Nonrelativistic superstring theories

Bom Soo Kim*

Department of Physics, University of California, Berkeley, California 94720
and Theoretical Physics Group, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA
(Received 22 October 2007; published 27 December 2007)

We construct a supersymmetric version of the critical nonrelativistic bosonic string theory [B. S. Kim, Phys. Rev. D **76**, 106007 (2007).] with its manifest global symmetry. We introduce the anticommuting bc conformal field theory (CFT) which is the super partner of the $\beta\gamma$ CFT. The conformal weights of the b and c fields are both $1/2$. The action of the fermionic sector can be transformed into that of the relativistic superstring theory. We explicitly quantize the theory with manifest $SO(8)$ symmetry and find that the spectrum is similar to that of type IIB superstring theory. There is one notable difference: the fermions are nonchiral. We further consider noncritical generalizations of the supersymmetric theory using the superspace formulation. There is an infinite range of possible string theories similar to the supercritical string theories. We comment on the connection between the critical nonrelativistic string theory and the lightlike linear dilaton theory.

DOI: 10.1103/PhysRevD.76.126013 PACS numbers: 11.25.−w

I. INTRODUCTION

Time-dependent backgrounds in string theory are hard to analyze [1]. Perturbative string theory breaks down in some spacetime regions due to a large string coupling, and it appears that a full nonperturbative string theory formulation is required. One clean example with the lightlike linear dilaton theory (LDT) is proposed in [2]. On the other hand, there are some interesting developments which emphasize the role of perturbative string theory in the analysis of time-dependent backgrounds [3,4]. But the complete understanding of time-dependent backgrounds is still out of reach in string theory.

It turns out that many interesting cosmological solutions have broken Lorentz symmetry. It is interesting to consider these solutions with their manifest global symmetry. Furthermore, fundamental issues related to time, especially to "emergent time," is not clear (see, e.g., [5]). Thus it is interesting to consider alternative approaches, which can shed light on time-dependent backgrounds and on fundamental issues of time.

Recently a bosonic string theory with manifest Galilean symmetry in target space was constructed in an elementary fashion [6], motivated by earlier works [7–9]. These nonrelativistic string theories clearly treat time differently than relativistic string theory. In nonrelativistic string theories, time in target space can be described by the first order nonunitary $\beta\gamma$ conformal field theory (CFT), while second order X^0 CFT plays the role of time in the relativistic theory. Thus we can hope to obtain some insights on the issues of time-dependent backgrounds in string theory from this very different approach. As we mention in the final section of this paper, there are some intriguing pieces of evidence that these nonrelativistic string theories can be connected to known time-dependent backgrounds in string theory. This possibility opens up a new framework for addressing the issues related to time and to time-dependent string solutions.

With these motivations, we briefly review the construction of the bosonic nonrelativistic string theory, which has a manifest Galilean symmetry in target space. Compared to earlier works, the theory does not assume a compact coordinate and has a simpler action, a $\beta\gamma$ CFT in addition to the usual bosonic X CFTs. The first order $\beta\gamma$ CFT is directly related to time and energy in target space. Time in target space is parametrized by a one-parameter family of selection sector and is explicitly realized through the generalized Galilean boost symmetry of the action. We quantize the theory in an elementary fashion which reveals many interesting features. The spectrum is very similar to the relativistic bosonic string theory, except for the overall motion of the string which is governed by a nonrelativistic energy dispersion relation. The ground state has the energy

$$E = \frac{1}{pp'}\left(\frac{\alpha'}{4}k^i k_i - 1\right), \tag{1}$$

where p and p' are the parameters which specify the selection sector and the ground state vertex operator, respectively, and k^is are the transverse momenta. The particle corresponding to the ground state is still "tachyonic" because it is possible to have negative energy for the range $\frac{\alpha'}{4}k^i k_i \leq 1$. Thus it is desirable to remove this state from the spectrum. The first excited state has 24 degrees of freedom which transform into each other under $SO(24)$ rotations.

The world sheet constraint algebra imposes strong restrictions on the spectrum of string theories. We can enlarge the world sheet constraint algebra by adding the supercurrents to construct nonrelativistic superstring theories. We start with the nonrelativistic superstring action in terms of the component fields in the critical case, which

*bskim@socrates.berkeley.edu

reveals an interesting simplification in the fermionic sector. The fermionic sector can be rewritten in the same form as in the relativistic superstring theory with a simple transformation. The rest of the quantization is very similar to that of the relativistic superstring theory, except for a different global symmetry structure. We explicitly construct the vertex operators using the bosonization technique, then we quantize the theory and check the modular invariance. We encounter a nonrelativistic analogue of the Dirac equation in the ground state of the R sector. By solving the equation we show that the fermionic sector has eight physical degrees of freedom which transform in the spinor representation **8** of $SO(8)$. But there is one clear difference: the fermions in this theory are nonchiral. We contrast this to the relativistic case. This is done in Sec. II.

In Sec. III, we consider the "noncritical" version of nonrelativistic superstring theories. We present the superspace formulation of the new first order matter $\mathbf{\Sigma\Gamma}$ CFT in detail. There exists an infinite range of possible string theories for the general conformal weights of the $\mathbf{\Sigma\Gamma}$ CFT. There are two different categories in the noncritical theories distinguished by the conformal weight of the $\beta\gamma$ CFT: those with integer weight and those with half-integer conformal weight. The former case is similar to the case we quantize in this paper. The latter case seems more exotic and it is expected to give us a rather different view on the geometric interpretation of target space.

Using the world sheet constraint algebra, we construct all possible string theories with extended supersymmetry in Sec. IV. The bosonic and supersymmetric nonrelativistic string cases are presented here. We comment on some immediate observations. We conclude in Sec. V. In Sec. VI, we mention possible intriguing applications of this nonrelativistic string theory to time-dependent string backgrounds such as the lightlike linear dilaton theory.

II. CRITICAL NONRELATIVISTIC SUPERSYMMETRIC STRING

A. New matter $\beta\gamma$ CFT and bc CFT

We start with a full nonrelativistic superstring action of component fields in the conformal gauge

$$S = \int \frac{d^2z}{2\pi}\Big(\beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + \frac{1}{\alpha'}\partial X^i\bar{\partial}X_i + b_g\bar{\partial}c_g + \bar{b}_g\partial\bar{c}_g\Big) + \int\frac{d^2z}{2\pi}\Big(b\bar{\partial}c + \bar{b}\partial\bar{c} + \frac{1}{2}(\psi^i\bar{\partial}\psi_i + \bar{\psi}^i\partial\bar{\psi}_i) + \beta_g\bar{\partial}\gamma_g + \bar{\beta}_g\partial\bar{\gamma}_g\Big), \tag{2}$$

where i runs from 2 to 9 for X^i and ψ^i for the critical nonrelativistic superstring theory. The commuting matter $\beta\gamma$ CFT has weights, $h(\beta) = 1$ and $h(\gamma) = 0$, and has its central charge, $c_{\beta\gamma} = 2$. The anticommuting matter bc CFT, whose central charge is $c_{bc} = 1$, has weight $h(b) = 1/2$ and $h(c) = 1/2$. In conventional notation for the superstring case, the total central charge of the matter sector is $\hat{c}^{\mathbf{m}} = \frac{2}{3}c^{\mathbf{m}} = \frac{2}{3}(3 + \frac{3}{2}D)$, which cancels the central charge from the ghost sector $\hat{c}^{\mathbf{gh}} = \frac{2}{3}c^{\mathbf{gh}} = \frac{2}{3}\times(-26 + 11) = -10$. Thus this theory is anomaly free if $D = 8$. This is indicated above by the spatial index i which runs from 2 to 9. We consider the cases with general conformal weights in the matter $\beta\gamma$ and bc CFTs in the next section. The case with conformal weight of β as 1 is rather special and we will call it the "critical" case as in bosonic nonrelativistic theory.

We briefly comment on the new matter $\beta\gamma$ and bc CFTs. Their operator product expansions (OPEs) are

$$\gamma(z_1)\beta(z_2) \sim \frac{1}{z_{12}} \sim -\beta(z_1)\gamma(z_2), \tag{3}$$

$$b(z_1)c(z_2) \sim \frac{1}{z_{12}} \sim c(z_1)b(z_2). \tag{4}$$

The bosonic and the fermionic energy-momentum tensors and their mode expansions are

$$T_b^{\beta\gamma bc} = -(\partial\gamma)\beta - \frac{1}{2}c(\partial b) + \frac{1}{2}(\partial c)b = \sum_{m\in\mathbf{Z}}\frac{L_m}{z^{m+2}}, \tag{5}$$

$$T_f^{\beta\gamma bc} = \frac{1}{2}c\beta - \frac{1}{2}(\partial\gamma)b = \sum_{r\in\mathbf{Z}+\nu}\frac{G_r}{2\cdot z^{r+3/2}}. \tag{6}$$

