Superinflation in loop quantum cosmology

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We investigate the dynamics of superinflation in two versions of loop quantum cosmology, one in which the Friedmann equation is modified by the presence of inverse volume corrections, and one in which quadratic corrections are important. Computing the tilt of the power spectrum of the perturbed scalar field in terms of fast-roll parameters, we conclude that the first case leads to a power spectrum that is scale invariant for steep power law negative potentials and for the second case, scale invariance is obtained for positive potentials that asymptote to a constant value for large values of the scalar field. It is found that in both cases, the horizon problem is solved with only a few e-folds of superinflationary evolution.

I. INTRODUCTION

An inflationary epoch is currently the most promising model for the origin of large-scale structure in the universe [1]. The predictions of inflation are fully compatible with the most recent observations suggesting that structure originated from a pattern of near scale-invariant, Gaussian and adiabatic primordial density fluctuations [2]. Despite these successes, however, a number of important questions remain. In particular, in what fundamental theory will inflation be seen to arise? Having asked the question, it is worth noting that there have been successful implementations of inflation both in the context of ordinary field theory [3,4] as well as in the context of string and M-theory [3,5–14]. Given the importance of inflation and the need to explore all possibilities of accommodating it in alternative theories of quantum gravity, in this paper we turn our attention to inflation in the context of loop quantum gravity (LQG) [15].

LQG is a background independent and nonperturbative canonical quantisation of general relativity based on Ashtekar variables: su(2) valued connections and conjugate triads. The variables used in the quantisation scheme are then holonomies of the connection and fluxes of the triad. The restriction of LQG to symmetric states gives rise to loop quantum cosmology (LQC) [16]. Although it is a particular limit of the more general LQG, and therefore can not be said to have generic features, LQC has produced a number of intriguing results and resolved many problematic issues present in the earlier Wheeler de Witt quantum cosmology. In particular, LQC can lead to a nonsingular quantum evolution [17], with the origin of the nonsingular behavior being traced to the methods used to quantize inverse triad operators in LQG [18,19].

While the consequences of this quantum evolution are fascinating, it is difficult to connect it to existing theories of the early universe which tend to be based on classical dynamics, and, in particular, to inflation. Therefore, an approach to LQC has been developed in which effective or "semiclassical" equations are derived and studied. In the isotropic setting, the effective equations which have been studied predominately to date include high energy modifications to the classical dynamics which originate from the spectra of quantum operators related to the inverse scale factor [19–22]. In the context of scalar field driven inflation, a number of important effects follow from these modifications. These include the possibility that the field can be excited up its self-interaction potential [23–28], leading to a subsequent period of slow-roll inflation. Most intriguing of all, however, is the presence of a superinflationary phase which occurs during the early phases of the universe's evolution irrespective of the form of the field's potential [21,29].

More recently, however, a further semiclassical modification, which arises from the use of holonomies as a basic variable in the quantization scheme, has been derived in the isotropic setting [30]. The modification is remarkably simple and takes the form of an additional negative ρ^2 term in the effective Friedmann equation, which appears in addition to the usual positive ρ term. Such a term has a number of effects, it forces a collapsing universe to undergo a nonsingular bounce once a critical density is reached, and immediately after this bounce it also causes the universe to undergo a period of superinflation. It is intriguing to note that such a term also appears in braneworld models with an extra timelike dimension [31].

While the two sets of modifications discussed above have rather different origins, it appears that the qualitative effects of both the inverse scale factor effects and the ρ^2 term are rather similar. In particular, they both give rise to a period of superinflation during which the Hubble factor rapidly increases, rather than remaining nearly constant as is the case during standard slow-roll inflation. Given that such a superaccelerating phase appears to be a robust prediction of LQC, it is important to study both the background dynamics, and particularly the cosmological per-

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turbations, which such a phase gives rise to. Considering a universe sourced by a scalar field, a number of important results have already been obtained. A scaling solution for the effective equations which arise from the inverse scale factor modifications has been derived [32], and a number of attempts made at studying perturbations in the super-inflationary regime [33,34]. Similarly the scaling solution for the ρ^2 effective equation has also been derived [35,36].

It is also interesting to note the close connections between the superinflationary phases in LQC and the evolution of a universe sourced by a phantom field, and with the ekpyrotic evolution of a collapsing universe [37–40]. In all these cases the magnitude of the Hubble rate grows with time. Moreover, the scale factor duality discussed in [41] maps the ekpyrotic collapse onto the superinflationary scaling solution for the inverse scale factor modified equations. On the other hand, another duality maps the ekpyrotic collapse phase onto the dynamics of a universe sourced by a phantom field [41]. These three regimes are therefore all related to one another. Furthermore, given that the ekpyrotic collapse is thought to offer a method for the generation of scale-invariant perturbations [42] (as is the dual superinflationary phase sourced by a phantom field [43]), it is reasonable to expect that a similar mechanism may operate in the superinflationary phases of LQC. Indeed such a mechanism has been discussed previously [34], though its relation to ekpyrotic and phantom models was not emphasized.

In this study we aim to explore further the phenomenology of superinflation in LQC. One complication, however, is that the relative status of the two sets of modifications discussed is at present unclear. We therefore take a pragmatic approach and study the dynamics when each of the modifications is considered in turn, but not including both sets of modifications simultaneously, although we believe it should not be too difficult to incorporate them both. A further difficulty is that despite considerable progress the understanding of metric perturbations in LQC is at present incomplete [44]. We therefore restrict our attention to perturbations in the scalar field as a first approximation. This approach allows us to establish a framework for dealing with perturbations in LQC in which metric perturbations can be incorporated as our understanding advances.

The paper is organised as follows. In Sec. II, we introduce the cosmological evolution equations which arise in LQC including the inverse volume corrections. Solutions are obtained including those showing scaling behavior, and the primordial spectrum of scalar perturbations is calculated for each solution in terms of "fast-roll" parameters. The stability of these solutions is then discussed. In Sec. III, we analyze the dynamics when the modification is induced by a ρ^2 correction to the Friedmann equation. Concentrating on the evolution just after the bounce, we demonstrate the existence of the superinflationary solution, obtain the scaling dynamics of the system, the primordial

spectrum of scalar perturbations as well as the stability of the background solutions. Section IV discusses the way in which superinflation in LQC can solve the horizon problem with a small number of e-foldings and we conclude in Sec. V.

