Chemical potentials in real-time thermal field theory

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In the functional integral formulation of real-time thermal field theory, a time-dependent canonical transformation of the integration variables can remove the chemical potential from the action. The transformation eliminates the chemical potential from the differential equation satisfied by the propagator, but the chemical potential appears in the transformed boundary conditions and the final result for the perturbative Green's functions is unchanged.

I. INTRODUCTION

When quantum field theories are investigated at fixed temperature and chemical potential, there is a small puzzle that has not been discussed in the literature. If the calculations are done using the real-time functional integral formalism, it appears that the chemical potential can be completely eliminated by a time-dependent canonical transformation. This paper shows that appearances are deceiving; the chemical potential cannot actually be eliminated from the problem. If it is transformed away from the action then it reappears in the boundary conditions that must be satisfied by the functional integral in such a way as to always give the same answer.

A chemical potential is introduced in the grand canonical ensemble as a Lagrange multiplier that controls the amount of a particular conserved charge in the environment [1–4]. The conserved charge may correspond to either a global symmetry or a local symmetry. The global symmetry can be Abelian or non-Abelian; the local symmetry can be Abelian or non-Abelian. Regardless of how complicated the theory, if calculations are done in real time then it always appears possible to transform away the chemical potential. The clearest example is that of a global Abelian symmetry, and that case will be treated explicitly in what follows.

Thus consider a charged scalar field with a free Hamiltonian density

$$\mathcal{H} = \pi_H^\dagger \pi_H + \vec{\nabla}\phi_H^\dagger \cdot \vec{\nabla}\phi_H + m^2 \phi_H^\dagger \phi_H. \qquad (1)$$

The subscript H denotes field operators in the Heisenberg picture. The charge density

$$\rho = i(\pi_H^\dagger \phi_H^\dagger - \pi_H \phi_H), \qquad (2)$$

when integrated over the volume, gives the charge operator $Q = \int d^3x \rho$. The charge commutes with the Hamiltonian $[Q, H] = 0$. The complex field ϕ_H has negative charge,

$$[Q, \phi_H] = -\phi_H, \qquad (3)$$

and the adjoint field has positive charge. The charge will continue to be conserved when there is an interaction of the form $V(\phi^\dagger \phi)$ added to Eq. (1). For a self-interacting

theory, V could be $\lambda(\phi^\dagger \phi)^2$. Another possibility is to introduce a real scalar field σ with the interaction $g\sigma\phi^\dagger \phi$, a model that has been recently studied and that has nontrivial μ dependence [5]. Neither the form of the interaction nor the possible occurrence of spontaneous symmetry breaking will have any bearing on the question at hand, and consequently, it is sufficient to consider the free Hamiltonian density (1).

In the operator formulation, the density operator at fixed temperature and chemical potential is

$$\exp\{-\beta(H - \mu Q)\}. \qquad (4)$$

In the real-time functional integral formulation, the generating functional with external sources J^*, J for a system at fixed temperature and chemical potential requires integrating over complex c-number momenta π and π^* and complex c-number fields ϕ and ϕ^*:

$$Z[J^*, J] = \int [d\Omega] \exp\left\{ i \int_C d^4x (I_\mu + J^*\phi + J\phi^*) \right\}. \qquad (5)$$

Here $[d\Omega] = [d\pi d\pi^*][d\phi d\phi^*]$ and the integrand is

$$I_\mu = \pi \frac{\partial \phi}{\partial t} + \pi^* \frac{\partial \phi^*}{\partial t} - \mathcal{H} + \mu\rho. \qquad (6)$$

$\mathcal{H}$ and ρ are the c-number forms of the Hamiltonian density and charge density:

$$\mathcal{H} = \pi^*\pi + |\vec{\nabla}\phi|^2 + m^2|\phi|^2,$$
$$\rho = i(\pi^*\phi^* - \pi\phi). \qquad (7)$$

The contour C in Eq. (5) lies in the complex-time plane, beginning at some real initial time t_i and ending at $t_i - i\beta$. Appendix A provides a derivation of Eq. (5).

