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QCD thermodynamics with 2 + 1 flavors at nonzero chemical potential

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We present results for the QCD equation of state, quark densities, and susceptibilities at nonzero chemical potential, using 2 + 1 flavor asqtad ensembles with $N_t = 4$. The ensembles lie on a trajectory of constant physics for which $m_{ud} \approx 0.1 m_s$. The calculation is performed using the Taylor expansion method with terms up to sixth order in μ/T.

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I. INTRODUCTION

The equation of state (EOS) of QCD is of special interest to the interpretation of data from heavy-ion collision experiments and to the development of nuclear theory and cosmology. The EOS at zero chemical potential ($\mu = 0$) has been extensively studied on the lattice. However, to approximate most closely the conditions of heavy-ion collision experiments (for example RHIC has $\mu \sim$ 15 MeV [1]) or of the interior of dense stars, the inclusion of nonzero chemical potential is necessary. Unfortunately, as is well known, inclusion of a nonzero chemical potential makes the fermion determinant in numerical simulations complex and straightforward Monte Carlo simulation not applicable. Several methods have been developed to overcome or circumvent this problem. They include the reweighting techniques [2,3], simulations with an imaginary chemical potential combined with analytical continuation [4,5] or canonical ensemble treatment [6], and lastly, the Taylor expansion method [7,8], which is employed here. In this method one Taylor expands the quantities needed for the computation of the EOS around the point $\mu = 0$ where standard Monte Carlo simulations are possible. The expansion parameter is the ratio μ/T, where T is the temperature. To ensure fast convergence of the Taylor series, the expansion parameter should be sufficiently small. Numerical calculations show satisfactory convergence for $\mu/T \lesssim 1$ (see reviews [9,10]).

In our simulations we use 2 + 1 flavors of improved staggered fermions. In such simulations where the number of flavors is not equal to a multiple of four, the so-called ‘‘fourth root trick’’ is employed to reduce the number of ‘‘tastes.’’ While this trick is still somewhat controversial, there is a growing body of numerical [11] and analytic [12] evidence that it leads to the correct continuum limit. For simulations at nonzero chemical potential the problems of rooting are much more severe [13]. However, the Taylor expansion method is not directly affected by this additional problem with rooting since the coefficients in the Taylor series are calculated in the theory with zero chemical potential. The Taylor expansion method is generally considered reliable in regions where the studied physics quantities are analytic.

The Taylor expansion method has been used to study the phase structure and the EOS of two-flavor QCD [7,14–17]. Our work improves on the previous studies by the addition of the strange quark to the sea. Our calculations are performed on 2 + 1 flavor ensembles generated with the R algorithm [18] and using the asqtad quark action [19] and a one-loop Symanzik improved gauge action [20]. These improved actions have small discretization errors of $O(\alpha_s a^2, a^4)$ and $O(\alpha_s^2 a^2, a^4)$, respectively. This is very important since we study the $N_t = 4$ case, where the lattice spacing $[a = 1/(TN_t)]$ is quite large, especially at low temperatures. Our ensembles lie along a trajectory of constant physics for which the ratio of the heavy quark mass and the light quark mass is $m_{ud}/m_s \approx 0.1$, and the heavy quark mass itself is tuned approximately to the physical value of the strange quark mass. The determination of the Taylor expansion coefficients, other than the zeroth order ones computed already previously, is necessary only on the finite-temperature ensembles (for our study $N_t = 4$). No zero-temperature subtractions are needed for them. We have determined the contributions to the energy density, pressure, and interaction measure due to the presence of a nonzero chemical potential. We also present results for the quark susceptibilities and densities. In addition, we have calculated the isentropic EOS, which is highly relevant for the heavy-ion collision experiments, where, after thermalization, the created matter is supposed to expand without further increase in entropy or change in the baryon number. All the results are obtained with the strange quark density fixed to $n_s = 0$ regardless of temperature, appropriate for the experimental conditions. This requires the tuning of the strange quark chemical potential along the trajectory of constant physics.

II. THE TAYLOR EXPANSION METHOD

In this section we give a brief description of the Taylor expansion method for the thermodynamic quantities we study and as applied to the asqtad fermion formulation.

A. Calculating the pressure

The asqtad quark matrix for a given flavor with nonzero chemical potential is:

$$\begin{aligned} M_{l,h} = M^{\text{spatial}}_{l,h} &+ \tfrac{1}{2}\eta_0(x)[U_0^{(F)}(x)\mathrm{e}^{\mu_{l,h}}\delta_{x+\hat{0},y} \\ &- U_0^{(F)\dagger}(x-\hat{0})\mathrm{e}^{-\mu_{l,h}}\delta_{x,y+\hat{0}} + U_0^{(L)}(x)\mathrm{e}^{3\mu_{l,h}}\delta_{x+3\hat{0},y} \\ &- U_0^{(L)\dagger}(x-3\hat{0})\mathrm{e}^{-3\mu_{l,h}}\delta_{x,y+3\hat{0}}], \end{aligned} \tag{1}$$

where $\mu_l = \mu_{ud}$ and $\mu_h = \mu_s$ are the quark chemical potentials in lattice units for the light (u and d) quarks and the heavy (strange s) quark, respectively. In the above

$$\begin{aligned} M^{\text{spatial}}_{l,h} = am_{l,h}\delta_{x,y} &+ \sum_{k=1}^{3}\frac{1}{2}\eta_k(x)[U_k^{(F)}(x)\delta_{x+\hat{k},y} \\ &- U_k^{(F)\dagger}(x-\hat{k})\delta_{x,y+\hat{k}} + U_k^{(L)}(x)\delta_{x+3\hat{k},y} \\ &- U_k^{(L)\dagger}(x-3\hat{k})\delta_{x,y+3\hat{k}}], \end{aligned} \tag{2}$$

with $m_{l,h}$ the light and strange quark masses. The superscripts F and L on the links U_μ denote the type of links, "fat" and "long"; appropriate weights and factors of the tadpole strength u_0 are included in $U_\mu^{(F)}$ and $U_\mu^{(L)}$. The partition function based on the asqtad quark matrix is

$$Z = \int \mathcal{D}U \mathrm{e}^{(n_l/4)\,\mathrm{ln}\,\mathrm{det}M_l}\mathrm{e}^{(n_h/4)\,\mathrm{ln}\,\mathrm{det}M_h}\mathrm{e}^{-S_g}, \tag{3}$$

where $n_l = 2$ is the number of light quarks and $n_h = 1$ is the number of heavy quarks. The pressure p can be obtained from the identity

$$\frac{p}{T^4} = \frac{\ln Z}{T^3 V}, \tag{4}$$

where T is the temperature and V the spatial volume. It can be Taylor expanded in the following manner

$$\frac{p}{T^4} = \sum_{n,m=0}^{\infty} c_{nm}(T)\left(\frac{\bar{\mu}_l}{T}\right)^n\left(\frac{\bar{\mu}_h}{T}\right)^m, \tag{5}$$

where $\bar{\mu}_{l,h}$ is the nonzero chemical potential in physical units. Because of the CP symmetry of the partition function, only the terms with $n + m$ even are nonzero. The expansion coefficients are defined by

$$c_{nm}(T) = \frac{1}{n!}\frac{1}{m!}\frac{N_t^3}{N_s^3}\frac{\partial^{n+m}\ln Z}{\partial(\mu_l N_t)^n\partial(\mu_h N_t)^m}\Bigg|_{\mu_{l,h}=0}, \tag{6}$$

with $\mu_{l,h} = a\bar{\mu}_{l,h}$ and N_s and N_t the spatial and temporal extents of the lattice. All coefficients need to be calculated on the finite-temperature ensembles only, except for $c_{00}(T)$. The latter is the pressure divided by T^4 at $\mu_{l,h} = 0$, which needs a zero-temperature subtraction. It should be calculated by other means, such as the integral method, which we have already done in [21]. The $c_{nm}(T)$ coefficients are linear combinations of observables $\mathcal{A}_{nm}$ and are given in Appendix B. The $\mathcal{A}_{nm}$ observables are obtainable as linear combinations of various products of the operators

$$L_n = \frac{n_l}{4}\frac{\partial^n \mathrm{ln}\,\mathrm{det}M_l}{\partial\mu_l^n}, \tag{7}$$

$$H_m = \frac{n_h}{4}\frac{\partial^m \mathrm{ln}\,\mathrm{det}M_h}{\partial\mu_h^m}, \tag{8}$$

evaluated at $\mu_{l,h} = 0$. For the definitions and explicit forms of the $\mathcal{A}_{nm}$ see Appendix B.

Figure 1 compares the cutoff effects due to the finite temporal extent N_t in the free theory case for the coefficients c_{00}, c_{20}, c_{40}, and c_{60} for three different staggered fermion actions: the standard, the Naik (asqtad), and the p4 action. The results for the first three coefficients are normalized to their respective Stefan-Boltzmann (SB) values. The SB value for c_{60} is zero (and the same holds for c_{06}). In the SB limit, the c_{0n} coefficients are, of course, equal to half of the SB values of c_{n0} for $0 < n \leq 4$. All other coefficients with n, $m \neq 0$ are zero in the SB limit. In the interacting case, the coefficients which are zero in the SB limit can acquire nonzero values. Figure 1 shows that the asqtad action has better scaling properties than the standard (unimproved) staggered action at $N_t = 4$, but it

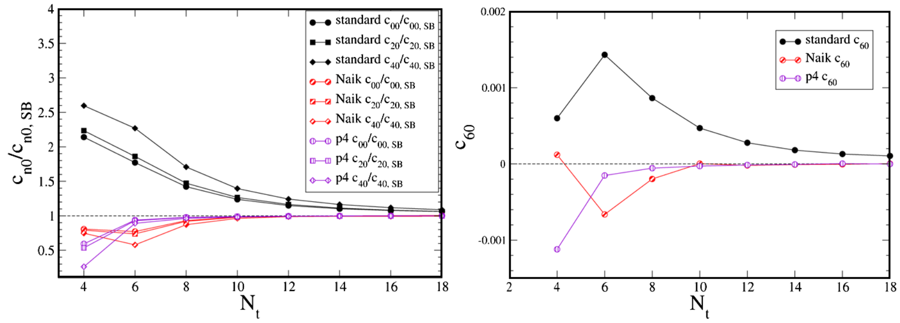

FIG. 1 (color online). The expansion coefficients c_{00}, c_{20}, c_{40}, and c_{60} for the pressure in the free theory case as a function of N_t.

is clear that a study at larger N_t is important for further reduction of the discretization errors.

B. Calculating the interaction measure and energy density

The interaction measure I can be Taylor expanded in a manner similar to the pressure

$$\frac{I}{T^4} = -\frac{N_t^3}{N_s^3}\frac{d\ln Z}{d\ln a} = \sum_{n,m}^{\infty} b_{nm}(T)\left(\frac{\bar{\mu}_l}{T}\right)^n\left(\frac{\bar{\mu}_h}{T}\right)^m, \quad (9)$$

where again only terms even in $n + m$ are nonzero and

$$b_{nm}(T) = -\frac{1}{n!m!}\frac{N_t^3}{N_s^3}\frac{\partial^{n+m}}{\partial(\mu_l N_t)^n\partial(\mu_h N_t)^m}\bigg|_{\mu_{l,h}=0}\left(\frac{d\ln Z}{d\ln a}\right). \quad (10)$$

The derivative with respect to $\ln a$ is taken along a trajectory of constant physics. The fermionic part of $\frac{d\ln Z}{d\ln a}$, considering the form of the asqtad action, is

$$\left\langle\frac{dS_f}{d\ln a}\right\rangle = \sum_{f=h,l}\frac{n_f}{4}\left[\frac{d(m_f a)}{d\ln a}\,\mathrm{tr}\langle M_f^{-1}\rangle + \frac{du_0}{d\ln a}\,\mathrm{tr}\left\langle M_f^{-1}\frac{dM_f}{du_0}\right\rangle\right]. \quad (11)$$

No volume normalization of the various traces is assumed in the above. The gauge part, taking into account the explicit form of the Symanzik gauge action, is

$$\left\langle\frac{-dS_g}{d\ln a}\right\rangle = \langle G\rangle = \left\langle 6\frac{d\beta}{d\ln a}P + 12\frac{d\beta_{\mathrm{rt}}}{d\ln a}R + 16\frac{d\beta_{\mathrm{pg}}}{d\ln a}C\right\rangle, \quad (12)$$

where P, R, and C are the appropriate sums of the plaquette, rectangle, and parallelogram terms, respectively (here they are not normalized to the volume). Thus the $b_{nm}(T)$ coefficients become

$$\begin{aligned} b_{nm}(T) = &-\frac{1}{n!m!}\frac{N_t^3}{N_s^3}\sum_{f=l,h}\frac{n_f}{4}\left[\frac{d(m_f a)}{d\ln a}\bigg|_{\mu_{l,h}=0}\right.\\ &\times\mathrm{tr}\frac{\partial^{n+m}\langle M_f^{-1}\rangle}{\partial(\mu_l N_t)^n\partial(\mu_h N_t)^m}\bigg|_{\mu_{l,h}=0}\\ &\left.+\frac{du_0}{d\ln a}\bigg|_{\mu_{l,h}=0}\mathrm{tr}\frac{\partial^{n+m}\langle M_f^{-1}\frac{dM_f}{du_0}\rangle}{\partial(\mu_l N_t)^n\partial(\mu_h N_t)^m}\bigg|_{\mu_{l,h}=0}\right]\\ &-\frac{1}{n!m!}\frac{N_t^3}{N_s^3}\frac{\partial^{n+m}\langle G\rangle}{\partial(\mu_l N_t)^n\partial(\mu_h N_t)^m}\bigg|_{\mu_{l,h}=0}. \end{aligned} \quad (13)$$

The explicit forms of the $b_{nm}(T)$ coefficients are more complex than those for $c_{nm}(T)$ and we save them for Appendix C. The SB limit of all b_{nm} coefficients is zero. In the presence of interactions their values can become different from zero. For the computation of the $b_{nm}(T)$ coefficients, in addition to the derivatives of the fermion matrix and the gauge action with respect to the chemical potentials, we have to know the derivatives of the action parameters with respect to $\ln a$ along the trajectory of constant physics. The latter have been determined in our previous work on the EOS at zero chemical potential [21], along with the coefficient $b_{00}(T)$, which is the interaction measure divided by T^4 in that case. The coefficients $c_{nm}(T)$ can be obtained from $b_{nm}(T)$ by integration along the trajectory of constant physics. This can serve as a consistency check of the calculation.