II. EFFECTIVE FIELD EQUATIONS WITH LQC INVERSE VOLUME CORRECTIONS

The first set of modified equations which we consider are those which incorporate two functions, $D_{j,l}(a)$ and $S_j(a)$, into the dynamics. These functions arise because of the presence of powers of the inverse scale factor in the Hamiltonian constraint for an isotropic and homogeneous universe. A full discussion of the origin of these terms can be found in Refs. [20,22] (a summary can be found in appendix B of Ref. [45]), but here we simply state their basic properties. We are implicitly considering either positively curved or topologically compact models. This ensures that the size of the fiducial cell does not enter in the equations of motion. D and S are both functions of the scale factor, and their form changes depending on the values of two ambiguity parameters: l which takes values in the range $0 < l < 1$, and j which takes half integer values. When the scale factor approaches zero, D and S also approach zero, whereas as a increases above the critical value $a_\star$ which depends on j, they both tend to unity.

The modified Friedmann equation is given by

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}S\left(\frac{\dot{\phi}^2}{2D} + V(\phi)\right), \tag{1}$$

where a dot denotes differentiation with respect to cosmic time t, and $\kappa^2 = 8\pi G$. In what follows we choose units in which $\kappa = 1$. We have omitted the curvature contribution as we assume either a compact flat universe or that the curvature term rapidly becomes subdominant and that it can be safely neglected. The equation of motion for the scalar field takes the form

$$\ddot{\phi} + 3H\left(1 - \frac{1}{3}\frac{d\ln D}{d\ln a}\right)\dot{\phi} + DV_{,\phi} = 0, \tag{2}$$

A subscript ϕ means differentiation with respect to the field. These equations can also be combined to give the Raychaudhuri equation

$$\dot{H} = -\frac{S\dot{\phi}^2}{2D}\left[1 - \frac{1}{6}\frac{d\ln D}{d\ln a} - \frac{1}{6}\frac{d\ln S}{d\ln a}\right] + \frac{SV}{6}\frac{d\ln S}{d\ln a}. \tag{3}$$

A. Scaling dynamics

We will be interested in the regime $a \ll a_\star$, where the function $D_{j,l}(a)$ may be approximated by a power law of the form $D(a) = D_\star a^n$, with $D_\star = (3/(3 + 2l))^{3/2(1-l)}a_\star^{3(l-3)/(1-l)}$ and $n = 3(3 - l)/(1 - l)$ takes values in the range $9 < n < \infty$. Likewise, the function $S(a)$

may be similarly approximated by $S(a) = S_\star a^r$, where $S_\star = (3/2)a_\star^{-3}$ and $r = 3$, though we keep r arbitrary in our calculations for generality. For $a \gg a_\star$, $S_\star \approx D_\star \approx 1$ and $r = n = 0$. Inserting this form for the functions S and D into Eq. (3), we can clearly see that for an expanding universe, and with $n > 6 + r$ which occurs for all l, $\dot{H}$ is necessarily positive (assuming that the potential is either positive or the term involving SV can be neglected). Hence superinflation is occurring. We will confine ourselves to these situations in what follows.

To study this regime further, it proves convenient to introduce the variables

$$x \equiv \frac{\dot{\phi}}{\sqrt{2D\rho}}, \qquad y \equiv \frac{\sqrt{|V|}}{\sqrt{\rho}}, \qquad (4)$$

where $\rho \equiv \dot{\phi}^2/2D + V(\phi)$. Using these definitions, the equation of motion for the scalar field (2) can be written for an expanding universe in terms of a system of first order differential equations as

$$x_{,N} = -3\alpha x \pm \sqrt{\tfrac{3}{2}}\lambda y^2 + 3\alpha x^3, \qquad (5)$$

$$y_{,N} = -\sqrt{\tfrac{3}{2}}\lambda xy + 3\alpha x^2 y, \qquad (6)$$

$$\lambda_{,N} = -\sqrt{6}\lambda^2(\Gamma - 1)x + \tfrac{1}{2}(n - r)\lambda, \qquad (7)$$

where

$$\lambda \equiv -\sqrt{\frac{D}{S}}\frac{V_{,\phi}}{V}, \qquad \Gamma \equiv \frac{VV_{,\phi\phi}}{V_{,\phi}^2}, \qquad (8)$$

with $\alpha = 1 - n/6 < 0$ and $N = \ln a$. These variables are subject to the constraint equation

$$x^2 \pm y^2 = 1. \qquad (9)$$

The plus and minus signs correspond to positive and negative potentials, respectively. Using the constraint equation to substitute for y in Eq. (5) renders Eq. (6) redundant.

The resulting system defined by Eqs. (5) together with the constraint equation and (7) has three fixed points for $\lambda \neq 0$. Two of them represent kinetic energy dominated solutions, valid for all values of λ:

$$x = -1, \qquad y = 0, \qquad \Gamma = 1 - \frac{\sqrt{6}}{12\lambda}(n - r), \quad (10)$$

$$x = +1, \qquad y = 0, \qquad \Gamma = 1 + \frac{\sqrt{6}}{12\lambda}(n - r), \quad (11)$$

and the third point is a scaling solution for which the kinetic and potential energies evolve in a constant ratio to one another:

$$x = \frac{\lambda}{\sqrt{6}\alpha}, \qquad y = \sqrt{\pm\left(1 - \frac{\lambda^2}{6\alpha^2}\right)}, \qquad (12)$$

$$\Gamma = 1 + \frac{\alpha}{2\lambda^2}(n - r). \qquad (13)$$

The scaling solution is therefore well defined for $\lambda^2 < 6\alpha^2$ for positive potentials and for $\lambda^2 > 6\alpha^2$ for negative potentials. In the remainder of this analysis, we will focus mainly on the scaling solution for negative potentials as this is the case that, as we shall see, leads to a scale-invariant power spectrum of the perturbed field. For this case one can check that the universe is undergoing super-inflationary expansion.

Considering the fixed point for the scaling solution (12), one can write

$$\frac{\dot{\phi}}{\sqrt{2D\rho}} = \sqrt{\frac{S}{D}}\frac{\phi_{,N}}{\sqrt{6}} = \frac{\lambda}{\sqrt{6}\alpha}, \qquad (14)$$

which upon integration gives

$$\phi = \frac{2\lambda}{(n - r)\alpha}\sqrt{\frac{D}{S}}, \qquad (15)$$

where we have set the integration constant to zero without loss of generality.

Then inserting this relation into the definition of λ in (8) gives

$$V = V_0\phi^\beta, \qquad (16)$$

where $\beta = -2\lambda^2/(n - r)\alpha > 0$.

