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Stability in the instantaneous Bethe-Salpeter formalism: Harmonic-oscillator reduced Salpeter equation

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A popular three-dimensional reduction of the Bethe-Salpeter formalism for the description of bound states in quantum field theory is the Salpeter equation, derived by assuming both instantaneous interactions and free propagation of all bound-state constituents. Numerical (variational) studies of the Salpeter equation with confining interaction, however, observed specific instabilities of the solutions, likely related to the Klein paradox and rendering (part of the) bound states unstable. An analytic investigation of the problem by a comprehensive spectral analysis is feasible for the reduced Salpeter equation with only harmonic-oscillator confining interactions. There we are able to prove rigorously that the bound-state solutions correspond to real discrete spectra bounded from below and are thus free of all instabilities.

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I. INTRODUCTION

The most widely explored three-dimensional reduction of the Bethe-Salpeter formalism [1] for the description of bound states within quantum field theory is the Salpeter equation [2]. The Salpeter equation controls the Salpeter amplitude, which in momentum space encodes the distribution of relative momenta of all bound-state constituents. In elementary particle physics its application to quantum electrodynamics (QED) and quantum chromodynamics (QCD) has met considerable success. In particular, within the latter realm it has evolved to a well-established standard tool for describing from first principles hadrons as bound states of quarks, confined by the strong interactions. Surprisingly or not, however, the solutions of Salpeter's equation with *confining* interactions have been *numerically* shown to develop for some Lorentz structures of the Bethe-Salpeter kernel representing all interactions between the bound-state constituents instabilities which cause states expected to be stable to decay.

In contrast, the *reduced* Salpeter equation [3–7], derived from the full Salpeter equation by neglecting some of the interaction terms (Sec. II), offers the chance to study the question of stability *analytically*. Such instabilities should arise first for pseudoscalar states (Sec. III); there, stripping off all angular variables simplifies [8–15] the reduced Salpeter equation to a single integral equation (Sec. IV). Harmonic-oscillator confining interactions (Sec. V) have a big advantage: In momentum space, they are represented by a simple Laplacian, converting thus our integral to differential equations (Sec. VI). Their analysis proves that, for all famous Lorentz structures, including one proposed by Böhm, Joos, and Krammer (BJK hereafter) [16], studied in Ref. [17] and more recently used, among others, by a group in Bonn [18–21], any bound-state solution is related to a real (Sec. VII) and discrete (Sec. VIII) energy eigenvalue, and thus stable. Similar considerations can be applied to the full Salpeter equation (Sec. IX).

II. BETHE-SALPETER FORMALISM IN INSTANTANEOUS LIMIT

We are interested in instantaneous approximations to Bethe-Salpeter equations describing bound states composed of a fermion and an antifermion. Let these constituents, denoted by $i = 1, 2$, carry the momenta p_1 and p_2, which will enter also in terms of the total momentum $P \equiv p_1 + p_2$ and, for a real parameter η or ζ, the relative momentum $p \equiv p_1 - \eta P \equiv \zeta P - p_2$.

A. Salpeter equation

Any derivation of the Salpeter equation [2] as one of a variety of possible three-dimensional reductions of the Bethe-Salpeter formalism is based on just two fundamental assumptions: First, one obtains the *instantaneous Bethe-Salpeter equation* if all interactions between the bound-state constituents are instantaneous in the center-of-momentum frame of the bound state. In the Bethe-Salpeter equation [1] all interactions between bound-state constituents are encoded in their integral kernel, K. The instantaneous approximation then implies that K depends

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only on the spatial components of the relative momenta involved: $K = K(\boldsymbol{p}, \boldsymbol{q})$. Second, one arrives at the *Salpeter equation* if every bound-state constituent propagates as a free particle with effective mass, m. Salpeter's equation adopts the free fermion propagator

$$S_0(p, m) = \frac{\mathrm{i}}{\not{p} - m + \mathrm{i}\varepsilon} \equiv \mathrm{i}\frac{\not{p} + m}{p^2 - m^2 + \mathrm{i}\varepsilon}, \qquad \varepsilon \downarrow 0.$$

Let us present those aspects of the Salpeter equation which will be relevant for stability. We introduce the one-particle energy $E_i(\boldsymbol{p})$, the one-particle Dirac Hamiltonian $H_i(\boldsymbol{p})$, and the energy projection operators $\Lambda_i^{\pm}(\boldsymbol{p})$ for positive or negative energy of particle $i = 1, 2$ by

$$\begin{aligned} E_i(\boldsymbol{p}) &\equiv \sqrt{\boldsymbol{p}^2 + m_i^2}, \qquad i = 1, 2, \\ H_i(\boldsymbol{p}) &\equiv \gamma_0(\boldsymbol{\gamma}\cdot\boldsymbol{p} + m_i), \qquad i = 1, 2, \\ \Lambda_i^{\pm}(\boldsymbol{p}) &\equiv \frac{E_i(\boldsymbol{p}) \pm H_i(\boldsymbol{p})}{2E_i(\boldsymbol{p})}, \qquad i = 1, 2; \end{aligned}$$

related, for $i = 1, 2$, by $H_i^2(\boldsymbol{p}) = E_i^2(\boldsymbol{p})$ and $\Lambda_i^{\pm}(\boldsymbol{p})H_i(\boldsymbol{p}) = H_i(\boldsymbol{p})\Lambda_i^{\pm}(\boldsymbol{p}) = \pm E_i(\boldsymbol{p})\Lambda_i^{\pm}(\boldsymbol{p})$, all operators $\Lambda_i^{\pm}(\boldsymbol{p})$ fulfilling $\Lambda_i^{\pm}(\boldsymbol{p})\Lambda_i^{\pm}(\boldsymbol{p}) = \Lambda_i^{\pm}(\boldsymbol{p})$, $\Lambda_i^{\pm}(\boldsymbol{p})\Lambda_i^{\mp}(\boldsymbol{p}) = 0$, $\Lambda_i^{+}(\boldsymbol{p}) + \Lambda_i^{-}(\boldsymbol{p}) = 1$. In terms of the above abbreviations, the Salpeter equation [2], for bound states of a fermion (of mass m_1 and momentum $\boldsymbol{p}_1$) and an antifermion (of mass m_2 and momentum $\boldsymbol{p}_2$), reads

$$\Phi(\boldsymbol{p}) = \int \frac{\mathrm{d}^3q}{(2\pi)^3}\left(\frac{\Lambda_1^{+}(\boldsymbol{p}_1)\gamma_0[K(\boldsymbol{p}, \boldsymbol{q})\Phi(\boldsymbol{q})]\gamma_0\Lambda_2^{-}(\boldsymbol{p}_2)}{P_0 - E_1(\boldsymbol{p}_1) - E_2(\boldsymbol{p}_2)} - \frac{\Lambda_1^{-}(\boldsymbol{p}_1)\gamma_0[K(\boldsymbol{p}, \boldsymbol{q})\Phi(\boldsymbol{q})]\gamma_0\Lambda_2^{+}(\boldsymbol{p}_2)}{P_0 + E_1(\boldsymbol{p}_1) + E_2(\boldsymbol{p}_2)}\right). \tag{1}$$

Accordingly, every solution $\Phi(\boldsymbol{p})$ of the Salpeter equation has to satisfy the two constraints

$$\Lambda_1^{+}(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^{+}(\boldsymbol{p}_2) = \Lambda_1^{-}(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^{-}(\boldsymbol{p}_2) = 0,$$

which, by considering their sum or difference, prove to be equivalent to a single constraint:

$$\frac{H_1(\boldsymbol{p}_1)}{E_1(\boldsymbol{p}_1)}\Phi(\boldsymbol{p}) + \Phi(\boldsymbol{p})\frac{H_2(\boldsymbol{p}_2)}{E_2(\boldsymbol{p}_2)} = 0 \tag{2}$$

or, equivalently,

$$\Phi(\boldsymbol{p}) + \frac{H_1(\boldsymbol{p}_1)\Phi(\boldsymbol{p})H_2(\boldsymbol{p}_2)}{E_1(\boldsymbol{p}_1)E_2(\boldsymbol{p}_2)} = 0.$$

With the help of the decomposition of unity in terms of projection operators $\Lambda_i^{\pm}(\boldsymbol{p})$, that is, $1 \otimes 1 = \Lambda_1^{+}(\boldsymbol{p}_1) \otimes \Lambda_2^{+}(\boldsymbol{p}_2) + \Lambda_1^{+}(\boldsymbol{p}_1) \otimes \Lambda_2^{-}(\boldsymbol{p}_2) + \Lambda_1^{-}(\boldsymbol{p}_1) \otimes \Lambda_2^{+}(\boldsymbol{p}_2) + \Lambda_1^{-}(\boldsymbol{p}_1) \otimes \Lambda_2^{-}(\boldsymbol{p}_2)$, a Salpeter amplitude $\Phi(\boldsymbol{p})$ may be, in general, decomposed into the components $\Lambda_1^{\pm}(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^{\pm}(\boldsymbol{p}_2)$:

$$\begin{aligned} \Phi(\boldsymbol{p}) = {} & \Lambda_1^{+}(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^{+}(\boldsymbol{p}_2) + \Lambda_1^{+}(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^{-}(\boldsymbol{p}_2) \\ & + \Lambda_1^{-}(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^{+}(\boldsymbol{p}_2) + \Lambda_1^{-}(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^{-}(\boldsymbol{p}_2). \end{aligned}$$

The above constraint(s) on the solutions $\Phi(\boldsymbol{p})$ of the Salpeter equation (1), arising from its specific projector structure, halve, in fact, the number of independent components of $\Phi(\boldsymbol{p})$:

$$\Phi(\boldsymbol{p}) = \Lambda_1^{+}(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^{-}(\boldsymbol{p}_2) + \Lambda_1^{-}(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^{+}(\boldsymbol{p}_2). \tag{3}$$

Assuming the Lorentz structures of the effective couplings of the bound fermions $i = 1, 2$ to the corresponding interaction potentials to be identical, the Bethe-Salpeter kernel $K(\boldsymbol{p}, \boldsymbol{q})$ is, quite generally, the sum of products of a tensor product $\Gamma \otimes \Gamma$ of generic Dirac matrices Γ and a Lorentz-scalar associated interaction function $V_\Gamma(\boldsymbol{p}, \boldsymbol{q})$: $K(\boldsymbol{p}, \boldsymbol{q}) = \sum_\Gamma V_\Gamma(\boldsymbol{p}, \boldsymbol{q})\Gamma \otimes \Gamma$. More precisely, the action of the kernel $K(\boldsymbol{p}, \boldsymbol{q})$ on the Salpeter amplitude $\Phi(\boldsymbol{p})$ is given by

$$[K(\boldsymbol{p}, \boldsymbol{q})\Phi(\boldsymbol{q})] = \sum_\Gamma V_\Gamma(\boldsymbol{p}, \boldsymbol{q})\Gamma\Phi(\boldsymbol{q})\Gamma.$$

