Axial anomaly and magnetism of nuclear and quark matter

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We consider the response of the QCD ground state at finite baryon density to a strong magnetic field B. We point out the dominant role played by the coupling of neutral Goldstone bosons, such as π^0, to the magnetic field via the axial triangle anomaly. We show that, in vacuum, above a value of $B \sim m_\pi^2/e$, a metastable object appears—the π^0 domain wall. Because of the axial anomaly, the wall carries a baryon number surface density proportional to B. As a result, for $B \gtrsim 10^{19}$ G a stack of parallel π^0 domain walls is energetically more favorable than nuclear matter at the same density. Similarly, at higher densities, somewhat weaker magnetic fields of order $B \gtrsim 10^{17}$–10^{18} G transform the color-superconducting ground state of QCD into new phases containing stacks of axial isoscalar (η or η') domain walls. We also show that a quark-matter state known as "Goldstone current state," in which a gradient of a Goldstone field is spontaneously generated, is ferromagnetic due to the axial anomaly. We estimate the size of the fields created by such a state in a typical neutron star to be of order 10^{14}–10^{15} G.

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I. INTRODUCTION

There have been several studies of the structure of QCD vacuum in high magnetic fields [1–4]. The typical strength of a magnetic field which would change the structure of the QCD vacuum is very high and can be estimated as

$$B \sim \frac{m_\rho^2}{e} \sim 10^{20}\ \text{G}, \tag{1}$$

where $m_\rho = 770$ MeV is the typical energy scale of QCD. For example, the typical magnetic field that changes substantially the chiral condensate is $(4\pi f_\pi)^2/e$ [1], which is of the same order as in Eq. (1). In Ref. [2] it was argued that for $B \gtrsim 10\ \text{GeV}^2 \approx 5\cdot 10^{21}$ G a condensate of spin-polarized $u\bar{u}$ pairs appear.

The behavior of nuclear matter in strong magnetic fields has been studied more extensively. The motivation for such studies is the high magnetic field observed in magnetars [5]. On general grounds one expects (see, e.g., Ref. [6]) that the magnetic field affects significantly the structure of the matter once the synchrotron (Landau level) energy $\sqrt{eB}$ is comparable to the typical energy associated with charge excitations in the system, such as, e.g., proton Fermi energies in nuclear matter.

The response of color-superconducting quark matter to a strong magnetic field has also been studied [7–13]. Similarly, in all mechanisms studied so far, the ground state is affected above some value of the magnetic field determined by the superconducting gap Δ and/or the chemical potential μ. For example, fields of order $\mu\Delta/e$ or higher are needed to destroy color superconductivity [7].

In this paper we show that, due to the anomalous coupling of neutral pseudoscalar Golstone bosons to electromagnetism, the structure of the ground state is modified at much lower values of the magnetic field. In fact, these values are parametrically lower than (1) in the limit where the Golsdtone bosons become massless (e.g., the chiral limit).

For the low-density nuclear matter we find two scales of magnetic field that are relevant (see Sec. III):

$$B_0 = \frac{3m_\pi^2}{e}, \qquad B_1 = 16\pi\frac{f_\pi^2 m_\pi}{e m_N}. \tag{2}$$

In particular, above B_1 nuclear matter is replaced by a different state. The most striking feature of Eq. (2) is that both B_0 and B_1 *vanish* in the chiral limit: when $m_\pi = 0$, the structure of nuclear matter is altered at an arbitrarily small magnetic field. This is in sharp contrast to the previous estimates of the critical magnetic field, Eq. (1).

The state of QCD associated with scales (2) is a π^0 domain wall—a configuration in which the local expectation value of the π^0 field varies along the direction of the magnetic field $\boldsymbol{B}$ over a scale of pion Compton wavelength. We show that for $|\boldsymbol{B}| > B_0$ the domain wall becomes locally stable (metastable).

The central observation of this paper is that such a domain wall carries nonzero surface baryon charge density proportional to $|\boldsymbol{B}|$. As we show, this is a consequence of the quantum axial anomaly—the triangle anomaly involving the baryon, electromagnetic, and neutral axial currents.[1] When $|\boldsymbol{B}| > B_1$ the parallel stack of such domain walls is energetically more favorable at $\mu \approx m_N$ than low-density nuclear matter, as it carries less energy per baryon. That means nuclear matter turns into a stack of π^0 domain walls at such large magnetic fields. For larger magnetic fields this "wall state" should persist down to chemical potentials $\mu \gtrsim m_N B_1/|\boldsymbol{B}|$.

[1]The physics of triangle anomaly at finite density has also received some interest recently, see, e.g., [14–17].

We note right away that although both B_0 and B_1 vanish in the chiral limit $m_\pi \to 0$ (with $B_0 \ll B_1$), for the physical pion mass, these magnetic fields are of order 10^{19} G, smaller than the QCD scale (1), but still much larger than the fields typical of magnetars.

The crucial role in our analysis is played by the Wess-Zumino-Witten (WZW) term describing the anomalous interaction of the neutral pion field with the external electromagnetic field, and a related pion contribution to the baryon current. For example, the WZW term describes the anomalous $\pi^0 \to 2\gamma$ decay. We review the prerequisite basics of the WZW action in Sec. II. We then derive the scales (2) in Sec. III.