It appears that the chemical potential can be eliminated by making a time-dependent change of integration variables:

$$\phi = e^{i\mu t}\phi', \qquad \phi^* = e^{-i\mu t}\phi'^*,$$
$$\pi = e^{-i\mu t}\pi', \qquad \pi^* = e^{i\mu t}\pi'^*. \qquad (8)$$

$\mathcal{H}$ and ρ have the same form in the new variables:

$$\mathcal{H}' = |\pi'|^2 + |\vec{\nabla}\phi'|^2 + m^2|\phi'|^2,$$
$$\rho' = i(\pi'^*\phi'^* - \pi'\phi'). \tag{9}$$

However, the integrand $\mathcal{I}_\mu$ of the action changes inhomogeneously because

$$\pi\frac{\partial\phi}{\partial t} + \pi^*\frac{\partial\phi^*}{\partial t} = \pi'\frac{\partial\phi'}{\partial t} + \pi'^*\frac{\partial\phi'^*}{\partial t} \\ + i\mu(\pi'\phi' - \pi'^*\phi'^*). \tag{10}$$

The last term is $-\mu\rho$ and cancels the $+\mu\rho$ in Eq. (6). Thus, in the new variables the action is independent of the chemical potential:

$$\mathcal{I}' = \pi'\frac{\partial\phi'}{\partial t} + \pi'^*\frac{\partial\phi'^*}{\partial t} - \mathcal{H}'. \tag{11}$$

The generating functional in Eq. (5) now appears to have only a trivial dependence on μ:

$$Z[J^*, J] = \int [d\Omega']\exp\Big\{i\int_C d^4x(\mathcal{I}' + e^{i\mu t}J^*\phi \\ + e^{-i\mu t}J\phi^*)\Big\}. \tag{12}$$

An alternative procedure would be to first perform the functional integration over π and π^* and then perform the change of variables on the remaining ϕ and ϕ^*. The chemical potential would still appear only in the source terms.

The question that will be investigated is whether Eq. (5), with μ present in the Lagrangian, gives the same result as Eq. (12), where μ is absent from the Lagrangian. The calculation of (5) is performed in Sec. II; the calculation of (12) is performed in Sec. III. Some concluding remarks appear in Sec. IV.

II. ACTION CONTAINING μ

Neither functional integral, (5) or (12), can be computed without an understanding of the kernel that inverts the respective Lagrangians. Specification of the kernels comes from the Kubo-Martin-Schwinger (KMS) conditions [6] that the kernels must satisfy. These conditions will be developed within the functional integral formalism without appeal to the underlying operator theories.

It is well known, and reviewed in Appendix A, that the generating functional $Z[J, J^*]$ in Eq. (5) represents the following trace of an operator product:

$$\text{Tr}\Big[e^{-\beta(H-\mu Q)}T_C\exp\Big\{i\int_C d^4x(J^*\phi_H + J\phi_H^\dagger)\Big\}\Big]. \tag{13}$$

Appendix B shows that the operator ϕ_H in this formula has a time dependence determined by $H - \mu Q$ instead of by the Hamiltonian alone. The thermal average in Eq. (13) is of the ordered exponential in which the ordering is along a time contour C that begins at t_i and ends at $t_i - i\beta$. The

standard real-time contour is a union of four straight-line paths: C_1 runs along the real axis from a large negative time t_i to a large positive time t_p; C_2 runs antiparallel to the real-time axis from $t_p - i\sigma$ to $t_i - i\sigma$, where $0 \le \sigma \le \beta$; C_3 connects t_p to $t_p - i\sigma$; and C_4 connects $t_i - i\sigma$ to $t_i - i\beta$. The numbering is conventional [1,2] and chosen because C_3 and C_4 make no contribution when the limits $t_i \to -\infty$ and $t_p \to \infty$ are taken at the end of the calculation.

The functional integral representation (5) is based on the Feynman-Matthews-Salam formula [7,8] for computing transition matrix elements between eigenstates of the Heisenberg field operator $\phi_H(x)$. (See Appendix A.) To incorporate the thermal weighting, the initial eigenstate is $|\phi_i, t_i\rangle$ and the final eigenstate is $|\phi_f, t_i - i\beta\rangle$. To compute the trace over states requires setting $\phi_f = \phi_i$ and then integrating over all ϕ_i. Consequently the functional integral is only over c-number fields that satisfy the periodicity conditions

$$\phi(\vec{x}, t_i - i\beta) = \phi(\vec{x}, t_i), \qquad \phi^*(\vec{x}, t_i - i\beta) = \phi^*(\vec{x}, t_i). \tag{14}$$

These must hold only for one value of t_i, not for all real times.

