Loop quantum gravity corrections to gravitational wave dispersion

Martin Bojowald* and Golam Mortuza Hossain†

Institute for Gravitation and the Cosmos, The Pennsylvania State University, 104 Davey Lab, University Park, Pennsylvania 16802, USA

(Received 17 September 2007; published 11 January 2008)

Cosmological tensor perturbations equations are derived for Hamiltonian cosmology based on Ashtekar's formulation of general relativity, including typical quantum gravity effects in the Hamiltonian constraint as they are expected from loop quantum gravity. This translates to corrections of the dispersion relation for gravitational waves. The main application here is the preservation of causality which is shown to be realized due to the absence of anomalies in the effective constraint algebra used.

DOI: 10.1103/PhysRevD.77.023508 PACS numbers: 98.80.Cq, 04.60.Pp, 98.80.Bp

I. INTRODUCTION

In cosmology, the study of gravitational waves created through physical processes in the early universe provides a unique window to initial stages of the universe. Significant efforts are being made to detect possible signatures of tensor mode perturbations of space-time geometry through measurements of the polarization in the cosmic microwave background (CMB). Since quantum gravity effects could play a significant role in the very early universe, it is of interest to study possible quantum gravity effects on gravitational wave propagation during these periods. In the last few years, applications of the method used in loop quantum gravity (LQG) [1–3], a candidate quantum theory of gravity, to early universe cosmology have led to significant progress. In particular, the quantization of homogeneous cosmological models known as loop quantum cosmology (LQC) [4] has led to a resolution of the big bang singularity [5–9], and techniques have become available to include inhomogeneous perturbations [10,11].

In this paper we study typical quantum gravity effects for tensor modes that are expected from loop quantum gravity.[1] In particular, we consider the effects on gravitational wave dynamics expected from corrections to classically divergent inverse powers of metric components and from the use of holonomies in the quantum theory instead of connection components. To study the dynamics we compute gravitational wave equations together with their dispersion relations.

In Sec. III, we present a derivation of tensor mode equations in Ashtekar variables. The calculations are purely canonical and split off the tensor mode in the metric from the outset. This mimics the usual covariant derivations as far as possible in a way accessible to canonical quantizations. Other canonical derivations exist [13] which due to their explicit use of Dirac observables appear more difficult to use in quantizations. In the following section, we consider the effects of quantum corrections to the inverse volume in Hamiltonian and compute the correspondingly corrected tensor mode equation. After that, we consider a second quantum effect due to the use of holonomies in a loop quantization. From the corrected wave equations one can easily derive the corresponding dispersion relations. Quantum gravity corrections are sensitive to the underlying discreteness of a quantum state, which in general changes as the universe expands. Thus, also propagation speeds derived from the corrected dispersion relations are functions of time, providing, in particular, a varying speed of light scenario.

Both corrections are typical of loop quantum gravity and thus test its basic features. The dispersion relations, in particular, allow one to investigate possible violations of causality which would arise if the propagation velocity of gravitational waves would turn out to be larger than the speed of light. In fact, we will see that gravitational waves travel faster than the classical speed of light, but not faster than the physical speed of light which is also subject to quantum corrections from an underlying discrete geometry. Quantum corrections to the gravitational and electromagnetic dynamics are related by the requirement of anomaly freedom, which can directly be implemented at the effective level and implies that physical causality is preserved.

II. CANONICAL FORMULATION

We consider linear tensor mode perturbations around spatially flat Friedmann-Robertson-Walker (FRW)

*bojowald@gravity.psu.edu
†hossain@gravity.psu.edu

[1]Possible cosmological implications for primordial gravitational waves with input from loop quantum cosmology have recently been discussed in [12]. However, only corrections to the background dynamics were considered while perturbation equations for the tensor mode were otherwise left unchanged. Moreover, the main analysis there focuses on nonperturbative quantum effects in the background dynamics which makes a rigorous inclusion of perturbative inhomogeneities around the background difficult; see e.g. the discussion in [10]. In the present paper, by contrast, we provide a consistent perturbative setting in which quantum corrections of the inhomogeneities themselves are included in the equations. As we will see, this by itself provides important effects which are not mimicked by corrections to the background dynamics.

spacetimes.[2] The general form of a perturbed metric around the isotropic background FRW background containing only the tensor mode is

$$g_{00} = -N^2 + q_{ab}N^aN^b = -a^2; \qquad g_{0a} = q_{ab}N^b = 0;$$
$$g_{ab} = q_{ab} = a^2[\delta_{ab} + h_{ab}], \tag{1}$$

where $a(t)$ is the scale factor of the FRW space-time. This notation is adapted to a canonical formulation, where the space-time metric $g_{\mu\nu}$ is decomposed in terms of the spatial metric q_{ab}, the lapse function N, and the shift vector N^a. Here we use the convention that Greek letters denote space-time indices whereas small Latin letters denote spatial indices. The symmetric metric perturbation field h_{ab} is transverse and traceless, i.e. it satisfies $\partial^a h_{ab} = 0$ and $\delta^{ab}h_{ab} = 0$. This removes any vectorial or scalar contributions from gradient terms $\partial_{(a}v_{b)}$ or $\partial_a\partial_b v$ or from the trace $u\delta_{ab}$ which rather contribute to the vector and scalar modes. Also the lapse N and shift N^a, being scalar and vectorial, respectively, do not contribute to tensor perturbations. Thus, in a canonical formulation tensor perturbations are generated through perturbations of the spatial metric q_{ab} alone.

A. Background

In Ashtekar's formulation of general relativity [15,16], the spatial metric as a canonical field is replaced by the densitized triad E^a_i, defined as

$$E^a_i := |\det(e^j_b)|e^a_i. \tag{2}$$

Here, e^a_i as a matrix is the inverse of the cotriad e^i_a whose relation to the spatial metric is $q_{ab} = e^i_a e^i_b$. The canonically conjugate variable to the densitized triad is the Ashtekar connection $A^i_a := \Gamma^i_a + \gamma K^i_a$, where K^i_a is the extrinsic curvature and γ is the so-called Barbero-Immirzi parameter [16,17]. The spin connection Γ^i_a is defined such that it leaves the triad covariantly constant and has the explicit form

$$\Gamma^i_a = -\epsilon^{ijk}e^b_j(\partial_{[a}e^k_{b]} + \tfrac{1}{2}e^c_k e^l_a \partial_{[c}e^l_{b]}). \tag{3}$$

As we perturb basic variables around a spatially flat FRW background, our background variables denoted by a bar are

$$\bar{E}^a_i = \bar{p}\delta^a_i; \qquad \bar{\Gamma}^i_a = 0; \qquad \bar{K}^i_a = \bar{k}\delta^i_a;$$
$$\bar{N} = \sqrt{\bar{p}}; \qquad \bar{N}^a = 0, \tag{4}$$

where $\bar{p} = a^2$ and the spatial metric is $\bar{q}_{ab} = a^2\delta_{ab}$.[3] The choice of $\bar{N} = a$ leads to conformal time which is used in what follows.

[2]The procedure follows that used for scalar [11] and vector modes [14] but is simpler at several places in the derivation of their equations of motion as well as for gauge issues.

[3]Compared to [18] we drop an additional tilde on $\bar{p}$ to keep the notation simple.

B. Perturbed canonical variables

The perturbed densitized triad E^a_i and Ashtekar connection A^i_a around a spatially flat background are given by

$$E^a_i = \bar{p}\delta^a_i + \delta E^a_i;$$
$$A^i_a = \Gamma^i_a + \gamma K^i_a = \gamma\bar{k}\delta^i_a + (\delta\Gamma^i_a + \gamma\delta K^i_a), \tag{5}$$

where $\bar{p}$ and $\gamma\bar{k}$ are the background densitized triad and Ashtekar connection, using the fact that $\bar{\Gamma} = 0$ for a spatially flat isotropic model. The general form of a cotriad corresponding to a spatial metric as in (1) is

$$e^i_a = a[\delta^i_a + \tfrac{1}{2}h^i_a], \tag{6}$$

where $h^i_a := \delta^{ib}h_{ab}$. The densitized triad (2) then has the perturbation

$$\delta E^a_i = -\tfrac{1}{2}\bar{p}h^a_i, \tag{7}$$

where we have used the fact that tensor mode perturbations are traceless, i.e. $\delta^i_a\delta E^a_i = 0$. For a general perturbed densitized triad (5) the linearized spin connection (3) becomes

$$\delta\Gamma^i_a = \frac{1}{\bar{p}}\epsilon^{ije}\delta_{ac}\partial_e\delta E^c_j. \tag{8}$$

As perturbations of lapse N and shift N^a do not contribute to the tensor mode, we can set $\delta N = 0$ and $\delta N^a = 0$ when studying tensor mode dynamics.

As described in more detail for scalar and vector modes in [11,14], the symplectic structure splits into one for the background variables and one for perturbations,

$$\{\bar{k}, \bar{p}\} = \frac{8\pi G}{3V_0}, \qquad \{\delta K^i_a(x), \delta E^b_j(y)\} = 8\pi G\delta^3(x,y)\delta^b_a\delta^i_j. \tag{9}$$

Here, G is the gravitational constant and V_0 is a fiducial volume introduced to arrive at a finite symplectic structure for the background variables by integrating the action only over a finite cell rather than all of $\mathbb{R}^3$. Since this background is homogeneous, no information is lost by the restriction to a cell. However, a fiducial quantity enters the formalism which must disappear from final physical results.