B. Reduced Salpeter equation

The projection operators $\Lambda_i^{\pm}(\boldsymbol{p})$, $i = 1, 2$, for positive or negative energy satisfy the identity

$$[\Lambda_i^{\pm}(\boldsymbol{p})]^{\mathrm{c}} \equiv [C^{-1}\Lambda_i^{\pm}(\boldsymbol{p})C]^{\mathrm{T}} = \Lambda_i^{\mp}(\boldsymbol{p}), \qquad i = 1, 2,$$

where C denotes the usual Dirac-space charge-conjugation matrix. Accordingly, the second term on the right-hand side of the Salpeter equation, Eq. (1), is that interaction term which is related to the *negative-energy components* $\Lambda_1^{-}(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^{+}(\boldsymbol{p}_2) = \Lambda_1^{-}(\boldsymbol{p}_1)\Phi(\boldsymbol{p})[\Lambda_2^{-}(\boldsymbol{p}_2)]^{\mathrm{c}}$ of the Salpeter amplitude $\Phi(\boldsymbol{p})$. Assuming that this term's contribution may be reasonably ignored relative to that of the first term yields the so-called reduced Salpeter equation [3–7]

$$\begin{aligned} & [P_0 - E_1(\boldsymbol{p}_1) - E_2(\boldsymbol{p}_2)]\Phi(\boldsymbol{p}) \\ & \quad = \int \frac{\mathrm{d}^3q}{(2\pi)^3}\Lambda_1^{+}(\boldsymbol{p}_1)\gamma_0[K(\boldsymbol{p}, \boldsymbol{q})\Phi(\boldsymbol{q})]\gamma_0\Lambda_2^{-}(\boldsymbol{p}_2). \end{aligned} \tag{4}$$

This neglect might be justifiable for nonrelativistic and weakly bound systems composed of heavy constituents, such that, on the average, $P_0 - E_1(\boldsymbol{p}_1) - E_2(\boldsymbol{p}_2) \ll P_0 + E_1(\boldsymbol{p}_1) + E_2(\boldsymbol{p}_2)$. More rigorously, one would like to be sure, at least, that, for appropriate *expectation values*,

$$\frac{1}{P_0 + E_1(\boldsymbol{p}_1) + E_2(\boldsymbol{p}_2)} \ll \frac{1}{P_0 - E_1(\boldsymbol{p}_1) - E_2(\boldsymbol{p}_2)}.$$

Formally, this reduction of the (full) Salpeter equation (1) to the reduced Salpeter equation (4) can be accomplished by subjecting the Salpeter amplitude $\Phi(\boldsymbol{p})$ to *any* of the equalities

$$\Lambda_1^{-}(\boldsymbol{p}_1)\Phi(\boldsymbol{p}) = 0, \qquad \Phi(\boldsymbol{p})\Lambda_2^{+}(\boldsymbol{p}_2) = 0,$$

or, equivalently,

$$H_1(\boldsymbol{p}_1)\Phi(\boldsymbol{p}) = E_1(\boldsymbol{p}_1)\Phi(\boldsymbol{p}),$$
$$\Phi(\boldsymbol{p})H_2(\boldsymbol{p}_2) = -E_2(\boldsymbol{p}_2)\Phi(\boldsymbol{p});$$

because of the particular projector structure of the resulting reduced Salpeter equation (4), imposition of one of these two constraints automatically implies the other for the solutions. These equalities entail the constraints $\Lambda_1^+(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^+(\boldsymbol{p}_2) = \Lambda_1^-(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^-(\boldsymbol{p}_2) = 0$ also satisfied by each solution of the Salpeter equation (1), as well as, in addition, the constraint $\Lambda_1^-(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^+(\boldsymbol{p}_2) = 0$. As a trivial consequence, any solution $\Phi(\boldsymbol{p})$ of the reduced Salpeter equation (4) is necessarily of the unique component structure $\Phi(\boldsymbol{p}) = \Lambda_1^+(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^-(\boldsymbol{p}_2)$.

III. PSEUDOSCALAR BOUND STATES

The states easiest to investigate are bound states composed of a fermion and the associated antifermion, which guarantees a well-defined behavior with respect to charge conjugation. The masses of a particle and the corresponding antiparticle are, of course, identical, that is, $m_1 = m_2 = m$. Now, in the *center-of-momentum* (or rest) *frame* of the two-particle system under study, defined by $\boldsymbol{P} = \boldsymbol{0}$, which implies $\boldsymbol{p} = \boldsymbol{p}_1 = -\boldsymbol{p}_2$, the time component P_0 of the total momentum P reduces to the bound-state mass eigenvalue M, i.e., $P_0 = M$. All indices $i = 1, 2$, distinguishing the two bound-state constituents, may then be dropped throughout the following analysis: $E_1(\boldsymbol{p}) = E_2(\boldsymbol{p}) = E(\boldsymbol{p}) = E(p) \equiv \sqrt{p^2 + m^2}$, where $p \equiv |\boldsymbol{p}| \equiv \sqrt{\boldsymbol{p}^2}$; and, *mutatis mutandis*, for the Hamiltonians $H_i(\boldsymbol{p})$ and energy projection operators $\Lambda_i^\pm(\boldsymbol{p})$.

For the sake of definiteness we focus, in what follows, to the case of fermion-antifermion bound states of total spin J, parity $P = (-1)^{J+1}$, and charge-conjugation quantum number $C = (-1)^J$ (which entails $CP = -1$ for all J). In usual spectroscopic notation, these states are denoted by $n\,{}^1J_J$ ($n = 1, 2, 3, \ldots$). As the perhaps simplest example within this context, precisely these physical systems have been studied rather frequently (cf., e.g., Refs. [8–15]).

More specifically, we consider, for reasons of simplicity, bound states of spin $J = 0$, that is, pseudoscalar bound states, with spin-parity-charge-conjugation assignment $J^{PC} = 0^{-+}$, spectroscopically called ${}^1\mathrm{S}_0$. In nature such systems are realized and observed, for instance, in the realm of quark-antiquark bound states, in the form of the pion and its radial excitations.

In the case of the *Salpeter equation* (1), as a consequence of the constraints discussed in Subsection II A the most general expansion of the Salpeter amplitude $\Phi(\boldsymbol{p})$, over a complete set of Dirac matrices, involves not the full 16 but only eight *independent* Salpeter components. For the description of ${}^1J_J$ states, only two of the latter, called $\phi_1(\boldsymbol{p})$ and $\phi_2(\boldsymbol{p})$, are relevant. With our notation for one-particle energy $E(\boldsymbol{p})$ and Dirac Hamiltonian $H(\boldsymbol{p})$ introduced in Subsection II A, full ${}^1J_J$ Salpeter amplitudes $\Phi(\boldsymbol{p})$ for fermion and antifermion of equal mass m and internal momentum $\boldsymbol{p}$ thus read, in the center-of-momentum frame of the bound state,

$$\Phi(\boldsymbol{p}) = \left[\phi_1(\boldsymbol{p})\frac{H(\boldsymbol{p})}{E(\boldsymbol{p})} + \phi_2(\boldsymbol{p})\right]\gamma_5.$$

In the case of the *reduced Salpeter equation* (4), the one additional, two-faced constraint analyzed in Subsection II B entails, possibly taking into account $\gamma_5 H(-\boldsymbol{p}) = -H(\boldsymbol{p})\gamma_5$, for the two independent Salpeter components $\phi_1(\boldsymbol{p})$ and $\phi_2(\boldsymbol{p})$ in the ${}^1J_J$ Salpeter amplitude $\Phi(\boldsymbol{p})$, $\phi_1(\boldsymbol{p}) = \phi_2(\boldsymbol{p}) \equiv \phi(\boldsymbol{p})$. Consequently, the generic ${}^1J_J$ reduced Salpeter solutions $\Phi(\boldsymbol{p})$ read

$$\Phi(\boldsymbol{p}) = \phi(\boldsymbol{p})\frac{H(\boldsymbol{p}) + E(\boldsymbol{p})}{E(\boldsymbol{p})}\gamma_5 \equiv 2\phi(\boldsymbol{p})\Lambda^+(\boldsymbol{p})\gamma_5.$$

IV. RADIAL EIGENVALUE EQUATIONS

For any (instantaneous) Bethe-Salpeter interaction kernel $K(\boldsymbol{p}, \boldsymbol{q})$ of *convolution type*, i.e., $K(\boldsymbol{p}, \boldsymbol{q}) = K(\boldsymbol{p} - \boldsymbol{q})$ and therefore $V_\Gamma(\boldsymbol{p}, \boldsymbol{q}) = V_\Gamma(\boldsymbol{p} - \boldsymbol{q})$, by factorizing off all dependence on angular variables encoded in corresponding (vector) spherical harmonics both the Salpeter equation (1) [8,10] and its reduced version (4) [11] can be converted into equivalent systems of coupled equations for the radial factors of all relevant independent Salpeter components. For a fixed Lorentz structure of the kernel, the interactions experienced by the bound-state constituents enter in such a set of equations in the form of Fourier-Bessel transforms $V_L(p, q)$ ($L = 0, 1, 2, \ldots$) of some spherically symmetric static potential $V(r)$ in configuration space:

$$V_L(p, q) \equiv 8\pi \int_0^\infty \mathrm{d}r r^2 j_L(pr) j_L(qr) V(r), \qquad L = 0, 1, 2, \ldots,$$

where $j_n(z)$, for $n = 0, \pm 1, \pm 2, \ldots$, label the spherical Bessel functions of the first kind [22].

