the particle's rest plus kinetic energy. This justifies the normalization we selected for the Lagrangian. We can thus identify $-e\tilde{A}_t e^{-\phi_c}$ as the physical value of the particle's electric energy. The physical electric potential of the black hole is inferred from this last energy at the horizon, $\Phi_{\rm BH} = -\tilde{A}_t(r_h)e^{-\phi_c}$.

We calculate $\tilde{A}_t$ by integrating the electric field (46) from infinity to r_h,

$$\tilde{A}_t(r_h) = \int_{r_h}^{\infty}\tilde{F}_{tr}dr = -\int_{r_h}^{\infty}\frac{Qe^{(1/2)(\tilde{\nu}-\tilde{\zeta})}}{r^2}dr$$
$$= -\frac{Qe^{2\phi_c}}{2r_h + M}. \qquad (100)$$

Substituting here the expression (86) for r_h in terms of M and Q, and switching to physical mass using Eq. (81), we finally get

$$\tilde{A}_t(r_h) = -\frac{Qe^{\phi_c}}{G_N m + \sqrt{(G_N m)^2 - G_N Q^2}}. \tag{101}$$

But we found the physical black hole electric potential to be $-\tilde{A}_t(r_h)e^{-\phi_c}$, so the present method of calculation gives exactly the same result, Eq. (97), as the thermodynamic computation.

The above calculations serve as a consistency check of the physical values for mass and charge which we attributed to the black hole. They also demonstrate the physical consistency of a thermodynamical description of spherical black holes in TeVeS. In particular, they justify our guess (84) for the form of the black hole entropy, and verify that the first law of black hole thermodynamics [12,13] holds for the TeVeS spherical black holes. All this is accomplished by referring all physics to the physical metric. However, there is one issue for which one must consider the role of the Einstein metric.

Recently DS [8] showed that, in a theory with Lorentz symmetry breaking via a time-dependent scalar field, in which there is more than one maximal propagation speed, it would be theoretically possible to construct a perpetuum mobile that would transfer heat from a colder to a hotter region. This would be accomplished, via Hawking radiation, by exploiting the different temperatures of the nested horizons corresponding to massless fields with different propagation speeds.

DS consider a static spherical black hole, and two fields, ψ_1 and ψ_2, which do not interact with each other except through gravity, and propagate at different speeds c_1 and c_2. Consequently there exist two distinct horizons, one for field ψ_1 that radiates *à la* Hawking with temperature T_1, and the second for ψ_2 radiating at temperature T_2. It is assumed that $c_2 > c_1$; the model then gives $T_2 > T_1$. DS assume the black hole is surrounded by two nested shells, shell A, which interacts with ψ_1 but is transparent to ψ_2, and shell B, which interacts only with ψ_2. Shell A has temperature T_A, and shell B is hotter at T_B. It is also assumed that $T_A > T_1$ and $T_B < T_2$. DS make the innocuous assumption that heat flows from higher to lower temperature, with the heat flow increasing monotonically with temperature difference, and vanishing only when the two temperatures are equal. Then they point out that heat will flow from A to the black hole via quanta of ψ_1 and from the black hole to B via ψ_2 particles. It is possible to adjust the shell temperatures so that the two mentioned flows become equal, in which case the black hole is in a steady state. Then the only overall effect is heat flow from A to B, that is, from cold to hot. The second law thus appears to be violated.

TeVeS also breaks local Lorentz symmetry, albeit by a different mechanism: it is equipped with a timelike nonvanishing future-pointing vector field. Further, TeVeS possesses two metrics; this feature implies different propagation velocities for light and for gravitational waves. Does the second law fail in TeVeS within some version of the DS scenario? In order to construct the appropriate version, we evidently first need to identify the distinct horizons associated with light and with gravitational waves. In TeVeS light propagates on the null cone of the physical metric $\tilde{g}_{\alpha\beta}$, and it is evident immediately from Eq. (68) that the horizon for light is at $r = r_h$. By contrast, tensor gravitational waves propagate on the null cone of the Einstein metric $g_{\alpha\beta}$ [3].

With ϕ given by the upper sign alternative of Eq. (69), the Einstein metric corresponding to the physical metric (68) is

$$g_{tt} = -e^{\tilde{\nu}-2\phi} = -\frac{(r-r_h)^{2-2\delta_-}(r+r_h)^{2-2\delta_+}}{(r^2+r_h^2+Mr)^{(2-\delta_- -\delta_+)}}, \tag{102}$$

$$g_{rr} = e^{\tilde{\zeta}+2\phi} = \frac{(r-r_h)^{2\delta_-}(r+r_h)^{2\delta_+}}{r^4(r^2+r_h^2+Mr)^{(\delta_- +\delta_+ -2)}}. \tag{103}$$

For this metric the horizon (for gravitational waves) could only be at $r = r_h$, the same location as the horizon for light. However, as measured with respect to $g_{\alpha\beta}$ that surface's area diverges: since $\delta_- \le 0$ for the physical K, k region, g_{rr} blows up at horizon, and since the metric is isotropic, this alone causes the area of the surface $r = r_h$ to blow up. The curvature scalars calculated with $g_{\alpha\beta}$ also diverge at r_h, revealing this location to be an essential singularity of the Einstein metric, and not a horizon. Thus we are in no position to form a well-defined entropy while working in the geometry perceived by gravitational waves. Likewise, we cannot obtain a black hole temperature: no entropy, no thermodynamic temperature. A similar problem arises in trying to calculate the temperature by applying the Euclidean prescription to the Einstein metric: the g_{tt} does not behave like l^2.