Applying $\delta^2/\delta J^*(x)\delta J(0)$ to Eq. (5) and then setting the sources to zero gives the functional integral representation of the contour-ordered propagator:

$$D_C(x - y) = \int[d\Omega]\exp\Big\{i\int_C d^4x\mathcal{I}_\mu\Big\}\phi(x)\phi^*(y). \tag{15}$$

Along the contour $C = C_1 \cup C_2 \cup C_3 \cup C_4$ the propagator is defined as

$$D_C(x - y) = \begin{cases} D_>(x - y) & \text{if } x^0 \text{ later than } y^0, \\ D_<(x - y) & \text{if } x^0 \text{ earlier than } y^0. \end{cases} \tag{16}$$

Now set $\vec{y} = 0$ and consider two sets of times:

$$\text{(a) } x^0 = t_i \quad \text{and} \quad y^0 = t_i + t, \\ \text{(b) } x^0 = t_i - i\beta \quad \text{and} \quad y^0 = t_i + t \tag{17}$$

where $t > 0$. For choice (a), x^0 occurs earlier than y^0 and so the propagator is $D_<(\vec{x}, -t)$; whereas in (b), x^0 occurs later on the contour than y^0 and so the propagator is $D_>(\vec{x}, -t - i\beta)$. In the functional integral (15) for case (a), the integrand is $\phi(\vec{x}, t_i)\phi^*(\vec{0}, t_i + t)$; and for (b), the integrand in (15) is $\phi(\vec{x}, t_i - i\beta)\phi^*(\vec{0}, t_i + t)$. Because $\phi(\vec{x}, t_i) = \phi(\vec{x}, t_i - i\beta)$ the integrand of (15) is the same in both cases. Thus the propagator satisfies

$$D_<(\vec{x}, -t) = D_>(\vec{x}, -t - i\beta). \tag{18}$$

This is the KMS condition, which was originally derived by operator methods [6].

To complete the calculation of the generating functional (5) requires a simple Gaussian integration over the momenta to obtain

$$Z[J^*, J] = \int [d\phi d\phi^*] \exp\left\{ i \int_C d^4x (\mathcal{L}_\mu + J^*\phi + J\phi^*) \right\}.$$
(19)

The μ dependence of the integrand appears in

$$\mathcal{L}_\mu = \left| \frac{\partial \phi}{\partial t} - i\mu\phi \right|^2 - |\vec{\nabla}\phi|^2 - m^2 |\phi|^2.$$
(20)

Another Gaussian integration gives

$$Z[J, J^*] = \exp\left\{ -\int_C d^4x d^4y J^*(x) D_C(x - y) J(y) \right\},$$
(21)

where D_C is the contour-ordered propagator. When $t_i \to -\infty$ and $t_p \to \infty$, the contours C_3 and C_4 do not contribute [1,2]. In the above formula, x is on either C_1 or C_2 and similarly for y. This makes matrix labeling convenient:

$$Z[J, J^*] = \exp\left\{ -\int_{C_1 \cup C_2} d^4x d^4y J_a^*(x) D_{ab}(x - y) J_b(y) \right\}.$$
(22)

When x and y are both on C_1, contour ordering is the same as time ordering; when both are on C_2, contour ordering is anti-time-ordering:

$$D_{11}(x) = \theta(t) D_>(x) + \theta(-t) D_<(x),$$
(23)

$$D_{22}(x) = \theta(t) D_<(x) + \theta(-t) D_>(x).$$
(24)

If x is on C_1 and y on C_2, then $D_{12}(x - y) = D_<(x - y)$ and the choice $\vec{y} = 0$ and $y^0 = -i\sigma$ gives

$$D_{12}(\vec{x}, t + i\sigma) = D_<(\vec{x}, t + i\sigma).$$
(25)

If x is on C_2 and y on C_1, then $D_{21}(x - y) = D_>(x - y)$ and the choice $x^0 = t - i\sigma$, $\vec{y} = 0$, $y^0 = 0$ gives

$$D_{21}(\vec{x}, t - i\sigma) = D_>(\vec{x}, t - i\sigma).$$
(26)

These results are standard [1–3]. The Lagrangian (20) determines the differential equation for the propagator matrix:

$$\left[\left(i\frac{\partial}{\partial t} + \mu \right)^2 + \nabla^2 - m^2 \right] D_{ab}(x) = i\delta_{ab}\delta^4(x).$$
(27)