As is well known there are two possible sectors for the fields with the half-integer conformal weight. These are $\nu = 0$ and $\nu = 1/2$ cases corresponding to the R sector and NS sector, respectively. We can also find mode expansions and their Hermiticity properties of the fields

$$\gamma(z) = \sum_{n\in\mathbf{Z}}\frac{\gamma_n}{z^n}, \qquad \gamma_n^\dagger = \gamma_{-n}, \qquad \beta(z) = \sum_{n\in\mathbf{Z}}\frac{\beta_n}{z^{n+1}}, \qquad \beta_n^\dagger = -\beta_{-n}, \tag{7}$$

$$c(z) = \sum_{r\in\mathbf{Z}+\nu}\frac{c_r}{z^{r+1/2}}, \qquad c_r^\dagger = c_{-r}, \qquad b(z) = \sum_{r\in\mathbf{Z}+\nu}\frac{b_r}{z^{r+1/2}}, \qquad b_r^\dagger = b_{-r}. \tag{8}$$

The mode expansions for the energy-momentum tensors are

$$L_m^{\beta\gamma bc} = \sum_{n\in\mathbf{Z}} n\beta_{m-n}\gamma_n + \sum_{s\in\mathbf{Z}+\nu}(s - m/2)b_{m-s}c_s + a\delta_{m,0}, \tag{9}$$

$$G_r^{\beta\gamma bc} = \sum_{m\in\mathbf{Z}}(c_{r-m}\beta_m + m\gamma_m b_{r-m}). \tag{10}$$

There is a normal ordering constant for L_0 in each sector

$$a_R^{\beta\gamma bc} = \tfrac{1}{8}, \qquad a_{NS}^{\beta\gamma bc} = 0. \tag{11}$$

This is only from the new matter sector, $\beta\gamma$ and bc CFTs, and is one part of the total normal ordering constant.[1]

B. Fermionic sector and its symmetry

The fermionic bc CFT is a new ingredient of this nonrelativistic superstring theory. There are immediate observations which are rather interesting. As we briefly mentioned at the beginning of this section, the conformal weights of the fields b, c and all the other fermionic fields ψ^i are equal and the value is $1/2$. From this observation, we can think about a transformation

$$c = \tfrac{1}{\sqrt{2}}(\psi^1 - \psi^0), \qquad b = \tfrac{1}{\sqrt{2}}(\psi^1 + \psi^0). \tag{12}$$

Combining these fields with the other fermionic fields ψ^i, we can see that the action of the fermionic sector is exactly the same as that of the relativistic one

$$\begin{aligned} S_F &= \int \frac{d^2z}{2\pi}\Big(b\bar{\partial}c + \bar{b}\partial\bar{c} + \frac{1}{2}(\psi^i\bar{\partial}\psi_i + \bar{\psi}^i\partial\bar{\psi}_i)\Big) \\ &= \int \frac{d^2z}{4\pi}(\psi^\mu\bar{\partial}\psi_\mu + \bar{\psi}^\mu\partial\bar{\psi}_\mu), \end{aligned} \tag{13}$$

where μ runs from 0 to 9. We can naively think that there is $SO(9,1)$ invariance in the fermionic sector of this nonrelativistic superstring theory. But as is obvious from the original action, there is no symmetry transformation which connects the fields ψ^0, ψ^1 and the other transverse fields ψ^i. The symmetry groups of the fermionic sector are the $SO(8)$ rotations among the ψ^is as well as a one-parameter family of superconformal symmetry which is related to rescaling $\beta \to x\beta$ and $\gamma \to \gamma/x$.[2] The latter is actually realized as the relative rescaling between k^γ and p' in the bosonic string case, related by rescaling $k^\gamma \to xk^\gamma$ and $p' \to p'/x$. We can denote this zero-dimensional conformal symmetry as "$SO(1,1)$," thus the symmetry group turns out to be $SO(1,1) \times SO(8)$. This symmetry group becomes important when we consider a nonrelativistic analogue of the Dirac equation. Even though we know there is no relativistic $SO(9,1)$ symmetry, we still use the relativistic notation to make the expression simple and to get some intuitions from the relativistic results.

[1]It is important to observe that the total normal ordering constant for nonrelativistic superstring theory is the same as that of the relativistic theory

$$a_R = 0, \qquad a_{NS} = -\tfrac{1}{2},$$

because there are other contributions from the X^i CFTs and the ghost CFT.

[2]We realize that there exists this symmetry when we have discussions with Professor Ori Ganor and with Professor Ashvin Vishwanath. We thank them for their questions and comments related to this.

C. Vertex operators

Most of the vertex operators for this theory are already known. The vertex operators of the X^i, ψ^i CFTs and of the superconformal ghost sector with the $b_g c_g$ and $\beta_g\gamma_g$ CFTs are already well understood and can be found in many places (see, e.g., [10–12]). Constructing vertex operators for the bosonic $\beta\gamma$ CFT is considered in [6,7].

Thus let us concentrate on the vertex operators of the fermionic bc CFT. The fermionic matter sector, in terms of the fermionic fields ψ^μ, $\mu = 0 \cdots 9$, has well-understood vertex operators in the relativistic string theory [10–12]. Thus we can just borrow the results from them with caution. In this section we will use both the notations ψ^0, ψ^1, and bc.

For the Neveu-Schwarz (NS) sector, there is no $r = 0$ mode and we can define the ground state to be annihilated by all $r > 0$ modes

$$\psi_r^\mu|0; k^\gamma, k^{\bar{\gamma}}, \vec{k}\rangle_{NS} = 0, \qquad r > 0. \tag{14}$$

This ground state is tachyonic. The vertex operator corresponding to NS ground state is

$$V_{NS,0}(k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z}) = e^{-\varphi}V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z}), \tag{15}$$

$$V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z}) = g : e^{ik^\gamma\gamma + ik^{\bar{\gamma}}\bar{\gamma} - ip'\int^z\beta - iq'\int^{\bar{z}}\bar{\beta} + ik^i\cdot X_i} :, \tag{16}$$

where the field φ comes from the bosonization of the superconformal ghost fields and has nothing to do with the selection parameter ϕ. The bosonic ground state vertex operator V_0 was considered in [6,7] with k^γ, $k^{\bar{\gamma}}$, and k^i representing the overall continuous momenta along the coordinates γ, $\bar{\gamma}$, and X^i, respectively.

The first excited state in the NS sector is a linear combination of the fermionic excitations $b_{-1/2}$, $c_{-1/2}$, and $\psi^i_{-1/2}$.

$$\begin{aligned} |e; k^\gamma, k^{\bar{\gamma}}, \vec{k}\rangle_{NS} = (&e_c c_{-1/2} + e_b b_{-1/2} \\ &+ e_i\psi^i_{-1/2})|0; k^\gamma, k^{\bar{\gamma}}, \vec{k}\rangle_{NS}. \end{aligned} \tag{17}$$

We use two different notations for the fermionic sector (i) $e_\mu\psi^\mu$ with $\mu = 0, \cdots, 9$ and (ii) $e_M\psi^M_{-1/2} = (e_c c_{-1/2} + e_b b_{-1/2} + e_i\psi^i_{-1/2})$ with $i = 2, \cdots, 9$. The vertex operator corresponding to the first excited state $V_{NS,1}(k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z})$ is

$$e^{-\varphi}\psi^M V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z}) \quad \text{or} \quad e^{-\varphi}\psi^\mu V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z}). \tag{18}$$

The modes with $r < 0$ for the fields ψ_r act as raising operators and each mode can be excited only once.

The Ramond (R) sector ground state is degenerate due to the zero modes ψ_0^μ (or ψ_0^M). We can define the R ground state to be those that are annihilated by all $r > 0$ modes. The zero modes satisfy the Dirac gamma matrix algebra with $\Gamma^\mu \cong \sqrt{2}\psi_0^\mu$. Since $\{\psi_r^\mu, \psi_0^\nu\} = 0$ for $r > 0$, the zero modes ψ_0^μ take ground states into ground states. Thus the ground states form a representation of the gamma matrix algebra. For the critical case with "10 dimensions" we can represent this as $|\mathbf{s}\rangle = |s_0\rangle \times |\vec{s}\rangle = |s_0\rangle \times |s_1, s_2, s_3, s_4\rangle$ with $s_0, s_a = \pm 1/2$. Here we separate s_0 from the others to indicate that there is no symmetry transformation between s_0 and $\vec{s}$.

It is convenient to combine two fermions, ψ^2 and ψ^3 for example, into a complex pair, $\psi \equiv \frac{1}{\sqrt{2}}(\psi^2 + i\psi^3)$ and $\psi^\dagger \equiv \frac{1}{\sqrt{2}}(\psi^2 - i\psi^3)$,[3] to consider a more general periodicity condition

$$\psi(w + 2\pi) = e^{2\pi i\nu}\psi(w), \tag{19}$$

for any real ν. Here we concentrate on two cases $\nu = 0$ and $\nu = 1/2$. The mode expansions are

$$\psi(z) = \sum_{r\in \mathbf{Z}+\nu} \frac{\psi_r}{z^{r+1/2}}, \qquad \psi^\dagger(z) = \sum_{s\in \mathbf{Z}-\nu} \frac{\psi_s^\dagger}{z^{s+1/2}}, \tag{20}$$

with a commutation relation $\{\psi_r, \psi_s^\dagger\} = \delta_{r,-s}$.