According to Sec. III, for pseudoscalar fermion-antifermion bound states each solution of the reduced Salpeter equation (4) involves only one independent Salpeter component, $\phi(p)$. Consequently, the aforementioned system of radial equations collapses to a single equation. The application of the radial reduction to the reduced Salpeter equation (4) then yields the radial eigenvalue equations, for interactions of Lorentz-scalar Dirac structure, $\Gamma \otimes \Gamma = 1 \otimes 1$,

$$2E(p)\phi(p) - \frac{1}{2}\int_0^\infty \frac{\mathrm{d}q\, q^2}{(2\pi)^2}\left[\left(1 + \frac{m^2}{E(p)E(q)}\right)V_0(p, q) - \frac{pq}{E(p)E(q)}V_1(p, q)\right]\phi(q) = M\phi(p),$$

for interactions of time-component Lorentz-vector Dirac structure, $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$,

$$2E(p)\phi(p)+\frac{1}{2}\int_0^\infty\frac{\mathrm{d}qq^2}{(2\pi)^2}\left[\left(1+\frac{m^2}{E(p)E(q)}\right)V_0(p,q)+\frac{pq}{E(p)E(q)}V_1(p,q)\right]\phi(q)=M\phi(p),$$

for interactions of Lorentz-vector Dirac structure, $\Gamma\otimes\Gamma=\gamma_\mu\otimes\gamma^\mu$,

$$2E(p)\phi(p)+\int_0^\infty\frac{\mathrm{d}qq^2}{(2\pi)^2}\left(2-\frac{m^2}{E(p)E(q)}\right)V_0(p,q)\phi(q)=M\phi(p),$$

for interactions of Lorentz-pseudoscalar Dirac structure, $\Gamma\otimes\Gamma=\gamma_5\otimes\gamma_5$,

$$2E(p)\phi(p)-\frac{1}{2}\int_0^\infty\frac{\mathrm{d}qq^2}{(2\pi)^2}\left[\left(1-\frac{m^2}{E(p)E(q)}\right)V_0(p,q)-\frac{pq}{E(p)E(q)}V_1(p,q)\right]\phi(q)=M\phi(p),$$

and, for interactions of BJK [16,17] Dirac structure, $\Gamma\otimes\Gamma=\frac{1}{2}(\gamma_\mu\otimes\gamma^\mu+\gamma_5\otimes\gamma_5-1\otimes 1)$,

$$2E(p)\phi(p)+\int_0^\infty\frac{\mathrm{d}qq^2}{(2\pi)^2}V_0(p,q)\phi(q)=M\phi(p).$$

For any mass eigenvalue M such that $M-2E(p)\neq 0$ the above radial eigenvalue equations are all of the form of a homogeneous linear (Fredholm) integral equation of the second kind.

V. HARMONIC-OSCILLATOR INTERACTION

The reduced Salpeter equations of Sec. IV may be discussed to a large extent analytically by focusing to harmonic-oscillator interactions described by the configuration-space potential

$$V(r)=ar^2,\qquad a=a^*\neq 0,\qquad r\equiv|\boldsymbol{x}|.$$

For this choice of $V(r)$ we are able to determine analytically all potential functions $V_L(p,q)$ entering in the radial eigenvalue equations, by taking advantage of the differential equation [22] satisfied by *all* spherical Bessel functions, generically called $w_n(z)$ ($n=0,\pm1,\pm2,\ldots$):

$$z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}w_n(z)+2z\frac{\mathrm{d}}{\mathrm{d}z}w_n(z)+[z^2-n(n+1)]w_n(z)=0.$$

Defining, as a radial relic of the Laplacian $\Delta\equiv\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$, the second-order differential operators

$$D_p^{(L)}\equiv\frac{\mathrm{d}^2}{\mathrm{d}p^2}+\frac{2}{p}\frac{\mathrm{d}}{\mathrm{d}p}-\frac{L(L+1)}{p^2},\qquad L=0,1,2,\ldots,$$

the spherical Bessel functions of the first kind, $j_L(pr)$, in the potentials $V_L(p,q)$ thus satisfy

$$D_p^{(L)}j_L(pr)=-r^2j_L(pr),\qquad L=0,1,2,\ldots$$

This relation allows to replace the harmonic-oscillator potential r^2 in the potential function $V_L(p,q)$ by the differential operator $D_p^{(L)}$. Hence, by means of the "orthogonality relations"

$$\int_0^\infty\mathrm{d}rr^2j_L(pr)j_L(qr)=\frac{\pi}{2p^2}\delta(p-q),\qquad L=0,1,2,\ldots,$$

involving Dirac's delta distribution the potential functions for harmonic oscillators become

$$V_L(p,q)=-\frac{(2\pi)^2a}{q^2}D_p^{(L)}\delta(p-q),\qquad L=0,1,2,\ldots \tag{5}$$

VI. ORDINARY DIFFERENTIAL EQUATIONS (OF SECOND ORDER)

For the potential functions (5) representing some harmonic-oscillator interaction, all radial integral eigenvalue equations derived in Sec. IV simplify to second-order homogeneous linear differential equations; these read, for kernels of Lorentz-scalar Dirac structure $\Gamma\otimes\Gamma=1\otimes 1$,

$$\left[2E(p)+a\left(\frac{1}{E^2(p)}+\frac{m^2(p^2-5m^2)}{2E^6(p)}-\frac{2m^2p}{E^4(p)}\frac{\mathrm{d}}{\mathrm{d}p}+\frac{m^2}{E^2(p)}D_p^{(0)}\right)\right]\phi(p)=M\phi(p)$$

or

$$\left[2E(p)+a\left(\frac{2p^2+3m^2}{2E^4(p)}+\frac{m^2}{E(p)}D_p^{(0)}\frac{1}{E(p)}\right)\right]\phi(p)=M\phi(p), \tag{6}$$

for kernels of a time-component Lorentz-vector Dirac structure $\Gamma\otimes\Gamma=\gamma^0\otimes\gamma^0$,

$$\left[2E(p)+a\left(\frac{2p^2+3m^2}{2E^4(p)}-D_p^{(0)}\right)\right]\phi(p)=M\phi(p), \tag{7}$$

for kernels of Lorentz-vector Dirac structure $\Gamma\otimes\Gamma=\gamma_\mu\otimes\gamma^\mu$,

$$\left[2E(p)-\frac{3am^4}{E^6(p)}-\frac{2am^2p}{E^4(p)}\frac{\mathrm{d}}{\mathrm{d}p}-a\left(2-\frac{m^2}{E^2(p)}\right)D_p^{(0)}\right]\phi(p)=M\phi(p)$$

or

$$\left[2E(p)+a\left(\frac{m^2}{E(p)}D_p^{(0)}\frac{1}{E(p)}-2D_p^{(0)}\right)\right]\phi(p)=M\phi(p), \tag{8}$$

for kernels of Lorentz-pseudoscalar Dirac structure $\Gamma\otimes\Gamma=\gamma_5\otimes\gamma_5$,

$$\left[2E(p) + a\frac{2p^2 + 3m^2}{2E^4(p)}\right]\phi(p) = M\phi(p), \tag{9}$$

and, for kernels of the BJK [16,17] Dirac structure $\Gamma \otimes \Gamma = \frac{1}{2}(\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$,

$$[2E(p) - aD_p^{(0)}]\phi(p) = M\phi(p). \tag{10}$$

Note that, for the Lorentz pseudoscalar $\Gamma \otimes \Gamma = \gamma_5 \otimes \gamma_5$, the reduced Salpeter equation with harmonic-oscillator potential is represented by a pure multiplication operator. This implies that the resulting spectrum is purely continuous. That is, there are no bound states at all. This fact is presumably not evident from the general representation of the reduced Salpeter equation with pseudoscalar Lorentz structure as a radial integral equation, as given in Sec. IV.

Transformation to a (zero-eigenvalue) Schrödinger equation

For $m \neq 0$, the ordinary differential equations that represent the reduced Salpeter equation (4) for a harmonic-oscillator interaction of *Lorentz-scalar* or *Lorentz-vector* Dirac structure do not (yet) resemble the familiar form of Schrödinger eigenvalue equations. However, both of them can be easily cast into the form of the ordinary differential equation of second order

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}p^2} - 2g(p)\frac{\mathrm{d}}{\mathrm{d}p} + h(p)\right]\phi(p) = 0, \tag{11}$$

involving two given functions, $g(p)$ and $h(p)$. Here, similar to the role of the mass m of the two bound-state constituents and the coupling a of the harmonic-oscillator interaction, the bound-state mass eigenvalue, M, enters—in the function $h(p)$ only—as a parameter. Then, by substitution of the amplitudes $\phi(p)$ by $\phi(p) = f(p)\psi(p)$, with the transforming function

$$f(p) = p\exp\left[-\int \mathrm{d}p\, g(p)\right],$$
$$\left[\frac{\mathrm{d}}{\mathrm{d}p} + g(p)\right]f(p) = \frac{f(p)}{p},$$

determined up to an irrelevant constant, Eq. (11) becomes the eigenvalue equation for $\psi(p)$

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}p^2} - \frac{2}{p}\frac{\mathrm{d}}{\mathrm{d}p} + U(p)\right]\psi(p) = 0,$$

corresponding to eigenvalue 0, of the Schrödinger operator $-D_p^{(0)} + U(p)$, with the potential

$$U(p) \equiv h(p) - \frac{1}{f(p)}\frac{\mathrm{d}^2 f}{\mathrm{d}p^2}(p) - 2\frac{g(p)}{f(p)}\frac{\mathrm{d}f}{\mathrm{d}p}(p)$$
$$= h(p) + \frac{\mathrm{d}g}{\mathrm{d}p}(p) + g^2(p); \tag{12}$$

here the second equality follows most easily from the differential equation satisfied by $f(p)$. Let us apply this transformation to the two cumbersome Lorentz structures under concern.

(i) In the case of a *Lorentz-scalar* Bethe-Salpeter kernel, $\Gamma \otimes \Gamma = 1 \otimes 1$, the integration of

$$g(p) = \frac{1}{p} - \frac{p}{E^2(p)}$$

gives $f(p) = E(p)$ [as may be guessed from Eq. (6)] whereas the potential $U(p)$ reads

$$U(p) = \frac{E^2(p)}{am^2}[M - 2E(p)] - \frac{1}{m^2} - \frac{1}{2E^2(p)}. \tag{13}$$

(ii) For the case of a *Lorentz-vector* interaction kernel, $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$, the integration of

$$g(p) = \frac{1}{p} + \frac{m^2 p}{E^2(p)[E^2(p) + p^2]}$$

(which can be performed most conveniently by partial fraction decomposition) yields

$$f(p) = \frac{E(p)}{\sqrt{E^2(p) + p^2}};$$

reading off $h(p)$ and inserting it into the expression (12) for the potential $U(p)$ entails

$$U(p) = \frac{E^2(p)[2E(p) - M]}{a[E^2(p) + p^2]} - \frac{2m^2p^2}{E^2(p)[E^2(p) + p^2]^2}. \tag{14}$$

With these formulations of harmonic-oscillator reduced Salpeter equations at our disposal, we may take advantage, in Sec. VIII, of some general results derived for Schrödinger operators.

VII. SELF-ADJOINTNESS OF "REDUCED SALPETER" OPERATORS

The main goal of our investigation is the thorough analysis of the qualitative features of the spectra of the reduced Salpeter equation (4) with harmonic-oscillator interaction of various Lorentz structures by close inspection of the resulting differential equations given in Sec. VI.