The energy density ε is simply obtained from the linear combination

$$\frac{\varepsilon}{T^4} = \frac{I + 3p}{T^4}. \quad (14)$$

C. Quark number densities and susceptibilities

The Taylor expansion for the quark number densities can be obtained from that for the pressure. For example, the light quark number density, n_{ud}, is

$$\frac{n_{ud}}{T^3} = \frac{\partial}{\partial\bar{\mu}_l/T}\left(\frac{\ln Z}{T^3 V}\right) = \sum_{n=1,m=0}^{\infty} n c_{nm}(T)\left(\frac{\bar{\mu}_l}{T}\right)^{n-1}\left(\frac{\bar{\mu}_h}{T}\right)^m, \quad (15)$$

and the heavy one, n_s, is

$$\frac{n_s}{T^3} = \frac{\partial}{\partial \bar{\mu}_h/T}\left(\frac{\ln Z}{T^3 V}\right) = \sum_{n=0,m=1}^{\infty} m c_{nm}(T)\left(\frac{\bar{\mu}_l}{T}\right)^n\left(\frac{\bar{\mu}_h}{T}\right)^{m-1}. \tag{16}$$

Similarly, the quark number susceptibilities are derivatives of the quark number densities with respect to the chemical potentials. Thus, the diagonal light-light quark susceptibility becomes

$$\frac{\chi_{uu}}{T^2} = \frac{\partial}{\partial \bar{\mu}_l/T}\left(\frac{n_{ud}}{T^3}\right) = \sum_{n=2,m=0}^{\infty} n(n-1)c_{nm}(T)\left(\frac{\bar{\mu}_l}{T}\right)^{n-2}\left(\frac{\bar{\mu}_h}{T}\right)^m, \tag{17}$$

and the heavy-heavy diagonal one is

$$\frac{\chi_{ss}}{T^2} = \frac{\partial}{\partial \bar{\mu}_h/T}\left(\frac{n_s}{T^3}\right) = \sum_{n=0,m=2}^{\infty} m(m-1)c_{nm}(T)\left(\frac{\bar{\mu}_l}{T}\right)^{n}\left(\frac{\bar{\mu}_h}{T}\right)^{m-2}. \tag{18}$$

Lastly, the mixed quark susceptibility has the form

$$\frac{\chi_{us}}{T^2} = \frac{\partial}{\partial \bar{\mu}_h/T}\left(\frac{n_{ud}}{T^3}\right) = \sum_{n=1,m=1}^{\infty} nmc_{nm}(T)\left(\frac{\bar{\mu}_l}{T}\right)^{n-1}\left(\frac{\bar{\mu}_h}{T}\right)^{m-1}. \tag{19}$$

III. SIMULATIONS

The asqtad-Symanzik gauge ensembles we use in this study have spatial volumes of 12^3 or 16^3 and $N_t = 4$, and are generated using the R algorithm. They are a subset of the ensembles in our EOS calculation at zero chemical potential [21]. The ensembles lie on an approximate trajectory of constant physics for which $m_{ud} \approx 0.1 m_s$, and m_s is tuned to the physical strange quark mass within 20%. Along the trajectory, the π to ρ mass ratio is $m_\pi/m_\rho \approx 0.3$. Table 1 in [21] contains the run parameters and trajectory numbers of the ensembles used here. They are the ones that have the gauge coupling values of $\beta = 6.0$, 6.075, 6.1, 6.125, 6.175, 6.2, 6.225, 6.25, 6.275, 6.3, 6.35, 6.6, and 7.08. The last column of that table shows the lattice scale. For explanation of the scale setting and other simulation details we refer the reader to Sec. 3 of [21]. The observables that need to be measured along the trajectory of constant physics in order to construct the Taylor coefficients in the expansion for the pressure are L_n and H_m defined by Eqs. (7) and (8). For the interaction measure determination the following observables have to be calculated in addition:

$$l_n = \frac{\partial^n \operatorname{tr} M_l^{-1}}{\partial \mu_l^n}, \qquad h_m = \frac{\partial^m \operatorname{tr} M_h^{-1}}{\partial \mu_h^m}, \tag{20}$$

$$\lambda_n = \frac{\partial^n \operatorname{tr}(M_l^{-1}\frac{dM_h}{du_0})}{\partial \mu_l^n}, \qquad \chi_m = \frac{\partial^m \operatorname{tr}(M_h^{-1}\frac{dM_h}{du_0})}{\partial \mu_h^m}, \tag{21}$$

and the gluonic observables P, R, and C. In Appendix C we show how they enter in the coefficients $b_{nm}(T)$. To sixth order in the Taylor expansion, the number of fermionic observables (L_n, H_m, l_n, h_m, λ_n, χ_m) that need to be determined is 40. We calculate them stochastically employing random Gaussian sources. In the region outside the phase transition or crossover we use 100 sources and double that number inside the transition/crossover region. This ensures that we work with statistical errors dominated by the gauge fluctuations and not by the ones coming from the stochastic estimators.

The ensembles we are working with have been generated using the inexact R algorithm which introduces finite step-size errors. In our previous study of these ensembles [21] we measured the step-size error in both gluonic and fermionic observables. The error was considerably less than 1% in the relevant gluonic and fermionic observables, measured on the high temperature ensembles. For the EOS at zero chemical potential it is necessary to subtract the high temperature and zero-temperature values. In the difference the effect of the step-size error becomes somewhat more pronounced. The contributions to the EOS due to nonzero chemical potential, computed here, do not require zero-temperature subtractions. Thus, based on the observations noted above, we expect any step-size errors in these contributions to be considerably smaller than our statistical errors.

IV. NUMERICAL RESULTS

Figure 2 shows our results for the temperature dependence of the $c_{n0}(T)$ and the $c_{0m}(T)$ coefficients. They all show rapid changes in the phase transition region and relatively quickly reach the Stefan-Boltzmann (SB) ideal gas values around $1.5T_c$–$2T_c$.

Unsurprisingly, the errors of the higher order coefficients are larger than the ones for the lowest order coefficients. They are worst for the sixth order coefficients $c_{60}(T)$ and $c_{06}(T)$. Although the magnitude of the coefficients decreases with each order in the Taylor expansion, for $\bar{\mu}/T \sim 1$ the sixth order terms contribute a great deal of noise in the thermodynamic quantities at the present level of statistics. Very similar conclusions can be made about the general behavior of the rest of the pressure coefficients, $c_{nm}(T)$ with both $n, m \neq 0$, shown in Fig. 3.

By comparison with the $c_{n0}(T)$ and $c_{0m}(T)$ coefficients, they are smaller and so are their contributions to the various thermodynamic quantities.

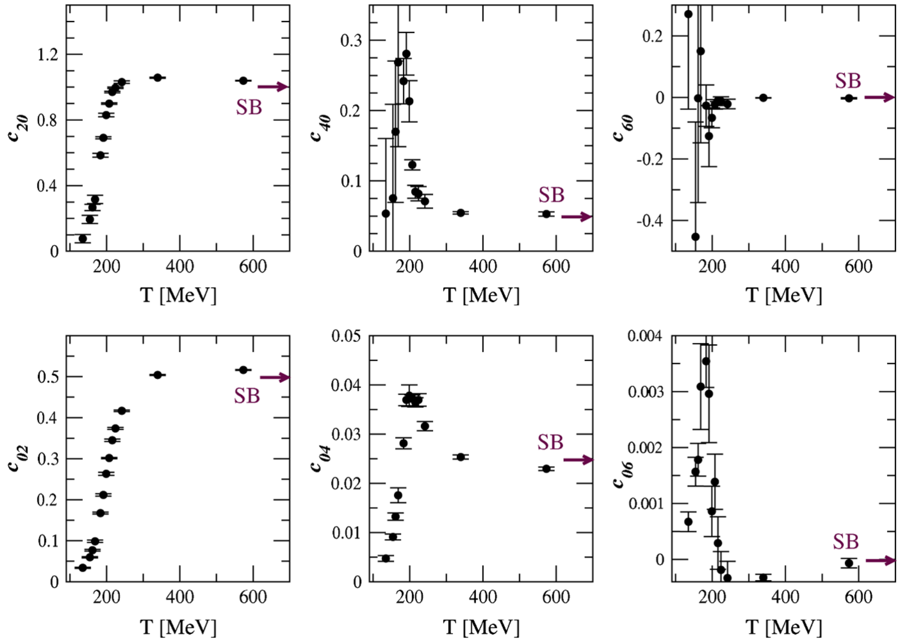

FIG. 2 (color online). Taylor expansion coefficients $c_{n0}(T)$ and $c_{0m}(T)$ for p/T^4.

Figures 4 and 5 show the coefficients in the Taylor expansion of the interaction measure.

Here again we see the rapid changes/large fluctuations around the transition region, the fast approach to the SB limit at high temperatures and the increase in magnitude of the errors and the decrease in magnitude of the coefficients with each successive order. In principle, each $c_{nm}(T)$ coefficient could be obtained from $b_{nm}(T)$ by integrating the latter along the trajectory of constant physics. For example, in Fig. 6 the $c_{20}(T)$ coefficient obtained directly using Eq. (5) is compared to its value calculated by integrating $b_{20}(T)$. The comparison shows that within the statistical errors the two results are the same.

Similar calculations were done for the rest of the coefficients and the consistency between the results from the two methods was satisfactory considering the large errors on the values obtained by integration.

Having determined the $c_{nm}(T)$ and $b_{nm}(T)$ coefficients we can now calculate the EOS to sixth order in the chemical potentials. We also determine the quark densities and

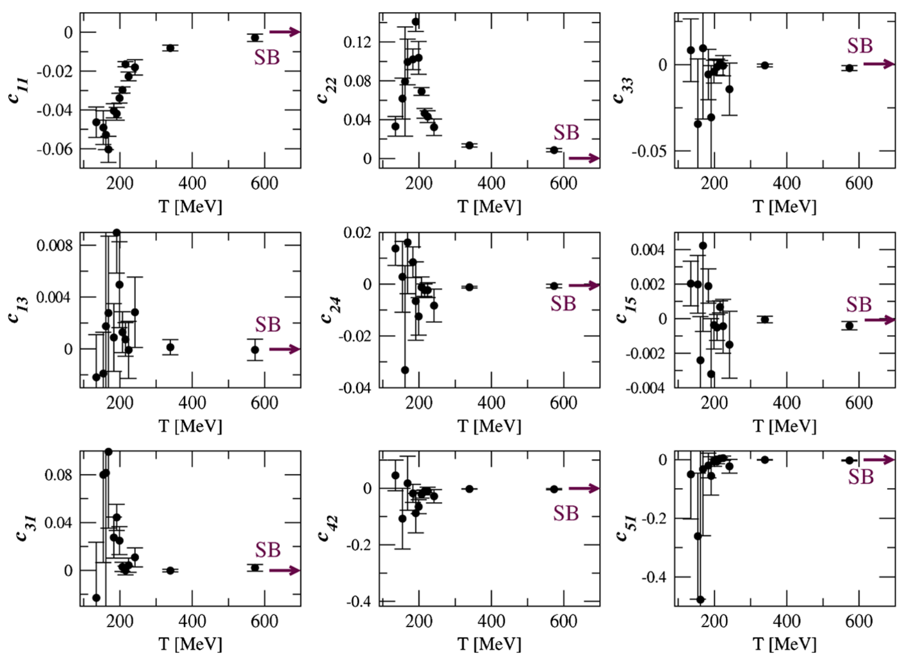

FIG. 3 (color online). Taylor expansion coefficients $c_{nm}(T)$ with $n, m \neq 0$ for p/T^4.