Although the differential equation contains μ, it does not contain the temperature. Temperature enters through the KMS boundary condition (18). The solution is summarized by

$$D_>(x) = \int \frac{d^3k}{(2\pi)^3 2E} \left[e^{i(\vec{k}\cdot\vec{x} - (E-\mu)t)}[1 + n(E - \mu)] \right.$$
$$\left. + e^{i(-\vec{k}\cdot\vec{x} + (E+\mu)t)} n(E + \mu) \right],$$
(28)

$$D_<(x) = \int \frac{d^3k}{(2\pi)^3 2E} \left[e^{i(\vec{k}\cdot\vec{x} - (E-\mu)t)} n(E - \mu) \right.$$
$$\left. + e^{i(-\vec{k}\cdot\vec{x} + (E+\mu)t)}[1 + n(E + \mu)] \right].$$
(29)

It is worth noting that $D_>$ and $D_<$ have both a trivial μ dependence in the multiplicative factor $e^{i\mu t}$ and a non-trivial dependence on μ that occurs in the Bose-Einstein functions $n(\omega) = 1/(\exp(\beta\omega) - 1)$.

III. ACTION INDEPENDENT OF μ

Now we compute the functional integral (12). The differential equation satisfied by the propagators will now be independent of the chemical potential. (Consistent with this, Appendix B shows that the time dependence of the field operator ϕ'_H is determined by the Hamiltonian H, rather than $H - \mu Q$ as was the case for ϕ_H in Sec. II.) However, the simpler differential equation for the propagators has boundary conditions that involve μ, and the result for the correlation functions will turn out the same.

The first step is to apply $\delta^2 / \delta J^*(x) \delta J(0)$ to the generating functional in Eq. (12) and then set the sources to zero. This gives the functional integral representation of a contour-ordered propagator, $D'_C(x - y)$. The question is whether this $D'_C(x - y)$ has the same value as obtained for $D_C(x - y)$ in Sec. II. The explicit factors of $\exp(\pm i\mu t)$ in Eq. (12) give

$$D'_C(x - y) = e^{i\mu x_0} G_C(x - y) e^{-i\mu y^0},$$
(30)

where G_C is defined by the function integral

$$G_C(x - y) = \int [d\Omega'] \exp\left\{ i \int_C d^4x I'_\mu \right\} \phi'(x) \phi'^*(y).$$
(31)

The periodicity conditions on the new field variables required by Eqs. (8) and (14) are

$$\phi'(\vec{x}, t_i - i\beta) = e^{-\beta\mu} \phi'(\vec{x}, t_i),$$
$$\phi'^*(\vec{x}, t_i - i\beta) = e^{\beta\mu} \phi'^*(\vec{x}, t_i).$$
(32)

Again, set $\vec{y} = 0$ and consider the same two sets of times given in Eq. (17). For (a), x^0 occurs earlier than y^0 and so the contour propagator in (31) is $G_<(\vec{x}, -t)$:

$$G_<(\vec{x}, -t) = \int [d\Omega'] \left[\exp\left\{ i \int_C d^4x I'_\mu \right\} \right.$$
$$\left. \times \phi'(\vec{x}, t_i) \phi'^*(\vec{0}, t_i + t) \right].$$
(33)

For (b), since x^0 occurs later on the contour than y^0, the contour propagator is $G_>(\vec{x}, -t - i\beta)$:

$$G_>(\vec{x}, -t - i\beta) = \int [d\Omega'] \left[\exp\left\{ i \int_C d^4x I'_\mu \right\} \right.$$
$$\left. \times \phi'(\vec{x}, t_i - i\beta) \phi'^*(\vec{0}, t_i + t) \right].$$
(34)

Because $\phi'(\vec{x}, t_i - i\beta) = e^{-\beta\mu}\phi'(\vec{x}, t_i)$ from (32), the modified KMS condition contains the chemical potential:

$$G_<(\vec{x}, -t) = e^{\beta\mu}G_>(\vec{x}, -t - i\beta). \qquad (35)$$

It is straightforward to complete the calculation of Eq. (12). Integration over the momenta π' and π'^* gives

$$Z[J^*, J] = \int [d\phi' d\phi'^*]$$
$$\times \exp\left\{i\int_C d^4x(\mathcal{L}' + e^{i\mu t}J^*\phi' + e^{-i\mu t}J\phi'^*)\right\},$$
$$(36)$$

where $\mathcal{L}'$ is independent of chemical potential:

$$\mathcal{L}' = \left|\frac{\partial\phi'}{\partial t}\right|^2 - |\vec{\nabla}\phi|^2 - m^2|\phi|^2. \qquad (37)$$

Integration over the fields gives

$$Z[J, J^*] = \exp\left\{-\int_C d^4x d^4y e^{-i\mu x^0}D'_C(x - y)J(y)e^{i\mu y^0}\right\}.$$
$$(38)$$