We can define a reference state $|0\rangle_\nu$ by

$$\psi_{n+\nu}|0\rangle_\nu = \psi_{n+1-\nu}^\dagger|0\rangle_\nu = 0, \qquad n = 0, 1, \cdots. \tag{21}$$

The first nonzero terms in the Laurent expansions are related to the indices $r = -1 + \nu$ and $s = -\nu$. These conditions can uniquely identify the state $|0\rangle_\nu$. Similarly for the corresponding vertex operator $\mathcal{A}_\nu$, the OPEs

$$\psi(z)\mathcal{A}_\nu(0) = \mathcal{O}(z^{-\nu+1/2}), \qquad \psi^\dagger(z)\mathcal{A}_\nu(0) = \mathcal{O}(z^{\nu-1/2}) \tag{22}$$

can determine the vertex operator as

$$\mathcal{A}_\nu \simeq e^{i(-\nu+1/2)H}. \tag{23}$$

This vertex operator has weight $h = \frac{1}{2}(\nu - \frac{1}{2})^2$. The boundary conditions are the same for ν and $\nu + 1$, but the reference states are not. The reference state is a ground state only for $0 \le \nu \le 1$. For the R sector with $\nu = 0$, there are two degenerate ground states which can be identified as $|s\rangle \cong e^{isH}$ with $s = 1/2$ and $s = -1/2$.

It is convenient to use bosonization to take care of the branch cut which arises in the fields with the half-integer conformal weight. The explicit bosonization expressions are

$$\tfrac{1}{\sqrt{2}}(\psi^1 - \psi^0) = c \cong e^{-iH^0}, \qquad \tfrac{1}{\sqrt{2}}(\psi^1 + \psi^0) = b \cong e^{iH^0}, \tag{24}$$

$$\tfrac{1}{\sqrt{2}}(\psi^{2a} \pm i\psi^{2a+1}) \cong e^{\pm iH^a}, \qquad a = 1, \cdots, 4, \tag{25}$$

where $H(z)$ fields are the holomorphic part of corresponding scalar fields. Then the corresponding vertex operator $\Theta_{\mathbf{s}}$ for an R sector ground state $|\mathbf{s}\rangle = |s_0, \vec{s}\rangle$ is

$$\Theta_{\mathbf{s}} \cong \exp[is_0H^0] \times \exp\left[i\sum_{a=1}^{4} s_aH^a\right]. \tag{26}$$

This spin field operator produces a branch cut in ψ^μ and needs to be combined with an appropriate antiholomorphic vertex operator.

Thus the R ground state vertex operators are

$$V_{R,0}(s_0, \vec{s}; k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z}) = e^{-\varphi/2}\Theta_{\mathbf{s}}V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z}), \tag{27}$$

where φ is related to the bosonization of the superconformal ghost fields and V_0 is given in Eq. (16). Now we are ready to quantize the theory.

D. Quantization

In the old covariant quantization procedure, we ignore the ghost excitations and concentrate on the matter sector, which has the X^i, ψ^i, $\beta\gamma$, and bc CFTs. We impose the physical states conditions

$$(L_n^{\mathbf{m}} + a\delta_{n,0})|\psi\rangle = 0, \qquad n \ge 0, \qquad G_r^{\mathbf{m}}|\psi\rangle = 0, \qquad r \ge 0, \tag{28}$$

where "$\mathbf{m}$" denotes the matter sector. We can construct spurious states which are orthogonal to all physical states such as

$$L_n^{\mathbf{m}}|\chi\rangle, \qquad n < 0, \qquad G_r^{\mathbf{m}}|\chi\rangle, \qquad r < 0. \tag{29}$$

These states satisfy $\langle\psi|L_n^{\mathbf{m}}|\chi\rangle = 0$ and $\langle\psi|G_r^{\mathbf{m}}|\chi\rangle = 0$. If these states satisfy the physical state conditions, then we call them null states. We need to impose equivalence relations to get a physical Hilbert space.

1. NS sector

The NS sector with $\nu = 1/2$ is simpler and we consider this first. For the ground state (with simplified notation $|0; k\rangle_{NS}$ instead of $|0; k^\gamma, k^{\bar{\gamma}}, \vec{k}\rangle_{NS}$), the physical state condition $(L_0^{\mathbf{m}} - \frac{1}{2})|0; k\rangle_{NS} = 0$ gives us the mass shell equation

$$\frac{\alpha'}{4}\vec{k}^2 - k^\gamma p' - \frac{1}{2} = 0. \tag{30}$$

The other physical state conditions, $L_n^{\mathbf{m}}|0; k\rangle_{NS} = 0$ for $n > 0$ and $G_r^{\mathbf{m}}|0; k\rangle_{NS} = 0$ for $r \ge 1/2$, are trivial. Thus there is one equivalence class, corresponding to a scalar particle.

The first excited level (with simplified notation $|e; k\rangle_{NS}$ instead of $|e; k^\gamma, k^{\bar{\gamma}}, \vec{k}\rangle_{NS}$) has 10 states

$$|e; k\rangle_{NS} = (e_cc_{-1/2} + e_bb_{-1/2} + e_i\psi_{-1/2}^i)|0; k\rangle_{NS}. \tag{31}$$

The nontrivial physical state conditions, $(L_0^{\mathbf{m}} - \frac{1}{2})|e; k\rangle_{NS} = 0$ and $G_{1/2}^{\mathbf{m}}|e; k\rangle_{NS} = 0$, give us

[3] Note that we use different notation for the complex field compared to [10].

$$\frac{\alpha'}{4}\vec{k}^2 - k^\gamma p' = 0, \qquad (32)$$

$$-p'e_c + k^\gamma e_b + (\alpha'/2)^{1/2}k^i e_i = 0, \qquad (33)$$

while a spurious state

$$G^{\mathbf{m}}_{-1/2}|0;k\rangle_{NS} = ((\alpha'/2)^{1/2}k^i\psi_{i,1/2} + k^\gamma c_{-1/2} - p'b_{-1/2})|0;k\rangle_{NS} \qquad (34)$$

is physical and null. Thus there is an equivalent relation

$$(e_c, e_b, e_i) \cong (e_c + k^\gamma, e_b - p', e_i + (\alpha'/2)^{1/2}k_i). \qquad (35)$$

Thus for the first excited state in the NS sector, there are only 8 independent degrees of freedom.

The global symmetries are the conformal scaling and the $SO(8)$ rotation, $SO(1,1)\times SO(8)$, as we point out above. At this stage, these symmetries are manifest in Eq. (32). But we show in the previous work [6] that the energy dispersion relation for the particle corresponding to this level is actually

$$E = p_t = \frac{1}{pp'}\left(\frac{\alpha'}{4}\vec{k}^2 - 1\right), \qquad (36)$$

where p and p' are parameters specifying a selection sector and the ground state vertex operator, respectively. Thus nonrelativistic particles have $SO(8)$ symmetry which is smaller than $SO(1,1)\times SO(8)$. The explicit dependence of energy on the parameter p' breaks $SO(1,1)$ scaling symmetry. Particularly, at the first excited level of the NS sector, these 8 degrees of freedom transform into each other in the vector representation of $\mathbf{8}_v$ of $SO(8)$ similar to the case of relativistic massless excitations.[4]

2. *R sector*

In the R sector, we have degenerate ground states $|v,u;k\rangle_R = |s_0,\vec{s};k\rangle_R(v_{s_0}\otimes u_{\vec{s}})$, where v and u are "polarizations" along bc and ψ^i, respectively. The nontrivial physical conditions are

$$0 = L^{\mathbf{m}}_0|v,u;k\rangle_R = \left(\frac{\alpha'}{4}\vec{k}^2 - k^\gamma p'\right)|v,u;k\rangle_R, \qquad (37)$$

$$0 = G^{\mathbf{m}}_0|v,u;k\rangle_R = \left(\left(\frac{\alpha'}{2}\right)^{1/2}k^i\psi_{0,i} + k^\gamma c_0 - p'b_0\right)|v,u;k\rangle_R. \qquad (38)$$

The first equation is the usual mass shell condition. The second equation is an analogue of the relativistic Dirac equation. We can check that $G_0^2 = L_0$. So the G_0 condition implies the mass shell condition.

The second equation is particularly important for us to investigate the difference between the spectrum of the nonrelativistic theory and that of the relativistic one. To make things more transparent, we can rewrite the equation in terms of the fields ψ^0 and ψ^1, which reads

$$\frac{1}{2^{1/2}}(\alpha'^{1/2}k^i\psi_{0,i} - (k^\gamma + p')\psi_{0,0} + (k^\gamma - p')\psi_{0,1}) = 0. \qquad (39)$$

This equation is the same as the relativistic one if we use $(\frac{\alpha'}{2})^{1/2}k^\mu\psi_{0,\mu} = 0$, with $(\alpha')^{1/2}k^0 = -k^\gamma - p'$ and $(\alpha')^{1/2}k^1 = k^\gamma - p'$. With an appropriate signature, we can get

$$k^\mu k_\mu = \frac{\alpha' k^i k_i}{2} - \frac{(k^\gamma + p')^2}{2} + \frac{(k^\gamma - p')^2}{2} = \frac{\alpha'}{2}k^i k_i - 2k^\gamma p' = 0. \qquad (40)$$

Particularly there is no further constraint in the vertex operators for the change of fields from bc to ψ^0, ψ^1, thus the fermionic sector has $SO(1,1)\times SO(8)$ symmetry,[5] where there is no connection between ψ^0, ψ^1 and the other ψ^is. It is interesting to observe that the $SO(1,1)$ has boost symmetry and is realized as the rescaling of the relative magnitude of k^γ and p' while keeping the magnitude of their product $k^\gamma p'$ fixed.

