Constraining the dark energy and smoothness parameter with supernovae

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The presence of inhomogeneities modifies the cosmic distances through the gravitational lensing effect, and, indirectly, must affect the main cosmological tests. Assuming that the dark energy is a smooth component, the simplest way to account for the influence of clustering is to suppose that the average evolution of the expanding Universe is governed by the total matter-energy density whereas the focusing of light is only affected by a fraction of the total matter density quantified by the α Dyer-Roeder parameter. By using two different samples of SNe type Ia data, the Ω_m and α parameters are constrained by applying the Zeldovich-Kantowski-Dyer-Roeder luminosity distance-redshift relation for a flat (ΛCDM) model. A χ^2-analysis using the 115 SNe Ia data of the Astier *et al.* sample (2006) constrains the density parameter to be $\Omega_m = 0.26^{+0.17}_{-0.07}$ (2σ) while the α parameter is weakly limited (all the values $\in [0, 1]$ are allowed even at 1σ). However, a similar analysis based the 182 SNe Ia data of Riess *et al.* (2007) constrains the pair of parameters to be $\Omega_m = 0.33^{+0.09}_{-0.07}$ and $\alpha \geq 0.42$ (2σ). Basically, this occurs because the Riess *et al.* sample extends to appreciably higher redshifts. As a general result, even considering the existence of inhomogeneities as described by the smoothness α parameter, the Einstein-de Sitter model is ruled out by the two samples with a high degree of statistical confidence (11.5σ and 9.9σ, respectively). The inhomogeneous Hubble-Sandage diagram discussed here highlights the necessity of the dark energy, and a transition deceleration/accelerating phase at $z \sim 0.5$ is also required.

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I. INTRODUCTION

The Hubble-Sandage diagram for type Ia supernovae (hereafter SNeIa), as measured by the Supernova Cosmology Project [1] and the High-z Supernova Search Team [2], provided the first evidence that the present Universe is undergoing a phase of accelerating expansion driven by an exotic component with negative pressure (in addition to the cold dark matter), usually called dark energy.

The idea of a dark energy-dominated universe is a direct consequence of a convergence of independent observational results, and constitutes one of the greatest challenges for our current understanding of fundamental physics [3]. Among a number of possibilities to describe this dark energy component, the simplest and most theoretically appealing way is by means of a cosmological constant Λ, which acts on the Einstein field equations as an isotropic and homogeneous source with a constant equation of state, $w \equiv p/\rho = -1$.

Although cosmological scenarios with a Λ term might explain most of the current astronomical observations, from the theoretical viewpoint they are plagued with at least a fundamental problem, namely, it is really difficult to reconcile the small value of the vacuum energy density required by observations ($\simeq 10^{-10}$ erg/cm^3) with estimates from quantum field theories ranging from 50–120 orders of magnitude larger [4]. This problem, sometimes called the cosmological constant problem, has inspired many authors to propose decaying Λ models [5] and other alternative approaches for describing dark energy [6]. Nevertheless, the present cosmic concordance model (CCM) which is supported by all the existing observations is a flat ΛCDM cosmology with a matter fraction of $\Omega_{\rm m} \sim 0.26$ and a vacuum energy contribution of $\Omega_\Lambda \sim 0.74$ [7–10].

On the other hand, the real Universe is not perfectly homogeneous, with light beams experiencing mass inhomogeneities along their way thereby producing many observable phenomena. For instance, light lines traversing in the Universe are attracted and refracted by the gravitational force of the galaxies on their path, which bring us the signal of lensing, one of which is the multiple images of a single far galaxy [11,12]. Nowadays, gravitationally lensed quasars and radio sources offer important probes of cosmology and the structure of galaxies. The optical depth for lensing depends on the cosmological volume element out to moderately high redshift. In this way, lens statistics can in principle provide valuable constraints on the cosmological constant or, more generally, on the dark energy density and its equation of state [13–15].

In this context, one of the most important issues in the modern cosmology is to quantify from the present observations the influence of such inhomogeneities on the evolution of the Universe. An interesting possibility to account for such effects is to introduce the smoothness parameter α which represents the magnification effects experienced by the light beam. When $\alpha = 1$ (filled beam), the Friedmann-Robertson-Walker (FRW) case is fully recovered; $\alpha < 1$ stands for a defocusing effect; $\alpha = 0$ represents a totally

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clumped universe (empty beam). The distance relation that takes the mass inhomogeneities into account is usually named Dyer-Roeder distance [16], although its theoretical necessity had been previously studied by Zeldovich [17] and Kantowski [18]. In this way, we label it here as the Zeldovich-Kantowski-Dyer-Roeder (ZKDR) distance formula (for an overview on cosmic distances taking into account the presence of inhomogeneities, see the paper by Kantowski [19]).