In Sec. IV we show that the same mechanism that leads to the formation of π^0 domain walls in vacuum also operates in color-superconducting phases of QCD at high baryon densities. Such phases could exist in the cores of dense neutron or quark stars. The Nambu-Goldstone bosons associated with broken symmetries in these phases are much lighter [18,19] than π^0 in vacuum. As a result, in these phases, the domain walls appear spontaneously at lower magnetic fields of order 10^{17}–10^{18} G, which decrease with increasing μ due to the decrease of the Nambu-Goldstone boson masses.

Finally, in Sec. V we consider another consequence of the anomaly: the spontaneous generation of magnetization, i.e., ferromagnetism, in dense QCD matter. Ferromagnetism of nuclear and quark matter, under various mechanisms, has been discussed in the literature [20–23]. It has been suggested that ferromagnetism may help explaining certain features of magnetars [24]. We point out that for such magnetization to appear, it is sufficient for a *pseudoscalar* Goldstone boson field to develop a nonzero average spatial gradient. Such a situation may indeed appear in the so-called "Goldstone boson current" phases of quark matter with mismatched quark Fermi surfaces. In the case when all gapless fermions are electrically neutral, we show that the magnitude of the magnetization is determined by the triangle anomalies. We estimate this magnitude in one particular scenario of Goldstone boson current in the color-flavor-locked phase with neutral kaon condensation (CFLK^0 phase) to be of order 10^{16} G. Since only a finite (and presumably small) region inside the neutron star is occupied by this current phase, we estimate the typical magnetic field generated by such a mechanism to be of order 10^{14}–10^{15} G. If such a mechanism indeed operates within the cores of some magnetars, it might account for their unusually large magnetic fields.

II. THE WZW ACTION IN ELECTROMAGNETIC FIELD

A. SU(3) case

We start from the SU(3) chiral perturbation theory, which describes the octet of pseudoscalar Nambu-Goldstone bosons in terms of a 3×3 unitary matrix Σ

$$\Sigma = \exp\left(\frac{i\lambda^a \varphi^a}{f_\pi}\right), \tag{3}$$

where λ^a are the 8 Gell-Mann matrices and

$$\frac{1}{\sqrt{2}}\lambda^a\varphi^a = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & K^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}. \tag{4}$$

Without the WZW term, the Lagrangian of the theory in an external electromagnetic field A_μ is

$$\mathcal{L} = \frac{f_\pi^2}{4}\,\mathrm{tr} D_\mu \Sigma^\dagger D_\mu \Sigma + \mathrm{tr}(M\Sigma + \mathrm{H.c.}), \tag{5}$$

where

$$D_\mu \Sigma = \partial_\mu \Sigma + ieA_\mu[Q, \Sigma], \tag{6}$$

with $Q = \mathrm{diag}(2/3, -1/3, -1/3)$. The Lagrangian is invariant under global $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry, and under the local $\mathrm{U}(1)_Q$ subgroup of this symmetry. Gauging the whole $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ in QCD is not possible due to the axial anomalies [25]. The anomalies are captured by the WZW term in the action [26,27]. We introduce the standard notations,

$$L_\mu = \Sigma \partial_\mu \Sigma^\dagger, \qquad R_\mu = \partial_\mu \Sigma^\dagger \Sigma. \tag{7}$$

In the background of the external electromagnetic field A_μ as well as an auxiliary gauge potential A_μ^B coupled to a baryon current, the WZW term is given by [26–29]

$$\begin{aligned} S_{\mathrm{WZW}}[\Sigma, A_\mu, A_\mu^B] = S_{\mathrm{WZW}}[0] - \int d^4x A_\mu^B j_B^\mu + \frac{\epsilon^{\mu\nu\alpha\beta}}{16\pi^2}\int d^4x \\ \times\Big[eA^\mu\,\mathrm{tr}(QL_\nu L_\alpha L_\beta + QR_\nu R_\alpha R_\beta) \\ - ie^2 F_{\mu\nu}A_\alpha\,\mathrm{tr}\Big(Q^2 L_\beta + Q^2 R_\beta \\ + \frac{1}{2}Q\Sigma Q \partial_\beta \Sigma^\dagger - \frac{1}{2}Q\Sigma^\dagger Q\partial_\beta\Sigma\Big)\Big]. \end{aligned} \tag{8}$$

Here $S_{\mathrm{WZW}}[0]$ is the WZW term without the gauge field (which can be written in the form of a five-dimensional integral). The additional terms in (8) make the action invariant with respect to local $\mathrm{U}(1)_B$ and $\mathrm{U}(1)_Q$ (baryon and electric charge) transformations.