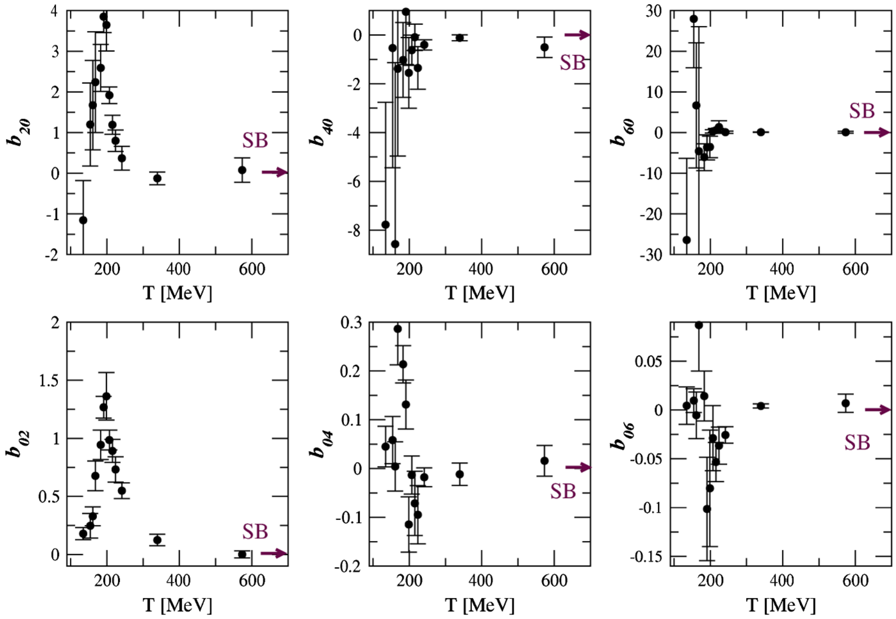

FIG. 4 (color online). Taylor expansion coefficients $b_{n0}(T)$ and $b_{0m}(T)$ for I/T^4.

various susceptibilities to fifth and fourth order, respectively. Since we want to work at strange quark density $n_s = 0$ to approximate the experimental conditions, we tuned $\bar{\mu}_h/T$ along the trajectory of constant physics in order to achieve that condition within the statistical error. Figure 7 (left) shows, for several values of $\bar{\mu}_l/T$, that with $\bar{\mu}_h/T = 0$ a slightly negative n_s is generated due to the nonzero $c_{n1}(T)$ terms. After the introduction of an appropriate nonzero $\bar{\mu}_h/T$ for each studied temperature and $\bar{\mu}_l/T$, Fig. 7 (right) shows our approximation of the condition $n_s = 0$. The effect of the tuning on thermodynamic quantities, other than n_s/T^3 itself, is small, because of the smallness of the "mixed expansion coefficients" $c_{nm}(T)$ and $b_{nm}(T)$ for $n, m \neq 0$. For our level of statistics the typical effect is within the statistical errors on the studied quantities.

Figures 8 and 9 show the corrections to the pressure, interaction measure, and energy density due to the presence of a nonzero $\bar{\mu}_l/T$. The correction to the pressure, for example, is the difference $\Delta p/T^4 = p(\mu_{l,h} \neq 0)/T^4 - p(\mu_{l,h} = 0)/T^4$, which is Eq. (5) minus the zeroth order

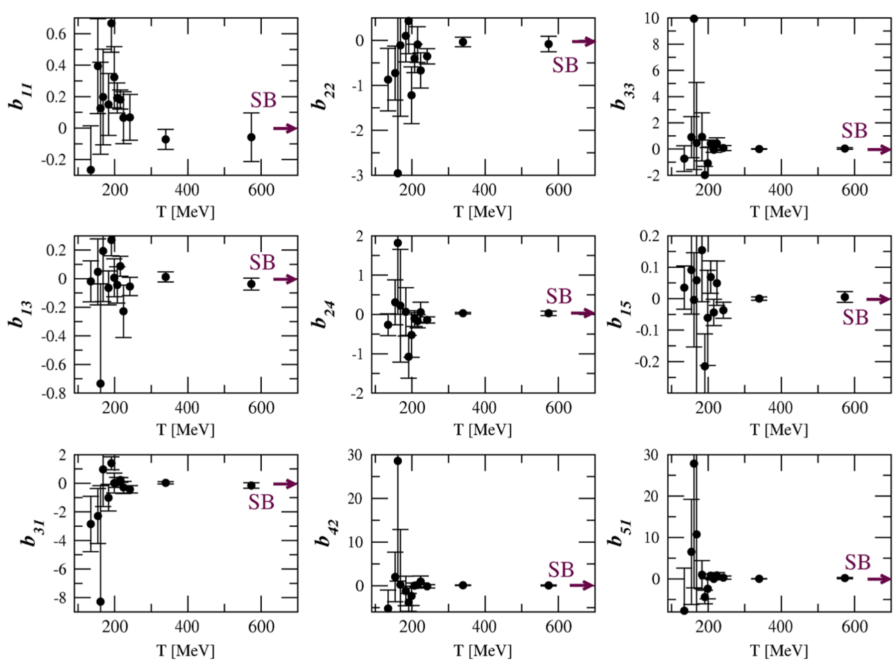

FIG. 5 (color online). Taylor expansion coefficients $b_{nm}(T)$ with $n, m \neq 0$ for I/T^4.

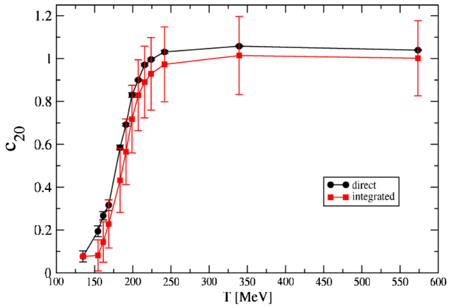

FIG. 6 (color online). Comparison between two different methods for calculating $c_{20}(T)$. The direct method uses Eq. (5) and the other method integrates $b_{20}(T)$ along the trajectory of constant physics. The integral method produces significantly larger errors than the direct one.

term $c_{00}(T) = p(\mu_{l,h} = 0)/T^4$. Similarly for the interaction measure and energy density, the corrections are $\Delta I/T^4 = I(\mu_{l,h} \neq 0)/T^4 - I(\mu_{l,h} = 0)/T^4$ and $\Delta\varepsilon/T^4 = \varepsilon(\mu_{l,h} \neq 0)/T^4 - \varepsilon(\mu_{l,h} = 0)/T^4$, which means again that the zeroth order terms are subtracted from the Taylor expansions for these quantities. Qualitatively, our EOS results are similar to the previous two-flavor studies [16]. The corrections to the thermodynamic quantities grow with increasing $\bar{\mu}_l/T$ and so do the statistical errors. The latter is due to the increasing contributions from higher order terms, which are noisier than the lowest order terms. Figures 10 and 11 show that similar observations are true

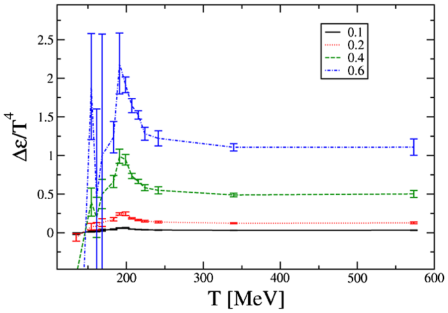

FIG. 9 (color online). Corrections to the energy density at several values of $\bar{\mu}_l/T$. $\bar{\mu}_h/T$ is tuned such that $n_s = 0$ along the trajectory.

for the rest of the studied quantities: the light quark density and the light-light, heavy-heavy, and light-heavy quark susceptibilities. Of these, the weakest dependence on $\bar{\mu}_l/T$ is shown by the heavy-heavy susceptibility χ_{ss}. A clear peak structure at the accessible $\bar{\mu}_l/T$ in the flavor diagonal light-light quark susceptibility χ_{uu} would be a sign of reaching the critical end point in the $\bar{\mu} - T$ plane. Our result does not show such a peak. Considering the significant errors for larger values of $\bar{\mu}_l/T$, it is difficult to say whether such a structure could be revealed with higher statistics or if the critical $\bar{\mu}_l/T$ has not been reached here. In any case, reducing the statistical errors and probably adding higher orders in the Taylor expansion would be the way to resolve that important problem.

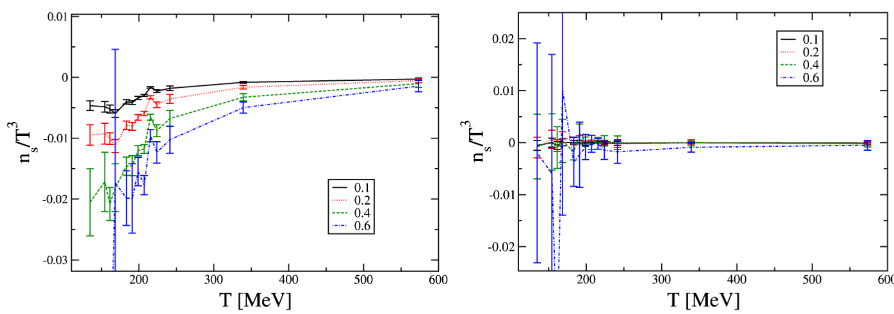

FIG. 7 (color online). The strange quark density n_s/T^3: left—results with $\bar{\mu}_h/T = 0$; right—tuned results. Different line styles denote different values of $\bar{\mu}_l/T$.

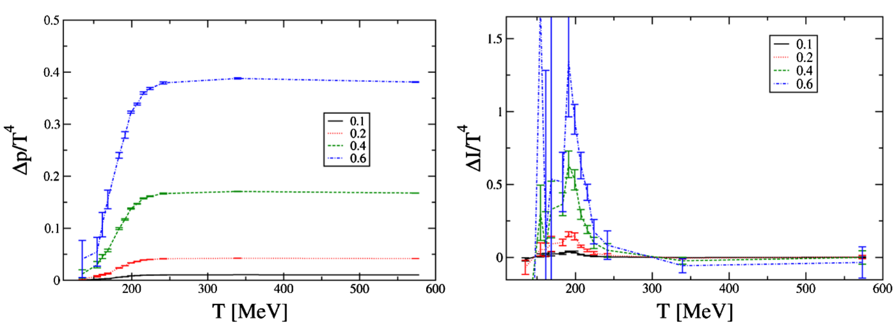

FIG. 8 (color online). Corrections to the pressure (left) and interaction measure (right) at several values of $\bar{\mu}_l/T$. $\bar{\mu}_h/T$ is tuned such that $n_s = 0$ along the trajectory.

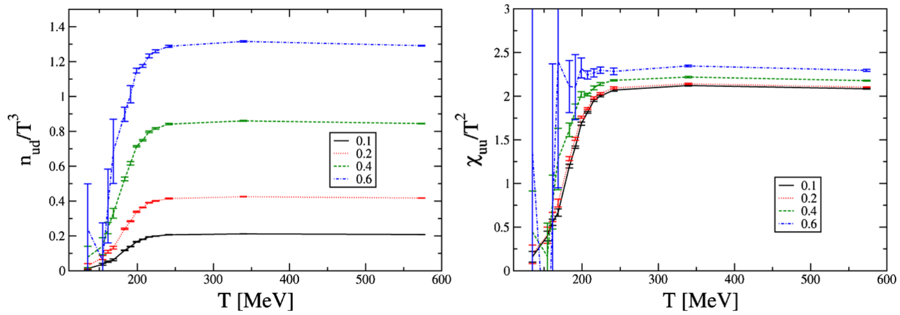

FIG. 10 (color online). Light quark density (left) and the light-light susceptibility (right) at several values of $\bar{\mu}_l/T$. $\bar{\mu}_h/T$ is tuned such that $n_s = 0$ along the trajectory.

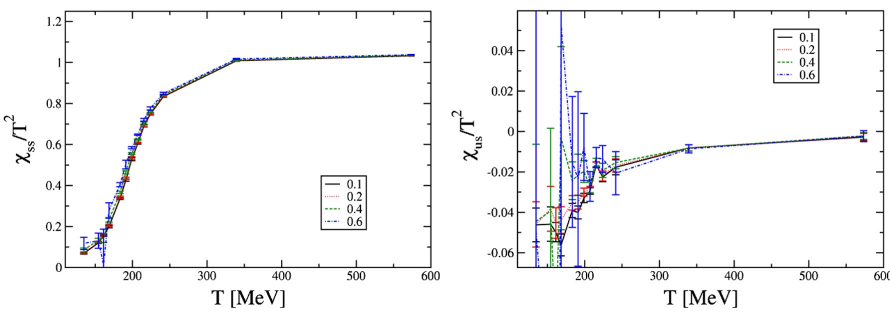

FIG. 11 (color online). Heavy-heavy (left) and heavy-light (right) susceptibilities at several values of $\bar{\mu}_l/T$. $\bar{\mu}_h/T$ is tuned such that $n_s = 0$ along the trajectory.