Several studies involving the ZKDR distances in dark energy models have been published in the past few years. Useful analytical expressions for ΛCDM models have been derived by Kantowski *et al.* [20,21] and Demianski *et al.* [22]. Working in the empty beam approximation ($\alpha = 0$), Sereno *et al.* [23] investigated some effects of the ZKDR distance for a general background. By assuming that both dominant components may be clustered, they also discussed the critical redshift, i.e., the value of z for which $d_A(z)$ is a maximum [or $\Theta(z)$ minimum], and compared to the homogeneous background results as given by Lima and Alcaniz [24], and further discussed by Lewis and Ibata [25]. Demianski and co-workers derived an approximate solution for a clumped concordance model valid on the interval $0 \leq z \leq 10$. Additional studies on this subject are related to time delays [25,26], gravitational lensing [27,28], and even accelerated models driven by particle creation have been investigated [29].

In a previous paper [30], we have applied the ZKDR equation in the framework of phantom cosmology in order to determine cosmological constraints from a sample of milliarcsecond compact radio sources. By assuming a Gaussian prior on the matter density parameter, i.e., $\Omega_m = 0.3 \pm 0.1$, the best fit model for a phantom cosmology with $\omega = -1.2$ occurs at $\Omega_m = 0.29$ and $\alpha = 0.9$ when we marginalize over the characteristic size of the compact radio sources. Such results suggested that the ZKDR distance can give important corrections to the so-called background tests of dark energy. In this article, the pair of cosmic parameters, $\Omega_m \equiv 1 - \Omega_\Lambda$ and α, are constrained from supernovae observations by applying the ZKDR distance-redshift relation for a flat (ΛCDM) model. As we shall see, the α parameter is not well constrained by the 115 SNe observed by Astier *et al.* (2006). However, the 182 SNe type Ia sample of Riess *et al.* (2007) constrains the pair of parameters to be $\Omega_m = 0.33^{+0.09}_{-0.07}$ and $\alpha \geq 0.42$ (2σ). As a general result, even considering the existence of inhomogeneities described by the α parameter, the Einstein-de Sitter model is ruled out by the two samples with a high degree of statistical confidence (11.5σ and 9.9σ, respectively).

The paper is organized as follows. In Sec. II, we present the basic equations and the distance description taking into account the inhomogeneities as described by the ZKDR equation. In Sec. III, we determine the constraints on the cosmic parameters from the two supernovae samples. Finally, we summarize the main conclusions in Sec. IV.

II. ZKDR EQUATION FOR LUMINOSITY DISTANCE

In a clumpy universe model, the local geometry is inhomogeneous, but its global aspect can be described by the FRW type geometry ($c = 1$)

$$ds^2 = dt^2 - R^2(t)(dr^2 + r^2 d\Omega^2), \tag{1}$$

where $R(t)$ is the scale factor and $d\Omega^2$ denotes the metric in the 2-sphere.

As it is widely known, the idea of a clumpy universe is still an ill-defined notion since we do not have a clear mathematical recipe to separate the global properties from the local inhomogeneous aspects of the Universe. After Dyer and Roeder [16], it is usual to introduce a phenomenological parameter, $\alpha = 1 - \frac{\rho_{cl}}{\langle\rho_m\rangle}$, called the "smoothness" parameter. Such a parameter quantifies the portion of matter in clumps (ρ_{cl}) relative to the amount of background matter which is uniformly distributed (ρ_m). In general, due to the structure formation process, it should be dependent of the redshift, as well as on the direction along the line of sight (see, for instance, [14,30] and references therein). However, in the majority of the works α is assumed to be a constant parameter. From a mathematical viewpoint the treatment is based on the optical-scalar equation for light propagation in the so-called geometric optics approximation [11,31]