The $\mathrm{U}(1)_B$ transformation is not a part of the $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ group and the fields Σ do not transform under it. However, the external $\mathrm{U}(1)_B$ gauge potential A_μ^B does couple to Σ via the Goldstone-Wilczek baryon current j_B^μ [27,30]. In the external electromagnetic field, the conserved and gauge-invariant baryon current j_B^μ can be found using the "trial and error" gauging, following Witten [27]

$$j^\mu_B = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\{\mathrm{tr}(L_\nu L_\alpha L_\beta) - 3ie\partial_\nu[A_\alpha\,\mathrm{tr}(QL_\beta + QR_\beta)]\}, \tag{9}$$

or the "covariant derivative" gauging, following Goldstone and Wilczek [30]

$$j^\mu_B = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\Big\{\mathrm{tr}[(\Sigma D_\nu\Sigma^\dagger)(\Sigma D_\alpha\Sigma^\dagger)(\Sigma D_\beta\Sigma^+)] - \frac{3ie}{2}F_{\nu\alpha}\,\mathrm{tr}[Q(\Sigma D_\beta\Sigma^+ + D_\beta\Sigma^\dagger\Sigma)]\Big\}. \tag{10}$$

In the form (9) both terms are obviously conserved, but not separately gauge invariant. In the form (10) both terms are obviously gauge invariant, but not separately conserved. It can be checked that the two forms are equivalent.

B. SU(2) case

If one specializes to the SU(2) case [i.e., only φ^1, φ^2, φ^3 are nonzero in Eq. (3)], then the previous formulas simplify. We can write

$$\Sigma = \frac{1}{f_\pi}(\sigma + i\tau^a\pi^a), \qquad \sigma^2 + \pi^a\pi^a = f_\pi^2, \tag{11}$$

and $Q = t^3 + 1/6$ $(t^3 = \tau^3/2)$ to verify, e.g., that $\mathrm{tr}(Q\Sigma Q\partial_\beta\Sigma^\dagger - Q\Sigma^\dagger Q\partial_\beta\Sigma) = (1/3)\,\mathrm{tr}[t^3(L_\beta + R_\beta)]$. The WZW action is zero in the absence of the external fields: $S_{\mathrm{WZW}}[0] = 0$. In the presence of external fields, it becomes

$$S_{\mathrm{WZW}} = \int d^4x\Big\{-A^B_\mu j^\mu_B + \frac{\epsilon^{\mu\nu\alpha\beta}}{16\pi^2}\Big(\frac{1}{3}eA_\mu\,\mathrm{tr}(L_\nu L_\alpha L_\beta) - \frac{ie^2}{2}F_{\mu\nu}A_\alpha\,\mathrm{tr}[t^3(L_\beta + R_\beta)]\Big)\Big\}, \tag{12}$$

and

$$j^\mu_B = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\{\mathrm{tr}(L_\nu L_\alpha L_\beta) - 3ie\partial_\nu[A_\alpha\,\mathrm{tr}(t^3L_\beta + t^3R_\beta)]\}, \tag{13}$$

or

$$j^\mu_B = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\Big\{\mathrm{tr}[(\Sigma D_\nu\Sigma^\dagger)(\Sigma D_\alpha\Sigma^\dagger)(\Sigma D_\beta\Sigma^+)] - \frac{3ie}{2}F_{\nu\alpha}\,\mathrm{tr}[t^3(\Sigma D_\beta\Sigma^+ + D_\beta\Sigma^\dagger\Sigma)]\Big\}. \tag{14}$$

The WZW action can therefore be written as

$$S_{\mathrm{WZW}} = -\int d^4x\Big(A^B_\mu + \frac{e}{2}A_\mu\Big)j^\mu_B. \tag{15}$$

The second term is the contribution of the baryon charge to the electric charge of a baryon as in the Gell-Mann-Nishijima formula $Q = I_3 + N_B/2$.

Consider one particular case, when Σ is restricted to the form

$$\Sigma = \exp\Big(\frac{i}{f_\pi}\tau_3\varphi_3\Big), \tag{16}$$

and the external field is chosen to be a constant magnetic field $B_i = \epsilon_{ijk}F_{jk}/2$ and baryon chemical potential $A^B_\nu = (\mu, \mathbf{0})$. In this case the WZW action assumes an even simpler form [only the last term in Eq. (13) survives]:

$$S_{\mathrm{WZW}} = \frac{e}{4\pi^2 f_\pi}\int d^4x\,\mu\boldsymbol{B}\cdot\boldsymbol{\nabla}\varphi_3. \tag{17}$$

This form of the magnetic effective action has been written down and discussed in Ref. [14], where it was interpreted as a nonzero magnetization of a π^0 domain wall at finite μ given by

$$\boldsymbol{M} = \frac{e}{4\pi^2 f_\pi}\mu\boldsymbol{\nabla}\varphi_3. \tag{18}$$

In this paper we point out that the same term is responsible for the nonzero baryon density of a domain wall in an external magnetic field:

$$n_B = \frac{e}{4\pi^2 f_\pi}\boldsymbol{B}\cdot\boldsymbol{\nabla}\varphi_3. \tag{19}$$

III. π^0 DOMAIN WALL IN A MAGNETIC FIELD

A. Local stability

To treat the π^0 domain wall and the fluctuations around it, it is most convenient to use the following parametrization:

$$\sigma = f_\pi\cos\chi\cos\theta, \qquad \pi^1 = f_\pi\sin\chi\cos\phi, \tag{20}$$

$$\pi^0 = f_\pi\cos\chi\sin\theta, \qquad \pi^2 = f_\pi\sin\chi\sin\phi. \tag{21}$$