This provides separate canonical structures for the background and perturbations, but these variables will be coupled dynamically. In particular, the homogeneous background dynamics would receive backreaction effects at quadratic or higher order.

III. CLASSICAL DYNAMICS

In canonical quantum gravity, dynamics is determined by a Hamiltonian (constraint) operator rather than a path integral. This implies that one obtains relevant quantum corrections at the level of an effective Hamiltonian as opposed to an effective action in a covariant quantization.

To study the effects of quantum corrections to the classical equations of motion one thus needs to derive these equations starting from an effective Hamiltonian. Here, we are interested in studying the quantum correction expected from loop quantum gravity which is based on Ashtekar variables in the classical formulation. As a preparation for an analysis of effective tensor mode Hamiltonian we thus derive in this section the classical gravitational wave equation in canonical gravity using Ashtekar variables.

In a canonical triad formulation of general relativity there are three types of constraints: the Gauss constraint which generates local rotations of the triad, the diffeomorphism constraint which generates spatial diffeomorphisms, and the Hamiltonian constraint which completes the space-time diffeomorphisms and is thus relevant for the dynamics. For linear perturbations including only the tensor mode, the corresponding Gauss constraint is trivially satisfied as the perturbation field $h^i_a = \delta^{ib}h_{ab}$ is symmetric. In fact, the triad perturbation (6) is symmetric, while su(2)-gauge transformations of the Gauss constraint could only generate antisymmetric contributions owing to the antisymmetry of the su(2)-structure constants. Also the diffeomorphism constraint is identically satisfied as $N^a = 0$ for the tensor mode as discussed before. Thus, solutions for tensor mode perturbations are completely governed by the Hamiltonian constraint.

A. Hamiltonian constraint

The Hamiltonian constraint generates "time evolution" of the spatial manifold in terms of a time coordinate. Its general expression is

$$H_{\mathrm{G}}[N] = \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^3x N \frac{E^c_j E^d_k}{\sqrt{|\det E|}}[\epsilon_i^{jk}F^i_{cd} - 2(1+\gamma^2)K^j_{[c}K^k_{d]}]. \tag{10}$$

Using the expression (5) of the perturbed basic variables and the curvature $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon_{ijk}A^j_a A^k_b$, one can simplify and expand (10) for linearized tensor modes. Up to quadratic terms in perturbations we have

$$H_{\mathrm{G}}[N] = \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^3x\bar{N}\bigg[-6\bar{k}^2\sqrt{\bar{p}} - \frac{\bar{k}^2}{2\bar{p}^{3/2}}(\delta E^c_j \delta E^d_k \delta^k_c \delta^j_d) + \sqrt{\bar{p}}(\delta K^j_c \delta K^k_d \delta^c_k \delta^d_j) - \frac{2\bar{k}}{\sqrt{\bar{p}}}(\delta E^c_j \delta K^j_c) + \frac{1}{\bar{p}^{3/2}}(\delta_{cd}\delta^{jk}\delta^{ef}\partial_e E^c_j \partial_f E^d_k)\bigg]. \tag{11}$$

As expected, γ dependent terms drop out of the Hamiltonian constraint when one uses the spin connection and the fact that densitized triad and extrinsic curvature are symmetric for the tensor mode.

B. Linearized equations

In the standard covariant formulation linearized equations for metric perturbations are derived by considering the variation of the action with respect to the perturbed metric. In a canonical formulation, the linearized equations are derived using Hamilton's equations of motion. For the perturbed densitized triad,

$$\delta\dot{E}^a_i = \{\delta E^a_i, H_{\mathrm{G}}[N] + H_{\mathrm{matter}}[N]\} \tag{12}$$

leads to the expression of extrinsic curvature. Here $H_{\mathrm{matter}}[N]$ denotes the matter Hamiltonian which together with the gravitational contribution (11) forms the total Hamiltonian. Also the matter Hamiltonian depends on the lapse function through the determinant of the space-time metric. The choice of the background lapse function then determines the time coordinate which the dot refers to, which from now on will be $\bar{N} = a$ for conformal time.

Using the expression (7) of the perturbed densitized triad and thus $\delta\dot{E}^a_i = -\frac{1}{2}(\bar{p}\dot{h}^a_i + \dot{\bar{p}}h^a_i)$, the equation of motion (12) then leads to the expression

$$\delta K_a{}^i = \tfrac{1}{2}[\dot{h}^i_a + \bar{k}h^i_a] \tag{13}$$

for the linearized extrinsic curvature, where we used the background extrinsic curvature $\bar{k} = \dot{\bar{p}}/2\bar{p}$ which follows in a similar way from the zero order Hamiltonian constraint.

The second Hamilton equation of motion

$$\delta\dot{K}^i_a = \{\delta K^i_a, H_{\mathrm{G}}[N] + H_{\mathrm{matter}}[N]\} \tag{14}$$

describes the evolution of perturbed extrinsic curvature. Using (13), one can derive the second order equation of motion for gravitational tensor mode perturbations:

$$\tfrac{1}{2}[\ddot{h}^i_a + 2\bar{k}\dot{h}^i_a - \nabla^2 h^i_a] = 8\pi G\Pi^i_a, \tag{15}$$

where

$$\Pi^i_a = \bigg[\frac{1}{3V_0}\frac{\partial H_{\mathrm{matter}}}{\partial\bar{p}}\bigg(\frac{\delta E^c_j \delta^j_a \delta^i_c}{\bar{p}}\bigg) + \frac{\delta H_{\mathrm{matter}}}{\delta(\delta E^a_i)}\bigg]. \tag{16}$$

As usual, in the absence of source terms (15) has propagating wave solutions which are the usual gravitational waves in the given cosmological background. Cosmological expansion leads to a friction term which is proportional to $\bar{k}$ and thus the Hubble parameter.

The quantity Π^i_a describes the linear transverse and traceless source terms that can be related to the transverse and traceless part of the perturbed stress-energy tensor as $\Pi^i_a = \bar{p}\delta T^{(t)i}{}_a$. For comparison, we now demonstrate the explicit relation between Π^i_a and the stress-energy tensor which is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{matter}}}{\delta g^{\mu\nu}} \tag{17}$$

for a given matter action S_{matter}. Including only tensor perturbations, the inverse spatial metric can be written as

$q^{ab} = g^{ab}$ since the shift vector N^a does not contribute to tensor perturbations, i.e. $N^a = 0$. Thus, space-space components of the stress-energy tensor are

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{ab}} = \frac{2}{N\sqrt{q}}\frac{\delta H_{\rm matter}[N]}{\delta q^{ab}}, \tag{18}$$

where q is the determinant of the spatial metric q_{ab}. The inverse spatial metric q^{ab} is related to the densitized triad by $qq^{ab} = E^a_i E^b_i$, for which we use the perturbed form $E^a_i = \bar{E}^a_i + \delta E^a_i$ with $\bar{E}^a_i = \bar{p}\delta^a_i$. In perturbation theory both $\bar{E}^a_i$ and δE^a_i are treated as independent degrees of freedom and one can express the stress-energy tensor (18) up to linear order in perturbations as

$$\begin{aligned}T_{ab} &= \frac{2}{N\sqrt{q}}\left[\left(\frac{\partial\bar{E}^c_j}{\partial q^{ab}}\right)_{\delta E^c_j}\frac{\delta H_{\rm matter}}{\delta\bar{E}^c_j} + \left(\frac{\partial(\delta E^c_j)}{\partial q^{ab}}\right)_{\bar{E}^c_j}\frac{\delta H_{\rm matter}}{\delta(\delta E^c_j)}\right]\\ &= \frac{2}{\bar{N}\bar{p}^{3/2}}\left[\left(\delta^j_c\frac{\partial\bar{E}^c_j}{\partial q^{ab}}\right)_{\delta E^c_j}\frac{1}{3V_0}\frac{\partial H_{\rm matter}}{\partial\bar{p}}\right.\\ &\quad\left. + \left(\frac{\partial(\delta E^c_j)}{\partial q^{ab}}\right)_{\bar{E}^c_j}\frac{\delta H_{\rm matter}}{\delta(\delta E^c_j)}\right].\end{aligned} \tag{19}$$

With the relation between the inverse spatial metric and the densitized triad, one can show that

$$\begin{aligned}\left(\delta^j_c\frac{\partial\bar{E}^c_j}{\partial q^{ab}}\right)_{\delta E^c_j} &= \bar{p}^2\left[-\delta_{ab} + \frac{5\delta_{ae}\delta_{bf}E^{(e}_i\delta^{f)}_i}{2\bar{p}}\right];\\ \left(\frac{\partial(\delta E^c_j)}{\partial q^{ab}}\right)_{\bar{E}^c_j} &= \frac{\bar{p}^2}{2}[\delta^{(c}_a\delta_{j)b} - \delta^c_j\delta_{ab}]\end{aligned} \tag{20}$$

using the fact that tensor perturbations are symmetric and traceless i.e. $\delta^j_c\delta E^c_j = 0$. While in the first equation we have kept terms up to first order in perturbations, in the second equation we have kept only the zeroth order terms as the term $(\delta H_{\rm m}/\delta(\delta E^c_j))$ itself is at least of first order in perturbations. We then compute the perturbed stress-energy tensor

$$\begin{aligned}\delta T^i_a &:= T^i_a - \bar{T}^i_a = \delta^i_c(q^{cb}T_{ab}) - \bar{T}^i_a\\ &= \frac{1}{\bar{N}\sqrt{\bar{p}}}\left[\frac{1}{3V_0}\frac{\partial H_{\rm matter}}{\partial\bar{p}}\frac{(\delta E^c_j\delta^i_c\delta^j_a)}{\bar{p}} + \frac{\delta H_{\rm matter}}{\delta(\delta E^a_i)}\right]\\ &= \frac{1}{\bar{N}\sqrt{\bar{p}}}\Pi^i_a,\end{aligned} \tag{21}$$

where we have used the requirement that for tensor perturbation, perturbed stress-energy tensor is trace-free, i.e. $\delta^c_j(\delta H_{\rm matter}/\delta(\delta E^c_j)) = 0$. The background stress-energy tensor $\bar{T}^i_a$ is given by

$$\bar{T}^{\,i}_{\ a} = -\frac{\delta^i_a}{\bar{N}V_0\bar{p}^{3/2}}\left(\frac{2\bar{p}}{3}\frac{\partial H_{\rm matter}}{\partial\bar{p}}\right). \tag{22}$$

This expression explicitly shows the relation between spatial components of the background stress-energy tensor and background pressure.