We can think about the nonrelativistic Dirac equation with manifest $SO(8)$ symmetry structure. For the spinors of $SO(8)$, we can impose the Majorana condition and the Weyl condition simultaneously, and there are two inequivalent irreducible spinor representations, $\mathbf{8}_c$ and $\mathbf{8}_s$. The description of Dirac matrices for $SO(8)$ requires a Clifford algebra with eight anticommuting matrices, which are 16-dimensional matrices corresponding to reducible $\mathbf{8}_c + \mathbf{8}_s$ representation of $SO(8)$. These matrices can be written in the block form

$$\gamma^i = \begin{pmatrix} 0 & \gamma^i_{a\dot{a}} \\ \gamma^i_{\dot{b}b} & 0 \end{pmatrix}, \qquad (41)$$

where the equations $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ are satisfied with $\gamma^i_{a\dot{a}}\gamma^j_{\dot{a}b} + \gamma^j_{a\dot{a}}\gamma^i_{\dot{a}b} = 2\delta^{ij}\delta_{ab}$ with $i, j = 2, \cdots, 9$. $\gamma^i_{\dot{a}a}$ is the transpose of $\gamma^i_{a\dot{a}}$ and can be expressed in terms of real components.

To apply these matrices to the nonrelativistic Dirac equation (39), we can construct the 10-dimensional Dirac matrices Γ^μ explicitly

$$\Gamma^0 = \sigma^3\otimes\mathbf{1}_{16}, \quad \Gamma^1 = \sigma^1\otimes\mathbf{1}_{16}, \quad \Gamma^0 = i\sigma^2\otimes\gamma^i, \qquad (42)$$

[4]There is another way to think about the expression (32). Rather than breaking $SO(1,1)$ symmetry, we can go to a frame, $k^i = 0$ for $i = 2, \cdots, 8$ and $k^9 \neq 0$, which is similar to the relativistic consideration and keeps the $SO(1,1)\times SO(7)$ symmetry. For further explanation, please see the appendix.

[5]It is interesting to observe that the one-parameter family of superconformal symmetry "$SO(1,1)$" can be transformed into $SO(1,1)$ Lorentz symmetry.

TABLE I. Spectrum of the holomorphic sector for ground and first excited level of the NS sector and ground state of the R sector. $\mathbf{8}_v$ is the fundamental representation of $SO(8)$ and $\mathbf{8}$ is one copy of the spinor representation of $SO(8)$.

Sector	$SO(8)$ spin	$-\frac{\alpha'}{4}\vec{k}^2 + k^\gamma p'$
NS_0	$\mathbf{1}$	$-1/2$
NS	$\mathbf{8}_v$	0
R	$\mathbf{8}$	0

where $\mathbf{1}_{16}$ is the 16×16 identity matrix and $i = 2, \cdots, 9$. Here all the Gamma matrices are real and thus it is possible to impose the Majorana condition for all the spinor fields. Using $\psi_0^\mu = \Gamma^\mu/\sqrt{2}$, we can rewrite Eq. (39) as $\frac{\alpha'^{1/2}}{2} k_\mu \Gamma^\mu = 0$. To go further we can use the basis

$$v_{s_0} \otimes u_{\vec{s}} = \begin{pmatrix} v_+ \\ v_- \end{pmatrix}_2 \otimes \begin{pmatrix} u^b \\ u^{\dot{a}} \end{pmatrix}_{16}. \tag{43}$$

We can explicitly write the nonrelativistic Dirac equation

$$\frac{\sqrt{\alpha'}}{2}\begin{pmatrix} v_+ \\ -v_- \end{pmatrix}_2 \otimes \begin{pmatrix} k_i \gamma^i_{a\dot{a}} u^{\dot{a}} \\ k_i \gamma^i_{\dot{b}b} u^b \end{pmatrix}_{16} + \begin{pmatrix} k^\gamma v_- \\ -p' v_+ \end{pmatrix}_2 \otimes \begin{pmatrix} u_a \\ u_{\dot{b}} \end{pmatrix}_{16} = 0. \tag{44}$$

To solve this problem we can go to a basis $v_+ = \sqrt{\frac{k^\gamma}{p'}} v_-$.[6] Then we have the equations

$$\frac{\sqrt{\alpha'}}{2} k_i \gamma^i_{a\dot{a}} u^{\dot{a}} + \sqrt{k^\gamma p'} u_a = 0, \tag{45}$$

$$\frac{\sqrt{\alpha'}}{2} k_i \gamma^i_{\dot{b}b} u^b + \sqrt{k^\gamma p'} u_{\dot{b}} = 0. \tag{46}$$

These equations are very similar to the relativistic Dirac equation presented in [11] with a definite chirality in the 10-dimensional fermion.[7] It is possible to satisfy the nonrelativistic Dirac equation with manifest $SO(8)$ symmetry by exploiting the superconformal rescaling symmetry. Furthermore this equation tells that there is no chiral property for the nonrelativistic fermions because these two inequivalent irreducible spinor representations $\mathbf{8}_c$ and $\mathbf{8}_s$ are connected by the nonrelativistic Dirac equation.[8] We will denote this as $\mathbf{8}$. Thus we can summarize the particle contents for the first two states in the NS sector and for the ground state of the R sector in Table I.

3. Closed string spectrum

The closed string spectrum has two copies of the above spectrum, each from holomorphic and antiholomorphic sectors. Because of the level matching condition the NS_0 sector can only combine with the other NS_0 sector $-\frac{\alpha'}{4}\vec{k}^2 + k^\gamma p' = -\frac{\alpha'}{4}\vec{k}^2 + k^{\bar{\gamma}} q' = -1/2$. This is a nondegenerate state of the nonrelativistic closed string. This state will be projected out due to the requirement of modular invariance which requires at least one R sector.

Now it is rather straightforward to construct the closed string spectrum at the next level because there is one copy of the vector representation $\mathbf{8}_v$ and one copy of the spinor representation $\mathbf{8}$ of $SO(8)$. The spinor representation $\mathbf{8}$ is nonchiral and it is expected that the whole theory is nonchiral. We can identify the spinor representation $\mathbf{8}$ as one of the two chiral representations $\mathbf{8}_c$ or $\mathbf{8}_s$ of $SO(8)$. Then the whole spectrum is similar to that of the relativistic type IIB superstring theory, which has the same spinor representations in both the holomorphic and the antiholomorphic sectors. This signals that the theory is modular invariant and consistent even before we actually check the modular invariance. We summarize the ground state and first excited states in Table II.

E. Partition function and modular invariance

To show that the theory is consistent, we need to check the modular invariance. The bosonic part of the modular invariance is already shown in the previous work [6]. Thus we can concentrate on the fermionic sector. As explained in the previous section, the field contents of the nonrelativistic superstring theory is the same as those of the relativistic IIB string theory. Thus the modular invariance can be proved in a similar way. For completeness we provide a very brief proof of the modular invariance of the fermionic sector by closely following [10].

For the complex fermion ψ, we can introduce a general periodicity $\alpha = 1 - 2\nu$ with

$$\psi(\omega + 2\pi) = e^{\pi i(1-\alpha)}\psi(\omega). \tag{47}$$

TABLE II. Closed superstring spectrum for the ground state and the first excited state of the NS sector and the ground state of the R sector. $\mathbf{8}_v$ is the fundamental representation and $\mathbf{8}$ is one copy of the spinor representation of $SO(8)$.

Sector	$SO(8)$ spin		Tensors		Dimensions
(NS_0, NS_0)	$\mathbf{1}\times\mathbf{1}$			=	$\mathbf{1}$
(NS, NS)	$\mathbf{8}_v \times \mathbf{8}_v$	=	$[0] + [2] + (2)$	=	$\mathbf{1} + \mathbf{28} + \mathbf{35}$
(NS, R)	$\mathbf{8}_v \times \mathbf{8}$			=	$\mathbf{8} + \mathbf{56}$
(R, NS)	$\mathbf{8} \times \mathbf{8}_v$			=	$\mathbf{8} + \mathbf{56}$
(R, R)	$\mathbf{8} \times \mathbf{8}$	=	$[0] + [2] + [4]$	=	$\mathbf{1} + \mathbf{28} + \mathbf{35}$

[6]This condition is actually equivalent to use the symmetry transformation of $SO(1, 1)$ to rescale $k^\gamma = p'$.

[7]We thank Professor Petr Hořava for discussions and comments on the nonrelativistic Dirac equation and interesting ideas related to the nonrelativistic system.

[8]Then why are there two inequivalent propagating degrees of freedom $\mathbf{8}_s$ and $\mathbf{8}_c$ in the relativistic case? These two inequivalent degrees of freedom come from the 10-dimensional Weyl conditions $\Gamma_{11}\lambda = \pm\lambda$, which are not available for the nonrelativistic theory. For $k^0 = k^9$, it is possible to impose $s_0 = 1/2$ and there is $\mathbf{8}_s$ spinor. For $k^0 = -k^9$, the other spinor $\mathbf{8}_c$ is available. (These two equations $k^0 = \pm k^9$ satisfy $k^\mu k_\mu = 0$.) This does not apply for the nonrelativistic theory. Because there is no 10-dimensional Weyl condition and the bosonic dispersion relation does not have two inequivalent choices for the relation k^γ and p'.