The Lagrangian (without the magnetic field) is given by

$$\mathcal{L} = \frac{f_\pi^2}{2}[(\partial_\mu\chi)^2 + \cos^2\chi(\partial_\mu\theta)^2 + \sin^2\chi(\partial_\mu\phi)^2] - f_\pi^2 m_\pi^2(1 - \cos\chi\cos\theta). \tag{22}$$

The π^0 domain wall corresponds to the following static solution to the field equations,

$$\chi = 0, \qquad \theta = 4\arctan e^{m_\pi z}. \tag{23}$$

Topologically, since Eq. (23) corresponds to a contractible loop in the SU(2) group manifold (S^3), the wall can be "unwound." Moreover, in the absence of a magnetic field the π^0 domain wall is not even *locally* stable. This can be seen by analyzing small fluctuations around the solution (23). For small π_1 and π_2 the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\pi_1)^2 + (\partial_\mu\pi_2)^2] - \frac{m_\pi^2}{2}\Big(1 - \frac{6}{\cosh^2 m_\pi z}\Big)(\pi_1^2 + \pi_2^2). \tag{24}$$

The equations of motion are

$$-(\partial_x^2+\partial_y^2)\pi^a-\partial_z^2\pi^a+m_\pi^2\left(1-\frac{6}{\cosh^2 m_\pi z}\right)\pi^a=E^2\pi^a. \tag{25}$$

The corresponding Schrödinger equation has two bound states. The lowest state is tachyonic,

$$E^2=k_x^2+k_y^2-3m_\pi^2, \tag{26}$$

so the wall is locally unstable. (The second bound state corresponds to a zero mode of the wall.)

In the magnetic field, the Laplacian in the (x, y) plane becomes the Hamiltonian of a particle in a magnetic field, whose spectrum (the Landau levels) is well known, leading to

$$E^2=(2n+1)eB-3m_\pi^2, \qquad n=0,1,\ldots \tag{27}$$

Therefore, when the magnetic field exceeds the value

$$B_0=\frac{3m_\pi^2}{e}\approx 1.0\times 10^{19}\ \mathrm{G}, \tag{28}$$

the π^0 domain wall becomes locally stable.

B. Global stability at finite μ

Substituting the configuration (23) into the Lagrangian (22), one finds the following energy density per unit area,

$$\frac{\mathcal{E}}{S}=8f_\pi^2 m_\pi. \tag{29}$$

At finite baryon chemical potential μ and in the presence of a magnetic field $F_{xy}=B$ (i.e., $B_z=-B$), the configuration (23) carries a baryon number according to Eq. (19) with $\varphi_3=f_\pi\theta$. The baryon number per unit surface area is thus given by

$$\frac{N_B}{S}=\frac{eB}{2\pi}. \tag{30}$$

Being a total derivative, the WZW term (17) does not affect the field equations.

The energy per baryon number of the π^0 domain wall is

$$\frac{\mathcal{E}}{N_B}=16\pi\frac{f_\pi^2 m_\pi}{eB}. \tag{31}$$

When the baryon chemical potential exceeds the value of that ratio, i.e., for $\mu>16\pi f_\pi^2 m_\pi/(eB)$, the wall becomes energetically more favorable than the vacuum, and the ground state must be a stack of parallel domain walls, (at least) as long as $\mu\lesssim m_N$—the energy per baryon number of the nuclear matter. In order to be more favorable than the nuclear matter at $\mu\approx m_N$ the ratio (31) must be less than m_N. This happens if the magnetic field exceeds

$$B_1=\frac{16\pi f_\pi^2 m_\pi}{em_N}\approx 1.1\times 10^{19}\ \mathrm{G}. \tag{32}$$

In the chiral limit $m_\pi\to 0$, $B_1\gg B_0$, but for the real-world pion mass B_1 is only slightly higher than B_0.

According to Eq. (15), the π^0 domain wall carries a finite surface electric charge density equal to half of the baryon charge density given by Eq. (30). Within QCD, this charge can be neutralized by the π^- bosons localized on the wall: according to Eq. (27) the energy cost of adding a π^- vanishes at $B=B_0$. The number of charged pions necessary to neutralize the wall fills exactly half of the first Landau level. This suggests that the electrically neutral ground state may show quantum Hall behavior. For $B>B_0$, each pion cost an energy of $(e(B-B_0))^{1/2}$. However, for $B>B_0$, within the full standard model (with electromagnetism), other mechanisms of neutralizing the electric charge of the wall may compete with adding charged pions (e.g., adding electrons). Since the energy of adding one electron to the system is only m_e (its lowest Landau level energy), our estimate for B_1 is largely unaffected.

C. Structure and baryon charge of a finite domain wall

So far we have considered an infinite domain wall. Let us now consider a large, but finite-size, domain wall. For the infinite wall, the baryon charge, given by Eq. (30), comes from the *second term* in the baryon current (13), which gives Eq. (17). This term is a full derivative, so for a *finite* wall it must vanish. Where does the baryon number come from in this case? We now demonstrate explicitly that the finite domain wall carries a baryon number that comes from the first term in Eq. (13).

We consider a flat domain wall with a circular boundary. We use cylindrical coordinates (ρ, φ, z) with the origin at the center of the wall. The boundary of the wall is chosen to be $z=0$, $\rho=R$. We assume the radius R is much larger than the thickness of the wall, $R\gg m_\pi^{-1}$.