$$\sqrt{A}'' + \tfrac{1}{2} R_{\mu\nu} k^\mu k^\nu \sqrt{A} = 0, \tag{2}$$

where a prime denotes differentiation with respect to the affine parameter λ, A is the cross-sectional area of the light beam, $R_{\mu\nu}$ the Ricci tensor, and k^μ the photon four-momentum. In this form, it is implicit that the influence of the Weyl tensor (shear) can be neglected. This means that the light rays are propagating far from the mass inhomogeneities so that the large-scale homogeneity implies that their shear contributions are canceled. The proportionality factor between the cross-sectional length $A^{1/2}$ and the angular distance d_A can be defined to be constant. Actually, the above optical-scalar equation is usually written in terms of the dimensionless angular-diameter distance $D_A = H_0 d_A$. Further, by recalling the existence of a simple relation between the luminosity distance, and the angular-diameter distance (from Etherington principle [32], $d_L = (1+z)^2 d_A$), it is easy to show that the ZKDR (dimensionless) luminosity distance for ΛCDM cosmology satisfies the following differential equation [20–23,26]:

$$(1+z)^2 \mathcal{F} \frac{d^2 D_L}{dz^2} - (1+z)\mathcal{G}\frac{dD_L}{dz} + \mathcal{H} D_L = 0, \tag{3}$$

which satisfies the boundary conditions:

$$\begin{cases} D_L(0) = 0, \\ \frac{dD_L}{dz}|_0 = 1. \end{cases} \tag{4}$$

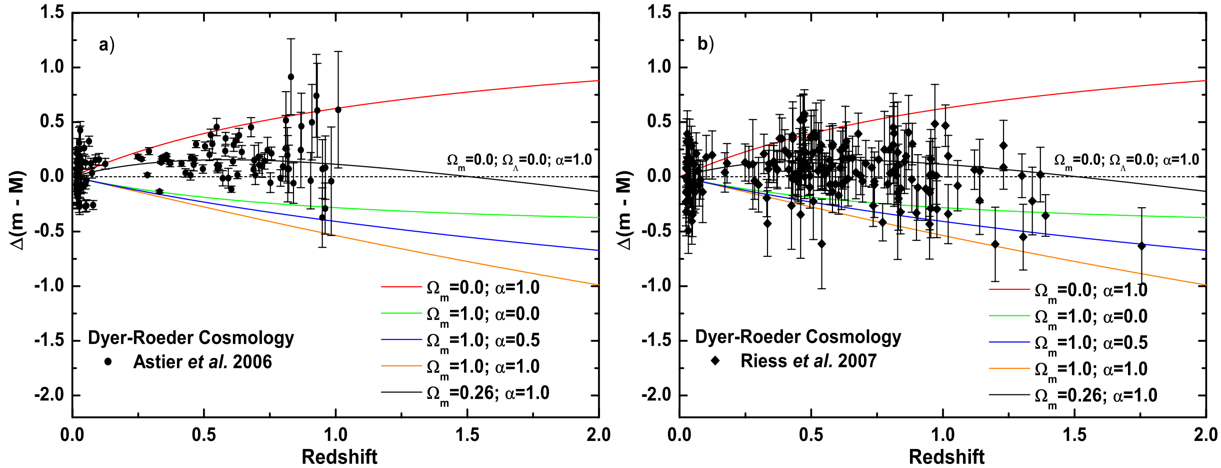

FIG. 1 (color online). The α-effect on the residual magnitudes. In (a) we show the 115 supernovae data from Astier *et al.* [7], and the predictions of the ZKDR luminosity distance for several values of α relative to an empty model ($\Omega_m = 0$, $\Omega_\Lambda = 0$, and $\alpha = 1$). In (b) we show the same graph but now for the 182 SNe type Ia from the Riess *et al.* sample [9]. For comparison, in both panels we see (black curves) the prediction of the cosmic concordance model ($\Omega_m = 0.26$, $\Omega_\Lambda = 0.74$, $\alpha = 1$).

where $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are functions of the cosmological parameters, expressed in terms of the redshift by

$$\begin{aligned} \mathcal{F} &= \Omega_m + (1-\Omega_m)(1+z)^{-3}, \\ \mathcal{G} &= \frac{\Omega_m}{2} + 2(1-\Omega_m)(1+z)^{-3}, \\ \mathcal{H} &= \left(\frac{3\alpha-2}{2}\right)\Omega_m + 2(1-\Omega_m)(1+z)^{-3}, \end{aligned} \tag{5}$$

where as remarked before, the α parameter appearing in the $\mathcal{H}$ expression (here assumed to be a constant) quantifies the clustered fraction of the pressureless matter.