As before, in the limit $t_i \rightarrow -\infty$ and $t_p \rightarrow \infty$ the contours C_3 and C_4 do not contribute and the result collapses to

$$Z[J^*, J] = \exp\left\{-\int_{C_1 \cup C_2} d^4x d^4y[e^{i\mu x_0}J_a^*(x)\right.$$
$$\left.\times G_{ab}(x - y)J_b(y)e^{-i\mu y_0}]\right\}.$$

The Lagrangian density $\mathcal{L}'$ requires the kernel G to satisfy

$$\left[-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\right]G_{ab}(x) = i\delta_{ab}\delta^4(x). \qquad (39)$$

The G_{ab} have the same decomposition in terms of $G_>$ and $G_<$ as in Sec. III. The solutions satisfying the modified KMS condition (35) are

$$G_>(x) = \int\frac{d^3k}{(2\pi)^3 2E}[e^{i(\vec{k}\cdot\vec{x} - Et)}[1 + n(E - \mu)]$$
$$+ e^{i(-\vec{k}\cdot\vec{x} + Et)}n(E + \mu)], \qquad (40)$$

$$G_<(x) = \int\frac{d^3k}{(2\pi)^3 2E}[e^{i(\vec{k}\cdot\vec{x} - Et)}n(E - \mu)$$
$$+ e^{i(-\vec{k}\cdot\vec{x} + Et)}[1 + n(E + \mu)]]. \qquad (41)$$

From Eq. (30), $D'_{ab}(x) = e^{i\mu t}G_{ab}(x)$. Consequently $D'_>(x) = e^{i\mu t}G_>(x)$ which is exactly the same as $D_>(x)$ in Eq. (28). And $D'_<(x) = e^{i\mu t}G_<(x)$ which is the same as $D_<(x)$ in Eq. (29).

IV. CONCLUSIONS

The previous calculations show that, in the functional integral formulation of real-time thermal field theory, the

physical consequences of a chemical potential cannot be transformed away. When any interaction that conserves charge is included, then order-by-order in perturbation theory, the physical effects of the chemical potential cannot be transformed away.

Appendix A shows that the functional integral representation requires the underlying field operator ϕ_H introduced in Sec. I to have a time dependence determined not by H but by $H - \mu Q$. Appendix B shows that this time dependence agrees with the canonical quantization procedure.

There is an interesting question that arises regarding the change of variables from π, ϕ to π', ϕ' that eliminates the chemical potential from the action. Since the functional integral is quantum mechanical, a change of integration variables is not required to preserve the classical equations of motion. Nevertheless, Appendix C shows that if π, ϕ obey the classical equations of Hamilton, then the canonical transformation that implements the change to π', ϕ' will change the classical Hamiltonian in such a way that π', ϕ' satisfy the classical equations with the new Hamiltonian. This holds regardless of the form of the Hamiltonian, and it holds even if the charge is not conserved by the Hamiltonian dynamics.

Appendix D shows that in the imaginary-time formulation of thermal field theory, where real field components are used, the transformation question does not arise because it is not possible to make a change of variables that removes the chemical potential from the action. In this sense the imaginary-time calculation is more straightforward. One could calculate the effects of a chemical potential and then analytically continue the results to real time. The equivalence of quantum field theories in imaginary time and in real time relies on the Osterwalder-Schrader theorem [9,10]. However this theorem requires $SO(4)$ invariance of the Euclidean theory and $SO(3, 1)$ invariance of the Minkowski theory. Since the chemical potential term explicitly breaks the invariance, the theorem is not applicable.

APPENDIX A: BRIEF DERIVATION OF THE FUNCTIONAL INTEGRAL REPRESENTATION FOR THE GENERATING FUNCTIONAL

1. Action containing μ

The presence of the chemical potential in Eq. (5) requires starting with the underlying operator theory in which $\mathcal{H}$ and ρ are given by Eqs. (1) and (2). To relate the operator theory to the functional integral, it is necessary that the time dependence of the Heisenberg field operator ϕ_H be given by

$$[H - \mu Q, \phi_H] = -i\frac{\partial\phi_H}{\partial t}. \qquad (A1)$$

This is a bit surprising. Appendix B employs canonical quantization to verify this time dependence. The immediate consequence of (A1) is