Then the raising operators can be written as $\psi_{-m+(1-\alpha)/2}$ and $\psi^{\dagger}_{-m+(1+\alpha)/2}$ with positive integer m. In the bosonized language given in (23), the weight of the vertex operator is $\alpha^2/8$.[9] Using this result we can calculate

$$\mathrm{Tr}_\alpha(q^{L_0-c/24}) = q^{(3\alpha^2-1)/24}\prod_{m=1}^{\infty}(1+q^{m-(1-\alpha)/2}) \times (1+q^{m-(1+\alpha)/2}). \tag{48}$$

To accommodate this general boundary condition, we join the fermions into complex pairs in (20). Then a fermion number Q can be defined as $+1$ for ψ and -1 for $\psi^\dagger$. Q corresponds to be H momentum in the bosonization formula. The ground state has a Q charge as $\alpha/2$. Thus we can define the more general trace

$$Z^\alpha_\beta(\tau) = \mathrm{Tr}_\alpha(q^{L_0-c/24}\exp(\pi i\beta Q)) = q^{(3\alpha^2-1)/24}\exp(\pi i\alpha\beta/2) \tag{49}$$

$$\times\prod_{m=1}^{\infty}(1+\exp(\pi i\beta)q^{m-(1-\alpha)/2}) \times (1+\exp(-\pi i\beta)q^{m-(1+\alpha)/2}) \tag{50}$$

$$= \frac{1}{\eta(\tau)}\vartheta\begin{bmatrix}\alpha/2\\ \beta/2\end{bmatrix}(0,\tau). \tag{51}$$

Here α and β can have 0 and 1. We have the relevant traces Z^0_0, Z^1_0, Z^0_1, and Z^1_1. The holomorphic part of the partition function for the fermionic sector is

$$Z_\psi(\tau) = \tfrac{1}{2}[Z^0_0(\tau)^4 - Z^0_1(\tau)^4 - Z^1_0(\tau)^4 - Z^1_1(\tau)^4], \tag{52}$$

where the first sign comes from the ghost contribution and the last two signs come from the spacetime spin statistics. The total partition function is

$$Z_{\mathrm{total}} = \frac{V_8 V_{\beta\gamma}}{2p'q'}\int_F \frac{d^2\tau}{16\pi^2\alpha'\tau_2^2}(Z^8_X Z_\psi(\tau)Z_\psi(\tau)^*). \tag{53}$$

This short explanation proves the modular invariance and it is the same as that of the type IIB string.

III. GENERAL NONRELATIVISTIC SUPERSYMMETRIC STRING

In this section we consider the $\beta\gamma$ and bc CFTs with general conformal weights. First we explain the new matter sector in the superspace formulation. Then we construct a noncritical version of the nonrelativistic superstring theories.

[9]We can get the same result from the fermionic language, where the normal ordering constant can be calculated by the zero point mnemonic given in [10].

A. Matter ΣΓ CFT

Let us start with supersymmetric string theory action with a matter $\mathbf{\Sigma\Gamma}$ CFT in addition to the usual $\mathbf{X}^i$ CFT and the ghost $\mathbf{BC}$ CFT in the conformal gauge

$$S_{\mathrm{susy}} = \int\frac{d^2zd^2\theta}{2\pi}(\mathbf{\Sigma}\bar{\mathbf{D}}_{\bar\theta}\mathbf{\Gamma}). \tag{54}$$

The equations of motion for the fields are $\bar{\mathbf{D}}_{\bar\theta}\mathbf{\Gamma} = 0 = \bar{\mathbf{D}}_{\bar\theta}\mathbf{\Sigma}$. There is a similar action and equations of motion for the antiholomorphic part of $\mathbf{\Sigma\Gamma}$ and $\mathbf{BC}$ CFTs.

OPEs of new $\mathbf{\Sigma\Gamma}$ CFT are given by

$$\mathbf{\Gamma}(z_1,\theta_1)\mathbf{\Sigma}(z_2,\theta_2) \sim \frac{\theta_{12}}{\hat{z}_{12}} \sim \mathbf{\Sigma}(z_1,\theta_1)\mathbf{\Gamma}(z_2,\theta_2), \tag{55}$$

where $\theta_{12} = \theta_1 - \theta_2$ and $\hat{z}_{12} = z_1 - z_2 - \theta_1\theta_2$. The super energy-momentum tensor[10] is a chiral superfield of dimension 3/2 with the ordinary energy-momentum tensor of dimension 2 in it $\mathbf{T}(\mathbf{z}) = T_F(z) + \theta T_B(z)$

$$\mathbf{T} = (\lambda-1)\mathbf{\Gamma}\partial\mathbf{\Sigma} + \tfrac{1}{2}(\mathbf{D\Gamma})(\mathbf{D\Sigma}) + (\lambda-\tfrac{1}{2})\partial\mathbf{\Gamma\Sigma}. \tag{56}$$

For the $\lambda = 1$ case, the super energy-momentum tensor simplifies further and has the form

$$\mathbf{T}_{\lambda=1} = \tfrac{1}{2}(\mathbf{D\Gamma})(\mathbf{D\Sigma}) + \tfrac{1}{2}\partial\mathbf{\Gamma\Sigma}, \tag{57}$$

which is very simple and we concentrate on the previous section as a critical case. It is simple to verify that this reduces to the component forms of the energy-momentum tensor (5) and (6), which are presented below. The case with $\lambda = 1/2$ also simplifies and corresponds to the critical case in a sense we explain in the next subsection.

The super energy-momentum tensor is itself an anomalous superconformal field

$$\mathbf{T}(z_1,\theta_1)\mathbf{T}(z_2,\theta_2) \sim \frac{8\lambda-6}{4\hat{z}^3_{12}} + \frac{3}{2}\frac{\theta_{12}}{\hat{z}_{12}}\mathbf{T}(z_2,\theta_2) + \frac{1}{2}\frac{1}{\hat{z}_{12}}\mathbf{D}_2\mathbf{T}(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}}\partial_2\mathbf{T}(z_2,\theta_2), \tag{58}$$

which tells us the central charge of the super energy-momentum tensor is $\hat{c} = \frac{2}{3}c = 8\lambda - 6$ and the conformal weight of the tensor is 3/2.

OPEs of the energy-momentum tensor with the super fields can be calculated

[10]This can be contrasted to the energy-momentum tensor of $\mathbf{BC}$ super ghost CFT

$$\mathbf{T}^{\mathbf{BC}}_{\mathrm{ghost}} = -(\lambda_g-1)\mathbf{C}(\mathbf{D}^2\mathbf{B}) + \tfrac{1}{2}(\mathbf{DC})(\mathbf{DB}) - (\lambda_g-\tfrac{1}{2})(\mathbf{D}^2\mathbf{C})\mathbf{B}.$$

The ghost energy-momentum tensor has the same form as that of the matter $\mathbf{\Sigma\Gamma}$ CFT except the sign differences. The conformal weights of the ghost super fields with $\lambda_g = 2$ are $h(\mathbf{B}) = \lambda_g - 1/2$, $h(\mathbf{C}) = 1 - \lambda_g$. Those of the component fields are $h(\beta_g) = \lambda_g - 1/2$, $h(c_g) = 1 - \lambda_g$, $h(b_g) = \lambda_g$, $h(\gamma_g) = 3/2 - \lambda_g$.

$$\begin{aligned}\mathbf{T}(z_1,\theta_1)\mathbf{\Gamma}(z_2,\theta_2) &\sim (1-\lambda)\frac{\theta_{12}}{\hat{z}_{12}^2}\mathbf{\Gamma}(z_2,\theta_2) + \frac{1}{2}\frac{1}{\hat{z}_{12}}\mathbf{D}_2\mathbf{\Gamma}(z_2,\theta_2) + \frac{\theta_{12}}{\hat{z}_{12}}\partial_2\mathbf{\Gamma}(z_2,\theta_2),\\ \mathbf{T}(z_1,\theta_1)\mathbf{\Sigma}(z_2,\theta_2) &\sim (\lambda-\frac{1}{2})\frac{\theta_{12}}{\hat{z}_{12}^2}\mathbf{\Sigma}(z_2,\theta_2) + \frac{1}{2}\frac{1}{\hat{z}_{12}}\mathbf{D}_2\mathbf{\Sigma}(z_2,\theta_2) + \frac{\theta_{12}}{\hat{z}_{12}}\partial_2\mathbf{\Sigma}(z_2,\theta_2).\end{aligned} \tag{59}$$

These equations tell us that the new fields $\mathbf{\Gamma}$ and $\mathbf{\Sigma}$ have conformal weights $h(\mathbf{\Gamma}) = 1-\lambda$ and $h(\mathbf{\Sigma}) = \lambda - 1/2$, respectively.

The dimensions of the component fields are

$$\mathbf{\Gamma} = -\gamma + \theta c, \qquad h(\gamma) = 1-\lambda, \qquad h(c) = 3/2 - \lambda, \tag{60}$$

$$\mathbf{\Sigma} = b + \theta\beta, \qquad h(b) = \lambda - 1/2, \qquad h(\beta) = \lambda. \tag{61}$$

γ, β, and $\mathbf{\Gamma}$ are commuting fields and b, c, and $\mathbf{\Sigma}$ are anticommuting fields.