IV. QUANTUM DYNAMICS

In the previous section, we have seen how the tensor mode equation is derived from canonical classical cosmology. We will now include two basic types of quantum corrections that are expected from the Hamiltonian of loop quantum gravity. These corrections arise for inverse powers of the densitized triad, which when quantized becomes an operator with zero in the discrete part of its spectrum thus lacking a direct inverse [19], and from the fact that a loop quantization is based on holonomies, i.e. exponentials of the connection rather than direct connection components. There is an additional source of corrections due to backreaction effects of quantum fluctuations on expectation values of the basic variables [20,21]. This is more complicated to derive and not included in the present analysis. We need to consider these corrections only in the Hamiltonian constraint because the full diffeomorphism constraint does not receive quantum corrections. It thus remains trivial for the tensor mode dynamics as in the classical case. We now consider these two basic types of corrections to Hamiltonian constraint separately, which is justified because they have different origins in properties of quantum geometry. Keeping them separate provides valuable insights in physical consequences of these different geometrical effects.

A. Inverse volume corrections

In loop quantum gravity, the factor $E^c_jE^d_k/\sqrt{|\det E|}$, which appears in the Hamiltonian constraint (10) and contains inverse powers of the densitized triad, cannot be quantized directly but only after it is reexpressed as a Poisson bracket not involving an inverse [19]. In homogeneous models, explicit calculations show that eigenvalues of the resulting operator approximate the classical expression for large values of densitized triad components, but do provide quantum corrections which become larger for small components [22–24]. One can include these corrections as one of the new terms in effective expressions by introducing a factor $\bar{\alpha}$ whose generic form in the large volume regime is

$$\bar{\alpha}(\bar{p}) = 1 + c\left(\frac{\ell_{\rm P}^2}{\bar{p}}\right)^n, \tag{23}$$

where n and c are positive numbers. Anticipating similar quantum corrections even for the inhomogeneous case, the effects of such a correction have already been studied for scalar and vector mode perturbations [11,14]. Here, we provide an analysis for tensor mode perturbations, starting with a corrected Hamiltonian constraint

$$H_{\rm G}^{\rm phen}[N] = \frac{1}{16\pi G}\int_\Sigma {\rm d}^3x \bar{N}\alpha(\bar{p},\delta E^a_i)\Big[-6\bar{k}^2\sqrt{\bar{p}} - \frac{\bar{k}^2}{2\bar{p}^{3/2}}(\delta E^c_j\delta E^d_k\delta^k_c\delta^j_d) + \sqrt{\bar{p}}(\delta K^j_c\delta K^k_d\delta^c_k\delta^d_j) - \frac{2\bar{k}}{\sqrt{\bar{p}}}(\delta E^c_j\delta K^j_c) + \frac{1}{\bar{p}^{3/2}}(\delta_{cd}\delta^{jk}\delta^{ef}\partial_e E^c_j\partial_f E^d_k)\Big]. \tag{24}$$

(We indicate quantum corrected expressions by a superscript "phen" to indicate that such terms are introduced for a phenomenological analysis while a systematic effective analysis is still outstanding.) This is to be used in a perturbative inhomogeneous context and is thus not set in a purely minisuperspace model. In this case, $\alpha(\bar{p}, \delta E^a_i)$ also depends on triad perturbations and is in general more complicated to compute from an underlying Hamiltonian operator than in homogeneous models. Moreover, since the function α comes from the quantized inverse densitized triad where the tensorial term $E^c_jE^d_k/\sqrt{|\det E|}$ is quantized as a whole, it could be tensorial in nature. However, later we will see that its leading effect on perturbation dynamics comes from the background corrections $\alpha(\bar{p}, \delta E^a_i = 0) = \bar{\alpha}$.

The only background variable determining the geometry is $\bar{p}$, as a function of which the corrections are expressed. The appearance of such a scale factor dependent function in dynamical equations has occasionally led to concerns that quantum gravity might break the scale invariance of flat isotropic models, or even introduce gauge artefacts. Alternatively, one can absorb the rescaling freedom in a redefinition of the fiducial volume V_0 encountered earlier, but then the dynamical equations as well as their solutions seem to depend on this fiducial volume. None of these problems occurs in genuine inhomogeneous models. The dependence of a correction function α in an inhomogeneous Hamiltonian constraint is through elementary area variables whose values are determined by an underlying inhomogeneous state. (Areas, or more precisely fluxes, are elementary because they are directly related to the densitized triad as a canonical variable.) These elementary areas build up the quantum geometry of space in a discrete manner and their sizes determine the degree of discreteness involved. The scale of corrections, too, is determined by the underlying state and thus depends on the size of discreteness.

More specifically, correction functions only seem to depend directly on the scale factor a because other parameters, most importantly the number of lattice sites $\mathcal{N}$ per volume in the underlying state, have been suppressed (see also [25]). This parameter rescales in the same way as the scale factor such that the whole expression is scaling invariant. Elementary areas are the primary object appearing in corrections and they are, on average, of the geometrical size $F = a^2\ell_0^2$ where ℓ_0 is the average coordinate length of lattice links. This quantity is certainly scaling independent. Moreover, ℓ_0 is related to $\mathcal{N}$ and thus depends on the precise quantum state and has to be determined from the underlying theory. The parameter $\mathcal{N}$, however, also depends on the chosen volume V_0 in which one counts the number of lattice sites: $\mathcal{N} = V_0/\ell_0^3$. One can identify V_0 with the fiducial volume introduced earlier. Then, an alternative worry has been voiced, namely, that a scaling invariant quantum correction would depend explicitly on the fiducial volume. Also this is not true: One simply rewrites the quantity F as before in a different way, $F = a^2V_0^{2/3}/\mathcal{N}^{2/3}$. Numerator and denominator are now scaling independent but V_0 dependent. Nevertheless, the total quantity F which appears in quantum corrections from the inverse volume is V_0 independent. Such quantum corrections are thus consistent and do not depend on any gauge or other choices.

Extrinsic curvature is derived using Hamilton's equation of motion (12). With quantum corrections in the Hamiltonian, also extrinsic curvature should receive quantum corrections. In our case of a Hamiltonian (24), this leads to

$$\delta K_a{}^i = \frac{1}{2}\Big[\frac{1}{\bar{\alpha}}\dot{h}^i_a + \bar{k}h^i_a\Big]. \tag{25}$$

Here one can see that the leading correction due to the background correction function is $\bar{\alpha}$ as inhomogeneous contributions to α will contribute only higher order terms. The second Hamilton's equation together with the just derived expression of extrinsic curvature (25) then provides a second order equation

$$\frac{1}{2}\Big[\frac{1}{\bar{\alpha}}\ddot{h}^i_a + 2\bar{k}\Big(1 - \frac{\bar{\alpha}'\bar{p}}{\bar{\alpha}}\Big)\dot{h}^i_a - \bar{\alpha}\nabla^2 h^i_a\Big] + \mathcal{A}^i_a = 8\pi G\Pi^i_a \tag{26}$$

for the dynamics of tensor mode perturbations, where the prime denotes a derivative by $\bar{p}$ and

$$\mathcal{A}^{\ i}_a = 3\bar{N}\bar{k}^2\sqrt{\bar{p}}\Big[\frac{\partial\alpha}{\partial(\delta E^a_i)} + \frac{1}{3\bar{p}}\frac{\partial\alpha}{\partial\bar{p}}(\delta E^d_k\delta^k_a\delta^i_d)\Big]. \tag{27}$$

Inverse densitized triad corrections lead to several significant changes in the wave equation (26) compared to its classical counterpart (15). First, there are corrections in the coefficient of $\ddot{h}^i_a$ and the coefficient of the Laplacian term $\nabla^2 h^i_a$. Second, there are additional contributions to the friction term and, third, an entirely new term $\mathcal{A}^i_a$. In the context of vector mode dynamics [14], the same term $\mathcal{A}^i_a$ appears in the equation of motion but it also presents an anomaly term in the constraint algebra between the perturbed Hamiltonian and diffeomorphism constraints. Requiring an anomaly-free constraint algebra in the presence of quantum corrections then implies that $\mathcal{A}^i_a$ must vanish and leads to restrictions on the possible functional form of the quantum correction function $\alpha(\bar{p}, \delta E^a_i)$. While there are no such anomalies in the constraint algebra for tensor modes as the diffeomorphism constraint is trivial here, the same quantum correction function $\alpha(\bar{p}, \delta E^a_i)$ as for the vector mode must occur since there is only one Hamiltonian constraint which is just split into different mode contributions to simplify the analysis. Thus, we must set $\mathcal{A}^i_a$ to zero, which we will do in the subsequent analysis.