Considering now the fixed point for y we have

$$\frac{V}{\rho} = \frac{VS}{3H^2} = 1 - \frac{\lambda^2}{6\alpha^2}. \qquad (17)$$

Differentiating Eq. (15) and eliminating $\dot{\phi}$ using Eq. (12), then substituting for ρ in terms of V using Eq. (17) and finally substituting for V in terms of a using Eqs. (15) and (16), one obtains an expression between a and $\dot{a}$. Then by integrating this expression we can determine that the scale factor evolves as a power law in time while the universe evolves according to the scaling solution. In terms of conformal time $dt = ad\tau$, we find

$$a(\tau) = (-\tau)^p, \qquad (18)$$

where for an expanding universe τ is negative and increasing towards zero and

$$p = \frac{2\alpha}{2\bar{\epsilon} - (2 + r)\alpha}, \qquad (19)$$

where, for direct comparison with previous literature, we have introduced the slow-roll parameter $\bar{\epsilon} \equiv \lambda^2/2$ and $\lambda = -\sqrt{2\bar{\epsilon}}$ for $\dot{\phi} > 0$. Using this form of a we find that $H = p/a\tau$, and it is straightforward to show that

$$\phi'(\tau) = -\frac{2\sqrt{2\bar{\epsilon}}}{2\bar{\epsilon} - (2 + r)\alpha}\sqrt{\frac{D}{S}}\frac{1}{\tau}, \qquad (20)$$

$$V(\tau) = \frac{4(3\alpha^2 - \bar{\epsilon})}{(2\bar{\epsilon} - (2 + r)\alpha)^2} \frac{1}{S(a\tau)^2}, \qquad (21)$$

where a prime means differentiation with respect to conformal time, τ. Equations (18)–(21) form the basis of our analysis. This scaling relation (for $S = 1$) was first uncovered in Ref. [32] using a different procedure.

B. Power spectrum of the perturbed field

For a universe which evolves according to the scaling solution, the primordial spectrum of scalar perturbations produced by this superinflationary phase was previously calculated in Ref. [34]. It was found that the spectrum tends to exact scale invariance for $\beta \gg 1$ (i.e. $\bar{\epsilon} \gg 1$), without any fine tuning of the quantization parameters of LQC. The purpose of this section is to review how scale invariance arises for the scaling solution with $\beta \gg 1$, and to generalize the analysis of [34] in order to allow for potentials which do not give rise to exact scaling solutions.

In order to calculate the spectrum of perturbations, we now perturb the scalar field equation. The perturbation in the field $\delta\phi$ then satisfies the equation

$$\delta\phi'' = \left[-2\frac{a'}{a} + \frac{D_{,\tau}}{D}\right]\delta\phi' + D[\nabla^2 - a^2 V_{,\phi\phi}]\delta\phi, \quad (22)$$

which can be written in the form [34]

$$u'' + (-D\nabla^2 + m_{\text{eff}}^2)u = 0, \qquad (23)$$

where u is defined as $u = aD^{-1/2}\delta\phi$ and the effective mass of the field u is given by

$$m_{\text{eff}}^2 = -\frac{(aD^{-1/2})''}{aD^{-1/2}} + a^2 DV_{,\phi\phi}. \qquad (24)$$

Decomposing u in Fourier modes w_k, that satisfy

$$w_k'' + (-Dk^2 + m_{\text{eff}}^2)w_k = 0, \qquad (25)$$

the power spectrum is then given by

$$\mathcal{P}_u = \frac{k^3}{2\pi^2}|w_k|^2. \qquad (26)$$

It was shown in Ref. [34] that the general solution to Eq. (25) is

$$w_k(\tau) = \sqrt{\frac{\pi}{2|2 + np|}}(d_1\sqrt{-\tau}H_{|\nu|}^{(1)}(x) + d_2\sqrt{-\tau}H_{|\nu|}^{(2)}(x)) \qquad (27)$$

whenever $m_{\text{eff}}\tau$ is constant, where

$$\nu = -\frac{\sqrt{1 - 4m_{\text{eff}}^2\tau^2}}{2 + np}, \qquad (28)$$

and d_1 and d_2 are constants subject to the condition $|d_1|^2 - |d_2|^2 = 1$ and $H_{|\nu|}^{(1)}(x)$ and $H_{|\nu|}^{(2)}(x)$ are Hankel functions of the first and second kind, respectively. In the long wave-length limit, the power spectrum yields

$$\mathcal{P}_u \propto k^{3-2|\nu|}(-\tau)^{1-|\nu|(np+2)}. \qquad (29)$$

Scale invariance of the power spectrum is then attained when the spectral tilt $\Delta n_u \equiv 3 - 2|\nu|$ is zero. Since for a universe evolving according to the scaling solution Eq. (18)

$$m_{\text{eff}}^2\tau^2 = -2 + (3 - 2n)p + \tfrac{1}{2}(6 + 2n - n^2)p^2, \quad (30)$$

we can see from Eq. (28) that scale invariance occurs whenever $p \rightarrow 0$, which, as we referred to, does indeed imply that $\bar{\epsilon} \gg 1$ and consequently $V < 0$ from Eq. (21). There is one other value of p for which scale invariance is attained, $p = -4/(n + 4)$, however, we will not consider it any further.

We would now like to generalize the form of the potential we are dealing with so that it no longer has to be of exactly the form which gives rise to the scaling solution. For standard slow-roll inflation, where the kinetic energy is small compared with the potential energy, it is possible to account for potentials of a form more general than a scaling potential by introducing slow-roll parameters. Such parameters parametrize the steepness of the potential, and how this steepness evolves as the field moves along the potential. For a given field potential they also allow the dynamics which follow from a more general potential to be expanded locally about the dynamics which follow from a scaling potential with the same local steepness. The power spectrum which follows from the general potential can then also be written in terms of the slow-roll parameters.

For the case at hand we would like to develop a similar expansion scheme. However, for the regime which we are considering in which $\bar{\epsilon} \gg 1$, it is clear that the kinetic energy is of approximately the same magnitude as the potential energy, and therefore the slow-roll approximation is inadequate. Indeed in this case the field is evolving rapidly along a steep negative potential, and we refer to the evolution as the ''fast-roll'' regime. Our strategy will therefore be to determine other suitable small parameters which characterise the steepness and curvature of the potential, and which we will refer to as ''fast-roll'' parameters. The derived parameters we will arrive at are similar to those obtained in Ref. [42], where fast-roll parameters were required to parametrize general potentials in the ekpyrotic scenario. This similarity is natural since, as we have already mentioned in the introduction, the evolution of the superinflationary scaling solution in LQC is dual to the ekpyrotic collapse.