$$\phi_H(\vec{x}, t) = e^{i(H-\mu Q)t}\phi_S(\vec{x})e^{-i(H-\mu Q)t},$$
$$\phi_H^\dagger(\vec{x}, t) = e^{i(H-\mu Q)t}\phi_S^\dagger(\vec{x})e^{-i(H-\mu Q)t}, \quad \text{(A2)}$$

where $\phi_S(\vec{x}) = \phi_H(\vec{x}, 0)$ is the Schroedinger-picture operator. The functional integral is built on eigenstates of the Schroedinger-picture operators:

$$\phi_S(\vec{x})|\phi\rangle = |\phi\rangle\phi(\vec{x}), \qquad \phi_S^\dagger(\vec{x})|\phi\rangle = |\phi\rangle\phi^*(\vec{x}). \quad \text{(A3)}$$

Then define time-dependent states

$$|\phi; t\rangle = e^{i(H-\mu Q)t}|\phi\rangle. \quad \text{(A4)}$$

It is important that the time dependence of the state is opposite to that of a Schroedinger-picture state vector. In particular, the time dependence is not the result of time evolution. Rather, the time dependence is mandated by the necessity to form eigenstates of the time-dependent Heisenberg field operators:

$$\phi_H(\vec{x}, t)|\phi; t\rangle = |\phi; t\rangle\phi_S(\vec{x}),$$
$$\phi_H^\dagger(\vec{x}, t)|\phi; t\rangle = |\phi; t\rangle\phi_S^*(\vec{x}). \quad \text{(A5)}$$

The eigenvalues ϕ_S and ϕ_S^* are time independent. With these states, the Feynman-Matthews-Salam [7,8] formula provides a way to compute matrix elements of time-ordered operators:

$$\langle\phi_f; t_f|T\exp\left\{i\int_{t_i}^{t_f} d^4x(J^*\phi_H + J\phi_H^\dagger)\right\}|\phi_i; t_i\rangle$$
$$= \int[d\pi d\pi^*]\int_{\phi_i}^{\phi_f}[d\phi d\phi^*]$$
$$\times \exp\left\{i\int_{t_i}^{t_f} d^4x(\mathcal{I} + J^*\phi + J\phi^*)\right\}, \quad \text{(A6)}$$

with $\mathcal{I}$ given by Eq. (6). The initial and final values of the field variables in the functional integration must satisfy $\phi(\vec{x}, t_i) = \phi_i(\vec{x})$ and $\phi(\vec{x}, t_f) = \phi_f(\vec{x})$.

The time dependence (A4) implies that

$$\langle\phi_f; t_f| = \langle\phi_f, t_i|\exp[i(H - \mu Q)(t_i - t_f)]. \quad \text{(A7)}$$

To make this into a thermal average requires choosing $t_i - t_f = i\beta$. Then choosing $\phi_f = \phi_i$ gives a diagonal matrix element of the form $\langle\phi_i; t_i|\ldots|\phi_i; t_i\rangle$. To compute the trace requires integrating over all ϕ_i. Thus the functional integration is over fields that satisfy the boundary conditions

$$\phi(\vec{x}, t_i - i\beta) = \phi(\vec{x}, t_i), \qquad \phi^\dagger(\vec{x}, t_i - i\beta) = \phi^\dagger(\vec{x}, t_i). \quad \text{(A8)}$$

The result of this prescription is the thermal average of the ordered exponential in which the ordering is along a time contour C that begins at t_i and ends at $t_i - i\beta$:

$$\text{Tr}\left[e^{-\beta(H-\mu Q)}T_C\exp\left\{i\int_C d^4x(J^*\phi_H + J\phi_H^\dagger)\right\}\right]$$
$$= \int[d\pi d\pi^*]\int_{\text{BC}}[d\phi d\phi^*]$$
$$\times \exp\left\{i\int_C d^4x(\mathcal{I} + J^*\phi + J\phi^*)\right\}, \quad \text{(A9)}$$

where BC refers to the boundary condition (A8). This is the generating function given in Eq. (5).