Using the component fields we can rewrite the supersymmetric action

$$S_1 = \int\frac{d^2z}{2\pi}(\beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + b\bar{\partial}c + \bar{b}\partial\bar{c}). \tag{62}$$

Given the conformal weights of the component fields, the central charge of the $\beta\gamma$ CFT and the bc CFT are $3(2\lambda-1)^2 - 1$ and $-3(2\lambda-2)^2+1$, respectively. Thus the total central charge is $c = 12\lambda - 9$, which agrees with the result from the OPE of the energy-momentum tensor.

The OPEs of the component fields are

$$\gamma(z_1)\beta(z_2) \sim \frac{1}{z_{12}} \sim -\beta(z_1)\gamma(z_2), \tag{63}$$

$$b(z_1)c(z_2) \sim \frac{1}{z_{12}} \sim c(z_1)b(z_2). \tag{64}$$

The energy-momentum tensor in the component form can be written

$$T_b = \left(\lambda-\frac{3}{2}\right)c(\partial b) + \left(\lambda - \frac{1}{2}\right)(\partial c)b - (\lambda-1)\gamma(\partial\beta) - \lambda(\partial\gamma)\beta = \sum_{m\in\mathbf{Z}}\frac{L_m}{z^{m+2}}, \tag{65}$$

$$T_f = -(\lambda-1)\gamma(\partial b) + \frac{1}{2}c\beta - \left(\lambda-\frac{1}{2}\right)(\partial\gamma)b = \sum_{r\in\mathbf{Z}+\nu}\frac{G_r}{2\cdot z^{r+3/2}}. \tag{66}$$

As is well known, the fields with the half-integer conformal weight have both NS and R sectors. To make the expressions simple, we concentrate on the case of integer λ. The mode expansions and the Hermiticity properties are

$$\begin{aligned}\gamma(z) &= \sum_{n\in\mathbf{Z}}\frac{\gamma_n}{z^{n+1-\lambda}}, \qquad \gamma_n^\dagger = \gamma_{-n},\\ \beta(z) &= \sum_{n\in\mathbf{Z}}\frac{\beta_n}{z^{n+\lambda}}, \qquad \beta_n^\dagger = -\beta_{-n},\end{aligned} \tag{67}$$

$$\begin{aligned}c(z) &= \sum_{r\in\mathbf{Z}+\nu}\frac{c_r}{z^{r+3/2-\lambda}}, \qquad c_r^\dagger = c_{-r},\\ b(z) &= \sum_{r\in\mathbf{Z}+\nu}\frac{b_r}{z^{r+\lambda-1/2}}, \qquad b_r^\dagger = b_{-r}.\end{aligned} \tag{68}$$

There are two possible values for ν. For the NS sector $\nu = 1/2$ and for the R sector $\nu = 0$. The mode expansions for the energy-momentum tensors are

$$L_m^{\beta\gamma bc} = \sum_{n\in\mathbf{Z}}(n-(1-\lambda)m)\beta_{m-n}\gamma_n - \sum_{s\in\mathbf{Z}+\nu}(s-(3/2-\lambda)m)b_{m-s}c_s + a\delta_{m,0}, \tag{69}$$

$$G_r^{\beta\gamma bc} = \sum_{n\in\mathbf{Z}}(c_{r-n}\beta_n + (n+2r(\lambda-1))\gamma_n b_{r-n}). \tag{70}$$

There is a normal ordering constant in each sector, $a_R^{\beta\gamma bc} = \frac{4\lambda-3}{8}$ and $a_{NS}^{\beta\gamma bc} = \frac{\lambda-1}{2}$.

B. Possible nonrelativistic superstring theories

It is interesting to construct a noncritical version of the nonrelativistic superstring theory. The central charge of the ghost part is $\hat{c}_{\mathbf{BC}} = -10$ and that of the matter CFT is $\hat{c}_{\mathbf{\Sigma\Gamma}} = 8\lambda - 6$. Thus to be consistent the dimension D of the spatial directions in target space is

$$D = 8(2-\lambda). \tag{71}$$

We summarized the interesting portion of theories in Table III.

Here we comment on the immediate observations of these possible consistent noncritical nonrelativistic superstring theories. These theories have the same actions and the $SO(1,1)\times SO(D)$ symmetries in addition to Galilean symmetry. There exists an infinite range of possible consistent theories with geometric interpretation, for which we mean it is possible to have a positive number of spatial coordinates.

It will be interesting to quantize them explicitly. We can divide them in two categories, (i) with integer λ cases and (ii) with half-integer λ cases, because there are two sectors for the fields with half-integer conformal weight. For the integer λ cases (i) with $D = 0, 8, 16, \cdots$, the bosonic commuting $\beta\gamma$ CFT has only one bosonic coordinate. From the explicit quantization of the previous section

TABLE III. Table for the superstring case. Conformal weight of the supersymmetric $\beta\gamma$ CFT and the number of spatial dimensions of target space are presented. For $\lambda > 2$, the geometric interpretation is not possible. As the parameter λ is decreasing, the number of spatial dimensions is growing indefinitely and linearly.

λ	$\cdots$	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$\cdots$
$\hat{c}_{\Sigma\Gamma} = 8\lambda - 6$	$\cdots$	10	6	2	-2	-6	-10	-14	$\cdots$
$D = 8(2-\lambda)$	$\cdots$	0	4	8	12	16	20	24	$\cdots$

and from [6], we know that it is relatively easy to quantize and establish the spacetime interpretation. On the other hand, there are two commuting bosonic sectors, NS and R, for the half-integer λ cases (ii) with $D = 4, 12, 20, \cdots$. Of course, in case (ii) the zero modes of the R sector of the $\beta\gamma$ CFT have a space and time interpretation. The case (ii) seems rather peculiar and it looks harder to quantize them. But these theories are expected to provide a different perspective for a space and time interpretation.

The challenges of establishing the zero modes of $\beta\gamma$ CFT in the new matter sector can be easily seen by the total normal ordering constant. As usual, the normal ordering constant for the R sectors is 0 due to the cancellation between the bosonic contribution and the fermionic contribution. Those of the NS sectors are

$$a_{NS}^{(i)} = \frac{\lambda - 2}{2}, \qquad a_{NS}^{(ii)} = \frac{2\lambda - 3}{4}. \tag{72}$$

Thus the total normal ordering constant for the NS sector depends on the parameter λ and there is nontrivial mapping between the unit vertex operator 1 and the corresponding state. We can see that the case with $\lambda = 1$, we considered in the previous section, is critical in the sense that the normal ordering constants $a_{NS}^{(i)} = -\frac{1}{2}$ recover those of the critical relativistic string theory. It is interesting to comment that there is another critical case for the case (ii) with $\lambda = \frac{1}{2}$. Thus the cases with $\lambda = 1$ and $\lambda = \frac{1}{2}$ tie together in a sense and we expect that the space and time interpretation is rather similar. This observation extends to all the other cases. The case with $\lambda = n$ and $\lambda = n + \frac{1}{2}$ tie together for integer n. Quantization of the theory with $\lambda = \frac{1}{2}$ and comparison to the critical case with $\lambda = 2$ will be very interesting.

In the case $\lambda = 2$ with $D = 0$, there are only $\mathbf{\Sigma\Gamma}$ CFT and $\mathbf{BC}$ CFT. Upon quantization, only the zero modes are present without oscillator excitations. The theory is topological. Furthermore there is a possible unification of these CFTs in a simple fashion. We comment on this at the end of this section. As explained in the previous paragraph, this case is tied with the $\lambda = \frac{3}{2}$ case in a sense that the normal ordering constant is the same and thus the zero modes have similar roles. But this is not a "topological" case because there are an additional 4 spatial coordinates.

TABLE IV. Table for the various properties of the first order matter CFT and the ghost CFT. We list the conformal weight, $U(1)$ charge of the matter $\beta\gamma$ CFT, and $U(1)$ charge of the ghost CFT.

Field	Weight	$U(1)^{\mathbf{m}}$	$U(1)^{\mathbf{gh}}$
b_g	λ_g	0	-1
β_g	$\lambda_g - 1/2$	0	-1
β	λ	-1	0
b	$\lambda - 1/2$	-1	0
c_g	$1 - \lambda_g$	0	1
γ_g	$3/2 - \lambda_g$	0	1
γ	$1 - \lambda$	1	0
c	$3/2 - \lambda$	1	0

Unification of all the first order CFTs

There is a curiosity related to a possible interesting $\mathbf{Z}_2$ graded algebra involving the nonzero conformal weight, the $U(1)$ ghost number, and the $U(1)$ number of the matter $\mathbf{\Sigma\Gamma}$ CFT. We can make a table for basic properties of the first order matter CFT and the ghost CFT

From Table IV we can imagine that there are two grand supermultiplets $\mathbf{V}$ and $\mathbf{W}$ with new field Θ_{gh} which carries conformal weight, $U(1)$ ghost charge, and $U(1)$ matter charge

$$\begin{aligned}\mathbf{V} &= \mathbf{\Sigma} + \Theta_{gh}\mathbf{B} = b + \theta\beta + \Theta_{gh}(\beta_g + \theta b_g)\\ &= b + \Theta_{gh}\beta_g + \theta(\beta + \Theta_{gh}b_g),\end{aligned} \tag{73}$$

$$\begin{aligned}\mathbf{W} &= \mathbf{C} + \Theta_{gh}\mathbf{\Gamma} = c_g + \theta\gamma_g + \Theta_{gh}(-\gamma + \theta c)\\ &= c_g - \Theta_{gh}\gamma + \theta(\gamma_g + \Theta_{gh}c).\end{aligned} \tag{74}$$

If one investigates these grand multiplets a little further one can read off that Θ_{gh} is the anticommuting field with conformal weight $\lambda - \lambda_g$, matter $U(1)$ charge -1, and ghost number 1. $\mathbf{V}$ is an anticommuting multiplet with the conformal weight $\lambda - 1/2$, the $U(1)$ matter charge -1, and the ghost $U(1)$ number 0, whereas $\mathbf{W}$ is an anticommuting multiplet with the conformal weight $1 - \lambda_g$, the $U(1)$ matter charge 0, and the ghost $U(1)$ number 1. We comment on two cases with immediate interest. One is the $\lambda = 1$ case with the conformal weight of the field Θ_{gh} as -1. Then all the fields have uniform gaps of their conformal weights. This is the case we quantized in the previous section. For $\lambda = 2$, the field Θ_{gh} has no conformal weight. This is a topological case with these two multiplets only without another matter sector.