The first step in accommodating a more general class of potentials is to allow $\bar{\epsilon}$ to become time dependent. From its definition it then follows that

$$\bar{\epsilon}' = -(2\bar{\epsilon})^{3/2}\eta\sqrt{\frac{S}{D}}\phi', \qquad (31)$$

where we have defined

$$\eta \equiv 1 - \frac{V_{,\phi\phi}V}{V_{,\phi}^2} - \frac{1}{2}\frac{V}{V_{,\phi}}\left(\frac{D_{,\phi}}{D} - \frac{S_{,\phi}}{S}\right), \qquad (32)$$

which can also be written in terms of the background quantities as

$$\eta \equiv 1 - \frac{V_{,\phi\phi}V}{V_{,\phi}^2} - \frac{1}{2}(n-r)\frac{V}{V_{,\phi}}\frac{a'}{a\phi'}. \qquad (33)$$

Likewise, we can calculate η' in terms of a third parameter ξ^2,

$$\eta' = -\sqrt{2\bar{\epsilon}}\xi^2\sqrt{\frac{S}{D}}\phi', \qquad (34)$$

where

$$\begin{aligned}
\xi^2 \equiv & \left[1 + \frac{V_{,\phi\phi\phi}V}{V_{,\phi\phi}V_{,\phi}} - 2\frac{V_{,\phi\phi}V}{V_{,\phi}^2}\right]\frac{V_{,\phi\phi}V}{V_{,\phi}^2} \\
& - \frac{1}{2}\left[1 + \frac{D_{,\phi\phi}V}{D_{,\phi}V_{,\phi}} - \frac{D_{,\phi}V}{DV_{,\phi}} - \frac{V_{,\phi\phi}V}{V_{,\phi}^2}\right]\frac{D_{,\phi}V}{DV_{,\phi}} \\
& + \frac{1}{2}\left[1 + \frac{S_{,\phi\phi}V}{S_{,\phi}V_{,\phi}} - \frac{S_{,\phi}V}{SV_{,\phi}} - \frac{V_{,\phi\phi}V}{V_{,\phi}^2}\right]\frac{S_{,\phi}V}{SV_{,\phi}}. \qquad (35)
\end{aligned}$$

In particular, for the scaling solution where $\bar{\epsilon}$ is constant, one can verify that $\eta = \xi^2 = 0$ exactly. Since we are considering potentials which are close to the form of a scaling potential, we expect that there will be solutions to the equations of motion of a form very similar to that given by Eqs. (18)–(21), when $\bar{\epsilon}$ is slowly varying. Assuming that Eqs. (18)–(20) are indeed good approximations, we have in general that

$$\frac{d\ln\bar{\epsilon}}{d\ln a} \approx 4\frac{\bar{\epsilon}\eta}{\alpha}, \qquad \frac{d\ln\eta}{d\ln a} \approx 2\frac{\bar{\epsilon}\xi^2}{\eta\alpha}. \qquad (36)$$

Imposing that $\bar{\epsilon}$ and η are slowly varying, and since $\bar{\epsilon}$ is large in the regime which gives rise to scale invariance, requires η and ξ^2 to be small, i.e. the potential must be nearly power law in form which is in agreement with our assumption. We refer to η and ξ as the second and third fast-roll parameters, and for convenience introduce the first fast-roll parameter as $\epsilon = 1/2\bar{\epsilon}$, where in terms of relative magnitude we find $\epsilon \sim \eta$ and $\xi^2 \sim \mathcal{O}(\epsilon^2)$. Using these relations it is possible to verify by substituting Eqs. (18)–(21) into the Friedmann and equation of motion that these solutions are indeed valid up to second order in fast-roll parameters for a general negative potential confirming the consistency of our analysis.

Then using Eqs. (18)–(21) to substitute for the respective quantities in the effective mass (24), and expanding to first order in the parameters ϵ and η, we obtain

$$\Delta n_u \approx 4\epsilon\left[1 - \frac{n}{12}\left(1 + \frac{n}{6} - r\right) - \frac{r}{2}\right] - 4\eta. \qquad (37)$$

We note that although we have used the solution to Eq. (25) which is valid only when $m_{\rm eff}\tau$ is constant, the solution will remain sufficiently accurate provided that ϵ does not vary significantly as a given k mode crosses the horizon from the small wavelength to the long wavelength regime. This is simply the condition that η is small, which we have assumed already. The spectral tilt can therefore be calculated at any given scale by inserting into Eq. (37) the values that ϵ and η take as this scale crosses the horizon. In particular, to compare with observations we need to consider the mode which corresponds to the largest scales on the cosmic microwave background (CMB).

Finally we also note that we could have included time derivatives of the power n and r. The generalization to do so is straightforward but for simplicity the computation we have presented does not include this possibility.

C. Stability of the fixed points

The analysis we have performed so far is intriguing. It appears that in the $a \ll a_*$ regime, the scaling solution which follows from a steep negative potential can give rise to a scale-invariant power spectrum of scalar field perturbations, and moreover we can generalize the analysis to potentials which deviate from the scaling potential.

However, there is another element which is involved in building a convincing theory for the origin of scale-invariant perturbations. That is, it would be highly desirable if the scaling solution was an attractor, so that initial conditions not exactly on the solution would evolve towards it, and the solution's stability against small local perturbations would be assured.

To determine the stability of the scaling solution, we study the nature of the fixed points of Eqs. (5)–(7). We do this by linearizing the equations about the fixed points and determining the corresponding eigenvalues (ω) in each case. For the kinetic energy dominated solutions, valid for an arbitrary λ, in Eqs. (10) and (11), we find their respective eigenvalues to be

$$\omega_+ = 6\alpha + \sqrt{6}\lambda, \qquad \omega_- = -\frac{1}{2}(n-r), \qquad (38)$$

$$\omega_+ = 6\alpha - \sqrt{6}\lambda, \qquad \omega_- = -\frac{1}{2}(n-r). \qquad (39)$$