B. Holonomy corrections

A loop quantization represents holonomies as basic operators on a Hilbert space rather than connection components. Moreover, it is impossible to derive operators for connection components from holonomies and thus any quantized expression depending on the connection must do so through holonomies. This is especially true for the Hamiltonian constraint, which thus receives quantum corrections from higher powers of the connection. Holonomies are nonlinear as well as (spatially) nonlocal in connection components. Thus, they provide higher order and higher spatial derivative terms. Higher time derivatives, as they would also be provided by higher curvature terms, do not arise in this way but rather through the coupling of fluctuations and higher moments of a quantum state to the expectation values [20,21]. Here we focus on corrections from holonomies as a typical effect of a loop quantization, while the more complicated quantum backreaction effects are genuine and occur for any interacting quantum theory.

We start by recalling the situation for a homogeneous and isotropic model with a massless free scalar field. This model allows one to compute an exact effective Hamiltonian [26]

$$\bar{H}^{\rm eff}_{\rm G}[\bar{N}] = \frac{\bar{N}V_0}{16\pi G}\left[-6\sqrt{\bar{p}}\left(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2\right], \tag{28}$$

where higher order terms of extrinsic curvature (which is proportional to the Ashtekar connection in a spatially flat model) are explicit in the sine. Again, V_0 is the volume of the fiducial cell introduced to avoid the integration over spatial infinity in (10) for a homogeneous background. Moreover, $\bar{\mu}$ is a new parameter related to the action of the fundamental Hamiltonian on a lattice state. It can be understood as the coordinate size of a loop whose holonomy is used to quantized the Ashtekar curvature components F^i_{ab}. In the limit $\bar{\mu} \to 0$, the effective Hamiltonian reduces to the standard classical Hamiltonian. In general, $\bar{\mu}$ can even depend on the triad component $\bar{p}$ to reflect refinements of the discrete state during dynamics [25]. While the precise behavior is difficult to compute, general considerations restrict the dependence to $\bar{\mu}(\bar{p}) = \bar{p}^n$ where $0 < n < -1/2$. Only the limiting cases $n = 0$ [18,27] and $n = -1/2$ [28] have so far been discussed in the literature.

Variation of the Hamiltonian constraint with respect to the background lapse function $\bar{N}$ leads to the effective Friedmann equation

$$\frac{1}{\bar{p}}\left(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2 = \frac{8\pi G}{3}\rho, \tag{29}$$

where ρ is the energy density defined as

$$\rho := \frac{1}{(V_0 p^{3/2})}\frac{\delta\bar{H}_{\rm matter}}{\delta\bar{N}}.$$

In this form, the effective equation is precise only for the energy density of a free scalar, $H_{\rm matter} = \frac{1}{2}\bar{N}V_0\bar{p}^{-3/2}p_\phi^2$ with momentum p_ϕ. If a matter potential or anisotropies and inhomogeneities are added, additional corrections arise [29] from quantum backreaction.

While classical cosmological dynamics is in general singular, the effective dynamics is nonsingular. The singularity avoidance is achieved by exhibiting a bounce at small volume when the energy density reaches a critical value [8]. This can be seen by writing the effective Friedmann equation (29) as $(\sin\bar{\mu}\gamma\bar{k})^2 = \rho/\rho_{\rm c}$ where

$$\rho_{\rm c} = \frac{3}{8\pi G\bar{\mu}^2\gamma^2\bar{p}}. \tag{30}$$

The boundedness of the sine then implies a minimum p and thus a minimum nonzero volume, the bounce scale. For the case $\bar{\mu} = \sqrt{\Delta/\bar{p}}$ for instance, $\rho_{\rm c}$ is a constant. The critical energy density $\rho_{\rm c}$ then signifies the maximum energy density that is reached at the bounce point. This can be seen explicitly from the Hamilton's equations of motion which are

$$\dot{\bar{p}} = 2\bar{N}\sqrt{\bar{p}}\left(\frac{\sin 2\bar{\mu}\gamma\bar{k}}{2\bar{\mu}\gamma}\right) \tag{31}$$

and

$$\dot{\bar{k}} = -\bar{N}\frac{\partial}{\partial\bar{p}}\left[\sqrt{\bar{p}}\left(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2\right] + \frac{8\pi G}{3V_0}\frac{\partial\bar{H}_{\rm matter}}{\partial\bar{p}}. \tag{32}$$

Thus, for an isotropic model sourced by a massless, free scalar field the effective Hamiltonian can be obtained by simply replacing the background Ashtekar connection $\gamma\bar{k}$ by $\bar{\mu}^{-1}\sin\bar{\mu}\gamma\bar{k}$. This is no longer true for other models, especially when inhomogeneities are included. But to study the effects on inhomogeneous perturbations, one can substitute the appearance of $\bar{k}$ in the classical Hamiltonian by a general form $\frac{\sin m\bar{\mu}\gamma\bar{k}}{m\bar{\mu}\gamma}$ where m is a number. There may be additional corrections, but qualitative effects can already be read off from such a replacement. With this prescription, the Hamiltonian constraint becomes

$$H_{\mathrm{G}}^{\mathrm{phen}}[N] = \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^3x\bar{N}\bigg[-6\sqrt{\bar{p}}\bigg(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\bigg)^2 - \frac{1}{2\bar{p}^{3/2}} \times\bigg(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\bigg)^2(\delta E^c_j\delta E^d_k\delta^k_c\delta^j_d) + \sqrt{\bar{p}}(\delta K^j_c\delta K^k_d\delta^c_k\delta^d_j) - \frac{2}{\sqrt{\bar{p}}}\bigg(\frac{\sin 2\bar{\mu}\gamma\bar{k}}{2\bar{\mu}\gamma}\bigg) \times(\delta E^c_j\delta K^j_c) + \frac{1}{\bar{p}^{3/2}}(\delta_{cd}\delta^{jk}\delta^{ef}\partial_e E^c_j\partial_f E^d_k)\bigg]. \tag{33}$$

In writing the explicit coefficients we have required that the Hamiltonian has a "homogeneous" limit in agreement with what has been used in isotropic models (28). This fixes the parameter m to equal one in the first two terms. The parameter for the last term as chosen here is the one which leads to an anomaly-free constraint algebra in the context of vector modes [14].

One should keep in mind that, although we write explicit sines in this expression and thus arbitrarily high powers of curvature components, this is to be understood only as a short form to write the leading order corrections. This is more compact than writing the leading terms of a Taylor expansion of the sines. The expressions are, however, reliable only when the argument of the sines is small, which excludes the bounce phase itself. Moreover, higher orders are supplemented by further, yet to be computed higher curvature quantum corrections. (Such sine corrections can be used throughout the bounce phase only for exactly isotropic models sourced by a free, massless scalar [26,30].)

The expression for extrinsic curvature is again derived using one of Hamilton's equations of motion and thus receives quantum corrections also from the use of holonomies in loop quantum gravity,

$$\delta K_a{}^i = \frac{1}{2}\bigg[\dot{h}^i_a + \bigg(\frac{\sin 2\bar{\mu}\gamma\bar{k}}{2\bar{\mu}\gamma}\bigg)h^i_a\bigg]. \tag{34}$$

Along with Hamilton's equation for the perturbed extrinsic curvature, this Eq. (34) then yields the quantum corrected second order equation for tensor perturbations

$$\frac{1}{2}\bigg[\ddot{h}^i_a + \bigg(\frac{\sin 2\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\bigg)\dot{h}^i_a - \nabla^2 h^i_a + T_Q h^i_a\bigg] = 8\pi G\Pi^i_{Qa}. \tag{35}$$

This Eq. (35) describes propagating degrees of freedom which are the usual gravitational waves subject to quantum corrections. Unlike for inverse densitized triad corrections, the coefficients of $\ddot{h}^i_a$ and $\nabla^2 h^i_a$ take the classical form. On the other hand, the friction term does receive corrections. As a new feature, there is an additional term proportional to field perturbations h^i_a with coefficient

$$T_Q = -2\bigg(\frac{\bar{p}}{\bar{\mu}}\frac{\partial\bar{\mu}}{\partial\bar{p}}\bigg)\bar{\mu}^2\gamma^2\bigg(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\bigg)^4. \tag{36}$$

For any $\bar{\mu} \propto |\bar{p}|^n$ with $n < 0$, T_Q is positive definite. Finally, the source terms from the matter Hamiltonian take the form

$$\Pi_{Qa}{}^i = \bigg[\frac{1}{3V_0}\frac{\partial H_{\mathrm{matter}}}{\partial\bar{p}}\bigg(\frac{\delta E^c_j\delta^j_a\delta^i_c}{\bar{p}}\bigg)\cos 2\bar{\mu}\gamma\bar{k} + \frac{\delta H_{\mathrm{matter}}}{\delta(\delta E^a_i)}\bigg] \tag{37}$$

as the transverse and traceless part of the stress-energy tensor that sources gravitational waves. The additional cosine can be understood from the fact that the background geometry receives quantum corrections and is used to define the trace-free part of stress-energy. The source $\Pi_{Qa}{}^i$ vanishes when there is no matter field, and it reduces to the classical transverse and traceless part of the stress-energy tensor Π^i_a in the limit $\bar{\mu} \to 0$.