In this respect, the first problem that arises is the question of the *self-adjointness* of the operators in our (differential) Eqs. (6)–(10). It is straightforward to show that all these operators are self-adjoint: multiplication by a real-valued function clearly defines a self-adjoint operator and by integration by parts it is easy to convince oneself that $D_p^{(0)}$ and

$$m^2\left(\frac{1}{E^2(p)}D_p^{(0)} - \frac{2p}{E^4(p)}\frac{\mathrm{d}}{\mathrm{d}p}\right) = \frac{m^2}{E(p)}D_p^{(0)}\frac{1}{E(p)} + \frac{3m^4}{E^6(p)}$$

or

$$\frac{m^2}{E(p)}D_p^{(0)}\frac{1}{E(p)} = \frac{m^2}{E^2(p)}D_p^{(0)} - \frac{2m^2p}{E^4(p)}\frac{\mathrm{d}}{\mathrm{d}p} - \frac{3m^4}{E^6(p)}$$

also represent self-adjoint operators. This implies that the corresponding spectra are real.

The reality of all *eigenvalues* M can be also inferred from Eqs. (18) and (16) of Ref. [11] along the lines of argument presented (for the case of the full Salpeter equation, however) in Sec. II of Ref. [8] or Sec. II of Ref. [23]. From Eq. (4), all Salpeter amplitudes $\Phi(\boldsymbol{p})$ satisfy

$$M\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{Tr}[\Phi^\dagger(\boldsymbol{p})\Phi(\boldsymbol{p})] = \int\frac{\mathrm{d}^3p}{(2\pi)^3}[E_1(\boldsymbol{p}) + E_2(\boldsymbol{p})]\,\mathrm{Tr}[\Phi^\dagger(\boldsymbol{p})\Phi(\boldsymbol{p})] + \int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\sum_\Gamma V_\Gamma(\boldsymbol{p},\boldsymbol{q})\,\mathrm{Tr}[\Phi^\dagger(\boldsymbol{p})\gamma_0\Gamma\Phi(\boldsymbol{q})\Gamma\gamma_0].$$

In this relation, both the integral multiplied by M on the left-hand side, that is, the "norm"

$$\|\Phi\|^2 \equiv \int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{Tr}[\Phi^\dagger(\boldsymbol{p})\Phi(\boldsymbol{p})]$$

of each Salpeter amplitude $\Phi(\boldsymbol{p})$ emerging from the *reduced* Salpeter equation, and the first term on the right-hand side are certainly real; the second term on the right-hand side is real provided all Lorentz-scalar potential functions $V_\Gamma(\boldsymbol{p},\boldsymbol{q})$ satisfy $V_\Gamma^*(\boldsymbol{q},\boldsymbol{p}) = V_\Gamma(\boldsymbol{p},\boldsymbol{q})$ and the Dirac matrices Γ satisfy $\gamma_0\Gamma^\dagger\gamma_0 = \pm\Gamma$, which implies that the matrices $\tilde{\Gamma} \equiv \gamma_0\Gamma$ are (anti-) Hermitian, i.e., $\tilde{\Gamma}^\dagger = \pm\tilde{\Gamma}$. In contrast to the full Salpeter equation, for the reduced Salpeter equation the norm $\|\Phi\|^2$ of all nonzero solutions $\Phi(\boldsymbol{p})$ is definitely nonvanishing. Therefore, all mass eigenvalues M are guaranteed to be real for reasonable interaction kernels $K(\boldsymbol{p},\boldsymbol{q})$.

VIII. SPECTRA AND STABILITY OF THE BOUND STATES

A. General considerations

Now let us investigate in turn the spectra corresponding to the different Lorentz structures. Our task is considerably facilitated by making use of a fundamental theorem [24] about the spectra of Hamiltonians with potentials increasing without bounds: a Schrödinger operator $H \equiv -\Delta + V$, defined as a sum of quadratic forms, with locally bounded, positive, infinitely rising potential $V(x)$, that is, $V(x) \to \infty$ for $|x| \to \infty$, may be shown [24], by application of the well-known minimum-maximum principle [24–26], to have a purely discrete spectrum.

In the case of harmonic-oscillator interactions of *Lorentz-scalar* or *Lorentz-vector* Dirac structure, we did not succeed, for $m \neq 0$, to reformulate the radial eigenvalue equations (6) and (8), which fix the bound-state mass eigenvalues M, as standard Schrödinger equations. However, in Sec. VI we managed to cast such unruly eigenvalue equations into the form

$$[-D_p^{(0)} + U(p)]\psi(p) = 0,$$

with $U(p)$ given in terms of well-defined potentials $U_1(p)$, $U_2(p)$ by $U(p) = U_1(p) + MU_2(p)$.

Accordingly, let us analyze the Hamiltonian operator $H_U \equiv -\Delta + U$. For $U \equiv U_1 + MU_2$ satisfying the assumptions of the above "infinitely-rising-potential theorem," the spectrum of this operator H_U is, for any M, entirely discrete. In other words, it consists exclusively of isolated eigenvalues $\varepsilon(M)$ of finite multiplicity that depend, of course, on the parameter M. Every single zero of these functions $\varepsilon_i(M)$ $(i \in Z)$ defines a bound-state mass eigenvalue M of Eq. (6) or Eq. (8). The derivative of each such function $\varepsilon_i(M)$ with respect to M is given, in accordance with the Hellmann-Feynman theorem [27], by the expectation value over the associated eigenstate $|i\rangle$ ($\langle i|i\rangle = 1$) of the derivative of this operator H_U with respect to M:

$$\frac{\mathrm{d}\varepsilon_i}{\mathrm{d}M}(M) = \left\langle i\left|\frac{\partial H_U}{\partial M}\right|i\right\rangle = \langle i|U_2|i\rangle.$$

Then, *if*, for all eigenvalues $\varepsilon_i(M)$, this derivative is strictly definite, i.e., if for given i either

$$\frac{\mathrm{d}\varepsilon_i}{\mathrm{d}M}(M) > 0 \quad \forall\ M$$

or

$$\frac{\mathrm{d}\varepsilon_i}{\mathrm{d}M}(M) < 0 \quad \forall\ M$$

holds, the discreteness of the spectrum of the Hamiltonian H_U for appropriate potentials U translates into the discreteness of all eigenvalues M of the bound-state equations (6) or (8).

B. Lorentz-scalar kernel: $\Gamma \otimes \Gamma = 1 \otimes 1$

For massless bound-state constituents, that is, for $m = 0$, our harmonic-oscillator reduced Salpeter equation with

Lorentz-scalar interaction kernel $\Gamma \otimes \Gamma = 1 \otimes 1$, Eq. (6), simplifies to

$$\left(2p + \frac{a}{p^2}\right)\phi(p) = M\phi(p).$$

This relation involves a pure multiplication operator, which possesses a purely continuous spectrum but no eigenvalues at all. Consequently, the harmonic-oscillator reduced Salpeter equation with Lorentz-scalar kernel will not describe bound states of massless constituents.

In the case of nonvanishing masses $m \neq 0$ of the bound-state constituents, the auxiliary potential $U(p)$ of Eq. (13) satisfies, for negative harmonic-oscillator coupling $a = -|a| < 0$, all requirements of the "infinitely-rising-potential theorem." The term U_2 multiplied by M,

$$U_2(p) = \frac{E^2(p)}{am^2},$$

is, for $a < 0$, obviously negative definite. Hence, all our considerations of Subsection VIII A apply:

$$\frac{\mathrm{d}\varepsilon_i}{\mathrm{d}M} \leq \frac{1}{a} < 0 \quad \forall\ i.$$

Thus, for nonzero bound-state constituents' mass the resulting spectrum is purely discrete.

Figure 1 depicts, for a Lorentz-scalar kernel, the typical behavior of the discrete auxiliary eigenvalues $\varepsilon(M)$ for negative harmonic-oscillator couplings $a < 0$ producing bound states.

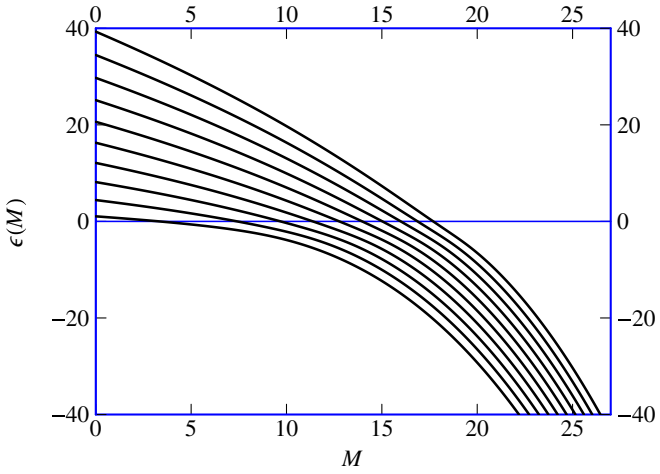

FIG. 1 (color online). First, lowest-lying eigenvalues $\varepsilon_i(M)$, $i = 0, 1, \ldots, 9$, of the auxiliary Hamiltonian $H_U \equiv -\Delta + U$ with effective potential U of Eq. (13), corresponding to the reduced Salpeter equation with harmonic-oscillator interaction of Lorentz-scalar structure, $\Gamma \otimes \Gamma = 1 \otimes 1$, for bound-state constituents' mass $m = 1$ and a "binding" coupling $a = -10$ (arbitrary units).

C. Time-component Lorentz-vector kernel: $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$

For arbitrary mass $m \geq 0$ of the bound-state constituents, our harmonic-oscillator reduced Salpeter equation with a time-component Lorentz-vector interaction kernel $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$, Eq. (7), yields a Schrödinger equation with an effective potential $V(p)$ in momentum space.