Since $n > r$ and $\alpha < 0$, the first fixed point is stable for $\lambda < -\sqrt{6}\alpha$ and the second for $\lambda > \sqrt{6}\alpha$. Turning to the scaling solution, Eq. (12), we find

$$\omega_\pm = -\frac{1}{4\alpha}(\theta \pm \sqrt{\theta^2 + 8\alpha(n-r)(\lambda^2 - 6\alpha^2)}, \qquad (40)$$

where $\theta = \alpha(6-r) - \lambda^2 < 0$. The scaling solution is therefore stable whenever $\lambda^2 > 6\alpha^2$ which coincides with the region of existence of this solution. In Fig. 1 we show the evolution of the ratio $-\dot{\phi}^2/2DV$ obtained by numerically solving the equations of motion. We can see that it approaches the value given by x^2/y^2 where x and y are
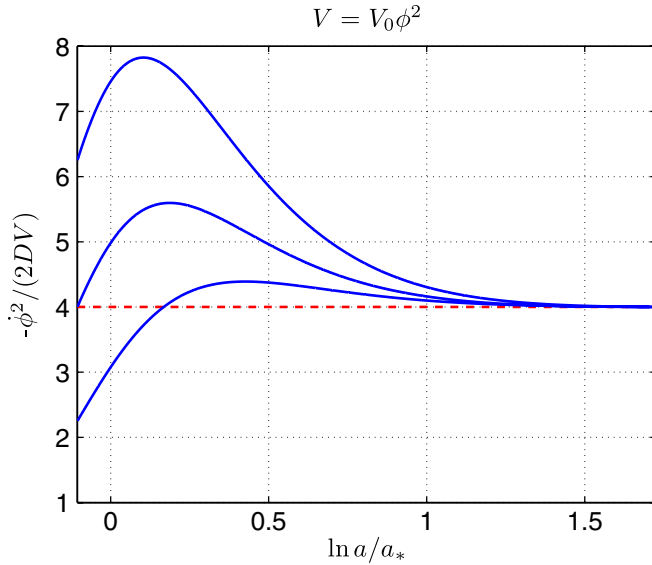
FIG. 1 (color online). The evolution of the ratio $-\dot{\phi}^2/2DV$ obtained by numerically solving the equations of motion for three different initial conditions (solid line). They approach the scaling solution given by x^2/y^2 where x and y are given by Eqs. (12) (dashed line). We used as parameters $V_0 = -10^{-20}$, $\phi_{\text{init}} = 1$, in Planck units and $n = 15$, $r = 3$ and $a_{\text{init}} = 0.9a_\star$, in a flat universe.

given by Eqs. (12). Typically the evolution only reaches the attractor when $a > a_\star$ which is far outside the region where the approximation $D \approx a^n$ is valid. We conclude that this solution must be extremely fine tuned in order to deliver the dynamics and power spectrum as described in the previous subsections.

In the second part of this article we will be dealing with a second possibility of obtaining a superinflationary regime that also leads to a scale-invariant power spectrum with the advantage that the stability of the scaling solution is no longer a dangerous issue.

III. EFFECTIVE DYNAMICS WITH QUADRATIC CORRECTIONS

The second modification to classical dynamics which we consider follows from considering that holonomies are the basic variables for quantisation in LQC. This modification gives rise to a Friedmann equation of the following form [30]:

$$H^2 = \frac{1}{3}\rho\left(1 - \frac{\rho}{2\sigma}\right). \tag{41}$$

Once again we are assuming either a flat universe or that the curvature contribution can be safely neglected. It is interesting that this form of the Friedmann equation is identical to the form which arises in braneworld scenarios with a single timelike extra dimension in the absence of a black hole in the bulk spacetime [31]. In the braneworld case σ represents the brane tension while for the LQC case

2σ represents the critical energy density arising from quantum geometry effects which leads to the scale factor undergoing a bounce as ρ approaches it.

We are interested in high density regimes where ρ approaches the bounding value of 2σ. In this case, the term within brackets tends to zero, and the behavior of the equations alters significantly compared with the classical behavior. Indeed in this regime we have $\dot{H} > 0$ and for an expanding universe superinflation takes place. We will again consider a scalar field dominated universe, hence, $\rho = \dot{\phi}^2/2 + V(\phi)$. We stress that we are studying inverse volume and quadratic corrections separately, hence we do not include the D and S functions in the Friedmann equation and in the definition of energy density. The scalar field equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \tag{42}$$

is unchanged from the classical form.

A. Scaling dynamics

It was shown in Ref. [36] (also see [35]) that the effective equations (41) and (42) also allow a scaling solution in which the kinetic and potential energy vary in proportion to one another. In this case the potential must be of the form $V = V_0 \cosh(\phi)$. However, while earlier works claimed otherwise, it can be shown that the scaling solution is not an attractor. Moreover, the scaling solution which was found implies an evolution for the scale factor which is not of a power law kind, and this means that it is unlikely to give rise to a scale-invariant spectrum of perturbations. Given these difficulties, instead of focusing on the scaling solution we will ask whether there is a form of the scalar field potential which does result in a power-law evolution of the scale factor. We are interested in the regime in which $\rho \approx 2\sigma$, when superinflation occurs, and where $H \approx 0$. Inserting the power-law ansatz,

$$a(t) = (-t)^m, \tag{43}$$

where $m < 0$, into the time derivative of the Hubble rate,

$$\dot{H} = -\frac{\dot{\phi}^2}{2}\left(1 - \frac{\rho}{\sigma}\right), \tag{44}$$

we see that for $\rho \approx 2\sigma$ the kinetic energy of the field is $\dot{\phi}^2/2 \approx -m/t^2$ which upon integration gives

$$\phi \approx \pm\sqrt{-2m}\,\ln t. \tag{45}$$

Using Eq. (41) and expanding in $6H^2/\sigma \ll 1$ we have that

$$\rho \approx 2\sigma - 3H^2, \tag{46}$$

and it follows from the definition of the energy density of the field and Eq. (45) that

$$V \approx 2\sigma - U_0 e^{-\lambda\phi}, \tag{47}$$

where $U_0 = 3m^2 - m > 0$ and $\lambda = 1/\sqrt{-2m}$. It is evident

that, in this regime, scaling exists between $V - 2\sigma$ and the kinetic energy $\dot{\phi}^2/2$. We now look for a more precise description of the dynamics. The form of the potential (47) motivates us to define the new variables:

$$x \equiv \frac{\dot{\phi}}{\sqrt{4\sigma - 2\rho}}, \qquad y \equiv \frac{\sqrt{U}}{\sqrt{2\sigma - \rho}}, \qquad (48)$$

such that $\rho \lesssim 2\sigma$ and $V(\phi) = 2\sigma - U(\phi)$. In terms of a system of first order differential equations, the equation of motion of the scalar field now reads,

$$x_{,N} = -3x - \sqrt{\tfrac{3}{2}}\lambda y^2 - 3x^3 \qquad (49)$$

$$y_{,N} = -\sqrt{\tfrac{3}{2}}\lambda xy - 3x^2 y \qquad (50)$$

$$\lambda_{,N} = -\sqrt{6}\lambda^2(\Gamma - 1)x + 3x^2\left(\frac{2\sigma}{\rho} - 1\right)\sqrt{\frac{2\sigma}{\rho}}, \qquad (51)$$

where λ and Γ are defined as

$$\lambda \equiv -\frac{U_{,\phi}}{U}\sqrt{\frac{2\sigma}{\rho}}, \qquad \Gamma \equiv \frac{UU_{,\phi\phi}}{U_{,\phi}^2}. \qquad (52)$$