V. DISPERSION RELATION

To study wave propagation it is often convenient to compute the relevant dispersion relation from the corresponding wave equation, presenting a relation between the frequency and the wave vector. In this section, we use dispersion relations for the quantum corrected gravitational wave equations to study some of their basic properties.

Starting with the classical dispersion relation to be able to contrast it with the corrected versions later on, we consider the source-free tensor mode perturbation equation by making a plane wave ansatz $h^i_a \propto \tilde{h}^i_a \exp(i\omega t - i\mathrm{k}.\mathbf{x})$. Here, the frequency ω corresponds to proper time t where the lapse function $\bar{N}$, in contrast to the previous section, is equal to unity. The classical tensor mode Eq. (15) then simply implies

$$\omega^2 = \bigg(\frac{\mathrm{k}}{a}\bigg)^2. \tag{38}$$

Here we have ignored the friction term in the equation of motion (15) since we are mainly interested in local propagation not involving cosmic scales. The dispersion relation (38) is, of course, the standard classical dispersion relation between the frequency ω and the proper wave number k/a. We further note that the corresponding group velocity of gravitational waves

$$v_{\mathrm{gw}} := \frac{\mathrm{d}\omega}{\mathrm{d}(k/a)}$$

is equal to 1 (in natural units).

A. Inverse volume corrections

Repeating the calculations of the classical case but using quantum corrections in the wave equation we obtain the corrected dispersion relations. In particular, the tensor mode Eq. (26) in the presence of inverse volume corrections leads to

$$\omega^2 = \bar{\alpha}^2\left(\frac{\mathrm{k}}{a}\right)^2 \tag{39}$$

as illustrated in Fig. 1.

As one can see, the quantum correction function multiplies the wave number k, thus affecting the mode on all scales. Moreover, given that $\bar{\alpha} > 1$, the corrected group velocity due to inverse volume corrections is greater than unity. This may appear as a violation of causality since gravitational waves would travel faster than with the speed of light. However, this refers to the classical speed of light, while a physical statement requires us to compare the velocity to the physical speed of light. This differs from the classical one because also the Maxwell Hamiltonian receives inverse volume corrections in loop quantum gravity [31]. In the regime of linear inhomogeneities such corrections have been computed in [32], and a derivation of the quantum corrected group velocity of electromagnetic waves, which we present in Sec. VI, shows that it is not smaller than that of gravitational waves. Thus, there are no violations of causality.

B. Holonomy corrections

We finally consider corrections to the dispersion relation of gravitational waves due to the appearance of holonomies. Again ignoring the friction term, a plane wave ansatz in the wave equation (35) leads to

$$\omega^2 = \left(\frac{\mathrm{k}}{a}\right)^2 + m_{\mathrm{g}}^2, \tag{40}$$

where

$$m_{\mathrm{g}}^2 := \frac{T_Q}{a^2} = \frac{1}{\Delta\gamma^2}\left(\frac{\rho}{\rho_{\mathrm{c}}}\right)^2. \tag{41}$$

One may note that holonomy corrections effectively contribute a new additive term m_{g}^2 in the dispersion relation (40) compared to the classical dispersion relation (38). With this quantum correction, the gravitational wave has

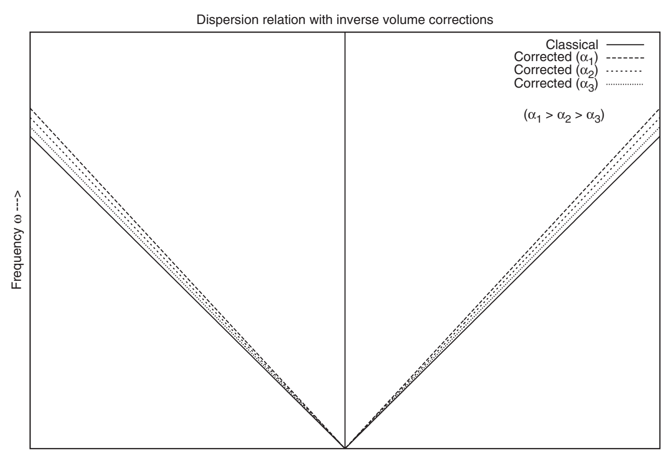

FIG. 1. Dispersion relation for gravitational waves in the presence of inverse volume corrections.

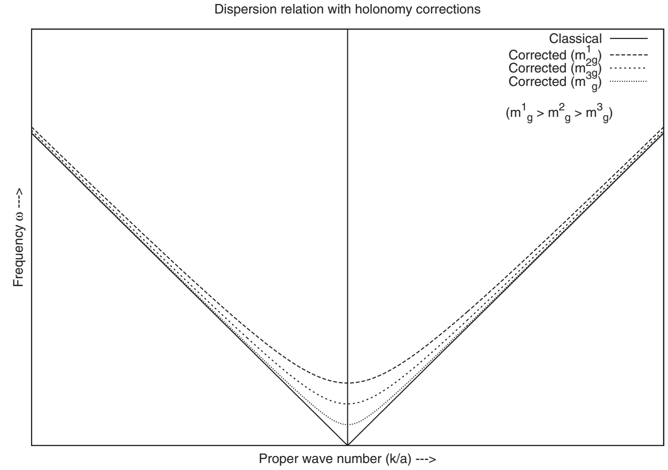

FIG. 2. Dispersion relation in the presence of holonomy corrections. The classical dispersion relation is approached as the background energy density decreases.

acquired an "effective mass." This, in turn, implies that the different modes of the gravitational waves propagate with different group velocities which are less than unity. Also here, causality is thus respected because the curvature independent electromagnetic Hamiltonian does not receive holonomy corrections. The corrected dispersion relation (40) is shown in Fig. 2.

Using expression (41), one can estimate the value of the "effective mass" of the graviton at the present epoch. Given the value of $\gamma \sim O(1)$, $\Delta \sim O(1)\ell_{\mathrm{P}}^2$, $\rho_{\mathrm{c}} \sim O(1)M_{\mathrm{P}}^4$ and the energy density $\rho \sim 10^{-120}M_{\mathrm{P}}^4$ of the present universe one obtains the value $m_{\mathrm{g}} \sim 10^{-120}M_{\mathrm{P}} = 10^{-92}$ eV. Here ℓ_{P} and M_{P} are Planck length and mass, respectively. Current observational bounds on the graviton mass from solar system measurements is $m_{\mathrm{g}} < 4.4 \times 10^{-22}$ eV and its accuracy could be lowered up to $m_{\mathrm{g}} < 10^{-26}$ eV from future gravitational wave measurements [33–36]. Thus, our estimated theoretical value of the "effective graviton mass" is well below the observational bound at present. It is unlikely that such a value could be tested observationally in the near future. However, given that the "effective mass" depends on the background energy density, such an effective mass could play a significant role in early universe physical phenomena such as inflation.

VI. CAUSALITY

To determine whether causality is respected by the quantum corrections, we have to compare the propagation speed of gravitational waves to the physical speed of light. Just as tensor perturbations of the metric receive quantum gravity corrections, the electromagnetic field also is corrected. Thus, the speed of its wave excitations may differ from the classical value in the same way in which the gravitational wave velocity differs from the classical one. For an analysis of causality the two corrected velocities have to be compared.

The basic field of Maxwell's theory of electromagnetism is the vector potential A_μ. Its source-free dynamics in a general space-time background is governed by the action

$$S_{\rm EM} = -\frac{1}{16\pi}\int {\rm d}^4x\sqrt{-g}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}, \tag{42}$$

where the background space-time is specified by the Lorentzian space-time metric $g_{\mu\nu}$. We again use the convention where Greek letters denote space-time indices whereas small Latin letters denote spatial indices.

A. Canonical formulation of the electromagnetic field

In a canonical formulation, as before, the space-time metric is decomposed into the spatial metric $q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ and normal components which provide the nondynamical lapse function and shift vector. Also the electromagnetic fields are decomposed, with the electric field π^a arising from the space-time components of the field strength, and the purely spatial components F_{ab} giving the magnetic field. The total Hamiltonian for the electromagnetic field corresponding to the action (42) can be written as

$$\mathcal{H}_{\rm EM} = H_{\rm EM}[N] + D_{\rm EM}[N^a] + G_{\rm EM}, \tag{43}$$

where $H_{\rm EM}[N]$ denotes the electromagnetic contribution

$$H_{\rm EM}[N] = \int_\Sigma {\rm d}^3xN\bigg[\frac{2\pi}{\sqrt{q}}\pi^c\pi^d q_{cd} + \frac{\sqrt{q}}{16\pi}F_{cd}F_{ef}q^{ce}q^{df}\bigg] \tag{44}$$

to the Hamiltonian constraint. Similarly, there is a contribution

$$D_{\rm EM}[N^a] = \int_\Sigma {\rm d}^3xN^c[F_{cd}\pi^d] \tag{45}$$

to the diffeomorphism constraint, and a U(1)-Gauss constraint

$$G_{\rm EM} = \int_\Sigma {\rm d}^3x[-A_0\partial_c\pi^c]. \tag{46}$$

As usual in canonical formulations, time and space components of the physical fields such as A_0 and A_a play different roles for the dynamics. Variation of $G_{\rm EM}$ with respect to A_0, the time component of A_μ, whose conjugate momentum is absent in (43) and which is thus a Lagrange multiplier, leads to the usual expression

$$\partial_a\pi^a = 0 \tag{47}$$

of the Gauss law.