III. SAMPLES AND RESULTS

The standard FRW models contain only homogeneously and isotropically distributed perfect fluid gravity sources, and the present CCM is assumed to represent both the "large-scale" geometry of the Universe and the matter content. However, the Universe appears homogeneous only in a statistical sense, when one is describing the largest scales. Therefore, although making very useful predictions, our cosmological models are somewhat inadequate at small and moderate scales. This means that relations like $\mu(H_0, \Omega_m, \Lambda; z)$, the distance modulus for a standard candle, commonly assumed to be valid on average can be incorrect even for observations including SNe Ia. In particular, if the underlying mass density approximately follows luminous matter (i.e., associated with bounded galaxies), the effects of inhomogeneities on relations like $\mu(H_0, \Omega_m, \Lambda, z)$ must be taken into account.

In Fig. 1, we display the effects of the inhomogeneities in the reduced Hubble-Sandage diagram for the Astier *et al.* (2006) and Riess *et al.* (2007) samples for some selected values of the smoothness parameter. The plots correspond to several values of Ω_m and α as indicated in the panels. The difference between the data and models from an empty universe case (OCDM) prediction is also displayed there. For the sake of comparison, we also show the Einstein-de Sitter model, i.e. $\Omega_m = 1$ and $\alpha = 1$, as well as the present cosmic concordance ($\Omega_m = 0.26$, $\Omega_\Lambda = 0.74$, $\alpha = 1$). Note that the α parameter contributes in the right direction i.e., the SNe type Ia become dimmer when it increases on the allowed range. In what follows, a χ^2 minimization will be applied for the two sets of SNe data with the parameters Ω_m and α spanning the interval [0,1] in steps of 0.01, for all numerical computations.

A. Astier *et al.* sample (2006)

Let us now discuss the bounds arising from SNe Ia observations on the pair of parameters (Ω_m, α) defining the ZKDR luminosity distance.

The current data from Supernova Legacy Survey (SNLS) collaboration correspond to the first year results of its planned five-year survey. The total sample includes 71 high-z SNe Ia in the redshift range $0.2 < z < 1$ plus 44 low-z SNe Ia as published by Astier *et al.* [7]. Although in a better agreement with WMAP 3-year results [8] than the *gold* sample [9] (for a more detailed discussion see e.g., Jassal *et al.* [33]), the most distant SNe Ia of these 115 events has redshift smaller than unity.

Following standard lines, the maximum likelihood estimator, $\mathcal{L}_{\rm SNIa} \propto \exp[-\chi^2_{\rm SNIa}(z;\mathbf{p})/2]$, is determined by a χ^2 statistics

$$\chi^2_{\rm SNIa}(z|\mathbf{p}) = \sum_i \frac{(\mu(z_i;\mathbf{p}) - \mu_{0,i})^2}{\sigma^2_{\mu_{0,i}} + \sigma^2_{\rm int}}, \tag{6}$$

where $\mathbf{p} \equiv (H_0, \alpha, \Omega_m)$ is the complete set of parameters that we want to fit, $\sigma_{\mu_{0,i}}$, $\sigma_{\rm int}$ are, respectively, the errors

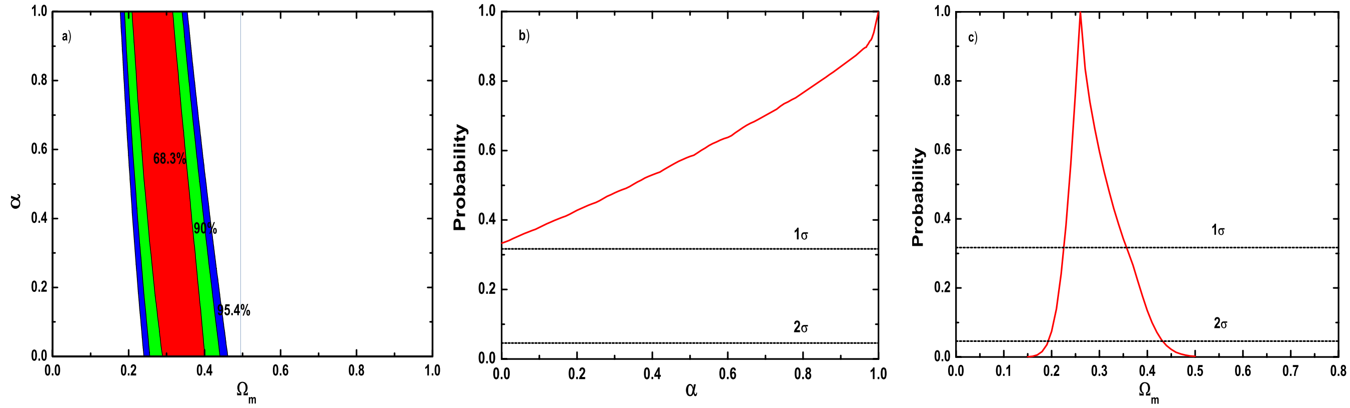

FIG. 2 (color online). (a) The $\Omega_m - \alpha$ plane for flat ΛCDM models obtained from 115 SNe Ia data Astier *et al.* [7]. Note that the α parameter is not well constrained by the data. (b) The likelihood for the α smoothness parameter. We see that even at 1σ the smoothness parameter is poorly restricted (all its admissible values are allowed). (c) Probability of the matter density parameter. We see that $0.19 \leq \Omega_m \leq 0.43$ with 2σ confidence level.