The variables x, and y are also related by the constraint condition

$$x^2 - y^2 = -1. \qquad (53)$$

Considering the regime discussed above where $2\sigma/\rho \approx 1$, we can see λ is a constant and by integrating λ, we see that the U part of the scalar potential is given by

$$U = U_0 e^{-\lambda\phi}, \qquad (54)$$

as we were expecting from Eq. (47).

We consider the section of the phase space in which $x < 0$, $y > 0$, and $\lambda > 0$. Taking λ to be constant, and substituting the constraint equation (53) into Eq. (49), results in an autonomous system with three fixed points. Two of them are nonphysical solutions with

$$x = \pm i, \qquad y = 0, \qquad (55)$$

and the third is a scaling solution with

$$x = -\frac{\lambda}{\sqrt{6}}, \qquad y = \sqrt{1 + \frac{\lambda^2}{6}}. \qquad (56)$$

The scaling solution is valid for all real values of λ.

As in the previous case, it is straightforward to show that as the universe evolves according to this solution, the scale factor undergoes a power-law evolution

$$a(\tau) = (-\tau)^p, \qquad (57)$$

where

$$p = -\frac{1}{\bar{\epsilon} + 1}, \qquad (58)$$

and $\bar{\epsilon}$ is here defined as $\bar{\epsilon} = (U_{,\phi}/U)^2/2 \approx \lambda^2/2$. This is of course what we expect since we began by searching for such a solution using the ansatz equation (43). The time derivative of the field and the potential yields,

$$\phi' = \frac{\sqrt{2\bar{\epsilon}}}{\bar{\epsilon} + 1}\frac{1}{\tau}, \qquad (59)$$

$$V = 2\sigma - \frac{3 + \bar{\epsilon}}{(1 + \bar{\epsilon})^2}\frac{1}{(a\tau)^2}. \qquad (60)$$

We are now ready to compute the spectrum of the scalar field perturbations produced by this power-law solution.

B. Power spectrum of the perturbed field

In this section we follow the same approach we took in the previous analysis of the scalar field perturbations. In terms of conformal time, the perturbation equation for the scalar field, ϕ, can be written as

$$\delta\phi'' = -2\frac{a'}{a}\delta\phi' + (\nabla^2 - a^2 V_{,\phi\phi})\delta\phi \qquad (61)$$

which in turn can be written in terms of $u \equiv a\delta\phi$ as

$$u'' + (-\nabla^2 + m_{\text{eff}}^2)u = 0, \qquad (62)$$

and the effective mass of the field u is

$$m_{\text{eff}}^2 = \left(-\frac{a''}{a} + a^2 V_{,\phi\phi}\right). \qquad (63)$$

Decomposing u in Fourier modes w_k, that satisfy

$$w_k'' + (-k^2 + m_{\text{eff}}^2)w_k = 0, \qquad (64)$$

the power spectrum is then given by

$$\mathcal{P}_u = \frac{k^3}{2\pi^2}|w_k|^2. \qquad (65)$$

The general solution to Eq. (25) is

$$w_k(\tau) = \sqrt{\frac{\pi}{4}}(d_1\sqrt{-\tau}H_{|\nu|}^{(1)}(x) + d_2\sqrt{-\tau}H_{|\nu|}^{(2)}(x)), \qquad (66)$$

where the subscript $|\nu|$ is

$$\nu = -\frac{\sqrt{1 - 4m_{\text{eff}}^2\tau^2}}{2}, \qquad (67)$$

and d_1 and d_2 are constants subject to the condition $|d_1|^2 - |d_2|^2 = 1$ and $H_{|\nu|}^{(1)}(x)$ and $H_{|\nu|}^{(2)}(x)$ are Hankel functions of the first and second kind, respectively. For large wavelength modes, the power spectrum can be approximated by

$$\mathcal{P}_u \propto k^{3-2|\nu|}(-\tau)^{1-2|\nu|}. \qquad (68)$$

By substituting Eqs. (57)–(60) into Eq. (63), we find

$$m_{\text{eff}}^2\tau^2 = -2 + 3p + 3p^2. \qquad (69)$$

Comparing this case to our previous results, we expect to

have scale invariance for $p \to 0$. Once again, therefore, scale invariance occurs when the field ϕ is rolling down a steep potential and the kinetic energy is not negligible, but comparable to $V - 2\sigma$. Hence the evolution should again be understood as a fast-roll regime.

Clearly, scale invariance is also obtained for $p \to -1$ or $\bar{\epsilon} \ll 1$ which corresponds to the standard slow-roll regime that we are not concerned with for the purposes of this work.

In order to extend this analysis to general potentials, as we did for the previous system of modified equations we studied, we allow $\bar{\epsilon}$ to depend on conformal time such that

$$\bar{\epsilon}' = -(2\bar{\epsilon})^{3/2}\eta\phi', \tag{70}$$

which defines the fast-roll parameter η,

$$\eta \equiv 1 - \frac{U_{,\phi\phi}U}{U_{,\phi}^2}. \tag{71}$$

Similarly, η' can be written in terms of the next order fast-roll parameter ξ^2 as

$$\eta' = -\sqrt{2\bar{\epsilon}}\xi^2\phi', \tag{72}$$

with

$$\xi^2 \equiv \left(1 + \frac{U_{,\phi\phi\phi}U}{U_{,\phi\phi}U_{,\phi}} - 2\frac{U_{,\phi\phi}U}{U_{,\phi}^2}\right)\frac{U_{,\phi\phi}U}{U_{,\phi}^2}. \tag{73}$$

For the scaling solution, it can be verified that both η and ξ^2 vanish. As in the previous case, we can use Eqs. (57)–(59) as approximate solutions, and hence we have

$$\frac{d\ln\bar{\epsilon}}{d\ln a} \approx 4\epsilon\eta, \qquad \frac{d\ln\eta}{d\ln a} \approx 2\frac{\bar{\epsilon}\xi^2}{\eta}, \tag{74}$$

which means that for a large and slowly varying $\bar{\epsilon}$ the parameter η must be small, and for a slowly varying η the parameter ξ^2 must also be small. Hence, the U part of the scalar potential must be close to exponential.