The isentropic EOS

The AGS, SPS, and RHIC experiments produce matter which is expected to expand isentropically, i.e., the entropy density s and baryon number $n_B = n_{ud}/3$ both remain unchanged during the expansion. This implies that s/n_B remains constant. For the experiments mentioned, s/n_B is approximately 30, 45, and 300 [17], respectively. In this subsection we present our results for the EOS and other thermodynamic quantities as calculated at nonzero chemical potential on trajectories in the $\bar{\mu} - T$ space with s/n_B fixed at the values relevant to these experiments. Figure 12 shows the trajectories in the (μ_l, μ_h, T) space, obtained by numerically solving the system

$$\frac{s}{n_B}(\mu_l, \mu_h) = C, \tag{22}$$

$$\frac{n_s}{T^3}(\mu_l, \mu_h) = 0, \tag{23}$$

with $C = 30$, 45, 300 for temperatures at which we have simulations. The tuning of the parameters μ_l and μ_h is done until the deviations from C and zero are no bigger than the statistical errors of s/n_B and n_s/T^3, respectively. After mapping the isentropic trajectories we use them to calculate the EOS, the results for which are shown in Figs. 13 and 14. For comparison, we also include the EOS result with $s/n_B = \infty$, which is the zero chemical

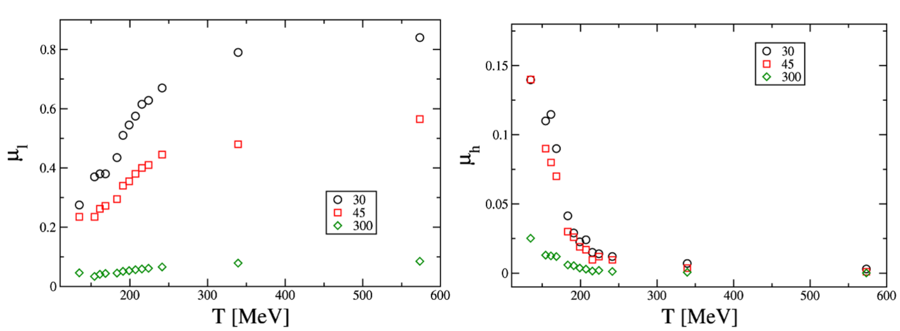

FIG. 12 (color online). The isentropic trajectories for different s/n_B.

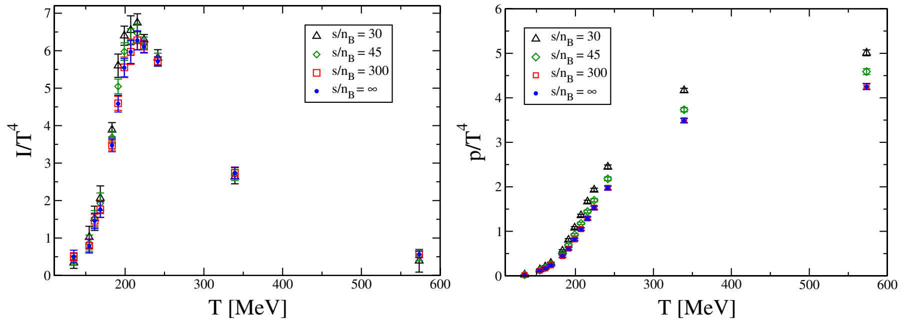

FIG. 13 (color online). Isentropic version of the interaction measure (left) and pressure (right) dependence on temperature at different finite values s/b_B as described in the text. The case of zero chemical potential ($s/n_B = \infty$) is also shown. These are the full results for the quantities, not only the correction due to the nonzero chemical potential.

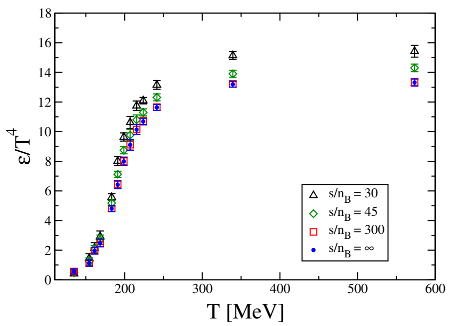

FIG. 14 (color online). Isentropic versions of the energy density dependence on temperature.

potential case ($\mu_l = \mu_h = 0$). From the EOS results we conclude that in the studied range of s/n_B the differences between the isentropic trajectories are not very large, with the interaction measure least affected by the change in s/n_B. Our results are again qualitatively very similar to the two-flavor isentropic EOS study from [17]. The isentropic results for n_{ud}, χ_{uu}, χ_{us}, and χ_{ss} are shown in Figs. 15 and 16.

It is interesting to note that χ_{uu} does not develop a peak structure on any of the isentropic trajectories. This means that all of the experiments work in the ranges of s/n_B far from the critical end point, if such an end point exists at all for physical quark masses [22]. The light quark density n_{ud}

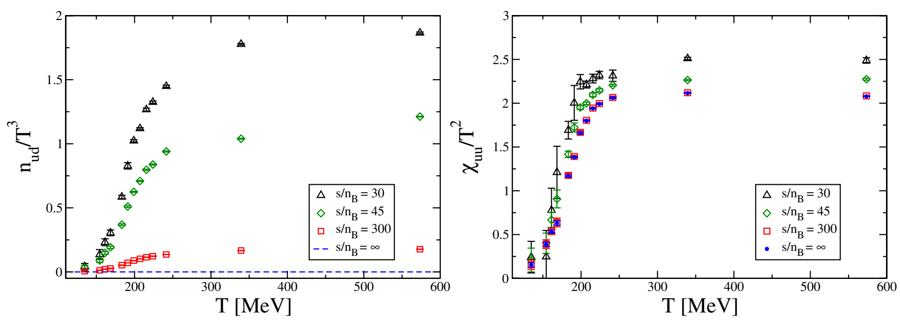

FIG. 15 (color online). Light quark density (left) and light-light susceptibility (right) for different s/n_B.

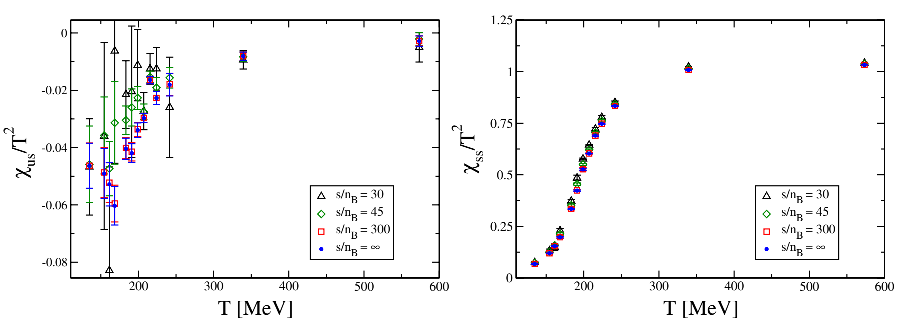

FIG. 16 (color online). Light-heavy (left) and heavy-heavy (right) susceptibilities for different s/n_B.

looks most affected by the value of s/n_B, and χ_{ss} is practically independent of it.

V. CONCLUSIONS

We have calculated the QCD equation of state for 2 + 1 flavors along a trajectory of constant physics and at nonzero chemical potential using the Taylor expansion method to sixth order in the chemical potential. The Taylor expansion coefficients for the pressure and the interaction measure were determined directly by measuring a set of fermionic and gluonic observables on the finite-temperature ensembles along the trajectory. We used Gaussian random sources in the calculation of the 40 fermionic observables. The higher the order of the coefficients the noisier they proved to be. Although the higher order coefficients have smaller magnitudes, for increasing values of the chemical potential they contribute significantly to the statistical errors. We tuned the heavy quark chemical potential at each temperature studied in order to keep a vanishing strange quark density and have determined a number of thermodynamic quantities at different values of the light quark chemical potential for which the ratio $\bar{\mu}_l/T \lesssim 1$. Our corrections to the EOS due to the nonzero chemical potential grow with the increasing values of $\bar{\mu}_l/T$. However, not all thermodynamic quantities are equally affected by the addition of a chemical potential. Indeed, the heavy-heavy quark susceptibility is practically independent of it.

We also have determined the isentropic versions of the EOS, the light quark densities, and quark number susceptibilities, which are supposedly most relevant for the current heavy-ion collision experiments. We found that the EOS is not strongly affected by changes in the ratio s/n_B, which is in agreement with previous two-flavor results [17].

ACKNOWLEDGMENTS

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APPENDIX A: PROPERTIES OF THE QUARK MATRIX DERIVATIVES

We use the following identities for the fermion matrix and its derivatives:

$$M^\dagger(\mu) = \gamma_5 M(-\mu)\gamma_5, \quad \text{and}$$
$$\frac{\partial^n M^\dagger}{\partial \mu^n}(\mu) = (-1)^n \gamma_5 \frac{\partial^n M}{\partial \mu^n}(-\mu)\gamma_5. \tag{A1}$$

Then, at $\mu = 0$

$$\mathrm{tr}\left(M^{-1}\frac{\partial^{n_1} M}{\partial \mu^{n_1}}M^{-1}\frac{\partial^{n_2} M}{\partial \mu^{n_2}}M^{-1}\cdots\right)^* = (-1)^{n_1+n_2+\cdots}\,\mathrm{tr}\left(M^{-1}\frac{\partial^{n_1} M}{\partial \mu^{n_1}}M^{-1}\frac{\partial^{n_2} M}{\partial \mu^{n_2}}M^{-1}\cdots\right). \tag{A2}$$

Because the terms in the nth derivative satisfy $n_1 + n_2 + \cdots = n$, we obtain

$$\left(\frac{\partial^n \ln\det M}{\partial \mu^n}\right)^* = (-1)^n \frac{\partial^n \ln\det M}{\partial \mu^n}; \tag{A3}$$

$$\left(\frac{\partial^n \mathrm{tr} M^{-1}}{\partial \mu^n}\right)^* = (-1)^n \frac{\partial^n \mathrm{tr} M^{-1}}{\partial \mu^n}; \tag{A4}$$

$$\left(\frac{\partial^n \mathrm{tr} M^{-1}\frac{dM}{du_0}}{\partial \mu^n}\right)^* = (-1)^n \frac{\partial^n \mathrm{tr} M^{-1}\frac{dM}{du_0}}{\partial \mu^n}; \tag{A5}$$

i.e. all even derivatives are real and all odd ones are purely imaginary. This means, for example, that

$$\mathrm{Re}\,\langle L_2 L_1 L_1\rangle = -\langle \mathrm{Re}(L_2)\,\mathrm{Im}(L_1)\,\mathrm{Im}(L_1)\rangle, \tag{A6}$$

and the real part of any observable containing odd number of odd derivatives is zero.

Explicitly the derivatives of the asqtad fermion matrix are

$$\begin{aligned}\frac{\partial^n M}{\partial \mu^n} = \frac{1}{2}\eta_0(x)[&U_0^{(F)}(x)\mathrm{e}^{\mu}\delta_{x+\hat{0},y} \\ &- (-1)^n U_0^{(F)\dagger}(x-\hat{0})\mathrm{e}^{-\mu}\delta_{x,y+\hat{0}} \\ &+ (3)^n U_0^{(L)}(x)\mathrm{e}^{3\mu}\delta_{x+3\hat{0},y} \\ &- (-3)^n U_0^{(L)\dagger}(x-3\hat{0})\mathrm{e}^{-3\mu}\delta_{x,y+3\hat{0}}].\end{aligned} \tag{A7}$$