2. Action independent of μ

Suppose instead that one is given the functional integral (12). The operators ϕ_H' and $\phi_H'^\dagger$ satisfy Eqs. (A1)–(A6) but with $\mu = 0$. Then Eq. (A7) becomes

$$\langle\phi_f'; t_f| = \langle\phi_f', t_i|\exp[iH(t_i - t_f)]. \quad \text{(A10)}$$

For this to be a thermal average requires $t_i - t_f = i\beta$ as usual. For the average to be performed at finite chemical potential requires

$$\langle\phi_f', t_i| = \langle\phi_i'; t_i|e^{\beta\mu Q}. \quad \text{(A11)}$$

This requires that the c-number fields satisfy the boundary conditions (32)

$$\phi'(\vec{x}, t_i - i\beta) = e^{-\beta\mu}\phi'(\vec{x}, t_i),$$
$$\phi'^*(\vec{x}, t_i - i\beta) = e^{\beta\mu}\phi'^*(\vec{x}, t_i). \quad \text{(A12)}$$

APPENDIX B: COMPARISON WITH CANONICAL QUANTIZATION

1. Action containing μ

It is simple to check that the functional integral calculation done in Sec. II corresponds to canonical quantization with the time dependence given by Eq. (A1). With a partition function $Z = \text{Tr}[\exp[-\beta(H - \mu Q)]]$, the two thermal correlators are

$$D_>(x) = -i\,\text{Tr}[e^{-\beta(H-\mu Q)}\phi_H(x)\phi_H^\dagger(0)]/Z,$$
$$D_<(x) = -i\,\text{Tr}[e^{-\beta(H-\mu Q)}\phi_H^\dagger(0)\phi_H(x)]/Z. \quad \text{(B1)}$$

The time dependence in (A1) implies

$$\phi_H(\vec{x}, -t - i\beta) = e^{\beta(H-\mu Q)}\phi_H(\vec{x}, -t)e^{-\beta(H-\mu Q)}, \quad \text{(B2)}$$

which leads to

$$D_<(\vec{x}, -t) = D_>(\vec{x}, -t - i\beta). \quad \text{(B3)}$$

This is the KMS relation in Eq. (18) that was determined directly from the functional integral.

2. Action independent of μ

The functional integral calculation performed in Sec. III corresponds to an operator theory in which ϕ_H is changed to $\phi_H' = e^{-i\mu t}\phi_H$. The time dependence that follows from

Eq. (A1) is

$$[H - \mu Q, \phi'_H] = -i\frac{\partial \phi'_H}{\partial t} + \mu \phi'_H. \quad \text{(B4)}$$

Since $[Q, \phi'_H] = -\phi'_H$, this implies

$$[H, \phi'_H] = -i\frac{\partial \phi'_H}{\partial t}. \quad \text{(B5)}$$

This is the usual time dependence of a field operator. The thermal correlation functions of this field are

$$G_>(x) = -i\,\text{Tr}[e^{-\beta(H-\mu Q)}\phi'_H(x)\phi'^\dagger_H(0)]/Z,$$
$$G_<(x) = -i\,\text{Tr}[e^{-\beta(H-\mu Q)}\phi'^\dagger_H(0)\phi'_H(x)]/Z. \quad \text{(B6)}$$

These satisfy

$$G_<(\vec{x}, -t) = e^{\beta\mu}G_>(\vec{x}, -t - i\beta), \quad \text{(B7)}$$

which is the second KMS relation (35), which was deduced in Sec. III from the functional integral.

APPENDIX C: ELIMINATION OF THE CHEMICAL POTENTIAL BY A CANONICAL TRANSFORMATION IN CLASSICAL MECHANICS

The change of variables from π, ϕ to π', ϕ' eliminates the chemical potential μ from the action. The functional integral, being quantum mechanical, is not limited to transformations that preserve the classical equations of motion. This appendix will show that the above transformation does preserve Hamilton's equations.

Let $\mathcal{H}(\pi, \pi^*, \phi, \phi^*)$ be any function of the momenta and the fields, which may or may not have a conserved charge. Define the Hamiltonian density

$$\mathcal{H}_\mu = \mathcal{H}(\pi, \pi^*, \phi, \phi^*) - i\mu(\pi^*\phi^* - \pi\phi). \quad \text{(C1)}$$

Hamilton's equations are

$$\frac{\partial \phi}{\partial t} = \frac{\partial \mathcal{H}_\mu}{\partial \pi}, \qquad \frac{\partial \pi}{\partial t} = -\frac{\partial \mathcal{H}_\mu}{\partial \phi}, \quad \text{(C2)}$$

and similarly for π^*, ϕ^*. Now consider new variables $\bar{\pi}$, $\bar{\phi}$ and their complex conjugates. We would like to find a new Hamiltonian $\bar{\mathcal{H}}$ such that

$$\frac{\partial \bar{\phi}}{\partial t} = \frac{\partial \bar{\mathcal{H}}}{\partial \bar{\pi}}, \qquad \frac{\partial \bar{\pi}}{\partial t} = -\frac{\partial \bar{\mathcal{H}}}{\partial \bar{\phi}}. \quad \text{(C3)}$$