We use the parametrization (20). We expect that when $\rho<R$ and $R-\rho>m_\pi^{-1}$, we are sufficiently far away from the boundary so that the domain wall is given by Eq. (23). In particular, when z varies from $-\infty$ to $+\infty$, θ jumps by 2π:

$$\theta(z=+\infty)-\theta(z=-\infty)=2\pi, \qquad \rho<R. \tag{33}$$

When $\rho>R$, one does not cross any domain wall as one moves along the z direction,

$$\theta(z=+\infty)-\theta(z=-\infty)=0, \qquad \rho>R. \tag{34}$$

We find that θ is a multiple-valued function: it changes by 2π when we move along a small loop around the boundary $\rho=R$, $z=0$. To avoid a singularity in the fields themselves, $\cos\chi$ has to vanish on the boundary. We can choose

$$\chi(\rho=R, z=0)=\frac{\pi}{2}. \tag{35}$$

We expect that χ is nonzero only near the boundary. So the π^1 and π^2 fields differ substantially from 0 only near

$\rho = R$. As these fields describe the charged pions, the boundary of the domain wall is a superconducting string [31]. At the boundary $\rho = R$, the charged pion condensate is largest, $(\pi^1)^2 + (\pi^2)^2 = f_\pi^2$. Moreover, the phase ϕ of the charged pion condensate has a nontrivial winding number around the circle $\rho = R$. Indeed, in order to minimize the kinetic energy, this winding number is equal to the magnetic flux that goes through the contour, in unit of the elementary flux:

$$\phi(\varphi = 2\pi) - \phi(\varphi = 0) = \frac{1}{2\pi} eB(\pi R^2) = \frac{1}{2} eBR^2. \tag{36}$$

Because of continuity, the phase ϕ has the same winding number on any contour that surrounds the z axis, $\rho = 0$. To avoid singularity on this axis, we must have $\sin\chi = 0$ at $\rho = 0$. We choose $\chi(\rho = 0) = 0$.

Thus we find that a finite π^0 domain wall has a peculiar feature: the phase ϕ makes $\frac{1}{2} eBR^2$ full circles on any contour that surrounds the axis $z = 0$, and the phase θ makes a full circle on any contour that has linking number one with the boundary $\rho = R$ of the wall. The phase χ changes from 0 on the z axis to $\pi/2$ on the boundary of the wall. It is easy to see that the configuration has the topology of a Skyrmion with the baryon charge $N_B = \frac{1}{2} eBR^2$. In can be already seen from Eq. (10) but it is instructive to check that Eq. (9) gives the same result. Indeed, the full derivative term in Eq. (9) does not contribute to the total baryon charge and we have

$$N_B = -\frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \operatorname{tr}(L_i L_j L_k). \tag{37}$$

Changing coordinate system to χ, θ, and ϕ, one finds that the baryon charge is equal to $\frac{1}{2} eBR^2$. The baryon charge per unit surface area is the same as in Eq. (30).

IV. COLOR-SUPERCONDUCTING PHASES

So far, we have considered the effect of the magnetic field on low-density matter. In this section, we consider the effect of the magnetic field on the structure of high-density quark matter. Such high-density matter may exist in one of the color-superconducting phases (see, e.g., Refs. [32–36] for reviews). We shall see that due to the existence of light pseudoscalar Nambu-Goldstone bosons, stacks of domain walls for such bosons can be generated, and because the corresponding bosons are light, the critical magnetic field can be much lower than in a vacuum.

A. 2SC phase in a magnetic field

Theoretically, the simplest color-superconducting phase is the two-flavor superconducting (2SC) phase [37,38]. On the phase diagram, this phase occupies a window of chemical potential next to low-density nuclear matter: right after the chiral symmetry is restored, but before the density of strange quarks becomes significant.

In this regime, the attraction between quarks in the color-triplet mutual state leads to an instability of the Fermi surface due to the familiar Cooper mechanism. The resulting Cooper pair condensate has the quantum numbers of a color triplet and an isospin singlet, and carries zero angular momentum.

Perturbatively, there are two such condensates: the left- and the right-handed quark pairs: $X \sim q_L q_L$ and $Y \sim q_R q_R$. The gauge-invariant (color singlet) order parameter is the singlet made out of X and Y color vectors: $\Sigma = XY^\dagger$. Like X and Y, Σ is also an isosinglet: the isospin $SU(2)_L \times SU(2)_R$ chiral symmetry is not broken in the 2SC phase. However, since the phases of X and Y change in opposite directions under the axial isospin singlet $U(1)_A$ symmetry, the phase of the order parameter $\Sigma = XY^\dagger$ changes under $U(1)_A$. This means that the $U(1)_A$ symmetry is broken by the condensate.

In reality, this $U(1)_A$ symmetry is not a true symmetry of QCD—it is violated by the quantum fluctuations of the gluon fields via an anomaly. However, the vacuum configurations of the gluon fields responsible for this violation, i.e., the instantons, are suppressed at large baryon density due to color Debye screening, and the $U(1)_A$ transformation can be treated as an approximate symmetry at large μ.