For massless particles, i.e., for $m = 0$, this potential $V(p)$ is singular at $p = 0$ (Fig. 2):

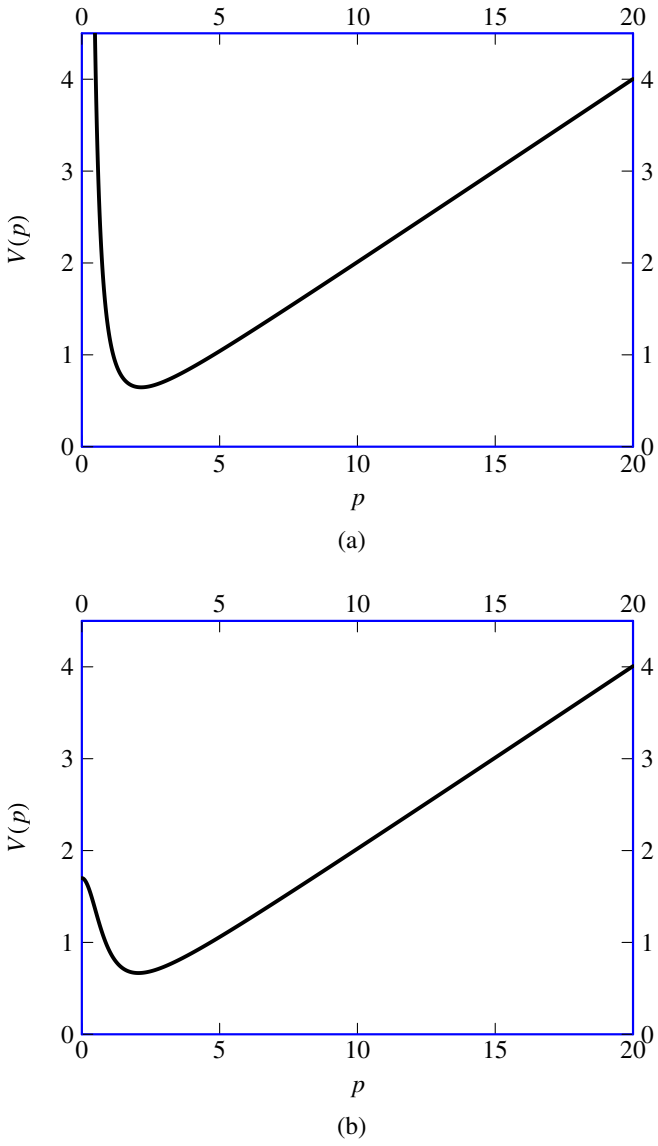

FIG. 2 (color online). Qualitative behavior of the effective potential, $V(p)$, in the differential-equation representation (7) of the reduced Salpeter equation with harmonic-oscillator interaction in the time-component Lorentz-vector kernel, $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$, for $a = 10$ and (a) vanishing mass $m = 0$ and (b) nonvanishing mass $m = 1$ of the bound-state constituents (arbitrary units).

$$V(p) = \frac{2p}{a} + \frac{1}{p^2}.$$

This potential, however, is the sum $V(p) = W(p) + 1/p^2$ of a linear potential $W(p) = 2p/a$, which is positive for a positive slope of this linear rise, that is, for all $a > 0$, and the singular but positive function $1/p^2$. Thus, the functions $V(p)$ and $W(p)$ are related by the inequality $V(p) \geq W(p)$. A suitable combination [28–32] of the minimum-maximum principle [24–26] with the resulting operator inequality $H_V \equiv -\Delta + V \geq H_W \equiv -\Delta + W$ allows to show that any discrete eigenvalue of H_V is bounded from below by a corresponding eigenvalue of H_W. Since, by the "infinitely-rising-potential theorem," the spectrum of H_W is purely discrete—see our detailed discussion in Subsection VIII D—, H_V necessarily has a purely discrete spectrum. Of course, the same result is obtained by suitable generalization of the theorem of Ref. [24].

For nonvanishing mass of the bound-state constituents, $m \neq 0$, the effective potential is fully compatible with the assumptions of the "infinitely-rising-potential theorem" (Fig. 2):

$$V(p) = \frac{2E(p)}{a} + \frac{2p^2 + 3m^2}{2E^4(p)} \quad (a > 0).$$

Here, too, our inevitable conclusion is that the emerging spectrum must be purely discrete.

D. Lorentz-vector kernel: $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$

For massless bound-state constituents, that is, for $m = 0$, our harmonic-oscillator reduced Salpeter equation with Lorentz-vector interaction kernel $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$, Eq. (8), becomes

$$2(p - aD_p^{(0)})\phi(p) = M\phi(p).$$

Of course, this is nothing but the standard nonrelativistic Schrödinger equation with linear potential, $V(p) = p/a$, which is equivalent to the Airy differential equation [22]. To see this, introduce a reduced radial wave function $\varphi(p) \equiv p\phi(p)$ and perform the change of variables

$$z = \frac{1}{a^{1/3}}\left(p - \frac{M}{2}\right)$$

to a convenient dimensionless variable, z, in order to arrive at the Airy differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z} w(z) = z w(z)$$

for $w(z) = \varphi(p)$. Thus, for $m = 0$ and a positive slope of this linear potential, i.e., for $a > 0$, the "infinitely-rising-potential theorem" guarantees the pure discreteness of this spectrum.

For nonvanishing bound-state constituents' masses, i.e., $m \neq 0$, we again have to invoke the transformation to an auxiliary Hamiltonian H_U performed in Sec. VI. The resulting effective potential $U(p)$ of Eq. (14) is, for positive harmonic-oscillator couplings $a > 0$, fully compatible with all the needs of the "infinitely-rising-potential theorem." Its "M-part" U_2,

$$U_2(p) = -\frac{E^2(p)}{a[E^2(p) + p^2]},$$

is, for all $a > 0$, negative definite: $U_2 < 0$. Following our line of argument of Subsection VIII A, one finds that, also for nonzero bound-state constituents' mass, the spectrum is purely discrete. Without surprise, the behavior of the discrete auxiliary eigenvalues $\varepsilon(M)$ for $a > 0$ is for a Lorentz-vector kernel rather similar to that of their Lorentz-scalar counterparts (cf. Fig. 3).

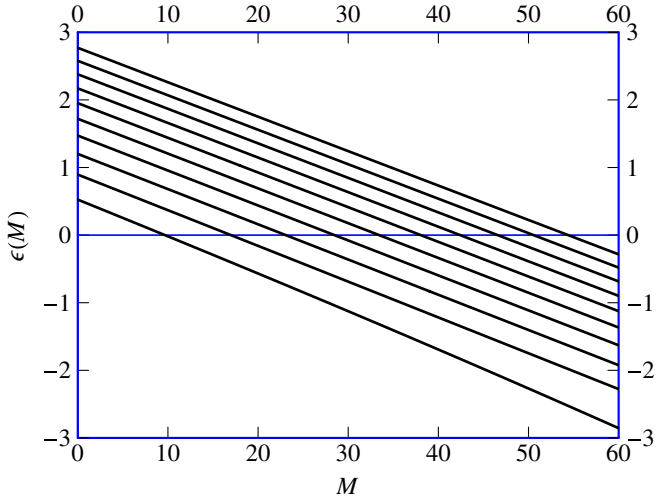

FIG. 3 (color online). First, lowest-lying eigenvalues $\varepsilon_i(M)$, $i = 0, 1, \ldots, 9$, of the auxiliary Hamiltonian $H_U \equiv -\Delta + U$ with effective potential U of Eq. (14), corresponding to the reduced Salpeter equation with a harmonic-oscillator interaction of Lorentz-vector structure $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$, for bound-state constituents' mass $m = 1$ and binding coupling $a = 10$ (arbitrary units).

E. Lorentz-pseudoscalar kernel: $\Gamma \otimes \Gamma = \gamma_5 \otimes \gamma_5$

Already in Sec. VI, our inspection of the differential equations revealed that for any kernel of Lorentz-pseudoscalar type the spectrum is purely continuous: there is no stability problem.

F. BJK Lorentz structure: $\Gamma \otimes \Gamma = \frac{1}{2}(\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$

For all $m \geq 0$, a harmonic-oscillator reduced Salpeter equation (10) for a kernel of the BJK Lorentz structure $\Gamma \otimes \Gamma = \frac{1}{2}(\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$ is just the eigenvalue equation of a Schrödinger operator with potential $V(p) = 2E(p)/a$. From the "infinitely-rising-potential theorem" we safely conclude that the spectrum of this operator is purely discrete for $a > 0$.

IX. DIGRESSION: INVESTIGATION OF FULL SALPETER EQUATION

It goes without saying that analogous spectral analyses [33,34] can be envisaged [35] for the Salpeter equation (1). There, however, we expect to face more severe problems because any solution $\Phi(\boldsymbol{p})$ of the Salpeter equation (1) involves more than one independent component. Thus a Salpeter equation with harmonic-oscillator interaction yields a *system* of more than one second-order differential equation or, equivalently, a higher-order differential equation.

Hence, let us recall some general features of the eigenvalues of the full Salpeter equation (1), regarded as an eigenvalue equation for the Salpeter amplitude $\Phi(\boldsymbol{p})$; these observations emerge either from the relationship, that is, more precisely, the equivalence, of the Salpeter equation to the well-known random phase approximation (RPA) familiar from the study of collective excitations in nuclear physics or from the inspection of the Salpeter equation (1). The normalization condition for Bethe-Salpeter amplitudes arising as a consequence of the inhomogeneous Bethe-Salpeter equation [1] motivates the introduction of a norm $\|\Phi\|$ of the Salpeter amplitude [8,10,23,36], given by (cf. Eq. (2.9) of Ref. [8] or Eq. (9) of Ref. [10])

$$\begin{aligned}\|\Phi\|^2 &\equiv \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\\ &\quad\times\mathrm{Tr}\left[\Phi^\dagger(\boldsymbol{p})\left(\frac{H_1(\boldsymbol{p})}{E_1(\boldsymbol{p})}\Phi(\boldsymbol{p})-\Phi(\boldsymbol{p})\frac{H_2(-\boldsymbol{p})}{E_2(\boldsymbol{p})}\right)\right]\\ &=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{Tr}\left[\Phi^\dagger(\boldsymbol{p})\frac{H_1(\boldsymbol{p})}{E_1(\boldsymbol{p})}\Phi(\boldsymbol{p})\right].\end{aligned}$$

The two expressions on the right-hand side of the above definition are, of course, equivalent by virtue of the constraint (2) satisfied by any solution of the Salpeter equation. Because of the Hermiticity $H_i^\dagger(\boldsymbol{p})=H_i(\boldsymbol{p})$ of the single-particle Dirac Hamiltonian $H_i(\boldsymbol{p})$, $i=1,2$, the *square* $\|\Phi\|^2$ of this norm is certainly real. It is, however, *not* necessarily positive definite. (Hence, the Salpeter norm $\|\Phi\|$ is not necessarily real, whence the quotation marks.) The spectrum of Salpeter's equation then exhibits the following characteristics [8,10,23,37–40].

(i) The Salpeter equation can be shown to be of the same algebraic structure as the RPA equation [23,37–40]. In other words, Salpeter's equation and the RPA equation prove to be equivalent [38–40]. Now, any RPA equation can always be rewritten in the form of a self-adjoint eigenvalue equation for the *square* of the energy [39,40]. Accordingly, the square of the energy is guaranteed to be real, which implies that all eigenvalues of such an RPA equation are either real or purely imaginary [37]. For the Salpeter equation (1) this means that the squares P_0^2 of the energy eigenvalues P_0 of a given bound state are real or (in the center-of-momentum or "rest" frame of the bound state, defined by $\boldsymbol{P}=\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{0}$) that the square M^2 of any bound-state mass M is real, respectively.