APPENDIX B: ALGEBRAIC TECHNIQUES FOR THE PRESSURE

The nonvanishing $c_{nm}(T)$ coefficients from second through sixth order are

$$
\begin{aligned}
c_{20} &\equiv \frac{1}{2}\frac{\partial^2(p/T^4)}{\partial(\mu_l/T)^2}\bigg|_{\mu_l=0} = \frac{1}{2}\frac{N_t}{N_s^3}\mathcal{A}_{20}\\
c_{40} &\equiv \frac{1}{4!}\frac{\partial^4(p/T^4)}{\partial(\mu_l/T)^4}\bigg|_{\mu_l=0} = \frac{1}{4!}\frac{1}{N_s^3N_t}(\mathcal{A}_{40}-3\mathcal{A}_{20}^2)\\
c_{60} &\equiv \frac{1}{6!}\frac{\partial^6(p/T^4)}{\partial(\mu_l/T)^6}\bigg|_{\mu_l=0} = \frac{1}{6!}\frac{1}{N_s^3N_t^3}(\mathcal{A}_{60}-15\mathcal{A}_{40}\mathcal{A}_{20}+30\mathcal{A}_{20}^3)\\
c_{02} &\equiv \frac{1}{2}\frac{\partial^2(p/T^4)}{\partial(\mu_h/T)^2}\bigg|_{\mu_h=0} = \frac{1}{2}\frac{N_t}{N_s^3}\mathcal{A}_{02}\\
c_{04} &\equiv \frac{1}{4!}\frac{\partial^4(p/T^4)}{\partial(\mu_h/T)^4}\bigg|_{\mu_h=0} = \frac{1}{4!}\frac{1}{N_s^3N_t}(\mathcal{A}_{04}-3\mathcal{A}_{02}^2),\\
c_{06} &\equiv \frac{1}{6!}\frac{\partial^6(p/T^4)}{\partial(\mu_h/T)^6}\bigg|_{\mu_h=0} = \frac{1}{6!}\frac{1}{N_s^3N_t^3}(\mathcal{A}_{06}-15\mathcal{A}_{04}\mathcal{A}_{02}+30\mathcal{A}_{02}^3)\\
c_{11} &\equiv \frac{1}{1!1!}\frac{\partial^2(p/T^4)}{\partial(\mu_l/T)\partial(\mu_h/T)}\bigg|_{\mu_{l,h}=0} = \frac{N_t}{N_s^3}\mathcal{A}_{11}\\
c_{31} &\equiv \frac{1}{3!1!}\frac{\partial^4(p/T^4)}{\partial^3(\mu_l/T)\partial(\mu_h/T)}\bigg|_{\mu_{l,h}=0} = \frac{1}{3!1!}\frac{1}{N_s^3N_t}(\mathcal{A}_{31}-3\mathcal{A}_{20}\mathcal{A}_{11})\\
c_{13} &\equiv \frac{1}{3!1!}\frac{\partial^4(p/T^4)}{\partial(\mu_l/T)\partial^3(\mu_h/T)}\bigg|_{\mu_{l,h}=0} = \frac{1}{3!1!}\frac{1}{N_s^3N_t}(\mathcal{A}_{13}-3\mathcal{A}_{02}\mathcal{A}_{11})\\
c_{22} &\equiv \frac{1}{2!2!}\frac{\partial^4(p/T^4)}{\partial^2(\mu_l/T)\partial^2(\mu_h/T)}\bigg|_{\mu_{l,h}=0} = \frac{1}{2!2!}\frac{1}{N_s^3N_t}(\mathcal{A}_{22}-\mathcal{A}_{20}\mathcal{A}_{02}-2\mathcal{A}_{11}^2),\\
c_{42} &\equiv \frac{1}{4!2!}\frac{\partial^6(p/T^4)}{\partial^4(\mu_l/T)\partial^2(\mu_h/T)}\bigg|_{\mu_{l,h}=0}\\
&= \frac{1}{4!2!}\frac{1}{N_s^3N_t^3}(\mathcal{A}_{42}-6\mathcal{A}_{20}\mathcal{A}_{22}-8\mathcal{A}_{11}\mathcal{A}_{31}-\mathcal{A}_{40}\mathcal{A}_{02}+24\mathcal{A}_{20}\mathcal{A}_{11}^2+6\mathcal{A}_{02}\mathcal{A}_{20}^2)\\
c_{24} &\equiv \frac{1}{4!2!}\frac{\partial^6(p/T^4)}{\partial^2(\mu_l/T)\partial^4(\mu_h/T)}\bigg|_{\mu_{l,h}=0}\\
&= \frac{1}{4!2!}\frac{1}{N_s^3N_t^3}(\mathcal{A}_{24}-6\mathcal{A}_{02}\mathcal{A}_{22}-8\mathcal{A}_{11}\mathcal{A}_{13}-\mathcal{A}_{04}\mathcal{A}_{20}+24\mathcal{A}_{02}\mathcal{A}_{11}^2+6\mathcal{A}_{20}\mathcal{A}_{02}^2)\\
c_{51} &\equiv \frac{1}{5!1!}\frac{\partial^6(p/T^4)}{\partial^5(\mu_l/T)\partial^1(\mu_h/T)}\bigg|_{\mu_{l,h}=0} = \frac{1}{5!1!}\frac{1}{N_s^3N_t^3}(\mathcal{A}_{51}-10\mathcal{A}_{31}\mathcal{A}_{20}-5\mathcal{A}_{40}\mathcal{A}_{11}+30\mathcal{A}_{11}\mathcal{A}_{20}^2)\\
c_{15} &\equiv \frac{1}{5!1!}\frac{\partial^6(p/T^4)}{\partial^1(\mu_l/T)\partial^5(\mu_h/T)}\bigg|_{\mu_{l,h}=0} = \frac{1}{5!1!}\frac{1}{N_s^3N_t^3}(\mathcal{A}_{15}-10\mathcal{A}_{13}\mathcal{A}_{02}-5\mathcal{A}_{04}\mathcal{A}_{11}+30\mathcal{A}_{11}\mathcal{A}_{02}^2)\\
c_{33} &\equiv \frac{1}{3!3!}\frac{\partial^6(p/T^4)}{\partial^3(\mu_l/T)\partial^3(\mu_h/T)}\bigg|_{\mu_{l,h}=0}\\
&= \frac{1}{3!3!}\frac{1}{N_s^3N_t^3}(\mathcal{A}_{33}-3\mathcal{A}_{31}\mathcal{A}_{02}-3\mathcal{A}_{13}\mathcal{A}_{20}-9\mathcal{A}_{11}\mathcal{A}_{22}+18\mathcal{A}_{20}\mathcal{A}_{02}\mathcal{A}_{11}+12\mathcal{A}_{11}^3).
\end{aligned}
$$

To generate the above expressions for c_{nm} we follow closely the technique given in [16]. Let

$$\frac{\partial \ln Z}{\partial \mu_l} \equiv \mathcal{A}_{01} = \langle L_1 \rangle, \tag{B1}$$

$$\frac{\partial \ln Z}{\partial \mu_h} \equiv \mathcal{A}_{01} = \langle H_1 \rangle. \tag{B2}$$

It can be shown that

$$\frac{\partial \mathcal{A}_{nm}}{\partial \mu_l} = \mathcal{A}_{n+1,m} - \mathcal{A}_{10}\mathcal{A}_{nm}, \tag{B3}$$

$$\frac{\partial \mathcal{A}_{nm}}{\partial \mu_h} = \mathcal{A}_{n,m+1} - \mathcal{A}_{01}\mathcal{A}_{nm}, \tag{B4}$$

where

$$\mathcal{A}_{nm} \equiv \left\langle e^{-L_0} e^{-H_0} \frac{\partial^n e^{L_0}}{\partial \mu_l^n} \frac{\partial^m e^{H_0}}{\partial \mu_h^m} \right\rangle. \tag{B5}$$

Higher order derivatives of $\ln Z$ at $\mu_{h,l} = 0$ are zero if $n + m$ is odd, which can be shown to mean that, in this case, $\mathcal{A}_{nm} = 0$. An example for getting a higher order derivative using either Eq. (B1) or Eq. (B2):

$$\frac{\partial^2 \ln Z}{\partial \mu_l \partial \mu_h} = \frac{\partial \mathcal{A}_{01}}{\partial \mu_l} = \frac{\partial \mathcal{A}_{10}}{\partial \mu_h} = (\mathcal{A}_{11} - \mathcal{A}_{10}\mathcal{A}_{01})|_{\mu_{h,l}=0} = \mathcal{A}_{11}. \tag{B6}$$

Once an expression for c_{nm} is obtained it is easy to get c_{mn} by just interchanging n and m in the former. The observables $\mathcal{A}_{nm}$ in terms of the operators

$$L_n = \frac{n_l}{4} \frac{\partial^n \ln\det M_l}{\partial \mu_l^n} \quad \text{and} \quad H_m = \frac{n_h}{4} \frac{\partial^m \ln\det M_h}{\partial \mu_h^m}, \tag{B7}$$

are

$$\begin{aligned}
\mathcal{A}_{20} &= \langle L_2 \rangle + \langle L_1^2 \rangle, \qquad \mathcal{A}_{40} = \langle L_4 \rangle + 4\langle L_3 L_1 \rangle + 3\langle L_2^2 \rangle + 6\langle L_2 L_1^2 \rangle + \langle L_1^4 \rangle,\\
\mathcal{A}_{60} &= \langle L_6 \rangle + 6\langle L_5 L_1 \rangle + 15\langle L_4 L_2 \rangle + 10\langle L_3^2 \rangle + 15\langle L_4 L_1^2 \rangle + 60\langle L_3 L_2 L_1 \rangle + 15\langle L_2^3 \rangle + 20\langle L_3 L_1^3 \rangle + 45\langle L_2^2 L_1^2 \rangle + 15\langle L_2 L_1^4 \rangle\\
&\quad + \langle L_1^6 \rangle,\\
\mathcal{A}_{02} &= \langle H_2 \rangle + \langle H_1^2 \rangle, \qquad \mathcal{A}_{04} = \langle H_4 \rangle + 4\langle H_3 H_1 \rangle + 3\langle H_2^2 \rangle + 6\langle H_2 H_1^2 \rangle + \langle H_1^4 \rangle,\\
\mathcal{A}_{06} &= \langle H_6 \rangle + 6\langle H_5 H_1 \rangle + 15\langle H_4 H_2 \rangle + 10\langle H_3^2 \rangle + 15\langle H_4 H_1^2 \rangle + 60\langle H_3 H_2 H_1 \rangle + 15\langle H_2^3 \rangle + 20\langle H_3 H_1^3 \rangle + 45\langle H_2^2 H_1^2 \rangle\\
&\quad + 15\langle H_2 H_1^4 \rangle + \langle H_1^6 \rangle,\\
\mathcal{A}_{11} &= \langle L_1 H_1 \rangle, \qquad \mathcal{A}_{22} = \langle L_2 H_2 \rangle + \langle L_2 H_1^2 \rangle + \langle L_1^2 H_2 \rangle + \langle L_1^2 H_1^2 \rangle, \qquad \mathcal{A}_{31} = \langle L_3 H_1 \rangle + 3\langle L_2 L_1 H_1 \rangle + \langle L_1^3 H_1 \rangle,\\
\mathcal{A}_{13} &= \langle H_3 L_1 \rangle + 3\langle H_2 H_1 L_1 \rangle + \langle H_1^3 L_1 \rangle,\\
\mathcal{A}_{42} &= \langle L_4 H_2 \rangle + 4\langle L_3 L_1 H_2 \rangle + 3\langle L_2^2 H_2 \rangle + 6\langle L_2 L_1^2 H_2 \rangle + \langle L_1^4 H_2 \rangle + \langle L_4 H_1^2 \rangle + 4\langle L_3 L_1 H_1^2 \rangle + 3\langle L_2^2 H_1^2 \rangle + 6\langle L_2 L_1^2 H_1^2 \rangle\\
&\quad + \langle L_1^4 H_1^2 \rangle,\\
\mathcal{A}_{24} &= \langle H_4 L_2 \rangle + 4\langle H_3 H_1 L_2 \rangle + 3\langle H_2^2 L_2 \rangle + 6\langle H_2 H_1^2 L_2 \rangle + \langle H_1^4 L_2 \rangle + \langle H_4 L_1^2 \rangle + 4\langle H_3 H_1 L_1^2 \rangle + 3\langle H_2^2 L_1^2 \rangle + 6\langle H_2 H_1^2 L_1^2 \rangle\\
&\quad + \langle H_1^4 L_1^2 \rangle,\\
\mathcal{A}_{51} &= \langle L_5 H_1 \rangle + 5\langle L_4 L_1 H_1 \rangle + 10\langle L_3 L_2 H_1 \rangle + 10\langle L_3 L_1^2 H_1 \rangle + 15\langle L_2^2 L_1 H_1 \rangle + 10\langle L_2 L_1^3 H_1 \rangle + \langle L_1^5 H_1 \rangle,\\
\mathcal{A}_{15} &= \langle H_5 L_1 \rangle + 5\langle H_4 H_1 L_1 \rangle + 10\langle H_3 H_2 L_1 \rangle + 10\langle H_3 H_1^2 L_1 \rangle + 15\langle H_2^2 H_1 L_1 \rangle + 10\langle H_2 H_1^3 L_1 \rangle + \langle H_1^5 L_1 \rangle,\\
\mathcal{A}_{33} &= \langle (L_3 + 3L_2 L_1 + L_1^3)(H_3 + 3H_2 H_1 + H_1^3) \rangle.
\end{aligned}$$

The observables L_n and H_m include the quark matrix $M(= M_{l,h})$ derivatives with respect to $\mu(= \mu_{l,h})$ which have the following form:

$$\frac{\partial \ln\det M}{\partial \mu} = \mathrm{tr}\left(M^{-1} \frac{\partial M}{\partial \mu}\right), \tag{B8}$$

$$\frac{\partial^2 \ln\det M}{\partial \mu^2} = \mathrm{tr}\left(M^{-1} \frac{\partial^2 M}{\partial \mu^2}\right) - \mathrm{tr}\left(M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial M}{\partial \mu}\right), \tag{B9}$$

$$\frac{\partial^3 \ln\det M}{\partial \mu^3} = \mathrm{tr}\left(M^{-1} \frac{\partial^3 M}{\partial \mu^3}\right) - 3\,\mathrm{tr}\left(M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial^2 M}{\partial \mu^2}\right) + 2\,\mathrm{tr}\left(M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial M}{\partial \mu}\right), \tag{B10}$$

$$\begin{aligned}
\frac{\partial^4 \ln\det M}{\partial \mu^4} &= \mathrm{tr}\left(M^{-1} \frac{\partial^4 M}{\partial \mu^4}\right)\\
&\quad - 4\,\mathrm{tr}\left(M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial^3 M}{\partial \mu^3}\right)\\
&\quad - 3\,\mathrm{tr}\left(M^{-1} \frac{\partial^2 M}{\partial \mu^2} M^{-1} \frac{\partial^2 M}{\partial \mu^2}\right)\\
&\quad + 12\,\mathrm{tr}\left(M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial^2 M}{\partial \mu^2}\right)\\
&\quad - 6\,\mathrm{tr}\left(M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial M}{\partial \mu} \times M^{-1} \frac{\partial M}{\partial \mu}\right),
\end{aligned} \tag{B11}$$

$$\begin{aligned}\frac{\partial^5 \ln\det M}{\partial\mu^5} &= \mathrm{tr}\Big(M^{-1}\frac{\partial^5 M}{\partial\mu^5}\Big) - 5\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^4 M}{\partial\mu^4}\Big) - 10\,\mathrm{tr}\Big(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\Big)\\
&\quad + 20\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\Big) + 30\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\Big)\\
&\quad - 60\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\Big) + 24\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\Big),\end{aligned} \tag{B12}$$

$$\begin{aligned}\frac{\partial^6 \ln\det M}{\partial\mu^6} &= \mathrm{tr}\Big(M^{-1}\frac{\partial^6 M}{\partial\mu^6}\Big) - 6\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^5 M}{\partial\mu^5}\Big) - 15\,\mathrm{tr}\Big(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^4 M}{\partial\mu^4}\Big) - 10\,\mathrm{tr}\Big(M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\Big)\\
&\quad + 30\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^4 M}{\partial\mu^4}\Big) + 60\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\Big)\\
&\quad + 60\,\mathrm{tr}\Big(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\Big) + 30\,\mathrm{tr}\Big(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\Big)\\
&\quad - 120\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\Big), -180\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\Big)\\
&\quad - 90\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\Big) + 360\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\Big)\\
&\quad - 120\,\mathrm{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\Big).\end{aligned} \tag{B13}$$

APPRENDIX C: ALGEBRAIC TECHNIQUES FOR THE INTERACTION MEASURE

Equation (13) for the coefficients $b_{nm}(T)$ contains three types of derivatives of the fermion matrix with respect to the chemical potentials. We tackle them separately in the following.