In the 2SC phase, where the $U(1)_A$ is spontaneously broken, the measure of the explicit violation of this symmetry by anomaly/instantons is the mass m_η of the Goldstone boson (which we call η). This mass decreases very fast with μ (see below and Ref. [18]). The smallness of m_η is what is responsible for the low value of the critical magnetic field.

The effective Lagrangian density for the η boson in the 2SC phase is [18]

$$\mathcal{L} = f^2[(\partial_0\varphi)^2 - u^2(\partial_i\varphi)^2 - m_\eta^2(1 - \cos\varphi)], \tag{38}$$

where φ is the local value of the $U(1)_A$ phase whose fluctuations generate Goldstone boson η. For asymptotically large $\mu \gg \Lambda_{\rm QCD}$ the low-energy constants in the effective Lagrangian (38) are calculable [19,39]:

$$f^2 = \frac{\mu_q^2}{8\pi^2}, \qquad u^2 = \frac{1}{3}. \tag{39}$$

and

$$m_\eta = \sqrt{\frac{a}{2}} \frac{\mu_q}{f} \qquad \Delta = 2\pi\sqrt{a}\Delta, \tag{40}$$

where Δ is the superconducting gap and a has been estimated in Ref. [18]

$$a = 5 \times 10^4 \left(\ln\frac{\mu_q}{\Lambda_{\rm QCD}}\right)^7 \left(\frac{\Lambda_{\rm QCD}}{\mu_q}\right)^{29/3}. \tag{41}$$

In Eqs. (39)–(41), μ_q denotes the quark chemical potential: $\mu_q \equiv \mu/3$.

The domain wall configuration $\varphi = 4\arctan[\exp\times(m_\eta z/u)]$ is a static solution of the equations of motion with energy per unit surface area given by

$$\frac{\mathcal{E}}{S} = 16uf^2 m_\eta. \tag{42}$$

Unlike the π^0 domain wall in Sec. III, it is locally stable because of the topology of $\mathrm{U}(1)_A$: the wall can be unwound only by changing the magnitude of Σ, which requires energies beyond the scale of the effective Lagrangian (38).

The interaction of φ with the magnetic field due to the axial anomaly is described by [14]

$$\mathcal{L} = \frac{e\mu}{36\pi^2}\nabla\varphi\cdot\boldsymbol{B}. \tag{43}$$

Being a total derivative, this term does not change the field equations for φ, but it does contribute to the total free energy of a domain wall. In particular, for the domain wall perpendicular to the homogeneous field $\boldsymbol{B}$, the magnetic free energy per unit area is given by $e\mu B/(18\pi)$, which can be interpreted as the surface density of dipole magnetic moment directed perpendicularly to the wall,

$$\frac{|\mathfrak{m}|}{S} = \frac{e\mu}{18\pi}. \tag{44}$$

For sufficiently large B, the free energy gain due to the interaction of the wall with the magnetic field outweighs the surface energy cost of creating a wall (42). Thus the critical field is

$$B_c = \frac{\mathcal{E}}{|\mathfrak{m}|} = 288\pi u\frac{f^2 m_\eta}{e\mu} = \frac{4}{\sqrt{3}\pi}\frac{\mu m_\eta}{e} \approx 1.2\cdot 10^{18}\ \mathrm{G}\left(\frac{\mu}{1\ \mathrm{GeV}}\right)\left(\frac{m_\eta}{10\ \mathrm{MeV}}\right). \tag{45}$$

For $B > B_c$, the domain walls are energetically favorable and (provided boundary conditions allow) they will stack up until their mean separation is of the order of their width $1/m_\eta$.

For comparison, the critical magnetic field needed to destroy superconductivity is at least of order $\mu\Delta/e$ [7]. Because of fast decrease of m_η with μ, the value of B_c is much lower than the critical field at large μ.

B. CFL

At large μ one eventually enters the regime where the mass of the strange quark can be neglected; the density of strange quarks is as large as that of up and down and the pairing involving all three flavors becomes energetically favorable. This pairing state is called color-flavor-locked (CFL) phase [40].

In the CFL phase, the Cooper pairs are both flavor and color triplets, i.e., $X \sim q_L q_L$ and $Y \sim q_R q_R$ each carry a color and a flavor index and transform as color-flavor matrices $X \to LXC^T$ and $Y \to RYC^T$ under the flavor and color $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R\times\mathrm{SU}(3)_C$ transformations. The gauge-invariant order parameter $\Sigma = XY^\dagger$ transforms in the same way as the ordinary chiral condensate in vacuum, $\Sigma\to L\Sigma R^\dagger$. Therefore, the chiral $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$ is broken, in the CFL phase, down to the vectorlike $\mathrm{SU}(3)_{L+R}$ as it is in the vacuum.

Similarly to the 2SC phase, the $\mathrm{U}(1)_A$ symmetry is also spontaneously broken in the CFL phase. The $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R\times\mathrm{U}(1)_A$ symmetry is explicitly violated by instantons and quark masses, so all Nambu-Goldstone bosons are massive. For simplicity, we consider the regime reached at asymptotically high μ where one can neglect the contribution of instantons to all masses. The lightest Nambu-Goldstone boson in this case is an isosinglet which has the quantum number of $\bar{s}s$, i.e., a mixture of η and η' [19]. Its mass square is given by [19]

$$m_{\bar{s}s}^2 = \frac{3\Delta^2 m_u m_d}{\pi^2 f^2} \tag{46}$$

where $f^2 \sim \mu^2$ is given below in Eqs. (48) and (49).