associated with the observational techniques in determining the distance moduli (includes a peculiar contribution) and the intrinsic dispersion of SNe Ia. The corresponding errors are reported in the paper by Astier *et al.* [7].

Marginalizing our likelihood function over the nuisance parameter, H_0, we obtain the likelihood function for the $\Omega_m - \alpha$ plane. In order to determine the cosmological parameters (Ω_m, α), a χ^2 minimization for the range of [0,1] in steps of 0.01 has been applied. The 68.3%, 90.0%, and 95.4% confidence levels are defined by the conventional two-parameters χ^2 levels 2.30, 4.61, and 6.17, respectively. It is very important to note that we do not consider any prior in Ω_m, as usually required by the SNe Ia test. The basic results are shown in Figs. 2(a)–2(c). From Fig. 2(a) we see that all of the range for α is accepted, while a $\Omega_m \approx 0.3 \pm 0.1$ is obtained. In Fig. 3(b) we see the likelihood for the smoothness parameter. The best fit adjustment occurs for values of $\alpha = 1.0$ and $\Omega_m = 0.26$ with $\chi^2_{\rm min} = 113.3$ and $\nu = 113$ degrees of freedom ($\chi^2/\nu = 1$), thereby showing that the model provides a good fit to these data. It is also interesting that, for any α value, we also find no evidence for a high Ω_m parameter as required by a flat Einstein-de Sitter universe ($\Omega_\Lambda = 0$). Actually, the Einstein-de Sitter scenario has a very small statistical significance $\chi^2 = 244.9$ (11.5σ outside). However, since the Astier *et al.* data are not restrictive for the α parameter, let us now consider the enlarged SNeIa sample observed by the High-z Supernovae Search Team [9].

B. Riess *et al.* sample (2007)

The so-called *gold* sample from the HZS team [9] is a selection of 182 SNe Ia events distributed over the redshift interval $0.01 \lesssim z \lesssim 1.755$, and constitutes the compilation

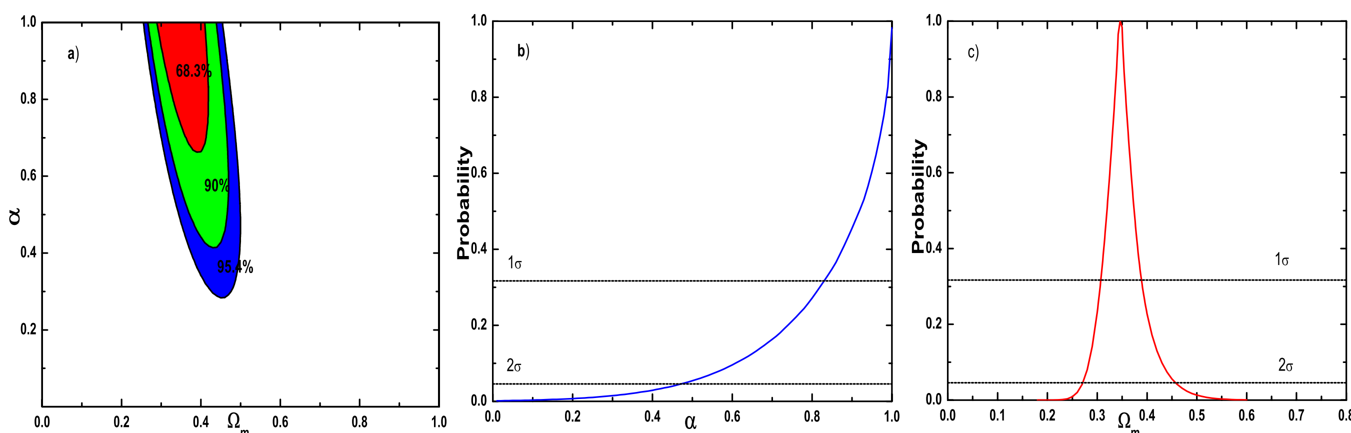

FIG. 3 (color online). (a) Confidence contours on the (Ω_m, α) plane for flat ΛCDM models as inferred from 182 SNe Ia measurement by Riess *et al.* [9]. (b) The likelihood function for the α smoothness parameter. We see that at 2σ the smoothness parameter is restricted on the interval ($0.42 \leq \alpha \leq 1.0$). (c) Probability of the matter density parameter. In this case a comparatively small region is permitted $0.25 \leq \Omega_m \leq 0.44$ with (2σ) confidence level.