1. First type of derivative

Here we give the method [16] for calculating the derivative

$$\frac{\partial^{n+m}\langle M_f^{-1}\rangle}{\partial(\mu_l N_t)^n\partial(\mu_h N_t)^m}\bigg|_{\mu_{l,h}=0}. \tag{C1}$$

A convenient place to start in this case is by defining the observables

$$\mathcal{B}_{nm} \equiv \Big\langle e^{-L_0}e^{-H_0}\frac{\partial^n(\mathrm{tr}M_l^{-1}e^{L_0})}{\partial\mu_l^n}\frac{\partial^m e^{H_0}}{\partial\mu_h^m}\Big\rangle, \tag{C2}$$

$$\mathcal{B}'_{nm} \equiv \Big\langle e^{-L_0}e^{-H_0}\frac{\partial^n e^{L_0}}{\partial\mu_l^n}\frac{\partial^m(\mathrm{tr}M_h^{-1}e^{H_0})}{\partial\mu_h^m}\Big\rangle. \tag{C3}$$

The above means

$$\mathcal{B}_{00} \equiv \langle\mathrm{tr}M_l^{-1}\rangle, \tag{C4}$$

$$\mathcal{B}'_{00} \equiv \langle\mathrm{tr}M_h^{-1}\rangle. \tag{C5}$$

It follows that

$$\frac{\partial\mathcal{B}_{nm}}{\partial\mu_l} = \mathcal{B}_{n+1,m} - \mathcal{A}_{10}\mathcal{B}_{nm}, \tag{C6}$$

$$\frac{\partial\mathcal{B}_{nm}}{\partial\mu_h} = \mathcal{B}_{n,m+1} - \mathcal{A}_{01}\mathcal{B}_{nm}, \tag{C7}$$

$$\frac{\partial\mathcal{B}'_{nm}}{\partial\mu_l} = \mathcal{B}'_{n+1,m} - \mathcal{A}_{10}\mathcal{B}'_{nm}, \tag{C8}$$

$$\frac{\partial\mathcal{B}'_{nm}}{\partial\mu_h} = \mathcal{B}'_{n,m+1} - \mathcal{A}_{01}\mathcal{B}'_{nm}. \tag{C9}$$

Using the above and then applying $\mu_{l,h} = 0$ we get

$$\frac{\partial^2\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_l^2} = \mathcal{B}_{20} - \mathcal{A}_{20}\mathcal{B}_{00},$$

$$\frac{\partial^4\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_l^4} = \mathcal{B}_{40} - 6\mathcal{A}_{20}\mathcal{B}_{20} + 6\mathcal{A}_{20}^2\mathcal{B}_{00} - \mathcal{A}_{40}\mathcal{B}_{00},$$

$$\frac{\partial^6\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_l^6} = \mathcal{B}_{60} - \mathcal{A}_{60}\mathcal{B}_{00} - 15\mathcal{A}_{20}\mathcal{B}_{40} - 15\mathcal{A}_{40}\mathcal{B}_{20} + 30\mathcal{A}_{20}\mathcal{A}_{40}\mathcal{B}_{00} + 90\mathcal{A}_{20}^2\mathcal{B}_{20} - 90\mathcal{A}_{20}^3\mathcal{B}_{00},$$

$$\frac{\partial^2\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_h^2} = \mathcal{B}_{02} - \mathcal{A}_{02}\mathcal{B}_{00},$$

$$\frac{\partial^4\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_h^4} = \mathcal{B}_{04} - 6\mathcal{A}_{02}\mathcal{B}_{02} + 6\mathcal{A}_{02}^2\mathcal{B}_{00} - \mathcal{A}_{04}\mathcal{B}_{00},$$

$$\frac{\partial^6\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_h^6} = \mathcal{B}_{06} - \mathcal{A}_{06}\mathcal{B}_{00} - 15\mathcal{A}_{02}\mathcal{B}_{04} - 15\mathcal{A}_{04}\mathcal{B}_{02} + 30\mathcal{A}_{02}\mathcal{A}_{04}\mathcal{B}_{00} + 90\mathcal{A}_{02}^2\mathcal{B}_{02} - 90\mathcal{A}_{02}^3\mathcal{B}_{00},$$

$$\frac{\partial^2\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_l\partial\mu_h} = \mathcal{B}_{11} - \mathcal{A}_{11}\mathcal{B}_{00},$$

$$\frac{\partial^4\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_l^2\partial\mu_h^2} = \mathcal{B}_{22} - \mathcal{A}_{22}\mathcal{B}_{00} + 2\mathcal{B}_{00}\mathcal{A}_{02}\mathcal{A}_{20} + 4\mathcal{B}_{00}\mathcal{A}_{11}^2 - 4\mathcal{B}_{11}\mathcal{A}_{11} - \mathcal{A}_{02}\mathcal{B}_{20} - \mathcal{A}_{20}\mathcal{B}_{02},$$

$$\frac{\partial^4\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_l^1\partial\mu_h^3} = \mathcal{B}_{13} - \mathcal{A}_{13}\mathcal{B}_{00} - 3\mathcal{A}_{02}\mathcal{B}_{11} - 3\mathcal{A}_{11}\mathcal{B}_{02} + 6\mathcal{A}_{11}\mathcal{A}_{02}\mathcal{B}_{00},$$

$$\frac{\partial^4\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_l^3\partial\mu_h^1} = \mathcal{B}_{31} - \mathcal{A}_{31}\mathcal{B}_{00} - 3\mathcal{A}_{20}\mathcal{B}_{11} - 3\mathcal{A}_{11}\mathcal{B}_{20} + 6\mathcal{A}_{11}\mathcal{A}_{20}\mathcal{B}_{00},$$

$$\begin{aligned}\frac{\partial^6\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_l^2\partial\mu_h^4} &= \mathcal{B}_{24} - \mathcal{A}_{24}\mathcal{B}_{00} - 8\mathcal{A}_{11}\mathcal{B}_{13} + 16\mathcal{A}_{13}\mathcal{A}_{11}\mathcal{B}_{00} - 8\mathcal{A}_{13}\mathcal{B}_{11} - \mathcal{A}_{04}\mathcal{B}_{20} - \mathcal{A}_{20}\mathcal{B}_{04} - 6\mathcal{A}_{02}\mathcal{B}_{22}\\ &\quad + 6\mathcal{A}_{02}^2\mathcal{B}_{20} - 6\mathcal{A}_{22}\mathcal{B}_{02} + 48\mathcal{A}_{02}\mathcal{A}_{11}\mathcal{B}_{11} + 12\mathcal{A}_{22}\mathcal{A}_{02}\mathcal{B}_{00} + 12\mathcal{A}_{20}\mathcal{A}_{02}\mathcal{B}_{02} + 24\mathcal{A}_{11}^2\mathcal{B}_{02}\\ &\quad - 72\mathcal{A}_{11}^2\mathcal{A}_{02}\mathcal{B}_{00} - 18\mathcal{A}_{20}\mathcal{A}_{02}^2\mathcal{B}_{00} + 2\mathcal{A}_{20}\mathcal{A}_{04}\mathcal{B}_{00},\end{aligned}$$

$$\begin{aligned}\frac{\partial^6\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_l^4\partial\mu_h^2} &= \mathcal{B}_{42} - \mathcal{A}_{42}\mathcal{B}_{00} - 8\mathcal{A}_{11}\mathcal{B}_{31} + 16\mathcal{A}_{31}\mathcal{A}_{11}\mathcal{B}_{00} - 8\mathcal{A}_{31}\mathcal{B}_{11} - \mathcal{A}_{40}\mathcal{B}_{02} - \mathcal{A}_{02}\mathcal{B}_{40} - 6\mathcal{A}_{20}\mathcal{B}_{22}\\ &\quad + 6\mathcal{A}_{20}^2\mathcal{B}_{02} - 6\mathcal{A}_{22}\mathcal{B}_{20} + 48\mathcal{A}_{20}\mathcal{A}_{11}\mathcal{B}_{11} + 12\mathcal{A}_{22}\mathcal{A}_{20}\mathcal{B}_{00} + 12\mathcal{A}_{02}\mathcal{A}_{20}\mathcal{B}_{20} + 24\mathcal{A}_{11}^2\mathcal{B}_{20}\\ &\quad - 72\mathcal{A}_{11}^2\mathcal{A}_{20}\mathcal{B}_{00} - 18\mathcal{A}_{02}\mathcal{A}_{20}^2\mathcal{B}_{00} + 2\mathcal{A}_{02}\mathcal{A}_{40}\mathcal{B}_{00},\end{aligned}$$

$$\begin{aligned}\frac{\partial^6\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_l^1\partial\mu_h^5} &= \mathcal{B}_{15} - \mathcal{A}_{15}\mathcal{B}_{00} - 10\mathcal{A}_{02}\mathcal{B}_{13} + 20\mathcal{A}_{13}\mathcal{A}_{02}\mathcal{B}_{00} - 5\mathcal{A}_{04}\mathcal{B}_{11} - 10\mathcal{A}_{13}\mathcal{B}_{02} - 5\mathcal{A}_{11}\mathcal{B}_{04} + 30\mathcal{A}_{02}^2\mathcal{B}_{11}\\ &\quad + 60\mathcal{A}_{11}\mathcal{A}_{02}\mathcal{B}_{02} + 10\mathcal{A}_{11}\mathcal{A}_{04}\mathcal{B}_{00} - 90\mathcal{A}_{11}\mathcal{A}_{02}^2\mathcal{B}_{00},\end{aligned}$$

$$\begin{aligned}\frac{\partial^6\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_l^5\partial\mu_h^1} &= \mathcal{B}_{51} - \mathcal{A}_{51}\mathcal{B}_{00} - 10\mathcal{A}_{20}\mathcal{B}_{31} + 20\mathcal{A}_{31}\mathcal{A}_{20}\mathcal{B}_{00} - 5\mathcal{A}_{40}\mathcal{B}_{11} - 10\mathcal{A}_{31}\mathcal{B}_{20} - 5\mathcal{A}_{11}\mathcal{B}_{40} + 30\mathcal{A}_{20}^2\mathcal{B}_{11}\\ &\quad + 60\mathcal{A}_{11}\mathcal{A}_{20}\mathcal{B}_{20} + 10\mathcal{A}_{11}\mathcal{A}_{40}\mathcal{B}_{00} - 90\mathcal{A}_{11}\mathcal{A}_{20}^2\mathcal{B}_{00},\end{aligned}$$

$$\begin{aligned}\frac{\partial^6\langle\mathrm{tr}M_l^{-1}\rangle}{\partial\mu_l^3\partial\mu_h^3} &= \mathcal{B}_{33} - \mathcal{A}_{33}\mathcal{B}_{00} - 3\mathcal{A}_{20}\mathcal{B}_{13} - 3\mathcal{A}_{02}\mathcal{B}_{31} - 9\mathcal{A}_{11}\mathcal{B}_{22} - 9\mathcal{A}_{22}\mathcal{B}_{11} - 3\mathcal{A}_{13}\mathcal{B}_{20} - 3\mathcal{A}_{31}\mathcal{B}_{02}\\ &\quad + 36\mathcal{A}_{11}^2\mathcal{B}_{11} - 36\mathcal{A}_{11}^3\mathcal{B}_{00} + 6\mathcal{A}_{13}\mathcal{A}_{20}\mathcal{B}_{00} + 6\mathcal{A}_{31}\mathcal{A}_{02}\mathcal{B}_{00} + 18\mathcal{A}_{02}\mathcal{A}_{20}\mathcal{B}_{11} + 18\mathcal{A}_{11}\mathcal{A}_{22}\mathcal{B}_{00}\\ &\quad + 18\mathcal{A}_{02}\mathcal{A}_{11}\mathcal{B}_{20} + 18\mathcal{A}_{11}\mathcal{A}_{20}\mathcal{B}_{02} - 54\mathcal{A}_{11}\mathcal{A}_{02}\mathcal{A}_{20}\mathcal{B}_{00}.\end{aligned}$$