The effective Lagrangian for this field, $\varphi_{\bar{s}s}$, is similar to the Lagrangian (38),

$$\mathcal{L} = f^2[(\partial\varphi_{\bar{s}s})^2 - u^2(\partial_i\varphi_{\bar{s}s})^2 - m_{\bar{s}s}^2(1-\cos\varphi_{\bar{s}s})]. \tag{47}$$

Since the boson is a mixture of the η and η', its decay constant is a linear combination of the singlet and the octet decay constants. One can easily derive

$$f^2 = \tfrac{1}{12}(f_{\eta'}^2 + 2f_\pi^2), \tag{48}$$

where $f_\eta'^2$ and f_π^2 have been computed in Ref. [19],

$$f_\eta'^2 = \frac{3}{4}\frac{\mu_q^2}{2\pi^2}, \qquad f_\pi^2 = \frac{21-8\ln 2}{18}\frac{\mu_q^2}{2\pi^2}. \tag{49}$$

The anomalous coupling of the $\varphi_{\bar{s}s}$ field to the magnetic field and baryon chemical potential is given by [14]

$$\mathcal{L}' = \frac{e\mu}{12\pi^2}\nabla\varphi_{\bar{s}s}\cdot\boldsymbol{B}. \tag{50}$$

Therefore, the critical magnetic field in CFL can be estimated as

$$B_c' = 96\pi u\frac{f^2 m_{\bar{s}s}}{e\mu} = \frac{111-32\ln 2}{81\sqrt{3}\pi}\frac{\mu m_{\bar{s}s}}{e} = \frac{8\sqrt{111-32\ln 2}}{3\sqrt{6}\pi}\Delta\sqrt{m_u m_d}. \tag{51}$$

Numerically, it can be written as

$$B_c' = 1.0\cdot 10^{17}\ \mathrm{G}\left(\frac{\mu}{1.5\ \mathrm{GeV}}\right)\left(\frac{m_{\bar{s}s}}{2\ \mathrm{MeV}}\right) = 8.3\cdot 10^{16}\ \mathrm{G}\left(\frac{\Delta}{30\ \mathrm{MeV}}\right)\left(\frac{\sqrt{m_u m_d}}{5\ \mathrm{MeV}}\right). \tag{52}$$

Numerically, the value obtained here is close to the theoretical upper limit of magnetic fields possible in neutron stars [5].

V. FERROMAGNETIC QUARK MATTER

The presence of the anomaly term $\mu\nabla\varphi\cdot\boldsymbol{B}$ in the Lagrangian implies that if a gradient of a pseudoscalar boson is spontaneously generated in the ground state, then the state will carry a spontaneous magnetization proportional to $\mu\nabla\varphi$—i.e., it will be ferromagnetic.[2] Such a phase has been discussed in the literature under the name "Goldstone boson current" or "supercurrent" phase. This phase becomes favorable in the range of chemical potentials between CFL and 2SC phases. If we start from the CFL phase and decrease the chemical potential μ, the splitting of the Fermi surfaces, $m_s^2/(2p_F)$, caused by strange quark mass m_s leads to an instability [45]. A similar instability occurs in the 2SC phase [46].

In the language of the effective theory (chiral perturbation theory with baryon excitations [47]), the instability arises when a fermion excitation mode (a baryon) is about to turn gapless [48,49] due to the effective chemical potential, $m_s^2/(2p_F)$, introduced by the strange quark mass. Because of the existence of a bilinear coupling $\nabla\varphi\cdot\boldsymbol{j}$ of the supercurrent $\nabla\varphi$ of a Goldstone boson to the normal current $\boldsymbol{j}=\psi^\dagger\boldsymbol{v}\psi$ of the fermion ψ, when the fermion is nearly gapless one can lower the energy by simultaneously generating the Goldston boson current $\nabla\varphi$ and the ordinary current $\boldsymbol{j}$ of opposite directions [50–52].

For definiteness, we shall discuss the Goldstone boson current state in the kaon-condensed CFL phase (CFLK^0) [53]. Most of the discussion is also relevant for the current phase in the CFL phase without kaon condensation [54] and in the 2SC phase [55].

As discussed in Ref. [54], to leading order in the strong-coupling constant α_s, there is a degeneracy between the "vector current" state and the "axial current" state. In the vector current state X and Y rotate in the same direction as one moves along the z direction, and in the axial current state they rotate in the opposite directions. We shall assume that the axial current state is favored. In this state, the gauge-invariant order parameter Σ varies in space.

We should stress that the term "current state" is somewhat misleading, as the total current in the ground state is zero. For example, in the axial current state the axial current from the condensate is compensated by the axial current of gapless fermions. However, in contrast to the conserved currents, there is no reason for the *magnetization* to vanish.