For both sets of Hamiltonian equations to come from a principle of least action, it is necessary that, for some function F of the various canonical variables,

$$\pi d\phi + \pi^* d\phi^* - \mathcal{H}_\mu dt = \bar{\pi}d\bar{\phi} + \bar{\pi}^*d\bar{\phi}^*$$
$$- \bar{\mathcal{H}}dt + dF. \quad \text{(C4)}$$

The relation of the new variables to the old variables is determined by the choice of F, and this choice then determines the dependence of $\bar{\mathcal{H}}$ on the new variables. To generate the transformation needed here, the appropriate choice is

$$F = \bar{\pi}(e^{-i\mu t}\phi - \bar{\phi}) + \bar{\pi}^*(e^{i\mu t}\phi^* - \bar{\phi}^*). \quad \text{(C5)}$$

Equating coefficients of $d\phi$ and $d\phi^*$ gives

$$\pi = \bar{\pi}e^{-i\mu t}, \qquad \pi^* = \bar{\pi}^*e^{i\mu t}. \quad \text{(C6)}$$

The coefficients of $d\bar{\pi}$ and $d\bar{\pi}^*$ give

$$e^{-i\mu t}\phi = \bar{\phi}, \qquad e^{i\mu t}\phi^* = \bar{\phi}^*. \quad \text{(C7)}$$

The coefficients of $d\bar{\phi}$ and $d\bar{\phi}^*$ give the identities $\bar{\pi} = \bar{\pi}$ and $\bar{\pi}^* = \bar{\pi}^*$. Equations (C6) and (C7) coincide with the change of integration variables in Eq. (8).

In the new variables, Hamilton's equations are Eq. (C3). The new Hamiltonian, $\bar{\mathcal{H}}$, is found by equating coefficients of dt in Eq. (C4):

$$\bar{\mathcal{H}} = \mathcal{H}_\mu + \frac{\partial F}{\partial t} = \mathcal{H}_\mu + i\mu(\bar{\pi}^*e^{i\mu t}\phi^* - \bar{\pi}e^{-i\mu t}\phi)$$
$$= \mathcal{H}_\mu + i\mu(\pi^*\phi^* - \pi\phi). \quad \text{(C8)}$$

The last term exactly cancels the μ dependence in (C1). Thus the new Hamiltonian is

$$\bar{\mathcal{H}} = \mathcal{H}(e^{-i\mu t}\bar{\pi}, e^{i\mu t}\bar{\pi}^*, e^{i\mu t}\bar{\phi}, e^{-i\mu t}\bar{\phi}^*). \quad \text{(C9)}$$

In the case of interest here the original Hamiltonian is

$$\mathcal{H}(\pi, \pi^*, \phi, \phi^*) = \pi^*\pi + |\vec{\nabla}\phi|^2 + m^2|\phi|^2, \quad \text{(C10)}$$

and consequently the new Hamiltonian is

$$\bar{\mathcal{H}} = \bar{\pi}^*\bar{\pi} + |\vec{\nabla}\bar{\phi}|^2 + m^2|\bar{\phi}|^2. \quad \text{(C11)}$$

APPENDIX D: CHEMICAL POTENTIALS IN IMAGINARY TIME

In the complex-time formulation of thermal field theory it is very awkward to remove the chemical potential from the action. With the complex field expressed in terms of real fields, $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, the Hamiltonian and charge density are

$$\mathcal{H} = \tfrac{1}{2}\pi_j^2 + \tfrac{1}{2}(\vec{\nabla}\phi_j)^2 + \tfrac{1}{2}m^2\phi_j^2,$$
$$\rho = \pi_1\phi_2 - \pi_2\phi_1. \quad \text{(D1)}$$

The fields are defined for imaginary time $t = -i\tau$ on the interval $0 \le \tau \le \beta$ and satisfy the periodicity condition

$$\phi_j(\vec{x}, \beta) = \phi_j(\vec{x}, 0). \quad \text{(D2)}$$

The chemical-potential dependent action in Eq. (6) becomes

$$I_\mu = i\pi_j \frac{\partial \phi_j}{\partial \tau} - \mathcal{H} + \mu\rho. \qquad (D3)$$

Both $\mathcal{H}$ and ρ are real, but the first term is pure imaginary. Thus no real transformation of the π_j and ϕ_j can produce a cancellation of the $\mu\rho$ term.

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