Replacing $\mathcal{B}$ with $\mathcal{B}'$ in the above we get the expressions for the derivatives of $\langle\mathrm{tr}M_h^{-1}\rangle$. Let

$$l_n = \frac{\partial^n \operatorname{tr} M_l^{-1}}{\partial \mu_l^n}, \tag{C10}$$

$$h_n = \frac{\partial^n \operatorname{tr} M_h^{-1}}{\partial \mu_h^n}, \tag{C11}$$

then explicitly we have

$$\mathcal{B}_{00} = \langle l_0 \rangle = \langle \operatorname{tr} M_l^{-1} \rangle, \qquad \mathcal{B}_{10} = \langle l_1 \rangle + \langle l_0 L_1 \rangle, \qquad \mathcal{B}_{20} = \langle l_2 \rangle + 2\langle l_1 L_1 \rangle + \langle l_0 L_2 \rangle + \langle l_0 L_1^2 \rangle,$$

$$\mathcal{B}_{30} = \langle l_3 \rangle + 3\langle l_2 L_1 \rangle + 3\langle l_1 L_2 \rangle + \langle l_0 L_3 \rangle + 3\langle l_1 L_1^2 \rangle + 3\langle l_0 L_1 L_2 \rangle + \langle l_0 L_1^3 \rangle,$$

$$\mathcal{B}_{40} = \langle l_4 \rangle + 4\langle l_3 L_1 \rangle + 6\langle l_2 L_2 \rangle + 4\langle l_1 L_3 \rangle + \langle l_0 L_4 \rangle + 6\langle l_2 L_1^2 \rangle + 12\langle l_1 L_1 L_2 \rangle + 3\langle l_0 L_2^2 \rangle + 4\langle l_0 L_1 L_3 \rangle + 4\langle l_1 L_1^3 \rangle + 6\langle l_0 L_1^2 L_2 \rangle + \langle l_0 L_1^4 \rangle,$$

$$\mathcal{B}_{50} = \langle l_5 \rangle + 30\langle l_2 L_1 L_2 \rangle + 30\langle l_1 L_1^2 L_2 \rangle + 20\langle l_1 L_1 L_3 \rangle + 10\langle l_0 L_1^3 L_2 \rangle + \langle l_0 L_5 \rangle + 5\langle l_1 L_1^4 \rangle + 10\langle l_2 L_3 \rangle + 5\langle l_4 L_1 \rangle + 10\langle l_3 L_2 \rangle + 10\langle l_3 L_1^2 \rangle + 5\langle l_1 L_4 \rangle + 15\langle l_1 L_2^2 \rangle + 10\langle l_2 L_1^3 \rangle + \langle l_0 L_1^5 \rangle + 10\langle l_0 L_2 L_3 \rangle + 5\langle l_0 L_1 L_4 \rangle + 15\langle l_0 L_1 L_2^2 \rangle + 10\langle l_0 L_1^2 L_3 \rangle,$$

$$\mathcal{B}_{60} = \langle l_6 \rangle + 60\langle l_1 L_2 L_3 \rangle + 15\langle l_0 L_1^4 L_2 \rangle + 20\langle l_0 L_1^3 L_3 \rangle + 90\langle l_2 L_1^2 L_2 \rangle + 90\langle l_1 L_1 L_2^2 \rangle + \langle l_0 L_6 \rangle + 6\langle l_1 L_1^5 \rangle + 45\langle l_2 L_2^2 \rangle + 15\langle l_2 L_1^4 \rangle + 20\langle l_3 L_3 \rangle + 15\langle l_2 L_4 \rangle + 6\langle l_5 L_1 \rangle + 15\langle l_4 L_2 \rangle + 15\langle l_4 L_1^2 \rangle + 20\langle l_3 L_1^3 \rangle + 10\langle l_0 L_3^2 \rangle + 15\langle l_0 L_2^3 \rangle + 6\langle l_1 L_5 \rangle + 60\langle l_3 L_1 L_2 \rangle + 60\langle l_2 L_1 L_3 \rangle + 60\langle l_1 L_1^2 L_3 \rangle + 30\langle l_1 L_1 L_4 \rangle + 60\langle l_1 L_1^3 L_2 \rangle + 45\langle l_0 L_1^2 L_2^2 \rangle + 15\langle l_0 L_2 L_4 \rangle + 6\langle l_0 L_1 L_5 \rangle + 15\langle l_0 L_1^2 L_4 \rangle + 60\langle l_0 L_1 L_2 L_3 \rangle + \langle l_0 L_1^6 \rangle,$$

$$\mathcal{B}_{02} = \langle l_0 H_2 \rangle + \langle l_0 H_1^2 \rangle,$$

$$\mathcal{B}_{04} = \langle l_0 H_4 \rangle + 4\langle l_0 H_3 H_1 \rangle + 3\langle l_0 H_2^2 \rangle + 6\langle l_0 H_2 H_1^2 \rangle + \langle l_0 H_1^4 \rangle,$$

$$\mathcal{B}_{06} = \langle l_0 H_6 \rangle + 6\langle l_0 H_5 H_1 \rangle + 15\langle l_0 H_4 H_2 \rangle + 10\langle l_0 H_3^2 \rangle + 15\langle l_0 H_4 H_1^2 \rangle + 60\langle l_0 H_3 H_2 H_1 \rangle + 15\langle l_0 H_2^3 \rangle + 20\langle l_0 H_3 H_1^3 \rangle + 45\langle l_0 H_2^2 H_1^2 \rangle + 15\langle l_0 H_2 H_1^4 \rangle + \langle l_0 H_1^6 \rangle,$$

$$\mathcal{B}_{11} = \langle l_1 H_1 \rangle + \langle l_0 L_1 H_1 \rangle,$$

$$\mathcal{B}_{22} = \langle (l_2 + 2 l_1 L_1 + l_0 L_2 + l_0 L_1^2)(H_2 + H_1^2) \rangle,$$

$$\mathcal{B}_{31} = \langle l_3 H_1 \rangle + 3\langle l_2 L_1 H_1 \rangle + 3\langle l_1 L_2 H_1 \rangle + \langle l_0 L_3 H_1 \rangle + 3\langle l_1 L_1^2 H_1 \rangle + 3\langle l_0 L_1 L_2 H_1 \rangle + \langle l_0 L_1^3 H_1 \rangle,$$

$$\mathcal{B}_{13} = \langle (l_1 + l_0 L_1)(H_3 + 3 H_2 H_1 + H_1^3) \rangle,$$

$$\mathcal{B}_{42} = \langle (l_4 + 4 l_3 L_1 + 6 l_2 L_2 + 4 l_1 L_3 + l_0 L_4 + 6 l_2 L_1^2 + 12 l_1 L_1 L_2 + 3 l_0 L_2^2 + 4 l_0 L_1 L_3 + 4 l_1 L_1^3 + 6 l_0 L_1^2 L_2 + l_0 L_1^4)(H_2 + H_1^2) \rangle,$$

$$\mathcal{B}_{24} = \langle (l_2 + 2 l_1 L_1 + l_0 L_2 + l_0 L_1^2)(H_4 + 4 H_3 H_1 + 3 H_2^2 + 6 H_2 H_1^2 + H_1^4) \rangle,$$

$$\mathcal{B}_{51} = \langle l_5 H_1 \rangle + 30\langle l_2 L_1 L_2 H_1 \rangle + 30\langle l_1 L_1^2 L_2 H_1 \rangle + 20\langle l_1 L_1 L_3 H_1 \rangle + 10\langle l_0 L_1^3 L_2 H_1 \rangle + \langle l_0 L_5 H_1 \rangle + 5\langle l_1 L_1^4 H_1 \rangle + 10\langle l_2 L_3 H_1 \rangle + 5\langle l_4 L_1 H_1 \rangle + 10\langle l_3 L_2 H_1 \rangle + 10\langle l_3 L_1^2 H_1 \rangle + 5\langle l_1 L_4 H_1 \rangle + 15\langle l_1 L_2^2 H_1 \rangle + 10\langle l_2 L_1^3 H_1 \rangle + \langle l_0 L_1^5 H_1 \rangle + 10\langle l_0 L_2 L_3 H_1 \rangle + 5\langle l_0 L_1 L_4 H_1 \rangle + 15\langle l_0 L_1 L_2^2 H_1 \rangle + 10\langle l_0 L_1^2 L_3 H_1 \rangle,$$

$$\mathcal{B}_{15} = \langle (l_1 + l_0 L_1)(H_5 + 5 H_4 H_1 + 10 H_3 H_2 + 10 H_3 H_1^2 + 15 H_2^2 H_1 + 10 H_2 H_1^3 + H_1^5) \rangle,$$

$$\mathcal{B}_{33} = \langle (l_3 + 3 l_2 L_1 + 3 l_1 L_2 + l_0 L_3 + 3 l_1 L_1^2 + 3 l_0 L_1 L_2 + l_0 L_1^3)(H_3 + 3 H_2 H_1 + H_1^3) \rangle.$$

From the above expressions it is easy to get the $\mathcal{B}'$ expressions by substitutions

$$\mathcal{B}'_{mn} = \mathcal{B}_{nm}(1 \to h, L \leftrightarrow H). \tag{C12}$$

Explicitly l_n and h_n are the derivatives below with $M = M_{l,h}$ and $\mu = \mu_{l,h}$.

$$\frac{\partial\, \mathrm{tr} M^{-1}}{\partial \mu} = -\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right), \tag{C13}$$

$$\frac{\partial^2\, \mathrm{tr} M^{-1}}{\partial \mu^2} = -\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\right) + 2\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right), \tag{C14}$$

$$\begin{aligned}\frac{\partial^3\, \mathrm{tr} M^{-1}}{\partial \mu^3} = &-\mathrm{tr}\left(M^{-1}\frac{\partial^3 M}{\partial \mu^3}M^{-1}\right) + 3\,\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right) + 3\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\right)\\ &- 6\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right),\end{aligned} \tag{C15}$$

$$\begin{aligned}\frac{\partial^4\, \mathrm{tr} M^{-1}}{\partial \mu^4} = &-\mathrm{tr}\left(M^{-1}\frac{\partial^4 M}{\partial \mu^4}M^{-1}\right) + 4\,\mathrm{tr}\left(M^{-1}\frac{\partial^3 M}{\partial \mu^3}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right) + 6\,\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\right)\\ &+ 4\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^3 M}{\partial \mu^3}M^{-1}\right) - 12\,\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right)\\ &- 12\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right) - 12\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\right)\\ &+ 24\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right),\end{aligned} \tag{C16}$$

$$\begin{aligned}\frac{\partial^5\, \mathrm{tr} M^{-1}}{\partial \mu^5} = &-\mathrm{tr}\left(M^{-1}\frac{\partial^5 M}{\partial \mu^5}M^{-1}\right) + 5\,\mathrm{tr}\left(M^{-1}\frac{\partial^4 M}{\partial \mu^4}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right) + 5\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^4 M}{\partial \mu^4}M^{-1}\right)\\ &+ 10\,\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^3 M}{\partial \mu^3}M^{-1}\right) + 10\,\mathrm{tr}\left(M^{-1}\frac{\partial^3 M}{\partial \mu^3}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\right)\\ &- 30\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\right) - 30\,\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right)\\ &- 30\,\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\right) - 20\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^3 M}{\partial \mu^3}M^{-1}\right)\\ &- 20\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^3 M}{\partial \mu^3}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right) - 20\,\mathrm{tr}\left(M^{-1}\frac{\partial^3 M}{\partial \mu^3}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right)\\ &+ 60\,\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right) + 60\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right)\\ &+ 60\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right) + 60\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\right)\\ &- 120\,\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\right),\end{aligned} \tag{C17}$$

$$
\begin{aligned}
\frac{\partial^6 \operatorname{tr} M^{-1}}{\partial \mu^6} = & -\operatorname{tr}\Big(M^{-1}\frac{\partial^6 M}{\partial\mu^6}M^{-1}\Big) + 6\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^5 M}{\partial\mu^5}M^{-1}\Big) + 6\operatorname{tr}\Big(M^{-1}\frac{\partial^5 M}{\partial\mu^5}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) \\
& + 15\operatorname{tr}\Big(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^4 M}{\partial\mu^4}M^{-1}\Big) + 15\operatorname{tr}\Big(M^{-1}\frac{\partial^4 M}{\partial\mu^4}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\Big) + 20\operatorname{tr}\Big(M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\Big) \\
& - 30\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^4 M}{\partial\mu^4}M^{-1}\Big) - 30\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^4 M}{\partial\mu^4}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) \\
& - 30\operatorname{tr}\Big(M^{-1}\frac{\partial^4 M}{\partial\mu^4}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) - 60\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\Big) \\
& - 60\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\Big) - 60\operatorname{tr}\Big(M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) \\
& - 60\operatorname{tr}\Big(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) - 60\operatorname{tr}\Big(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\Big) \\
& - 90\operatorname{tr}\Big(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\Big) - 60\operatorname{tr}\Big(M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\Big) \\
& + 120\operatorname{tr}\Big(M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) + 120\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) \\
& + 120\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) + 120\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\Big) \\
& + 180\operatorname{tr}\Big(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) + 180\operatorname{tr}\Big(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) \\
& + 180\operatorname{tr}\Big(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\Big) + 180\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) \\
& + 180\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\Big) + 180\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\Big) \\
& - 360\operatorname{tr}\Big(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) \\
& - 360\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) \\
& - 360\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) \\
& - 360\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big) \\
& - 360\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\Big) \\
& + 720\operatorname{tr}\Big(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\Big).
\end{aligned}
\tag{C18}
$$