Hamilton's equations of motion for the canonical fields A_a and π^a take the form $\dot{f} = \{f, \mathcal{H}_{\rm EM}\}$, explicitly given by

$$\dot{A}_a = \partial_a A_0 + N^cF_{ca} + \frac{4\pi N}{\sqrt{q}}\pi^c q_{ca}, \tag{48}$$

and

$$\dot{\pi}^a = \partial_c(N^c\pi^a) - \partial_d(N^a\pi^d) + \frac{1}{4\pi}\partial_c(N\sqrt{q}F_{ef}q^{ce}q^{af}). \tag{49}$$

For further details of the canonical analysis we refer to [32].

B. Classical propagation

As before, we analyze the propagation of linear electromagnetic waves on a spatially flat Friedmann-Robertson-Walker background. (Small perturbations of the electromagnetic wave will induce small perturbations for geometric variables as well. However, to linear order the perturbations are independent of each other and can thus be studied separately.) As before, the spatial metric q_{ab} and shift vector N^a then are

$$q_{ab} = a^2\delta_{ab}; \qquad N^a = 0, \tag{50}$$

where $a(t)$ is the scale factor.

The momentum can be eliminated from (48) by computing its divergence and using the Gauss constraint (47):

$$\partial_t(\partial^a A_a) - \nabla^2 A_t = 0, \tag{51}$$

where ∂_t refers to the time derivative according to Hamilton's equations of motion, $\partial^c = \delta^{ec}\partial_e$ and $\delta^{ec}\partial_e\partial_c = \nabla^2$. To satisfy this equation we make the standard gauge choices $A_0 = 0$ and $\partial^a A_a = 0$. The Eqs. (48) and (49) together then lead to the electromagnetic wave equation

$$\partial_t\bigg(\frac{a}{N}\partial_t A_a\bigg) - \bigg(\frac{N}{a}\bigg)\nabla^2 A_a = 0. \tag{52}$$

With the choice of proper time i.e. $N = 1$, the wave equation (52) has the usual friction term due to the evolving cosmological background whereas with the choice of conformal time i.e. $N = a$, the friction term will not be explicitly present.

From the equations of motion we can again compute the dispersion relation using the standard wave ansatz $A_a \sim \tilde{A}_a \exp i(\omega t + {\rm k}.\mathbf{x})$. Here, we will choose $N = 1$ so that the frequency ω corresponds to proper time, but we ignore the friction term which is justified for small wavelengths compared to cosmological scales. The classical wave equation (52) then leads to the standard dispersion relation

$$\omega^2 = \bigg(\frac{\rm k}{a}\bigg)^2. \tag{53}$$

Moreover, the group velocity of electromagnetic wave propagation is

$$v_{\rm EM} = \frac{{\rm d}\omega}{{\rm d}({\rm k}/a)} = 1 \tag{54}$$

which is constant in a classical cosmological background.

C. Propagation in the presence of quantum gravity corrections

Quantum gravity corrections mainly affect the Hamiltonian constraint, which becomes [31,32,37]

$$H^{\rm phen}_{\rm EM}[N] = \int_\Sigma {\rm d}^3x N\bigg[\alpha_{\rm EM}(q_{cd})\frac{2\pi}{\sqrt{q}}\pi^a\pi^b q_{ab} + \beta_{\rm EM}(q_{cd})\frac{\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}\bigg], \quad (55)$$

where $\alpha_{\rm EM}(q_{cd})$ and $\beta_{\rm EM}(q_{cd})$ are the correction functions due to quantum gravity effects in inverse triad components. This provides equations of motion

$$\dot{A}_a = \partial_a(t^\mu A_\mu) + N^c F_{ca} + \frac{4\pi N}{\sqrt{q}}\alpha_{\rm EM}\pi^c q_{ca}, \quad (56)$$

and

$$\dot{\pi}^a = \partial_c(N^c\pi^a) - \partial_d(N^a\pi^d) + \frac{1}{4\pi}\partial_c(N\beta_{\rm EM}\sqrt{q}F_{ef}q^{ce}q^{af}). \quad (57)$$

Using the same gauge fixing $A_t = 0$ and $\partial^a A_a = 0$, which is possible since the Gauss constraint does not receive quantum corrections, one obtains the corrected wave equation

$$\partial_t\bigg(\frac{a}{N\bar{\alpha}_{\rm EM}}\partial_t A_a\bigg) - \bigg(\frac{N\bar{\beta}_{\rm EM}}{a}\bigg)\nabla^2 A_a = 0, \quad (58)$$

where $\bar{\alpha}_{\rm EM} := \alpha_{\rm EM}|_{q_{cd}=a^2\delta_{cd}}$ and $\bar{\beta}_{\rm EM} := \beta_{\rm EM}|_{q_{cd}=a^2\delta_{cd}}$. This provides the dispersion relation

$$\omega^2 = \bar{\alpha}_{\rm EM}\bar{\beta}_{\rm EM}\bigg(\frac{\rm k}{a}\bigg)^2 \quad (59)$$

and group velocity

$$v_{\rm EM} = \frac{{\rm d}\omega}{{\rm d}({\rm k}/a)} = \sqrt{\bar{\alpha}_{\rm EM}\bar{\beta}_{\rm EM}}. \quad (60)$$

As in the case of gravitational waves, we see that $v_{\rm EM} > 1$ is larger than the classical value, since both $\bar{\alpha}_{\rm EM}$ and $\bar{\beta}_{\rm EM}$ are always greater than one in perturbative regimes [10]. Second, the group velocity is no longer constant but varies with time as the universe expands. Varying speed of light has been studied in literature mainly in the cosmological context [38,39], and motivated for instance from bimetric gravity [40] or noncommutative geometry [41]; see [42] for a review.

D. Relation between the speed of gravitational waves and the speed of light

In the electromagnetic Hamiltonian (55) we have seen two quantum correction functions $\alpha_{\rm EM}$ and $\beta_{\rm EM}$. For homogeneous situations, they have the generic feature of being greater than unity while they approach unity in a classical limit. Based on the kinematical quantization alone, their values are not fixed but subject to quantization ambiguities. Similarly, in the gravity sector we have seen a quantum correction function α subject to ambiguities. *A priori*, these quantum correction functions are independent. On the other hand, these functions change the dispersion relations of gravitational as well as electromagnetic waves, and the corresponding changes in propagation velocities may give rise to concerns regarding causality. In particular, the propagation of gravitational waves may become superluminal depending on the precise form of correction functions.

There are, however, further consistency conditions once the dynamics of the quantum fields is considered. In a canonical formulation of general relativity, the classical constraints C_I form a first class Poisson algebra, i.e. $\{C_I, C_J\} = f^K_{IJ}(A,E)C_K$ whose coefficients $f^K_{IJ}(A,E)$ can in general be structure functions. The first class nature, i.e. the fact that the Poisson brackets of constraints vanish on the constraint surface defined by $C_I = 0$, ensures that the transformations generated by the constraints are gauge and are tangential to the constraint surface. Quantum correction functions such as $\alpha(E^a_i)$ change the constraints and thus their algebra. Making sure that the corrected constraints remain first class, i.e. that there is no anomaly, provides additional consistency conditions beyond those following from the kinematical quantization. As we will see, closure of the corrected constraint algebra, in particular, for the Poisson bracket of $H^{\rm phen}[N] := H^{\rm phen}_{\rm G}[N] + H^{\rm phen}_{\rm EM}$ with itself, leads to a relation between all the quantum correction functions in the matter and gravity sectors.

Specifically, the classical Hamiltonian constraint satisfies

$$\{H[N_1], H[N_2]\} = \{H_{\rm G}[N_1], H_{\rm G}[N_2]\} + \{H_{\rm EM}[N_1], H_{\rm EM}[N_2]\}, \quad (61)$$

where cross terms between matter and gravity contributions drop out because $H_{\rm EM}[N]$ couples minimally to gravity. On the other, the gravitational Hamiltonian constraint itself satisfies

$$\{H_{\rm G}[N_1], H_{\rm G}[N_2]\} = D_{\rm G}[N_1\partial^a N_2 - N_2\partial^a N_1], \quad (62)$$

where, without loss of generality, we assume the gravitational Gauss constraint to be solved. The matter term of expression (44) of $H_{\rm EM}[N]$

$$\{H_{\rm EM}[N_1], H_{\rm EM}[N_2]\} = D_{\rm EM}[N_1\partial^a N_2 - N_2\partial^a N_1]. \quad (63)$$

The Eqs. (61)–(63) together thus lead to

$$\{H[N_1], H[N_2]\} = D[N_1\partial^a N_2 - N_2\partial^a N_1], \quad (64)$$

where $D[N^a]$ is the total diffeomorphism constraint.