of the best observations made so far by them and by the Supernova Cosmology Project events observed by the Hubble Space Telescope (HST). As before, constraints on the cosmological parameters (Ω_m, α), are determined from a χ^2 minimization within the range of [0,1] spanned by such parameters.

In Fig. 3(a), one can see that $0.42 \leq \alpha \leq 1.0$ and $0.25 \leq \Omega_m \leq 0.44$ with 90% statistical confidence. The best fit adjustment occurs for values of $\Omega_m = 0.33$ and $\alpha = 1$ with $\chi^2_{\rm min} = 158.6$ and $\nu = 180$ degrees of freedom the reduced ($\chi^2/\nu \sim 0.9$). Therefore, the model provides a very good fit to the Riess *et al.* sample. In Fig. 3(b) we see the likelihood for the smoothness parameter. As previously remarked, the Riess *et al.* data set is much more restrictive for the smoothness parameter than the Astier *et al.* sample. Within 2σ, the allowed range for the α falls on the interval $0.42 \leq \alpha \leq 1.0$ (cf. Fig. 2(a)]. In Fig. 3(c) we show the probability for the density matter parameter. In this case a small region is permitted $0.25 \leq \Omega_m \leq 0.44$ with of the confidence level (2σ). In the analysis for the Einstein-de Sitter universe ($\Omega_m = 1.0$, $\Omega_\Lambda = 0.0$), the $\chi^2 = 255.8$ is too bad (9.9σ C.L. outside for 1 degree of freedom), and guarantees us to exclude this model with high confidence.

IV. COMMENTS AND CONCLUSIONS

Cosmology is in an exciting period. A considerable set of rather sophisticated experiments, until a few years ago regarded as futuristic, have now been completed with spectacular success. The results of the first observations almost one decade ago have been confirmed what was long surmised, namely, that most of the matter is nonbaryonic and that we live in an accelerating expanding universe dominated by dark energy.

In this article, we have discussed the influence of inhomogeneities on the expansion rate of the Universe, and, in particular, if the smoothness α parameter could be constrained through a statistical analysis involving two large sets of SNe Ia data. As we have seen, in the case of the Astier *et al.* sample, the entire interval of α is allowed while a $\Omega_m \approx 0.3 \pm 0.1$ is obtained. Within the existing uncertainties, these results are consistent with the constraints obtained from the angular diameter of compact radio sources with a basis on the Gurvits *et al.* data [30,34]). Therefore, although in close agreement with rather different analysis, this SNe data set is incapable of constraining the smoothness parameter. Actually, at this moment, the sample of Riess *et al.* provides a more stringent constraint with the allowed range for α falling on the interval $0.42 \leq \alpha \leq 1.0$ (2σ). Basically, this occurs because the Riess *et al.* sample extends to appreciably higher redshifts. In general, both analyses suggest that a large range for α is permitted, and that the Einstein-de Sitter model is strongly excluded (11.5σ and 9.9σ, respectively). As we have seen, the necessity of the dark energy and a transition from deceleration to an accelerating phase is maintained even when one takes into account the clustering phenomenon. However, at the level of the SNe Ia observations discussed here, these results suggest that the clumpiness of matter distribution can mimic at least a small fraction of the dark energy component.

Finally, we would like to stress that measurements from SNe Ia combined with the ZKDR inhomogeneous approach adopted here may provide an independent and more rigorous cosmological test for the cosmic concordance model in the near future. In this concern, it should be very important to investigate whether the α parameter can be constrained using independent observations, among them: the cosmic microwave background anisotropies, the physics of galaxies clusters, Sunyaev-Zeldovich effect, time delay, and statistical of gravitational lensing. Some studies along these lines will be presented in a forthcoming communication. The present results based only on the Hubble-Sandage diagram show that the Riess *et al.* sample is more restrictive than the Astier *et al.* sample, thereby reinforcing the interest to observe more supernova events at higher redshifts.

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