Using now Eqs. (57)–(59) in the expression for the effective mass of u Eq. (63), where $\bar{\epsilon}$ is now time dependent, and then using Eq. (67), we find that $\Delta n_u \equiv 3 - 2|\nu|$ is to first order

$$\Delta n_u = -4(\epsilon - \eta) \tag{75}$$

where we have again defined a further fast-roll parameter by $\epsilon = 1/2\bar{\epsilon}$. Again the assumption that $m_{\text{eff}}\tau$ is nearly constant as a given mode evolves outside the cosmological horizon is valid. Hence the spectral tilt for a given k mode can be calculated by inserting the values the fast-roll parameters took as the mode crossed the horizon in Eq. (75). It is clear that the system we have investigated here results in a scale-invariant spectrum of scalar field perturbations for $\epsilon \ll 1$ and $\eta \ll 1$.
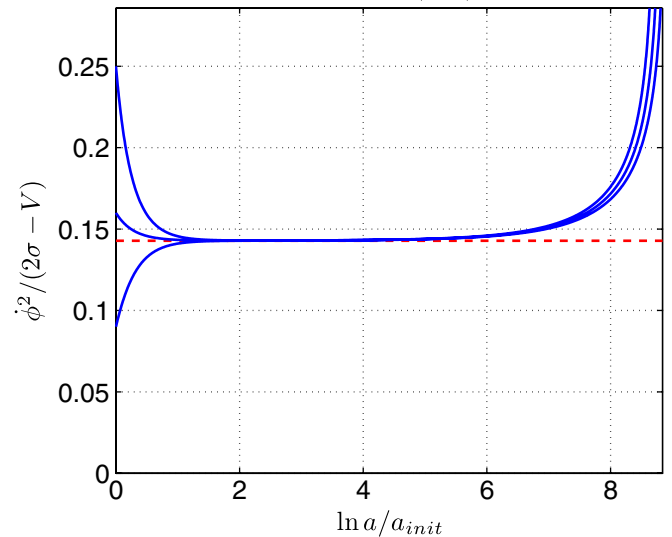

FIG. 2 (color online). The evolution of the ratio $\dot{\phi}^2/(2\sigma - V)$ obtained by numerically solving the equations of motion for three different initial conditions (solid lines). They approach the scaling solution given by $2x^2/y^2$ where x and y are given by Eqs. (56) (dashed line). We used as parameters $U_0 = 10^{-2}$, $\phi_{\text{init}} = 1$ and $\sigma = 0.41$ in Planck units, in a flat universe.

C. Stability of the fixed points

Linearizing the system (49), using Eq. (53) about the fixed points we find the following eigenvalues ω: for the unphysical kinetic energy dominated solution, Eq. (55),

$$\omega = 6 \pm i\sqrt{6}\lambda, \tag{76}$$

and hence this solution is unstable. While for the scaling solution, Eq. (56)

$$\omega = -\tfrac{1}{2}(6 + \lambda^2), \tag{77}$$

and hence this point is a *stable* attractor for *all* values of λ. A numerical analysis of this system shown in Fig. 2 supports our analytical results presented here. The figure shows the evolution of the ratio $\dot{\phi}^2/(2\sigma - V)$ obtained by numerically solving the equations of motion for three different initial conditions. They approach the scaling solution given by $2x^2/y^2$ where x and y are given by Eqs. (56). When $a \gg a_\star$ the quadratic corrections become negligible and the numerical evolution diverges from this attractor.

IV. NUMBER OF E-FOLDS

Before concluding, it is important to address the question of whether a sufficient amount of superinflation can occur in LQC to account for the largest scale perturbations observed on the CMB. This in turn is equivalent to asking whether the superinflationary phases can solve the cosmological horizon problem.

It is clear that only a small number of e-folds of super-inflation can be considered generic for either of the modified sets of evolution equations we have studied [29]. This might be considered disappointing, since experience from standard inflation suggests that approximately 60 e-folds of inflation are required for consistency with observations. However, the inflationary periods we have been studying are considerably different to standard inflation and this has a dramatic effect.

Solving the horizon problem is essentially the requirement that aH grows sufficiently during an early stage of the universe's evolution. While in standard inflation this is accomplished by a changing rapidly as H remains nearly constant, in our case the converse appears to be possible, that H increases sufficiently as a remains nearly constant. The number of e-folds usually only refers to the change in a, and so we only expect a small number of e-folds to be necessary in a superinflationary phase, provided that H changes sufficiently. To confirm the expectation that our superinflationary phases can indeed solve the horizon problem, let us now quantify our qualitative arguments.

During superinflation, perturbation modes exit the cosmological horizon, and once superinflation ends, modes start to reenter. Let us consider a perturbation mode with wave number k, such that k exited the cosmological horizon $N(k)$ e-folds before the end of the superinflationary phase, and reentered sometime later. The mode reentering the horizon today, k_*, must satisfy $k_* = a_0 H_0$, where subscript 0 indicates quantities at the present epoch. Comparing this with the generic k mode we have:

$$\frac{k}{a_0 H_0} = \frac{a_k H_k}{a_0 H_0} = \frac{a_k H_k}{a_{\text{end}} H_{\text{end}}} \frac{a_{\text{end}}}{a_{\text{reh}}} \frac{a_{\text{reh}}}{a_{\text{eq}}} \frac{a_{\text{eq}}}{a_0} \frac{H_{\text{end}}}{H_0}, \quad (78)$$

where subscript "end" labels quantities at the end of inflation, "reh" at reheating, and "eq" at matter radiation equality. Then employing the known evolution of the universe from reheating to the present day, together with the measured value of the Hubble rate at the present epoch, H_0, and for simplicity assuming that the universe behaves as if it is matter dominated between the end of inflation and reheating, we find:

$$\ln\left[\frac{k}{a_0 H_0}\right] \approx 68 + \ln\left[\frac{a_k H_k}{a_{\text{end}} H_{\text{end}}}\right] - \frac{1}{2}\ln\left[\frac{M_{\text{Pl}}}{H_{\text{end}}}\right] - \frac{1}{3}\ln\left[\frac{\rho_{\text{end}}}{\rho_{\text{reh}}}\right]^{1/4}. \quad (79)$$