With these observations we can rewrite the superstring action in a very simple form for the holomorphic part

$$\begin{aligned}S_{\mathbf{VW}} &= \int \frac{d^2z d^2\theta}{2\pi} d\Theta_{gh}(\mathbf{V}\bar{\mathbf{D}}_{\bar{\theta}}\mathbf{W})\\ &= \int \frac{d^2z d^2\theta}{2\pi}(\mathbf{\Sigma}\bar{\mathbf{D}}_{\bar{\theta}}\mathbf{\Gamma} + \mathbf{B}\bar{\mathbf{D}}_{\bar{\theta}}\mathbf{C}).\end{aligned} \tag{75}$$

TABLE V. Survey of possible string theory. The first five columns represent the number of reparametrization currents with corresponding spins as indicated in the subscript of $n_{\rm spin}$. $n_{3/2}$ represent the number of supersymmetry. c_{gh} is the total central charge of the supersymmetrized ghost CFT and $c^{\mathbf{m}}_{\beta\gamma,bc,\cdots}$ is the total central charge of the supersymmetrized $\beta\gamma$ CFT. The last two columns represent the symmetry and the representation of the supercharge.

	n_2	$n_{3/2}$	n_1	$n_{1/2}$	n_0	c_{gh}	$c^{\mathbf{m}}_{\beta\gamma,bc,\cdots}$	Symmetry	T_F Rep.
I	1	0	0	0	0	-26	$2(6\lambda^2-6\lambda+1)$		
II	1	1	0	0	0	-15	$12\lambda-9$		
III	1	2	1	0	0	-6	$+6$	$U(1)$	± 1
IV	1	3	3	1	0	0	0	$SU(2)$	$\mathbf{3}$
V	1	4	7	4	0	0	$24(\lambda-2)$	$SU(2)^2\times U(1)$	$(\mathbf{2},\mathbf{2},0)$
VI	1	4	6	4	1	0	0	$SU(2)^2$	$(\mathbf{2},\mathbf{2})$
VII	1	4	3	0	0	12	$36-24\lambda$	$SU(2)$	$\mathbf{2}$

Note that this action has still the derivative of the form $\bar{\mathbf{D}}_{\bar\theta}=\partial_{\bar\theta}+\bar\theta\partial_{\bar z}$ and we did not gauge the field Θ_{gh}. It will be interesting if we can gauge the field Θ_{gh}.

IV. NONRELATIVISTIC STRINGS WITH HIGHER SUPERSYMMETRY

Following Polchinski [10], we would like to survey possible superconformal algebras and their related nonrelativistic superstring theories. The basic idea is to find the sets of holomorphic and antiholomorphic currents, whose Laurent coefficients form a closed constraint algebra. This is motivated by the idea of enlarging the world sheet constraint algebra with supercurrents $T_F(z)$ and $\bar T_F(\bar z)$. Here the constraint is part of the symmetry singled out to be imposed on physical states in the old covariant quantization (OCQ) or Becci-Rouet-Stora-Tyupin (BRST) sense.

Here we assume that there is only one (2, 0) constraint current because the sum of the $\beta\gamma$, bc, and X^i energy-momentum tensors have geometric interpretation in terms of conformal invariance. This is similar to the relativistic case. Thus the result of the constraint current algebra in the world sheet is the same as the relativistic case. Concentrating on the holomorphic current with conformal weight as a multiple of a half-integer and less than and equal to 2,[11] there are very limited possible algebras and it is given in Table V.

The cases I and II are explained already in the bosonic string theory [6] and in the previous section, respectively. These theories are explicitly quantized and have the nonrelativistic dispersion relation. The cases III, IV, and VI are rather different from the other cases because both the supersymmetric ghost **BC** CFT and the $\Sigma\Gamma$ CFT have the central charges independent of λ, which are the same in magnitude with opposite sign. Thus there is no room for the spatial coordinates. But it is still possible to have some geometric interpretation from the matter $\Sigma\Gamma$ CFTs.

In addition to the II case, there are two possible cases with an infinite number of possible string theories, the cases V and VII. Both cases have 4 super charges in the world sheet CFT. For case V, the central charge of the superconformal ghost CFTs is 0 and the central charge of the matter $\mathbf{\Sigma\Gamma}$ CFTs is $24(\lambda-2)$. Thus for $\lambda\leq 2$ cases, it is possible to have spatial X CFTs. In the last case, VII, the central charge has positive contribution from the ghost CFTs. On the other hand, there are negative contributions from the matter $\Sigma\Gamma$ CFTs. We can make the parameter λ large and there is corresponding string theory. It will be interesting to quantize these sets of theories.

V. CONCLUSIONS

In this paper we construct a supersymmetric version of the recently constructed nonrelativistic string theory. The nonrelativistic superstring theory has a first order $\mathbf{\Sigma\Gamma}$ super conformal field theory (SCFT) on top of the usual eight second order $\mathbf{X}$ SCFTs. The fermionic sector has an anticommuting matter bc CFT in addition to the eight ψ^i fields. The component fields, b and c, have the conformal weights 1/2. These can be transformed into the ψ^0 and ψ^1 fields, and the fermionic action is the same as that of the relativistic superstring theory. The symmetry group is $SO(1,1)\times SO(8)$.

We quantize the theory in an elementary fashion. In addition to the physical state conditions imposed by the energy-momentum tensor, there exist other conditions from the super current. These give us a nonrelativistic analogue of the Dirac equation in the ground state of the R sector. This equation can be solved with the manifest $SO(8)$ symmetry by exploiting $SO(1,1)$ symmetry. The fermionic spectrum is nonchiral because the nonrelativistic Dirac equation connects the two irreducible spinor representations $\mathbf{8}_c$ and $\mathbf{8}_s$ for the $SO(8)$ group. For the closed string spectrum, modular invariance requires to project out

[11] For the ghost CFT, there are restrictions as we mentioned. But there is no restriction for the matter $\beta\gamma$ or bc CFT because they are part of the (2, 0) constraint current and they are a consistent part of the algebra as long as all the matter conformal weight sums up to satisfy the physical state conditions.

the ground state in the NS sector. The spectrum of this theory is very similar to that of the type IIB superstring theory, except for the chiral property and the energy dispersion relation. The one loop consistency check is straightforward and the theory is modular invariant.

We present a noncritical version of the nonrelativistic superstring theories by generalizing the conformal weight of the first order $\mathbf{\Sigma\Gamma}$ SCFT. It turns out that there is an infinite range of possible nonrelativistic superstring theories. We present some immediate observations related to these possible consistent string theories. We further survey possible nonrelativistic string theories with extended supersymmetry utilizing the world sheet constraint algebra. The matter $\beta\gamma$ CFT (and its supersymmetric partners) combined with the X CFT (and its partners) form a (2, 0) constraint current (and its partners) to have a geometric interpretation. Thus the matter first order CFTs are not constrained severely compared to the ghost sector. There are three infinite series of possible string theories: two with the four super charges and one with the one super charge, which is considered in the present work. It will be interesting to quantize these noncritical nonrelativistic string theories.

VI. FUTURE DIRECTIONS

Understanding cosmological singularities such as the big bang is an interesting and outstanding problem. It requires understanding time-dependent backgrounds in string theory, which are very difficult to analyze [1]. Perturbative string theory breaks down in some spacetime regions where the string coupling becomes large. One clean example with the lightlike linear dilaton theory was recently proposed in [2].[12] The dilaton is proportional to a light cone coordinate, $-X^+$, and the theory is defined as an exact CFT that describes string propagating in flat spacetime with the string coupling, $g_s = e^{-QX^+}$. Thus the spacetime is free at late times and strongly coupled at early times. At early times, there is a true singularity happening at a finite affine parameter, which requires a matrix string description as explained in [2]. It appears to be necessary to have a complete nonperturbative description of string theory to understand time-dependent backgrounds in string theory. There is an interesting nonperturbative formulation of noncritical M theory in (2 + 1) dimensions using the nonrelativistic Fermi liquid and its time-dependent solutions [14]. Earlier work with time-dependent background with closed string tachyon condensation can be found in the (1 + 1) noncritical string theory [15].

On the other hand there are very interesting developments which emphasize the role of perturbative string theory in the analysis of time-dependent backgrounds. It is claimed that a certain spacetime singularity can be replaced by a tachyon condensate phase within perturbative string theory [3]. Very recent papers [4] argue, using alternative gauge choices to free world sheet gravitino, that spacetime decay to nothing in string and M theory should be addressed at weak string coupling, where the nonperturbative instanton instability is expected to turn into a perturbative tachyon instability. See also [16]. Similar considerations in supercritical string theories can be found in [17,18].