(ii) In the center-of-momentum frame of the bound state, any solution $\Phi(\boldsymbol{p})$ of Salpeter's equation (1) is subject to the relation (cf. Eq. (2.17) of Ref. [8] or Eq. (11) of Ref. [10])

$$\begin{aligned}M\|\Phi\|^2 &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}[E_1(\boldsymbol{p})+E_2(\boldsymbol{p})]\,\mathrm{Tr}[\Phi^\dagger(\boldsymbol{p})\Phi(\boldsymbol{p})]\\ &\quad+\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\\ &\quad\times\mathrm{Tr}[\Phi^\dagger(\boldsymbol{p})\gamma_0[K(\boldsymbol{p},\boldsymbol{q})\Phi(\boldsymbol{q})]\gamma_0]\\ &=\int\frac{\mathrm{d}^3p}{(2\pi)^3}[E_1(\boldsymbol{p})+E_2(\boldsymbol{p})]\,\mathrm{Tr}[\Phi^\dagger(\boldsymbol{p})\Phi(\boldsymbol{p})]\\ &\quad+\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\sum_\Gamma V_\Gamma(\boldsymbol{p},\boldsymbol{q})\\ &\quad\times\mathrm{Tr}[\Phi^\dagger(\boldsymbol{p})\gamma_0\Gamma\Phi(\boldsymbol{q})\Gamma\gamma_0].\end{aligned}\tag{15}$$

For all interaction kernels such that $\int\mathrm{d}^3p\int\mathrm{d}^3q\,\mathrm{Tr}[\Phi^\dagger(\boldsymbol{p})\gamma_0[K(\boldsymbol{p},\boldsymbol{q})\Phi(\boldsymbol{q})]\gamma_0]$ is real,

$$\begin{aligned}&\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}(\mathrm{Tr}[\Phi^\dagger(\boldsymbol{p})\gamma_0[K(\boldsymbol{p},\boldsymbol{q})\Phi(\boldsymbol{q})]\gamma_0])^*\\ &\quad=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\\ &\qquad\times\mathrm{Tr}[\Phi^\dagger(\boldsymbol{p})\gamma_0[K(\boldsymbol{p},\boldsymbol{q})\Phi(\boldsymbol{q})]\gamma_0],\end{aligned}$$

the right-hand side of Eq. (15) is, of course, real. In this case, remembering the reality of the square of the Salpeter norm, the mass eigenvalues M of all bound states with nonvanishing norm $\|\Phi\|$ of the associated Salpeter amplitudes Φ, that is, $\|\Phi\|^2\neq 0$, are real: $M^*=M$ [8,23]. The needed reality of the interaction term in Eq. (15) holds, for instance, for all Bethe-Salpeter kernels which are sums of terms of Dirac structure $\Gamma\otimes\Gamma$ such that the Dirac matrices $\tilde{\Gamma}\equiv\gamma_0\Gamma$ are (anti-) Hermitian, $\tilde{\Gamma}^\dagger=\pm\tilde{\Gamma}$, with the associated Lorentz-scalar potential function $V_\Gamma(\boldsymbol{p},\boldsymbol{q})$ satisfying $V_\Gamma^*(\boldsymbol{q},\boldsymbol{p})=V_\Gamma(\boldsymbol{p},\boldsymbol{q})$.[1]

(iii) The comparison of Salpeter's equation (1) with its Hermitian conjugate (sandwiched between γ_0 from the left and γ_0 from the right) reveals [8,23] that the solutions of this eigenvalue equation always occur in pairs of the form (Φ, M) and $(\Psi, -M^*)$, with the Salpeter amplitude $\Psi(\boldsymbol{p})$ related to the Hermitian conjugate of its "partner solution" $\Phi(\boldsymbol{p})$, provided

[1]Thus, (at least) for the present class of interactions any solution of Salpeter's equation (1) with nonreal and therefore purely imaginary mass eigenvalue $M^*\neq M=\pm\mathrm{i}|M|$ must have vanishing norm $\|\Phi\|^2=0$.

the interaction kernel satisfies $[K(\boldsymbol{p},\boldsymbol{q})\Phi(\boldsymbol{q})]^\dagger = \gamma_0[K(\boldsymbol{p},\boldsymbol{q})\Psi(\boldsymbol{q})]\gamma_0$; the squares of the "norms" of conjugate solutions have opposite sign: $\|\Psi\|^2 = -\|\Phi\|^2$. For the case of equal masses $m_1 = m_2$ of the bound-state constituents [8], the relation between Φ and Ψ is particularly simple, viz., $\Psi(\boldsymbol{p}) \equiv \gamma_0\Phi^\dagger(\boldsymbol{p})\gamma_0$. For the general case of unequal masses $m_1 \neq m_2$ of the bound-state constituents [23] the relation of Φ and Ψ involves, in addition, the charge-conjugation matrix C defined, e.g., by $C \equiv \mathrm{i}\gamma^2\gamma^0$. This doubling of the eigenvalues to pairs of opposite sign is well known from solutions to the RPA equation [23,37,38]. The requirement on the interaction kernel holds, for example, for every Bethe-Salpeter kernel which is a sum of terms each of which is the product of a real Lorentz-scalar interaction function $V_\Gamma(\boldsymbol{p},\boldsymbol{q}) = V_\Gamma^*(\boldsymbol{p},\boldsymbol{q})$ and a Dirac structure $\Gamma\otimes\Gamma$ such that all Dirac matrices $\tilde\Gamma \equiv \gamma_0\Gamma$ are (anti-) Hermitian, $\tilde\Gamma^\dagger = \pm\tilde\Gamma$. Furthermore, the relation between $\Phi(\boldsymbol{p})$ and $\Psi(\boldsymbol{p})$ holding within the particular class of interactions defined by the requirement $[K(\boldsymbol{p},\boldsymbol{q})\Phi(\boldsymbol{q})]^\dagger = \gamma_0[K(\boldsymbol{p},\boldsymbol{q})\Psi(\boldsymbol{q})]\gamma_0$ also entails that a Salpeter amplitude Φ solving the Salpeter equation for a nondegenerate zero mass eigenvalue $M = 0$ necessarily has vanishing Salpeter "norm": $\|\Phi\| = 0$ [23].

In particular, any momentum-space interaction function $V_\Gamma(\boldsymbol{p},\boldsymbol{q})$ which can be represented as the Fourier transform $V_\Gamma(\boldsymbol{p},\boldsymbol{q}) = V_\Gamma(\boldsymbol{p}-\boldsymbol{q}) = \int \mathrm{d}^3p\,\exp[-\mathrm{i}(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}]V_\Gamma(r)$ of some (real) central potential $V_\Gamma(r) = V_\Gamma^*(r)$ in configuration space experienced by the two bound-state constituents at distance $r \equiv |\boldsymbol{x}|$ satisfies both $V_\Gamma^*(\boldsymbol{p},\boldsymbol{q}) = V_\Gamma(\boldsymbol{p},\boldsymbol{q})$ and $V_\Gamma(\boldsymbol{q},\boldsymbol{p}) = V_\Gamma(\boldsymbol{p},\boldsymbol{q})$. Thus, for Dirac matrices Γ in the kernel $K(\boldsymbol{p},\boldsymbol{q}) = \sum_\Gamma V_\Gamma(\boldsymbol{p},\boldsymbol{q})\Gamma\otimes\Gamma$ obeying $\gamma_0\Gamma^\dagger\gamma_0 = \pm\Gamma$ both conditions on the kernel $K(\boldsymbol{p},\boldsymbol{q})$ required for the above-mentioned spectral properties to hold are simultaneously fulfilled. In this case, the spectrum of mass eigenvalues M of the Salpeter equation (1) can only involve, in the complex-M plane, pairs $(M,-M)$ of opposite sign on the real axis (excluding the origin), with the associated Salpeter amplitudes having Salpeter norms squared of opposite sign, and/or points $M = -M^*$ on the imaginary axis (including the origin), with corresponding Salpeter solutions of vanishing Salpeter "norm."

Merely for illustrative purposes let us now generalize, for a few examples, the treatment applied to the reduced Salpeter equation (4) in Sec. III through Sec. VI to the case of the (full) Salpeter equation (1). As recalled in Sec. III, Salpeter amplitudes $\Phi(\boldsymbol{p})$ for bound states with spectroscopic label $n\,^1J_J$ and, therefore, all Salpeter amplitudes $\Phi(\boldsymbol{p})$ for bound states with spin-parity-charge-conjugation assignment $J^{PC} = 0^{-+}$ consist of two independent Salpeter components $\phi_1(\boldsymbol{p})$ and $\phi_2(\boldsymbol{p})$. The corresponding Salpeter equation will therefore reduce to a system of two coupled (integral) equations for two radial wave functions $\phi_1(p)$ and $\phi_2(p)$.[2]

In order to analyze simultaneously Bethe-Salpeter interaction kernels of Lorentz-scalar and time-component Lorentz-vector Dirac structure, we introduce a simple sign factor σ by

$$\sigma = \begin{cases} +1 & \text{for } \Gamma\otimes\Gamma = \gamma^0\otimes\gamma^0 \quad \text{(time-component Lorentz-vector interactions)},\\ -1 & \text{for } \Gamma\otimes\Gamma = 1\otimes 1 \quad \text{(Lorentz-scalar interactions)}. \end{cases}$$

(Lorentz-scalar and time-component Lorentz-vector confining interaction kernels and their linear combinations attracted, for purely phenomenological reasons, particular attention in the Bethe-Salpeter descriptions [8–15,39–41] of hadrons as bound states of quarks [42,43].) With the help of the parameter σ and the interaction functions $V_L(p,q)$ $(L = 0, 1)$ of Sec. IV, the radial remnants of the Salpeter equation (1) for *pseudoscalar* bound states are, for both Lorentz-scalar $(\Gamma\otimes\Gamma = 1\otimes 1)$ and time-component Lorentz-vector $(\Gamma\otimes\Gamma = \gamma^0\otimes\gamma^0)$ kernels,

$$2E(p)\phi_2(p) + \int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}\sigma V_0(p,q)\phi_2(q) = M\phi_1(p),$$

$$2E(p)\phi_1(p) + \int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}\left[\sigma\frac{m^2}{E(p)E(q)}V_0(p,q) + \frac{pq}{E(p)E(q)}V_1(p,q)\right]\phi_1(q) = M\phi_2(p).$$

This set of equations exhibits a very peculiar structure: For $M = 0$, the equations decouple. For $M \neq 0$, from one of the equations one of the two independent Salpeter components, say $\phi_1(p)$, may be expressed in terms of the other, $\phi_2(p)$, and inserted into the other equation in order to reformulate this eigenvalue problem for the square M^2 of the bound-state mass M:

[2] As recalled in Sec. III, for pseudoscalar bound states the constraints on the solutions $\Phi(\boldsymbol{p})$ of the reduced Salpeter equation (4) imply $\phi_1(\boldsymbol{p}) = \phi_2(\boldsymbol{p}) \equiv \phi(\boldsymbol{p})$. Consequently, for a given Lorentz structure $\Gamma\otimes\Gamma$ of the interaction kernel the radial eigenvalue equation representing the reduced Salpeter equation can be derived by adding the two radial eigenvalue equations related to Salpeter's equation (1) after letting $\phi_1(p) = \phi_2(p)$.

$$4E^2(p)\phi_2(p) + 2E(p)\int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}\sigma V_0(p,q)\phi_2(q) + \frac{2}{E(p)}\int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}[\sigma m^2 V_0(p,q) + pqV_1(p,q)]\phi_2(q)$$
$$+ \int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}\left[\sigma\frac{m^2}{E(p)E(q)}V_0(p,q) + \frac{pq}{E(p)E(q)}V_1(p,q)\right]\int_0^\infty \frac{\mathrm{d}kk^2}{(2\pi)^2}\sigma V_0(q,k)\phi_2(k) = M^2\phi_2(p).$$

Following Sec. V, specifying the interactions to harmonic-oscillator form allows to trade any integration over the potential functions $V_L(p,q)$, $L = 0, 1$, for the differential operator $D_p^{(0)}$:

$$\left[4E^2(p) - \frac{2a}{E(p)}(\sigma(p^2+2m^2)D_p^{(0)} + pD_p^{(0)}p - 2) + \frac{a^2}{E(p)}(m^2D_p^{(0)} + \sigma pD_p^{(0)}p - 2\sigma)\frac{1}{E(p)}D_p^{(0)}\right]\phi_2(p) = M^2\phi_2(p).$$

This is still a homogeneous linear differential equation but of fourth order. Thus its analysis requires techniques or tools beyond those familiar from the study of Schrödinger operators.

The Salpeter equation for a Bethe-Salpeter kernel of the BJK [16,17] Lorentz structure $\Gamma\otimes\Gamma = \frac{1}{2}(\gamma_\mu\otimes\gamma^\mu + \gamma_5\otimes\gamma_5 - 1\otimes 1)$ constitutes an exceptional case; there all interactions enter only in one of the two relations which form the set of coupled radial integral equations describing *pseudoscalar bound states*, the other relation being of "merely" algebraic nature:

$$2E(p)\phi_2(p) + 2\int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}V_0(p,q)\phi_2(q) = M\phi_1(p),$$
$$2E(p)\phi_1(p) = M\phi_2(p).$$

Merging these two relations generates an eigenvalue problem for M^2, posed equivalently by

$$4E^2(p)\phi_1(p) + 4\int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}V_0(p,q)E(q)\phi_1(q) = M^2\phi_1(p),$$
$$4E^2(p)\phi_2(p) + 4E(p)\int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}V_0(p,q)\phi_2(q) = M^2\phi_2(p).$$

For the *harmonic-oscillator interaction* $V(r) = ar^2$ these eigenvalue equations become, by virtue of Eq. (5), i.e., $q^2V_0(p,q) = -(2\pi)^2 aD_p^{(0)}\delta(p-q)$, the *ordinary differential equations*

$$4[E^2(p) - aD_p^{(0)}E(p)]\phi_1(p) = M^2\phi_1(p),$$
$$4[E^2(p) - aE(p)D_p^{(0)}]\phi_2(p) = M^2\phi_2(p).$$

Applying the substitution $E(p)\phi_1(p) \propto \phi_2(p)$ proves the equivalence of these formulations. In contrast to the case of an arbitrary interaction kernel, for the BJK Lorentz structure our harmonic-oscillator Salpeter problem reduces to a single second-order differential equation. The differential operator on the left-hand side of this eigenvalue problem is *not* self-adjoint. Nevertheless, according to the general properties of eigenvalues of the Salpeter equation (1) summarized at the beginning of this section, the spectrum of eigenvalue squares M^2 is real. Recalling, for $a > 0$, our reasoning of Subsection VIII A for $H_U \equiv -\Delta + U$ with auxiliary potential

$$U(p) = \frac{E(p)}{a} - \frac{M^2}{4aE(p)},$$

it is trivial to demonstrate, by similar arguments, that the spectrum of squared eigenvalues M^2 and, as a consequence thereof, the spectrum of *mass eigenvalues* M are purely discrete.

X. SUMMARY, CONCLUSIONS, AND OUTLOOK

The present investigation has been devoted to an exploration of the conditions under which the reduced Salpeter equation with *confining* interactions has *stable* bound-state solutions. For harmonic-oscillator interactions, the reduced Salpeter equation becomes in momentum space either an algebraic relation or a second-order ordinary differential equation involving the Laplacian Δ acting on states of angular momentum $\ell = 0$ (i.e., our differential operator $D_p^{(0)}$ introduced in Sec. V). For real harmonic-oscillator couplings, all corresponding spectra are real. For pseudoscalar states, where instabilities are expected to appear first, we showed that, depending on the Lorentz nature of the kernel, the resulting spectrum is either purely continuous or entirely discrete, consisting of isolated mass eigenvalues of finite multiplicity.

As a byproduct, the same analysis proves the *boundedness from below* of all the spectra for the appropriate choice of the respective sign of the harmonic-oscillator coupling constant a.

(i) For interaction kernels of the time-component Lorentz-vector structure ($\Gamma\otimes\Gamma = \gamma^0\otimes\gamma^0$), Lorentz-pseudoscalar structure ($\Gamma\otimes\Gamma = \gamma_5\otimes\gamma_5$), the (eventually simple) BJK [16,17] structure, $\Gamma\otimes\Gamma = \frac{1}{2}(\gamma_\mu\otimes\gamma^\mu + \gamma_5\otimes\gamma_5 - 1\otimes 1)$, and, if $m = 0$, of Lorentz-scalar structure ($\Gamma\otimes\Gamma = 1\otimes 1$) or Lorentz-vector structure ($\Gamma\otimes\Gamma = \gamma_\mu\otimes\gamma^\mu$), each harmonic-oscillator ordinary differential equation of Sec. VI is, for arbitrary $a > 0$, the eigenvalue equation of some *positive* operator. The entire spectrum of any such operator must be positive.

(ii) For an interaction kernel of Lorentz-pseudoscalar structure ($\Gamma \otimes \Gamma = \gamma_5 \otimes \gamma_5$), and any $m \neq 0$, Eq. (9) is, for finite $a < 0$, the eigenvalue equation of an operator which is not positive but *bounded from below*, by $2m + 3a/(2m^2)$, as is its (continuous) spectrum.

(iii) For $m \neq 0$, the ordinary differential equations, in Sec. VI, related to interaction kernels of Lorentz-scalar structure, $\Gamma \otimes \Gamma = 1 \otimes 1$, or Lorentz-vector structure, $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$, are not of the (standard) form of eigenvalue equations of some Schrödinger operators. In order to decide on the nature of their spectra we have to rely on the transformation of Sec. VI to Schrödinger-like auxiliary operators. For $a < 0$ in the Lorentz-scalar case and $a > 0$ in the Lorentz-vector case, all eigenvalues $\varepsilon_i(M)$, $i = 0, 1, 2, \ldots$, of the latter operators can be shown to be entirely discrete for all M, and strictly decreasing functions of M. Because of their strict decrease, with increasing bound-state mass M, the zeros of the lowest trajectories $\varepsilon_0(M)$ define not necessarily positive lower bounds on the spectra of mass eigenvalues M, proving both spectra to be *bounded from below*.

Altogether this provides a rigorous proof of the stability of the considered bound states: their energies form (for couplings of suitable sign) *real discrete* spectra *bounded from below*.

Our findings point in the same direction as the observations made in a purely numerical analysis [44] aiming at the description of quark-antiquark bound states with the help of the *full* Salpeter equation (1). In this investigation the (instantaneous) interaction between the bound-state constituents is modeled by the sum of a Coulomb-type short-range interaction (arising from one-gluon exchange between quark and antiquark) and a rather sophisticated confining interaction, which interpolates between a harmonic-oscillator-type behavior for small interquark distances r, or small masses of the bound-state constituents, and a linear rise for large interquark distances r, or large masses of the bound-state constituents. From a point of view of principle it is a pity that this study has been performed merely at a single point in free-parameter space, determined (for an equal-weight mixture of time-component Lorentz-vector and Lorentz-scalar Dirac structures of this particular confining interaction) from a fit of the mass spectrum of experimentally observed mesons. The authors of Ref. [44] arrive at the following conclusions (for bound states composed of equal-mass constituents):

(i) For bound states of heavy-mass constituents, the solutions prove to be stable for both time-component Lorentz-vector and Lorentz-scalar Dirac structures of the confining interaction and, consequently, for any linear combination of these Lorentz structures.

(ii) For bound states of light-mass constituents, the solutions are still all stable for a pure time-component Lorentz-vector Dirac structure of the confining interaction but turn out to be mostly unstable for a pure Lorentz-scalar spin structure of the confinement.

However, in Ref. [44] it was also (numerically) shown that neglecting the "negative-energy" components of the Salpeter amplitude $\Phi(\boldsymbol{p})$ by simply imposing onto $\Phi(\boldsymbol{p})$ the requirement $\Lambda_1^-(\boldsymbol{p}_1)\Phi(\boldsymbol{p})\Lambda_2^+(\boldsymbol{p}_2) = 0$ (almost totally) removes the instability of the bound-state energy levels otherwise showing up in the case of a Lorentz-scalar confining interaction. According to our discussion of the component contents of all solutions of both the full and the reduced Salpeter equations with respect to the energy projectors $\Lambda_i^\pm(\boldsymbol{p})$ in Sec. II, imposition of such a constraint on $\Phi(\boldsymbol{p})$ is tantamount to the consideration of the *reduced* Salpeter equation (4).

Our approach can be clearly generalized [35] to analyze not only Salpeter's equation (1) but, in a similar way, *three-dimensional reductions* of the Bethe-Salpeter equation different from the Salpeter equation; some of these reductions are reviewed, for instance, in Ref. [45].

The intention of the present study was to perform, as a purely theoretical investigation, a *rigorous* analysis of the spectral properties of the Salpeter equation (4), irrespective of its four-dimensional origin within the Bethe-Salpeter formalism. Our pragmatic point of view seems to be justified by the fact that in the past exactly this bound-state equation has been applied in numerous treatments of, for instance, quark-antiquark bound states in quantum chromodynamics (for details, consult the reviews in Refs. [42,43]). As already mentioned in the introduction, our interest in the problem of the particular type of instabilities discussed here has been aroused by their occasional observation [10,44], and the attempts to arrive at full *analytic* understanding of them [39–41]. Nevertheless, we feel obliged to add a few brief comments on the practical relevance of this framework and the implications of our findings.