The energy scale at the end of inflation must be determined by requiring that the magnitude of the curvature perturbation accounts for the temperature anisotropies in the CMB. Since we have only worked with the scalar field perturbation this is so far undetermined in our model and we therefore take H_{end} to be the highest possible scale, i.e. $H_{\text{end}} = M_{\text{Pl}}$. A lower scale would lead to fewer required e-folds. Further considering $k = k_*$ and assuming instant reheating, we find that

$$\ln\left(\frac{a_{\text{end}} H_{\text{end}}}{a_{k_*} H_{k_*}}\right) \approx 68. \quad (80)$$

In order to determine the number of e-folds of super-inflation required, we now consider the cases in which the scale factor is undergoing pure power-law behavior, $a \propto (-\tau)^p$. Using this together with Eq. (80), we find

$$\ln\left(\frac{\tau_{\text{end}}}{\tau_{k_*}}\right)^{-1} \approx 68, \quad (81)$$

and in turn

$$N(k_*) = \ln\left(\frac{a_{\text{end}}}{a_{k_*}}\right) = -68p. \quad (82)$$

Recalling that p must be small and negative for scale invariance, we see that only a small number of e-folds are required. Although considering behavior which deviates from pure power law behavior will alter Eq. (82), it is clear that the conclusion of only a small number of e-folds being necessary will remain valid.

V. DISCUSSION

In this paper we have investigated the nature of super-inflation in loop quantum cosmology. Considering two specific examples which lead to modifications of the Friedmann equation, one where the modification is due to the presence of inverse volume corrections, and the second where it is induced by the use of holonomies as the basic variable in the quantisation of Loop Quantum Gravity, we have demonstrated explicitly in both cases the existence of superinflation solutions defined by $\dot{H} > 0$ where H is the Hubble parameter. Through the use of phase plane analysis we have been able to discuss the nature of the attractor solutions and their stability in both cases. Further we determined the scalar perturbations arising in these models, and through the introduction of fast-roll parameters, showed that a class of potentials exhibit near scale-invariant spectra. To be more specific we have shown that for the case with inverse volume corrections, if we concentrate on the regime $D \propto a^n$, three solutions exist, two of them corresponding to kinetic energy dominated solutions and one to a superinflationary scaling solution where the kinetic energy scales in proportion to the potential energy. In this last case, when the potential is negative, scaling occurs when it is polynomial in form, and the perturbed scalar field equations for this potential are scale invariant (as first demonstrated in [34]). However, we have been able to go further. By introducing fast-roll parameters $\bar{\epsilon}$ and η in an analogous manner to the way introduced for the ekpyrotic scenario [42], we were able to extend our calculation of the scalar power spectrum beyond the case of the polynomial scaling potential. In other words we can determine the degree to which scale invariance is broken in terms of the fast-roll parameters. Through the stability analysis of these solutions we have seen that the scaling

solution which gives rise to the scale-invariant spectrum is a stable attractor. In general, however, the attractor is reached when $a \gg a_\star$ when $D \approx 1$ and therefore we concluded that in only a limited range of parameters we have an analytical understanding of the dynamics of the system in the semiclassical phase $a \ll a_\star$.

We considered a second set of corrections to the Friedmann equation, arising from the quantisation procedure in LQG where a ρ^2 term in the Friedmann equation, analogous to that found, in particular, braneworld models [31], is present. In this case the scaling solution found in [36], which involves a $\cosh(\phi)$ potential, is not an attractor (in contradiction to the claims in Refs. [35,36]) and we believe it is unlikely to lead to a scale-invariant spectrum. In order to obtain the solution with a scale-invariant spectrum we decided to take another route, namely, search for the potential with the correct time evolution in the scale factor which would lead to scale invariance. In these models superinflation occurs just after the bounce and using this fact we quickly arrive at the intriguing result that the corresponding potential is an uplifted negative exponential potential, and in this case the scaling solution is stable. Scale invariance easily follows as does the generalization to include a new set of fast-roll parameters, hence obtaining potentials which will yield small deviations from scale invariance.

In both cases, we note that we do not expect the scalar potential to remain of the form we have used to give scale invariance throughout the entire evolution of the universe as a negative value of the potential after the end of superinflation may lead to a subsequent recollapse. The full details of the exit from superinflation, the subsequent form of the potential, as well as the transition to the radiation epoch, are beyond the scope of the present work.

We emphasize that we only dealt with the evolution of the universe in the expanding phase, however, for closed models, the universe may have gone through a bounce from a collapsing phase into the superinflationary evolution. It is natural to question whether the bounce is symmetric or not, i.e. if the period of superinflation is nothing more than the counterpart of an identical deflationary period and an epoch of standard inflation is still necessary. Though this is a possible situation, we stress that the scaling solutions produced by the potentials we considered here are the attractors only during the expanding epoch, hence, the evolution through the bounce is indeed expected to be asymmetric.

Finally, we have discussed the horizon problem in this set up. We found that for the class of potentials studied that lead to scale invariance, the required number of e-folds of superinflation is of only a few, more specifically, $N \approx -68p$ where $-1 \ll p < 0$ defines the time dependence of the scale factor of the universe, $a = (-\tau)^p$. A previous criticism of inflation within the inverse volume corrections approach has been that $a_\star$ must be unphysically large to give rise to 60 e-folds [46]. We have seen in this work, however, that only a small number of e-folds of superinflation are required, which suggests that this bound on $a_\star$ can be evaded. We also note that previous studies found a small probability of sufficient standard inflation in the context of LQC [47]. However, their conclusions do not apply directly to our model of superinflation since, as we have discussed, only a small number of e-folds are required and moreover, in their study, the Hubble rate was assumed to be constant.

In our calculation we have neglected metric perturbations, instead we have only considered scalar perturbations and been able to find scale invariance through the scalar field perturbations. Given the similarity mentioned throughout the paper between our LQC results and the ekpyrotic mechanism, it seems likely that when metric perturbations are included in the LQC calculation, we will find that the scale-invariant spectrum occurs in the Newtonian potential and the scalar field perturbation, but not in the curvature perturbation, just like the ekpyrotic case (for example, see [42]). If this turns out to be the case, it would be interesting to consider whether the inclusion of a second scalar field in LQC, as in the ''new ekpyrotic'' mechanism, would allow the scale-invariant spectrum to be transferred from the scalar field perturbation into the curvature perturbation via an entropy perturbation [48–50].

ACKNOWLEDGMENTS

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