It turns out that many interesting cosmological solutions have broken Lorentz symmetry. It is interesting to consider these solutions with their manifest global symmetries. Furthermore fundamental issues related to time, especially to "emergent time," is not clear (see, e.g., [5]). Thus it is interesting to consider alternative approaches, which can shed light on time-dependent backgrounds and on fundamental issues of time.[13] Our current work and a previous paper [6], motivated by earlier works [7–9], provide examples for these alternative approaches.

As we saw in the main body, the nonrelativistic string theory shares many features with relativistic string theory. The difference between these two theories comes from the replacement of the X^0 and X^1 CFTs by $\beta\gamma$ CFT. This effect is minimal because these matter CFTs are part of the (2, 0) constraint current, which makes a geometric interpretation possible. As a result, the spectrum is very similar to that of type IIB superstring theory. On the other hand, these nonrelativistic string theories provide a very different perspective on time. Thus these nonrelativistic string theories appear to be ideal for investigating general issues related to time-dependent backgrounds with broken Lorentz symmetry, such as the lightlike linear dilaton theory and supercritical string theories.

We would like to comment on a few preliminary results for the correspondence between the critical nonrelativistic string theory and the lightlike LDT.[14] These two theories have the same set of global symmetries, which can be checked with the identification $X^+ = t$ in the lightlike LDT case. In the light cone gauge, the spectrum of the lightlike LDT can be checked to be the same as that of the nonrelativistic string theory. These equivalences are

[12]There are some direct generalizations of this simple solution [13]. We thank Professor Nobuyoshi Ohta for drawing our attention for these solutions.

[13]An example which motivates a different approach for time can be seen in the low energy limits of open string theory with magnetic and electric $NS-NS$ B-field. In the appropriate limits, the theory with electric $NS-NS$ B-field is reduced to noncommutative open string theory while the theory with magnetic $NS-NS$ B-field reduces to the noncommutative Yang-Mills theory. This suggests that time is rather different from space. This is motivated to consider nonrelativistic string theories in [6].

[14]This correspondence between the nonrelativistic string theory and the lightlike linear dilaton theory was pointed out by Professor Petr Hořava. We are grateful for his careful and extensive suggestions on this correspondence and on various references.

enough for us to be serious about investigating the exact mapping between these two theories. We hope to report these results in the near future.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor Ori Ganor for encouragements, Professor Petr Hořava for introducing crucial ideas and references, and Professor Ashvin Vishwanath for answering many questions related to properties of the nonrelativistic system. Their careful comments and extensive discussions were critical to make this work possible. I also thank Jordan Carlson, Sharon Jue, and Stefan Leichenauer for reading and commenting on the manuscript. This work was supported in part by the Center of Theoretical Physics at UC Berkeley, and in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

APPENDIX: PHYSICAL SPECTRUM WITH $SO(7)$ SYMMETRY

In this appendix, we consider a relativistic approach to investigate the spectrum of this nonrelativistic string theory. It is interesting to compare these results with those in the main text.

We have $SO(1,1)\times SO(8)$ symmetry and we want to analyze the nonrelativistic mass shell condition (32) and the nonrelativistic Dirac equation (39)

$$\frac{\alpha'}{4}\vec{k}^2 - k^\gamma p' = 0, \tag{A1}$$

$$\frac{1}{2^{1/2}}(\alpha'^{1/2}k^i\psi_{0,i} - (k^\gamma + p')\psi_{0,0} + (k^\gamma - p')\psi_{0,1}) = 0. \tag{A2}$$

Rather than breaking the $SO(1,1)$ symmetry, we can go to a frame, $k^i = 0$ for $i = 2,\cdots,8$ and $k^9 \neq 0$, which preserves the $SO(1,1)\times SO(7)$ symmetry, to solve these two equations (A1) and (A2). From the quantization procedure we know that there are 8 physical degrees of freedom. There is only the $SO(7)$ manifest symmetry in the first excited level of the NS sector, which has a vector representation **7** of $SO(7)$. Then where is one extra degree of freedom? It is a "Dilaton" originated from the conformal rescaling $SO(1,1)$, which transforms as a singlet under $SO(7)$. Thus the first excited level has 8 degrees of freedom which transform as **1** + **7** under the $SO(7)$ rotation.

Then we can solve the nonrelativistic Dirac equation (A2) by using the $SO(1,1)$ symmetry by picking particular values of k^γ and p'. Then the remaining symmetry group $SO(1,1)\times SO(7)$ is broken to $SO(7)$. The irreducible spinor representation of the $SO(7)$ group is **8** as is well known. Thus there are actually 8 independent degrees of freedom in the ground state of the R sector. It is obvious that there is no chance for the fermions to have any chiral property. We present the holomorphic spectrum with $SO(7)$ symmetry in Table VI.

TABLE VI. Spectrum of the holomorphic sector for the ground and first excited level of the NS sector and ground state of the R sector. **7** and **8** are the vector representation and the spinor representation of $SO(7)$, respectively.

Sector	$SO(7)$ spin	$-\frac{\alpha'}{4}\vec{k}^2 + k^\gamma p'$
NS_0	**1**	$-1/2$
NS	**1** + **7**	0
R	**8**	0

TABLE VII. Closed superstring spectrum for the ground and the first excited levels of the NS sector and ground state of the R sector. **1**, **7**, **27** are the tensor representations and **8**, **48** are the spinor representations of $SO(7)$.

Sector	$SO(7)$ spin		Dimensions
(NS_0, NS_0)	**1** × **1**	=	**1**
(NS, NS)	(**1** + **7**) × (**1** + **7**)	=	**1** + (**7** + **7**) + (**1** + **21** + **27**)
(NS, R)	(**1** + **7**) × **8**	=	**8** + (**8** + **48**)
(R, NS)	**8** × (**1** + **7**)	=	**8** + (**8** + **48**)
(R, R)	**8** × **8**	=	**1** + (**7** + **21**) + (**1** + **7** + **27**)

It is straightforward to construct the nonrelativistic closed superstring spectrum. They are presented in Table VII. We would like to have a few comments. Comparing the approach with the manifest $SO(8)$ symmetry, the $SO(7)$ symmetry is not efficient to describe the physical spectrum. Furthermore it is not clear how we can demonstrate the modular invariance at all. The field contents are very similar to the relativistic string theory with a circle compactification. But in that case there are discrete momentum modes and discrete winding modes in the twisted sector. On the other hand, we have just continuous momentum without compact coordinate or twisted sector.

[1] H. Liu, G. W. Moore, and N. Seiberg, J. High Energy Phys. 06 (2002) 045 (2002); H. Liu, G. W. Moore, and N. Seiberg, J. High Energy Phys. 10 (2002) 031.

[2] B. Craps, S. Sethi, and E. P. Verlinde, J. High Energy Phys. 10 (2005) 005; B. Craps, A. Rajaraman, and S. Sethi, Phys. Rev. D **73**, 106005 (2006).

[3] J. McGreevy and E. Silverstein, J. High Energy Phys. 08 (2005) 090 (2005).

[4] P. Hořava and C. A. Keeler, arXiv:0709.2162; arXiv:0709.3296.

[5] N. Seiberg, arXiv:hep-th/0601234.

[6] B. S. Kim, Phys. Rev. D **76**, 106007 (2007).

[7] J. Gomis and H. Ooguri, J. Math. Phys. (N.Y.) **42**, 3127 (2001).

[8] U. H. Danielsson, A. Guijosa, and M. Kruczenski, J. High Energy Phys. 10 (2000) 020.

[9] U. H. Danielsson, A. Guijosa, and M. Kruczenski, J. High Energy Phys. 03 2001, 041 (2001).

[10] J. Polchinski, *String Theory, Vol. I, II* (Cambridge University Press, Cambridge, England, 1998).

[11] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory, Vol. I, II* (Cambridge University Press, Cambridge, England, 1987).

[12] D. Friedan, E. Martinec, and S. Shenker, Nucl. Phys. **B271**, 93 (1986).

[13] M. Li, Phys. Lett. B **626**, 202 (2005); B. Chen, Phys. Lett. B **632**, 393 (2006); T. Ishino, H. Kodama, and N. Ohta, Phys. Lett. B **631**, 68 (2005); T. Ishino and N. Ohta, Phys. Lett. B **638**, 105 (2006); S. R. Das, J. Michelson, K. Narayan, and S. P. Trivedi, Phys. Rev. D **74**, 026002 (2006).

[14] P. Hořava and C. A. Keeler, J. High Energy Phys. 07 (2007) 059; Nucl. Phys. **B745**, 1 (2006); J. High Energy Phys. 06 (2007) 031.

[15] J. L. Karczmarek and A. Strominger, J. High Energy Phys. 05 (2004) 062.

[16] S. Hellerman and I. Swanson, arXiv:0710.1628.

[17] O. Aharony and E. Silverstein, Phys. Rev. D **75**, 046003 (2007).

[18] S. Hellerman and I. Swanson, arXiv:hep-th/0611317; J. High Energy Phys. 09 (2007) 096; arXiv:hep-th/0612116; Phys. Rev. Lett. **99**, 171601 (2007).