(1) From the very beginning, our considerations have been confined to three-dimensional reductions of the Bethe-Salpeter formalism, obtained by assuming all interactions to be *instantaneous* in the center-of-momentum frame of the two-particle systems under study. The relevance of all statements about the presence or absence of instabilities in any three-dimensional reduction (as given by Salpeter's equation) for the situation in the four-dimensional Bethe-Salpeter formalism remains unclear; its discussion would necessitate a spectral analysis of its own, which will become much more involved than our analysis reported above, but is well beyond the scope of the present investigation.

(2) As recalled in Sec. II, in addition to some instantaneous approximation any derivation of, in particular, the Salpeter equation relies on the assumption of free propagation of all bound-state constituents, with a constant (i.e., momentum-independent) effective or constituent mass which should adequately parametrize some significant part of the dynamical self-energy effects. Needless to say, the free-propagator assumption cannot be compatible with a confining interaction and is therefore conceptually problematic: In quantum field theory, the Dyson-Schwinger equations relate the propagators, that is, the two-point Green functions, to the n-point Green functions which represent the interactions in the Bethe-Salpeter equation. Therefore, propagators and interactions cannot be chosen independently from each other. Thus in non-Abelian gauge theories the simultaneous assumption of free propagation of the bound particles and confining interactions (as induced by quantum chromodynamics) are intrinsically inconsistent. Moreover, phenomena such as the dynamical breakdown of chiral symmetry can only be taken into account by retaining the exact propagators, obtained as the solutions of the Dyson-Schwinger equation for the corresponding two-point Green function of the bound-state constituents. Their proper incorporation is crucial for the interpretation of the lowest-lying pseudoscalar quark-antiquark bound states: as Goldstone bosons. To make a long story short, it is, of course, very desirable to have also in one's favorite *instantaneous* bound-state equation the exact fermion propagators at one's disposal.

One of the attempts to retain in some three-dimensional reduction (as far as possible) the original wave-function renormalization and mass functions parametrizing the full propagators of both fermionic bound-state constituents resulted in the instantaneous bound-state equation proposed in Ref. [46]. The first tentative exploration of some of the implications of this generalization [47] of Salpeter's equation for quark-antiquark bound states may be found in Ref. [48]. The latter study utilizes exact propagators of light quarks, as extracted within a "renormalization-group-improved rainbow-ladder truncation" (which scheme has the undeniable advantage to preserve the axial-vector Ward-Takahashi identity) applied to both the quark Dyson-Schwinger equation and the meson Bethe-Salpeter equation [49]. Given the formalism introduced in Ref. [46], our (obvious) next step is a similar study [34,50] for our *full-propagator* version of the Salpeter equation (4). Our preliminary results indicate that for reasonable behavior of both bound-state constituents' propagator functions, i.e., nontrivial wave-function renormalization and dynamical mass, stability (in our sense) can be achieved [34,50]. The significance of these findings for the four-dimensional Bethe-Salpeter formalism, which includes all the constituents' self-energy effects, may be judged by future work.

(3) Of course, one might argue that a sufficiently precise purely numerical solution of the bound-state equation in use may suffice to settle the stability issues once and forever. However, as demonstrated, e.g., by the not really conclusive findings of Refs. [10,44] a numerical analysis may give but a hint of potential problems with unstable solutions. Thus, it is our conviction that a *genuine* understanding of the origin of these troubles and a compelling solution of this problem may be gained only by some analytic proof.

Hence, as our conclusion let us stress once again that, for the solutions of the *instantaneous* Bethe-Salpeter formalism, at least, the kind of analysis proposed above provides a rigorous answer to the question of stability, defined by our requirement that each bound state found belongs to a *real* and *discrete* spectrum that is *bounded from below*. Note that the requested reality of the bound-state masses precludes, for example, any solutions of tachyonic nature.

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[1] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[2] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).
[3] A. B. Henriques, B. H. Kellett, and R. G. Moorhouse, Phys. Lett. B **64**, 85 (1976).
[4] S. Jacobs, M. G. Olsson, and C. J. Suchyta, III, Phys. Rev. D **35**, 2448 (1987).
[5] A. Gara, B. Durand, L. Durand, and L. J. Nickisch, Phys. Rev. D **40**, 843 (1989).
[6] A. Gara, B. Durand, and L. Durand, Phys. Rev. D **42**, 1651 (1990); **43**, 2447(E) (1991).
[7] W. Lucha, H. Rupprecht, and F. F. Schöberl, Phys. Rev. D **45**, 385 (1992).
[8] J.-F. Lagaë, Phys. Rev. D **45**, 305 (1992).

[9] J.-F. Lagaë, Phys. Rev. D **45**, 317 (1992).
[10] M. G. Olsson, S. Veseli, and K. Williams, Phys. Rev. D **52**, 5141 (1995).
[11] M. G. Olsson, S. Veseli, and K. Williams, Phys. Rev. D **53**, 504 (1996).
[12] W. Lucha, K. Maung Maung, and F. F. Schöberl, Phys. Rev. D **63**, 056002 (2001).
[13] W. Lucha, K. Maung Maung, and F. F. Schöberl, in *Proceedings of the International Conference on Quark Confinement and the Hadron Spectrum IV*, edited by W. Lucha and K. Maung Maung (World Scientific, Singapore, 2002), p. 340.
[14] W. Lucha, K. Maung Maung, and F. F. Schöberl, Phys. Rev. D **64**, 036007 (2001).
[15] W. Lucha and F. F. Schöberl, Int. J. Mod. Phys. A **17**, 2233 (2002).
[16] M. Böhm, H. Joos, and M. Krammer, Nucl. Phys. **B51**, 397 (1973).
[17] F. Gross and J. Milana, Phys. Rev. D **43**, 2401 (1991).
[18] M. Koll, R. Ricken, D. Merten, B. C. Metsch, and H. R. Petry, Eur. Phys. J. A **9**, 73 (2000); M. Koll, Ph.D. thesis, University of Bonn, 2001.
[19] R. Ricken, M. Koll, D. Merten, B. Ch. Metsch, and H. R. Petry, Eur. Phys. J. A **9**, 221 (2000); R. Ricken, Ph.D. thesis, University of Bonn, 2001.
[20] D. Merten, R. Ricken, M. Koll, B. Metsch, and H. Petry, Eur. Phys. J. A **13**, 477 (2002); D. Merten, Ph.D. thesis, University of Bonn, 2002.
[21] R. Ricken, M. Koll, D. Merten, and B. C. Metsch, Eur. Phys. J. A **18**, 667 (2003).
[22] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1964).
[23] J. Resag, C. R. Münz, B. C. Metsch, and H. R. Petry, Nucl. Phys. **A578**, 397 (1994); J. Resag, Ph.D. thesis, University of Bonn, 1994.
[24] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators* (Academic, New York, 1978).
[25] A. Weinstein and W. Stenger, *Methods of Intermediate Problems for Eigenvalues—Theory and Ramifications* (Academic, New York/London, 1972).
[26] W. Thirring, *A Course in Mathematical Physics 3: Quantum Mechanics of Atoms and Molecules* (Springer, New York, 1990).
[27] H. Hellmann, Acta Physicochim URSS **1**, 913 (1934); **4**, 225 (1936); Einführung in die Quantenchemie (F. Deuticke, Leipzig, 1937), p. 285; R. P. Feynman, Phys. Rev. **56**, 340 (1939).
[28] W. Lucha and F. F. Schöberl, Int. J. Mod. Phys. A **14**, 2309 (1999).
[29] W. Lucha and F. F. Schöberl, Fiz. B **8**, 193 (1999).
[30] W. Lucha and F. F. Schöberl, Phys. Rev. A **54**, 3790 (1996).
[31] W. Lucha and F. F. Schöberl, J. Math. Phys. (N.Y.) **41**, 1778 (2000).
[32] W. Lucha and F. F. Schöberl, Recent Res. Devel. Physics **5**, 1423 (2004).
[33] W. Lucha and F. F. Schöberl, in *QCD @ Work 2007, International Workshop on Quantum Chromodynamics: Theory and Experiment*, edited by P. Colangelo, D. Creanza, F. De Fazio, R. A. Fini, E. Nappi, and G. Nardulli, AIP Conf. Proc. No. 964 (AIP, New York, 2007), p. 318.
[34] W. Lucha and F. F. Schöberl, Report No. HEPHY-PUB 856/07, 2007 [arXiv:0711.1736] (to be published).
[35] W. Lucha and F. F. Schöberl (unpublished).
[36] A. Le Yaouanc, L. Oliver, S. Ono, O. Pène, and J.-C. Raynal, Phys. Rev. D **31**, 137 (1985).
[37] C. Long, Phys. Rev. D **30**, 1970 (1984).
[38] J. Resag and D. Schütte, Bonn Report No. TK-93-19, 1993.
[39] J. Parramore and J. Piekarewicz, Nucl. Phys. **A585**, 705 (1995).
[40] J. Parramore, H.-C. Jean, and J. Piekarewicz, Phys. Rev. C **53**, 2449 (1996).
[41] M. Uzzo and F. Gross, Phys. Rev. C **59**, 1009 (1999).
[42] W. Lucha, F. F. Schöberl, and D. Gromes, Phys. Rep. **200**, 127 (1991).
[43] W. Lucha and F. F. Schöberl, Int. J. Mod. Phys. A **7**, 6431 (1992).
[44] A. Archvadze, M. Chachkhunashvili, T. Kopaleishvili, and A. Rusetsky, Nucl. Phys. **A581**, 460 (1995).
[45] T. Kopaleishvili, Phys. Part. Nucl. **32**, 560 (2001).
[46] W. Lucha and F. F. Schöberl, J. Phys. G **31**, 1133 (2005).
[47] W. Lucha and F. F. Schöberl, in *Proceedings of 7th Conference on Quark Confinement and the Hadron Spectrum—QCHS7*, edited by J. E. F. T. Ribeiro, AIP Conf. Proc. (AIP, New York, 2007), Vol. 892, p. 524.
[48] Z.-F. Li, W. Lucha, and F. F. Schöberl, Mod. Phys. Lett. A **21**, 1657 (2006).
[49] P. Maris and C. D. Roberts, Phys. Rev. C **56**, 3369 (1997).
[50] Z.-F. Li, W. Lucha, and F. F. Schöberl, Report No. HEPHY-PUB 855/07 (unpublished).