With quantum corrections we have the gravitational Hamiltonian constraint

$$H_{\rm G}^{\rm phen}[N] = \frac{1}{16\pi G}\int_\Sigma {\rm d}^3x N\alpha(E^a_i)\frac{E^c_jE^d_k}{\sqrt{|\det E|}} \times (\epsilon_i^{jk}F^i_{cd} - 2(1+\gamma^2)K^j_{[c}K^k_{d]}) \quad (65)$$

which now satisfies

$$\{H_{\rm G}^{\rm phen}[N_1], H_{\rm G}^{\rm phen}[N_2]\} = D_{\rm G}[\alpha^2(N_1\partial^aN_2 - N_2\partial^aN_1)] \quad (66)$$

(for details see [43]). For the corrected Maxwell Hamiltonian (55), on the other hand, we have

$$\{H_{\rm EM}^{\rm phen}[N_1], H_{\rm EM}^{\rm phen}[N_2]\} = D_{\rm EM}[\alpha_{\rm EM}\beta_{\rm EM}(N_1\partial^aN_2 - N_2\partial^aN_1)]. \quad (67)$$

This can be combined to a first class algebra of the total constraints if and only if

$$\alpha^2 = \alpha_{\rm EM}\beta_{\rm EM}, \quad (68)$$

such that

$$\{H^{\rm phen}[N_1], H^{\rm phen}[N_2]\} = D[\alpha^2(N_1\partial^aN_2 - N_2\partial^aN_1)]. \quad (69)$$

For linear waves, it is sufficient to use the relation (68) between the homogeneous parts of quantum correction functions, i.e. $\bar\alpha^2 = \bar\alpha_{\rm EM}\bar\beta_{\rm EM}$. They appear in the group velocities

$$v_{\rm gw} = \frac{{\rm d}\omega}{{\rm d}(k/a)} = \bar\alpha \quad \text{and} \quad v_{\rm EM} = \sqrt{\bar\alpha_{\rm EM}\bar\beta_{\rm EM}} \quad (70)$$

for gravitational and electromagnetic waves. Thus, the requirement of a closed constraint algebra, implying (68), ensures that there is no violation of causality: the corrected speed of gravitational waves agrees with the physical speed of light, which itself is subject to corrections.[4]

[4]In vacuum, the gravitational correction function α could be absorbed into the lapse function, and even so in the presence of an electromagnetic field after first using a duality transformation making $\alpha_{\rm EM} = \beta_{\rm EM}$ and then referring to anomaly-freedom such that $\alpha_{\rm EM} = \beta_{\rm EM} = \alpha$. The correction would then merely appear as a change in what is proper time, and a preservation of causality in this case would not be surprising. In fact, if α changes proper time, the dispersion relation derived from Eq. (26) for the tensor mode would, when formulated in the new proper time, be free of quantum corrections. This reasoning is not correct, however, because it overlooks the triad-dependence of the correction functions as well as the fact that corrections are dynamical rather than kinematical. The triad-dependence prevents the use of a simple duality transformation in the electromagnetic field since the correction functions would mix up the new symplectic structure between gravitational and electromagnetic variables. Even in vacuum, the corrections are nontrivial because they arise in the Hamiltonian, i.e. in dynamics, but not in the canonical form of the line element used to infer the form of proper time. The time part $-N^2{\rm d}t^2$ of a general line element is not affected by the corrections being considered, and thus proper time remains defined by ${\rm d}\tau = N{\rm d}t$. Corrections appear only in the dynamics, where they become noticeable.

VII. WAVE PROPAGATION AND LATTICE REFINEMENTS OF QUANTUM GRAVITY

As a further fundamental application, we analyze holonomy corrections in more detail because they can give insights into the precise form in which an underlying discrete quantum gravity state is being refined during its evolution. Holonomies as multiplication operators in loop quantum gravity can create new edges and vertices of a lattice state, and thus can dynamically imply its refinements. This can also be described at the effective level where, however, the complicated relation to the full theory requires one to refer to several parameters describing this refining behavior and, in particular, the functional form of $\bar\mu(\bar p)$ used before. Here we show that tensor mode dynamics can be used to restrict the possible choices.

We parametrize the Hamiltonian constraint as

$$H_{\rm G}^{\rm phen}[N] = \frac{1}{16\pi G}\int_\Sigma {\rm d}^3x\bar N\Big[-6\sqrt{\bar p}\Big(\frac{\sin\bar\mu\gamma\bar k}{\bar\mu\gamma}\Big)^2 - \frac{1}{2\bar p^{3/2}}\times\Big(\frac{\sin\bar\mu\gamma\bar k}{\bar\mu\gamma}\Big)^2(\delta E^c_j\delta E^d_k\delta^k_c\delta^j_d) + \sqrt{\bar p}(\delta K^j_c\delta K^k_d\delta^c_k\delta^d_j) - \frac{2}{\sqrt{\bar p}}\Big(\frac{\sin m\bar\mu\gamma\bar k}{m\bar\mu\gamma}\Big)\times(\delta E^c_j\delta K^j_c) + \frac{1}{\bar p^{3/2}}(\delta_{cd}\delta^{jk}\delta^{ef}\partial_eE^c_j\partial_fE^d_k)\Big], \quad (71)$$

where one parameter is m, the other appears in the power law form $\bar\mu(\bar p) \propto |\bar p|^n$. Here we have already required that the effective Hamiltonian (71) has a homogeneous limit in agreement with what has been used in isotropic models. This fixes the parameters analogous to m in the first two terms to equal one. The parameter for the last term cannot be fixed by taking the homogeneous limit and is thus kept free for now.

Expression (71) provides corrected second order equations

$$\frac{1}{2}\Big[\ddot h^i_a + \Big(\frac{\sin 2\bar\mu\gamma\bar k}{\bar\mu\gamma}\Big)\dot h^i_a - \nabla^2h^i_a + T_Qh^i_a\Big] = 8\pi G\Pi_{Qa}{}^i, \quad (72)$$

where

$$T_Q = \frac{1}{2}\Big(\frac{\sin\bar\mu\gamma\bar k}{\bar\mu\gamma}\Big)^2(\cos m\bar\mu\gamma\bar k - \cos 2\bar\mu\gamma\bar k) - \Big(\frac{\sin m\bar\mu\gamma\bar k}{m\bar\mu\gamma} - \frac{\sin 2\bar\mu\gamma\bar k}{2\bar\mu\gamma}\Big)^2 - 2\Big(\frac{\bar p}{\bar\mu}\frac{\partial\bar\mu}{\partial\bar p}\Big)\Big[2\bar\mu^2\gamma^2\Big(\frac{\sin\bar\mu\gamma\bar k}{\bar\mu\gamma}\Big)^4 - \Big(\frac{\sin\bar\mu\gamma\bar k}{\bar\mu\gamma}\Big)\Big(\cos\bar\mu\gamma\bar k\frac{\sin m\bar\mu\gamma\bar k}{m\bar\mu\gamma} - \cos m\bar\mu\gamma\bar k\frac{\sin\bar\mu\gamma\bar k}{\bar\mu\gamma}\Big)\Big]. \quad (73)$$

As before, corrections to the dispersion relation take the form of an effective mass term,

$$\omega^2 = \left(\frac{\mathrm{k}}{a}\right)^2 + m_{\mathrm{g}}^2 \tag{74}$$

where

$$\begin{aligned} m_{\mathrm{g}}^2 &:= \frac{T_Q}{a^2} \\ &\simeq \left[-2n\left(\frac{7-m^2}{3}\right) - \left(\frac{m^2-4}{4}\right) - \left(\frac{m^2-4}{6}\right)^2\right] \\ &\quad \times \frac{1}{\bar{\mu}^2\gamma^2\bar{p}}\left(\frac{\rho}{\rho_c}\right)^2. \end{aligned} \tag{75}$$

As one can see, this effective mass squared is not guaranteed to be positive for all parameter values. Thus, stability of the perturbation can be used as a criterion to restrict the ambiguities.

One can use anomaly cancellation to relate the free parameters, for which we have to refer to vector modes since the tensor mode equations are automatically anomaly-free. Specifically, we use the Poisson bracket between the diffeomorphism and Hamiltonian constraints and ensure that it is again linear in the constraints. For simplicity we will consider here only effects of source-free vector perturbations, and correspondingly assume that matter constraints vanish. The perturbed Hamiltonian constraint including only vector mode perturbations is

$$\begin{aligned} H_{\mathrm{G}}^{\mathrm{phen}}[N] = \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^3x\bar{N}\bigg[&-6\sqrt{\bar{p}}\left(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2 \\ &- \frac{1}{2\bar{p}^{3/2}}\left(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2(\delta E^c_j\delta E^d_k\delta^k_c\delta^j_d) \\ &+ \sqrt{\bar{p}}(\delta K^j_c\delta K^k_d\delta^c_k\delta^d_j) \\ &- \frac{2}{\sqrt{\bar{p}}}\left(\frac{\sin m\bar{\mu}\gamma\bar{k}}{m\bar{\mu}\gamma}\right)(\delta E^c_j\delta K^j_c)\bigg], \end{aligned} \tag{76}$$

and the perturbed diffeomorphism constraint is

$$D_{\mathrm{G}}[N^a] = \frac{1}{8\pi G}\int_\Sigma \mathrm{d}^3x\delta N^c[-\bar{p}(\partial_k\delta K^k_c) - \bar{k}\delta^k_c(\partial_d\delta E^d_k)]. \tag{77}$$