2. Second type of derivative

The next term we are concerned with is the derivative

$$\frac{\partial^{n+m}\langle M_f^{-1}\frac{dM_f}{du_0}\rangle}{\partial(\mu_l N_t)^n \partial(\mu_h N_t)^m}\Bigg|_{\mu_{l,h}=0}. \tag{C19}$$

Here we start from the definitions

$$\mathcal{C}_{nm} \equiv \Big\langle e^{-L_0}e^{-H_0}\frac{\partial^n[\operatorname{tr}(M_l^{-1}\frac{dM_l}{du_0})e^{L_0}]}{\partial\mu_l^n}\frac{\partial^m e^{H_0}}{\partial\mu_h^m}\Big\rangle, \tag{C20}$$

$$\mathcal{C}'_{nm} \equiv \Big\langle e^{-L_0}e^{-H_0}\frac{\partial^n e^{L_0}}{\partial\mu_l^n}\frac{\partial^m[\operatorname{tr}(M_h^{-1}\frac{dM_h}{du_0})e^{H_0}]}{\partial\mu_h^m}\Big\rangle. \tag{C21}$$

From the above

$$\mathcal{C}_{00} \equiv \left\langle \mathrm{tr}\left(M_l^{-1}\frac{dM_l}{du_0}\right)\right\rangle, \tag{C22}$$

$$\mathcal{C}'_{00} \equiv \left\langle \mathrm{tr}\left(M_h^{-1}\frac{dM_h}{du_0}\right)\right\rangle. \tag{C23}$$

The following can be proven true:

$$\frac{\partial \mathcal{C}_{nm}}{\partial \mu_l} = \mathcal{C}_{n+1,m} - \mathcal{A}_{10}\mathcal{C}_{nm}, \tag{C24}$$

$$\frac{\partial \mathcal{C}_{nm}}{\partial \mu_h} = \mathcal{C}_{n,m+1} - \mathcal{A}_{01}\mathcal{C}_{nm}, \tag{C25}$$

$$\frac{\partial \mathcal{C}'_{nm}}{\partial \mu_l} = \mathcal{C}'_{n+1,m} - \mathcal{A}_{10}\mathcal{C}'_{nm}, \tag{C26}$$

$$\frac{\partial \mathcal{C}'_{nm}}{\partial \mu_h} = \mathcal{C}'_{n,m+1} - \mathcal{A}_{01}\mathcal{C}'_{nm}. \tag{C27}$$

The derivatives

$$\frac{\partial^n \langle \mathrm{tr}(M_{l,h}^{-1}\frac{dM_{l,h}}{du_0})\rangle}{\partial \mu_{l,h}^n} \tag{C28}$$

have the form of the derivatives of $\langle \mathrm{tr}(M_{l,h}^{-1})\rangle$ in the previous section with the substitutions $\mathcal{B}_{nm} \to \mathcal{C}_{nm}$ and $\mathcal{B}'_{nm} \to \mathcal{C}'_{nm}$. The explicit forms of $\mathcal{C}_{nm}$ and $\mathcal{C}'_{nm}$ are the same as for $\mathcal{B}_{nm}$ and $\mathcal{B}'_{nm}$ with the substitutions $l_n \to \lambda_n$ and $h_n \to \chi_n$, where

$$\lambda_n = \frac{\partial^n \mathrm{tr}(M_l^{-1}\frac{dM_h}{du_0})}{\partial \mu_l^n}, \tag{C29}$$

$$\chi_n = \frac{\partial^n \mathrm{tr}(M_h^{-1}\frac{dM_h}{du_0})}{\partial \mu_h^n}. \tag{C30}$$

These derivatives have the form below with $M = M_{l,h}$ and $\mu = \mu_{l,h}$:

$$\frac{\partial\, \mathrm{tr}(M^{-1}\frac{dM}{du_0})}{\partial \mu} = \mathrm{tr}\left(\frac{\partial M^{-1}}{\partial \mu}\frac{dM}{du_0} + M^{-1}\frac{\partial}{\partial \mu}\frac{dM}{du_0}\right)$$

$$\frac{\partial^2 \mathrm{tr}(M^{-1}\frac{dM}{du_0})}{\partial \mu^2} = \mathrm{tr}\left(\frac{\partial^2 M^{-1}}{\partial \mu^2}\frac{dM}{du_0} + 2\frac{\partial M^{-1}}{\partial \mu}\frac{\partial}{\partial \mu}\frac{dM}{du_0} + M^{-1}\frac{\partial^2}{\partial \mu^2}\frac{dM}{du_0}\right)$$

$$\frac{\partial^3 \mathrm{tr}(M^{-1}\frac{dM}{du_0})}{\partial \mu^3} = \mathrm{tr}\left(\frac{\partial^3 M^{-1}}{\partial \mu^3}\frac{dM}{du_0} + 3\frac{\partial^2 M^{-1}}{\partial \mu^2}\frac{\partial}{\partial \mu}\frac{dM}{du_0} + 3\frac{\partial M^{-1}}{\partial \mu}\frac{\partial^2}{\partial \mu^2}\frac{dM}{du_0} + M^{-1}\frac{\partial^3}{\partial \mu^3}\frac{dM}{du_0}\right)$$

$$\frac{\partial^4 \mathrm{tr}(M^{-1}\frac{dM}{du_0})}{\partial \mu^4} = \mathrm{tr}\left(\frac{\partial^4 M^{-1}}{\partial \mu^4}\frac{dM}{du_0} + 4\frac{\partial^3 M^{-1}}{\partial \mu^3}\frac{\partial}{\partial \mu}\frac{dM}{du_0} + 6\frac{\partial^2 M^{-1}}{\partial \mu^2}\frac{\partial^2}{\partial \mu^2}\frac{dM}{du_0} + 4\frac{\partial M^{-1}}{\partial \mu}\frac{\partial^3}{\partial \mu^3}\frac{dM}{du_0} + M^{-1}\frac{\partial^4}{\partial \mu^4}\frac{dM}{du_0}\right),$$

$$\begin{aligned}\frac{\partial^5 \mathrm{tr}(M^{-1}\frac{dM}{du_0})}{\partial \mu^5} = \mathrm{tr}\Big(&\frac{\partial^5 M^{-1}}{\partial \mu^5}\frac{dM}{du_0} + 5\frac{\partial^4 M^{-1}}{\partial \mu^4}\frac{\partial}{\partial \mu}\frac{dM}{du_0} + 5\frac{\partial M^{-1}}{\partial \mu}\frac{\partial^4}{\partial \mu^4}\frac{dM}{du_0} + 10\frac{\partial^3 M^{-1}}{\partial \mu^3}\frac{\partial^2}{\partial \mu^2}\frac{dM}{du_0} + 10\frac{\partial^2 M^{-1}}{\partial \mu^2}\frac{\partial^3}{\partial \mu^3}\frac{dM}{du_0}\\ &+ M^{-1}\frac{\partial^5}{\partial \mu^5}\frac{dM}{du_0}\Big)\end{aligned}$$

$$\begin{aligned}\frac{\partial^6 \mathrm{tr}(M^{-1}\frac{dM}{du_0})}{\partial \mu^6} = \mathrm{tr}\Big(&\frac{\partial^6 M^{-1}}{\partial \mu^6}\frac{dM}{du_0} + 6\frac{\partial^5 M^{-1}}{\partial \mu^5}\frac{\partial}{\partial \mu}\frac{dM}{du_0} + 6\frac{\partial M^{-1}}{\partial \mu}\frac{\partial^5}{\partial \mu^5}\frac{dM}{du_0} + 15\frac{\partial^4 M^{-1}}{\partial \mu^4}\frac{\partial^2}{\partial \mu^2}\frac{dM}{du_0} + 15\frac{\partial^2 M^{-1}}{\partial \mu^2}\frac{\partial^4}{\partial \mu^4}\frac{dM}{du_0}\\ &+ 20\frac{\partial^3 M^{-1}}{\partial \mu^3}\frac{\partial^3}{\partial \mu^3}\frac{dM}{du_0} + M^{-1}\frac{\partial^6}{\partial \mu^6}\frac{dM}{du_0}\Big).\end{aligned}$$

In the above the derivatives of M^{-1} can be taken from the previous subsection. The derivative of M with respect both to the chemical potential and the tadpole factor for the asqtad action, is

$$\begin{aligned}\frac{\partial^n}{\partial \mu^n}\frac{dM}{du_0} = \frac{1}{2}\eta_0(x)\Big[&\frac{dU_0^{(F)}(x)}{du_o}e^{\mu}\delta_{x+\hat{0},y}\\ &- (-1)^n\frac{dU_0^{(F)\dagger}}{du_o}e^{-\mu}\delta_{x,y+\hat{0}}\\ &+ (3)^n\frac{dU_0^{(L)}(x)}{du_o}\mathrm{e}^{3\mu}\delta_{x+3\hat{0},y}\\ &- (-3)^n\frac{dU_0^{(L)\dagger}(x-3\hat{0})}{du_o}\mathrm{e}^{-3\mu}\delta_{x,y+3\hat{0}}\Big].\end{aligned} \tag{C31}$$

3. Third type of derivative

The third type is the gauge derivative

$$\frac{\partial^{n+m}\langle \mathcal{G}\rangle}{\partial(\mu_1 N_t)^n \partial(\mu_h N_t)^m}\bigg|_{\mu_{l,h}=0}. \tag{C32}$$

In this case let

$$G_{nm} \equiv \left\langle \mathcal{G}\mathrm{e}^{-L_0}\mathrm{e}^{-H_0}\frac{\partial^n \mathrm{e}^{L_0}}{\partial \mu_l^n}\frac{\partial^m \mathrm{e}^{H_0}}{\partial \mu_h^m}\right\rangle, \tag{C33}$$

and similarly as before

$$\frac{\partial G_{nm}}{\partial \mu_l} = G_{n+1,m} - \mathcal{A}_{10}G_{nm}, \tag{C34}$$

$$\frac{\partial G_{nm}}{\partial \mu_h} = G_{n,m+1} - \mathcal{A}_{01} G_{nm}, \tag{C35}$$

with

$$G_{00} = \langle \mathcal{G} \rangle. \tag{C36}$$

This means that the necessary derivatives $\frac{\partial^{n+m}\langle \mathcal{G}\rangle}{\partial(\mu_l N_t)^n \partial(\mu_h N_t)^m}|_{\mu_{l,h}=0}$ have the same form as the derivatives $\frac{\partial^{n+m}\,\mathrm{tr}\langle M_f^{-1}\rangle}{\partial(\mu_l N_t)^n \partial(\mu_h N_t)^m}|_{\mu_{l,h}=0}$ with $\mathcal{B}_{nm} \to G_{nm}$. The G_{nm} observables have very similar form to the $\mathcal{A}_{nm}$ observables, but with an additional multiplication by $\mathcal{G}$ inside the ensemble average brackets of each term in them. For example,

$$\frac{\partial^2 \langle \mathcal{G} \rangle}{\partial \mu_l^2}\bigg|_{\mu_{l,h}=0} = G_{20} - \mathcal{A}_{20} G_{00} \tag{C37}$$

and

$$G_{20} = \langle \mathcal{G} L_2 \rangle + \langle \mathcal{G} L_1^2 \rangle, \tag{C38}$$

etc.

For example, combining the three types of terms for each flavor, one of the simplest of the Taylor coefficients in the interaction measure expansion, b_{20}, becomes

$$\begin{aligned} b_{20} = -\frac{1}{2!}\frac{N_t}{N_s^3}\bigg[&\frac{1}{2}\frac{d(m_l a)}{d\ln a}\bigg|_{\mu_{l,h}=0}(\mathcal{B}_{20} - \mathcal{A}_{20}\mathcal{B}_{00}) \\ &+ \frac{1}{2}\frac{du_0}{d\ln a}\bigg|_{\mu_{l,h}=0}(\mathcal{C}_{20} - \mathcal{A}_{20}\mathcal{C}_{00}) \\ &+ \frac{1}{4}\frac{d(m_h a)}{d\ln a}\bigg|_{\mu_{l,h}=0}(\mathcal{B}'_{20} - \mathcal{A}_{20}\mathcal{B}'_{00}) \\ &+ \frac{1}{4}\frac{du_0}{d\ln a}\bigg|_{\mu_{l,h}=0}(\mathcal{C}'_{20} - \mathcal{A}_{20}\mathcal{C}'_{00}) \\ &+ G_{20} - \mathcal{A}_{20} G_{00}\bigg]. \end{aligned} \tag{C39}$$

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