According to Ref. [53], the Goldstone boson current CFLK^0 phase appears when the effective chemical potential μ_s induced by the strange quark mass is in a narrow range

$$1.605\Delta < \mu_s \equiv \frac{m_s^2}{2p_F} < 1.615\Delta. \tag{53}$$

Here $p_F=\mu/3$ is the quark Fermi momentum.

The chiral field Σ in the CFLK^0 phase is

$$\begin{aligned}\Sigma &= \exp(-iczQ)\exp\left(\frac{i\pi}{2}\lambda_6\right)\exp(-iczQ)\\ &= \exp(-i2czQ)\exp\left(\frac{i\pi}{2}\lambda_6\right),\end{aligned} \tag{54}$$

where c is some constant that is determined by energy minimization. There is also a $\mathrm{U}(1)_A$ linear background but it does not contribute to the anomaly that we need (since $\mathrm{tr}\,Q=0$). It turns out [53] that the minimum of the energy is achieved when $c\approx\Delta$, so one is stretching the applicability of the effective theory. We are interested in rough estimates, so we shall use the effective theory extrapolation. In the ground state,

$$\Sigma\partial_z\Sigma^\dagger = \partial_z\Sigma^+\Sigma = 2icQ, \tag{55}$$

so the WZW term contribution to the Lagrangian is

$$\frac{e}{2\pi^2}\mu B\,\mathrm{tr}(cQ^2) = \frac{e}{3\pi^2}\mu Bc. \tag{56}$$

Putting $c=\Delta$, we find the magnetic moment density (magnetization)

$$M = \frac{e}{3\pi^2}\mu\Delta = 2.4\cdot 10^{16}\ \mathrm{G}\left(\frac{\mu}{1.5\ \mathrm{GeV}}\right)\left(\frac{\Delta}{30\ \mathrm{MeV}}\right). \tag{57}$$

An important point not to be overlooked in such a calculation of the magnetization is a possible contribution of the near-gapless fermions that are present in the system. In the particular case of CFLK^0 considered here, these fermions are electrically neutral and do not contribute.

What is a typical value of the magnetic field generated by this mechanism inside a neutron or quark star? The local baryon chemical potential is a function of the distance to the center of the star and is increasing towards the center of the star. Let us assume that it reaches the narrow range in which the Goldstone boson current CFLK^0 phase appears [53]

$$\frac{m_s^2}{2\Delta}(1.615)^{-1} < \frac{\mu}{3} < \frac{m_s^2}{2\Delta}(1.605)^{-1}, \tag{58}$$

before reaching the maximum at the star's center. This range maps onto a relatively thin shell inside the star, and we denote its mean radius as R and the thickness d (we

[2]The ferromagnetism of an axial domain wall in vacuum has been discussed in Refs. [41,42] using a microscopic approach in connection with the primordial magnetic field generation (see also Ref. [43]). It is worth pointing out that unlike the vacuum case, where the magnetization is forbidden by C parity [44], in the case we consider the C parity is explicitly broken by the background baryon charge density.

estimate below $d \sim 100$ m for a typical star of $R_* \sim 10$ km radius). Assuming that the magnetization in the shell is uniform, one finds that the magnetic field it creates outside is the same as that of a dipole moment equal to the total magnetic moment of the shell $M \cdot 4\pi R^2 d$. Near the surface of the shell this field is of order

$$B \sim M \frac{d}{R} \tag{59}$$

(within the shell the field is much larger $B \sim M$ and it is zero inside the nonferromagnetic region surrounded by the shell—the shell screens the field out of it). From Eq. (58) the width of the range in μ is of the order of 10 MeV. Taking the typical range of variation of μ in the star of order 500 MeV, we estimate $d/R \sim 10/500 = 0.02$. Using the estimate (57) for the magnetization M, we find from (59) that typical fields generated by such mechanism are of order $B \sim 10^{14}$–10^{15} G, which is the right order of magnitude to account for the observed magnetic fields of magnetars.

VI. CONCLUSION

In this paper we discussed the effects of the magnetic field on the ground state of QCD at different values of baryon density. The key mechanism which leads to the effects we describe is due to the axial anomaly. In the effective low-energy description of QCD—the chiral Lagrangian for the Goldstone bosons—this effect is represented by a term which appears when we gauge the topological (Goldstone-Wilczek) baryon current. On the microscopic level, it is given by the triangle diagram with the baryon, electromagnetic, and axial charge currents at the vertices.

We have demonstrated that in a sufficiently strong magnetic field the most stable state with finite baryon number is not nuclear matter, but a π^0 domain wall. Similarly, at higher baryon densities, the most stable state in a sufficiently strong magnetic field is that of an isoscalar axial (η or η') domain wall.

We also show that the states of quark matter with Goldstone boson current are ferromagnetic, and show that their magnetization is related to triangle anomalies. We estimate the magnetic field generated by such a mechanism in a typical neutron/quark star to be of order 10^{14}–10^{15} G, which is a relevant magnitude for neutron star phenomenology.

Further work is needed to understand if such ferromagnetic quark matter exists. In particular, one should understand whether the ‘‘vector current’’ or ‘‘axial current’’ state is favored. In addition, one should determine if the current states are favored compared to other candidate ground states (for example, the Fulde-Ferrell-Larkin-Ovchinnikov states with multiple plane waves) [36].

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