With the Hamiltonian constraint (76), we then have

$$\begin{aligned} \{H_{\mathrm{G}}^{\mathrm{phen}}[N], D_{\mathrm{G}}[N^a]\} = &\frac{\bar{N}}{\sqrt{\bar{p}}}\left(\bar{k} - \frac{m\sin 2\bar{\mu}\gamma\bar{k} - \sin m\bar{\mu}\gamma\bar{k}}{m\bar{\mu}\gamma}\right) \\ &\times D_{\mathrm{G}}[N^a] \\ &+ \frac{1}{8\pi G}\int_\Sigma \mathrm{d}^3x\bar{p}(\partial_c\delta N^j)\mathcal{A}^c_j, \end{aligned} \tag{78}$$

where

$$\begin{aligned} \mathcal{A}^c_j = \frac{\bar{N}}{\sqrt{\bar{p}}}\bigg[&\bar{p}\frac{\partial}{\partial\bar{p}}\left(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2 + \left(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2 - \bar{k}^2 \\ &+ \bar{k}\left(\frac{m\sin 2\bar{\mu}\gamma\bar{k} - 2\sin m\bar{\mu}\gamma\bar{k}}{m\bar{\mu}\gamma}\right)\bigg]\left(\frac{\delta E^c_j}{\bar{p}}\right). \end{aligned} \tag{79}$$

The Poisson bracket has terms which cannot be expressed through the constraints unless one imposes restrictions on the parameters. To evaluate this, we have to recall that even though we write sines in the expression (71) of quantum corrections, it is to be understood as a convenient notation to consider leading order quantum corrections. Anomaly cancellation up to order $\bar{k}^4$ then leads to the condition

$$m^2 = 5 + 2n \tag{80}$$

such that

$$m_{\mathrm{g}}^2 := \frac{T_Q}{a^2} \simeq \left(\frac{22n^2 - 35n - 5}{18}\right)\left(\frac{8\pi G}{3}\right)^2(\bar{\mu}^2\gamma^2\bar{p})\rho^2 \tag{81}$$

depends on only one remaining parameter n. We have also used the background Hamiltonian constraint to express T_Q in terms of the background energy density ρ.

The requirement of a positive "effective mass" squared now implies $-0.1319 > n \geq -5/2$, restricting the possible functional form of $\bar{\mu}$ as a function of $\bar{p}$. As one can see, some part of the otherwise allowed range $-1/2 < n < 0$ is ruled out here, including a nonrefining dynamics $n = 0$. The other limiting case, $n = -1/2$ of [28], on the other hand, is allowed.

VIII. DISCUSSIONS

We have considered tensor mode perturbation equations in Hamiltonian cosmology based on Ashtekar variables. In particular, we have derived possible effects of quantum gravity on the dispersion relation of gravitational wave propagation in a flat cosmological background. Included were typical corrections that one expects from loop quantum gravity, arising for inverse volume terms in the Hamiltonian constraint and from the use of holonomies. All final results are independent of gauge or other choices in the derivation.

This shows that inhomogeneities can be considered consistently within a perturbative framework of loop quantum gravity. So far, no complete effective Hamiltonian has been derived, but several separate effects are known and have at least partially been computed. Different types of quantum corrections can thus be studied separately to elucidate possible consequences, always keeping in mind that eventually all of them have to be combined for a complete picture. The two types of corrections considered here result in rather different correction terms in dispersion relations for gravitational waves, which indicates that it is reasonable to keep these corrections separate. Typically, only one of them will be dominant in a given cosmological regime, and the consequences have different physical consequences.

Since the magnitude of all the corrections depends on the precise form of a quantum state, such properties must be known for a precise quantitative estimate. Qualitative implications are, however, clear based on more general principles of loop quantum gravity. Also the rate of change of correction terms during cosmic evolution depends on the precise state and, in particular, its refinement. From the tensor mode analysis we have provided further evidence that discrete graph states of loop quantum gravity must be refined during evolution, supporting the results of [25,28,44–46]. Details will also determine the precise rate of varying speeds of light and gravitational waves.

The results provide a viability test of loop quantum gravity already in the absence of observations: no violations of causality occur even if quantum corrections in the dispersion relations are considered. Along similar lines one has to evaluate more general implications of Lorentz symmetries, especially in the context of potential Lorentz violating effects where anomaly issues have not yet been considered in the literature. While anomaly calculations are difficult for full quantum operators, we have illustrated that partial information can be gained economically at the effective level. A much more detailed analysis is required to see whether Lorentz symmetries are completely preserved once anomaly-freedom is implemented. An anomaly-free set of effective constraints would mean that quantum corrections implement a consistent deformation of the classical theory which preserves the number of symmetry generators although the form of symmetry transformations may be changed. This appears more like a deformation of the classical symmetry, rather than a breaking. Still, effective equations are approximations and thus make it difficult to derive an exact symmetry group.

As seen here, the requirement of anomaly-free equations, while allowing for nontrivial quantum corrections, eliminates one effect which would otherwise blatantly violate Lorentz invariance. This requires a close relation between quantizations of gravitational and matter (especially Maxwell) contributions to the Hamiltonian constraint, which is realized by the quantization procedures of loop quantum gravity [19,31] and tightened by the requirement of an anomaly-free constraint algebra. There is thus a weak sense of unification of gravity and matter since quantum corrections in the respective terms cannot be independent of each other.

ACKNOWLEDGMENTS

We thank Laurent Freidel, Mikhail Kagan, and Nico Yunes for discussions. This work was supported in part by NSF grants No. PHY0653127 and No. PHY0456913.

[1] C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, England, 2004).
[2] A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **21**, R53 (2004).
[3] T. Thiemann, arXiv:gr-qc/0110034.
[4] M. Bojowald, Living Rev. Relativity **8**, 11 (2005), http://relativity.livingreviews.org/Articles/lrr-2005-11/.
[5] M. Bojowald, Phys. Rev. Lett. **86**, 5227 (2001).
[6] M. Bojowald, Classical Quantum Gravity **20**, 2595 (2003).
[7] M. Bojowald, G. Date, and K. Vandersloot, Classical Quantum Gravity **21**, 1253 (2004).
[8] A. Ashtekar, T. Pawlowski, and P. Singh, Phys. Rev. Lett. **96**, 141301 (2006).
[9] M. Bojowald, Nature Phys. **3**, 523 (2007).
[10] M. Bojowald, H. Hernández, M. Kagan, and A. Skirzewski, Phys. Rev. D **75**, 064022 (2007).
[11] M. Bojowald *et al.*, Phys. Rev. D **74**, 123512 (2006).
[12] J. Mielczarek and M. Szydłowski, Phys. Lett. B **657**, 20 (2007).
[13] B. Dittrich and J. Tambornino, Classical Quantum Gravity **24**, 4543 (2007).
[14] M. Bojowald and G. Hossain, Classical Quantum Gravity **24**, 4801 (2007).
[15] A. Ashtekar, Phys. Rev. D **36**, 1587 (1987).
[16] J. F. Barbero G., Phys. Rev. D **51**, 5507 (1995).
[17] G. Immirzi, Classical Quantum Gravity **14**, L177 (1997).
[18] A. Ashtekar, M. Bojowald, and J. Lewandowski, Adv. Theor. Math. Phys. **7**, 233 (2003).
[19] T. Thiemann, Classical Quantum Gravity **15**, 839 (1998).
[20] M. Bojowald and A. Skirzewski, Rev. Math. Phys. **18**, 713 (2006).
[21] M. Bojowald and A. Skirzewski, Int. J. Geom. Methods Mod. Phys. **4**, 25 (2007).
[22] M. Bojowald, Phys. Rev. D **64**, 084018 (2001).
[23] M. Bojowald, Classical Quantum Gravity **19**, 5113 (2002).
[24] M. Bojowald, Pramana **63**, 765 (2004).
[25] M. Bojowald, Gen. Relativ. Gravit. **38**, 1771 (2006).
[26] M. Bojowald, Phys. Rev. D **75**, 081301(R) (2007).
[27] M. Bojowald, Classical Quantum Gravity **19**, 2717 (2002).
[28] A. Ashtekar, T. Pawlowski, and P. Singh, Phys. Rev. D **74**, 084003 (2006).
[29] M. Bojowald, H. Hernández, and A. Skirzewski, Phys. Rev. D **76**, 063511 (2007).
[30] M. Bojowald, Phys. Rev. D **75**, 123512 (2007).
[31] T. Thiemann, Classical Quantum Gravity **15**, 1281 (1998).
[32] M. Bojowald and R. Das, Phys. Rev. D **75**, 123521 (2007).
[33] C. Talmadge, J. P. Berthias, R. W. Hellings, and E. M. Standish, Phys. Rev. Lett. **61**, 1159 (1988).
[34] C. M. Will, Phys. Rev. D **57**, 2061 (1998).
[35] L. S. Finn and P. J. Sutton, Phys. Rev. D **65**, 044022 (2002).
[36] D. I. Jones, Astrophys. J. **618**, L115 (2005).

[37] C. N. Kozameh and M. F. Parisi, Classical Quantum Gravity **21**, 2617 (2004).
[38] J. W. Moffat, Int. J. Mod. Phys. D **2**, 351 (1993).
[39] A. Albrecht and J. Magueijo, Phys. Rev. D **59**, 043516 (1999).
[40] M. A. Clayton and J. W. Moffat, Phys. Lett. B **460**, 263 (1999).
[41] S. Alexander and J. Magueijo, arXiv:hep-th/0104093.
[42] J. Magueijo, Rep. Prog. Phys. **66**, 2025 (2003).
[43] M. Bojowald *et al.* (unpublished).
[44] M. Bojowald, D. Cartin, and G. Khanna, Phys. Rev. D **76**, 064018 (2007).
[45] W. Nelson and M. Sakellariadou, Phys. Rev. D **76**, 044015 (2007).
[46] W. Nelson and M. Sakellariadou, Phys. Rev. D **76**, 104003 (2007).