The appearance of a singularity of $g_{\alpha\beta}$ at the same surface as the physical metric's horizon does not pose an insurmountable problem. It was shown by Zlosnik, Ferreira, and Starkman [14] that TeVeS can be reformulated as a vector-tensor theory, with a single metric, the physical metric. Probably, the formal failure to bring out a thermodynamics in the Einstein metric reflects the fact that TeVeS is at the bottom a one-metric theory, with the Einstein metric being no more than a mathematical convenience.

The absence of a thermodynamic temperature for gravitational waves in the TeVeS black hole background most likely means that any Hawking-like emission of these waves is not thermal. The attempt, *à la* DS, to identify two distinct black hole temperatures for the same black hole, each tied to a different maximal propagation velocity, thus fails. However, it has been suggested to us that the paradox can still arise as follows. One associates with the black hole a *graviton* effective temperature τ in lieu of, say, T_2 in the DS scenario. This τ is defined as the temperature that the shell B, which interacts solely with gravitons, would have to possess in order to just balance the black hole's emission power in gravitons (though, of course, without any pretense of detailed balance). Now suppose τ exceeds T_1, the black hole's photon temperature. By suitably adjusting T_A and T_B while observing the ordering $\tau > T_B > T_A > T_1$, it should be possible to annul the overall energy gain of the black hole. Then we have a DS scenario where heat flows from the low temperature T_A to the higher T_B without any other change taking place. Of course if $\tau < T_1$, we have to arrange things with $\tau < T_B < T_A < T_1$ to get flow from T_B to the higher T_A. We emphasize that, with the introduction of the effective temperature τ, the propagation velocities and Lorentz symmetry violation no longer play the dominant role they played in the original DS argument.

In both of the above setups we seem to have a violation of the ordinary second law of thermodynamics. This can be avoided only if necessarily $\tau = T_1$, when the above schemes require $T_A = T_B$ so that no violation is possible. Our methods in this paper are not suitable for the study of Hawking-like radiation, so here we cannot affirm or exclude this possibility. But it seems a fair conjecture that, in fact, $\tau = T_1$.

Following the DS paper, Eling, Foster *et al.* [15] suggested a *classical* mechanism for violating the second law in its generalized form within a gravitational theory equipped with a timelike vector field that causes Lorentz symmetry violation. This mechanism also relies on two maximal propagation speeds, with $c_A > c_B$. It is implemented in a spherically symmetric static situation around a black hole. The scenario envisages a particle of type A and one of type B. They both fall through the horizon for particles of type A and interact with each other in a region still outside the horizon for type B particles, where the time Killing vector is spacelike with respect to the metric sensed by A particles. It is then possible for the interaction to make the A particle acquire negative energy while the B particle's energy remains positive. If now A falls through the B particle's horizon, and B escapes passing on its way out through A's horizon, the black hole's mass has been lowered in the process. Under mild assumptions, this means the black hole entropy decreases. But since the particle state may have been pure all along, there is no ordinary entropy to compensate, and so the generalized second law is violated.

We point out that in our black hole solution there is no intermediate region still outside the second horizon where the Killing vector already has positive norm. Equation (68) shows that in the region $r < r_h$ the Killing vector is indeed spacelike with respect to the physical metric. However, this region is not outside the null surface [also, $r = r_h$ but in the Einstein metric Eq. (102)] that might have been construed as the horizon for gravitons. Thus negative energy particles can be created, but only inside the black hole, and the scenario envisaged in Ref. [15] cannot be enacted.

VII. CONCLUSIONS AND SUMMARY

Here we have derived a pair of charged spherical static black hole solutions of TeVeS that, as far as the physical metric is concerned, resemble the RN solution of GR. The new features are the TeVeS vector field which points in the time direction, and the spherically symmetric TeVeS scalar field. We have shown that, for a wide range of TeVeS parameters, the scalar field of one solution is positive everywhere as long as the field has a modest positive cosmological value. This insures that superluminal propagation does not take place in that solution's background. Regarding the TeVeS analogue of Schwarzschild's black hole exhibited earlier by Giannios, we showed that there are actually two separate solutions here too. For one of them the evident positivity of the scalar field precludes superluminal propagation. This singles out the physical solution.

An element of guesswork entered in both Giannos' and our derivations. He had to guess Eq. (28); we had to guess the RN form of the physical metric. Thus in both cases it is not clear if the black hole solutions found are the unique ones. Proof of uniqueness in both cases is still at large.

By expressing the parameters of the charged black hole in terms of physical attributes measurable by material Minkowski observers, we calculated the entropy, temperature, and electric potential characterizing the black hole. They turn out to be the same as for GR's RN black hole. Black hole entropy and temperature cannot be consistently defined for the Einstein metric, which would have been the correct framework for studying Hawking emission of gravitational waves. We consider in this context a modified version of the Dubovsky-Sibiryakov [8] scenario for bringing about a violation of the second law of thermodynamics out of Lorentz symmetry breaking. This violation of the second law can be forestalled if a conjectured equality of the effective graviton radiation temperature and the photon Hawking temperature hold. The scenario described by Eling, Foster *et al.* [15] for bringing about classical violations of the second law in theories with Lorentz symmetry violation cannot be implemented with our TeVeS black hole solutions.

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