PHYSICAL REVIEW D **76,** 125021 (2007)

Fermionic vacuum polarization in a higher-dimensional global monopole spacetime

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(Received 21 August 2007; published 19 December 2007)

In this paper we analyze the vacuum polarization effects associated with a massless fermionic field in a higher-dimensional global monopole spacetime in the "braneworld" scenario. In this context we admit that our Universe, the bulk, is represented by a flat $(n-1)$-dimensional brane having a global monopole in an extra transverse three-dimensional submanifold. We explicitly calculate the renormalized vacuum average of the energy-momentum tensor, $\langle T_A^B(x)\rangle_{\rm Ren}$, admitting the global monopole as being a pointlike object. We observe that this quantity depends crucially on the value of n, and provide explicit expressions to it for specific values attributed to n.

DOI: 10.1103/PhysRevD.76.125021 PACS numbers: 11.10.Kk, 04.62.+v, 98.80.Cq

I. INTRODUCTION

Recently the braneworld model has attracted renewed interest. In this scenario our world is represented by a four-dimensional submanifold, a three-brane, embedded in a higher-dimensional spacetime [1]. Braneworlds naturally appear in the string/*M* theory context and provide a novel setting for discussing phenomenological and cosmological issues related to extra dimensions. The models introduced by Randall and Sundrum are particularly attractive [2,3]. The corresponding spacetime contains two (RSI), respectively, one (RSII), Ricci-flat brane(s) embedded on a five-dimensional anti-de Sitter (AdS) bulk. It is assumed that all matter fields are confined on the branes and only the gravity propagates in the five-dimensional bulk. The idea that matter is confined to a lower-dimensional manifold is not a new one. The localization of fermions on a domain wall has been discussed in [4].

The hierarchy problem between the Planck scale and the electroweak one is solved in the RSI model, if the distance between the two branes is about 37 times the AdS radius. The braneworld model also provides some alternative discussions about one of the most important problems in modern physics: the cosmological constant problem (see, for instance, Ref. [5]). In this way, the Casimir energy associated with quantum fields which propagates in the bulk obeying specific boundary conditions on the branes may contribute to both the brane and bulk cosmological constant. The Casimir energy associated with the scalar field on the five-dimensional Randall-Sundrum model is calculated in [6]. Surface Casimir densities and induced cosmological constants on the branes are calculated in [7] for a massive scalar quantum field obeying Robin boundary conditions on two parallel branes in a general $(D+1)$-dimensional anti-de Sitter bulk.

Although topological defects have first been studied in a four-dimensional spacetime [8], they have been considered in spacetimes of higher dimensions in the context of the braneworld. In this scenario the defects live in a d-dimensional submanifold, with their cores on the 3-brane. In this way, domain wall and cosmic string cases have been analyzed in [4] and [9,10], respectively, considering $d=1$ and 2. Local and global monopoles have also been analyzed in [11,12] and [13–17], respectively, considering $d=3$. Specifically, in [17] it was shown that, if η_0, the energy scale where the gauge symmetry of the global system is spontaneously broken, is smaller than the Planck mass, then the seven-dimensional Einstein equations admit a solution which for points outside the global monopole's core can be expressed by

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \frac{dr^2}{\alpha^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) = g_{AB}d^A dx^B, \quad (1)$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$ is the Minkowski metric, $\alpha^2 = 1 - \kappa^2\eta_0^2$, a parameter smaller than unity, and κ is related to the seven-dimensional Planck mass. In order to be more precise, in [13] the authors have obtained the solution to the Einstein equations considering a general n-dimensional Minkowski brane worldsheet and a $d \geq 3$ global monopole in the transverse extra dimensions. In this general case the metric is a generalization of (1) with μ, $\nu = 0, 1 \ldots, n-1$ and $\alpha^2 = 1 - \frac{\kappa^2\eta_0^2}{d-2}$. The solid angle associated with the 3-geometry above depends on the parameter α and reads $\Omega = 4\pi^2\alpha^2$, so it is smaller than the usual one. Consequently, in this submanifold there is a solid angle deficit, $\Delta\Omega = 4\pi^2\kappa^2\eta_0^2$.

Composite topological defects have also first been studied in a four-dimensional spacetime. Specifically, a composite monopole, i.e., a system composed of a local and a global monopole, was analyzed in [18–20] (for composite strings see [21]). More recently, the composite monopole has been analyzed in the braneworld scenario in [22].

In a previous publication [23], we have analyzed the vacuum polarization effects associated with a massless

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scalar quantum field in an $(n+3)$-dimensional bulk spacetime, which has the structure of an n-dimensional Minkowski brane with a global monopole in the transverse three-dimensional submanifold.[1] Specifically, we calculated the renormalized vacuum expectation value (VEV) of the square of the field, $\langle\Phi^2(x)\rangle_{\rm Ren}$, and have showed that this quantity depends crucially on the values attributed to n. We also analyzed the structure of the renormalized vacuum expectation value of the energy-momentum tensor, $\langle T_{AB}(x)\rangle_{\rm Ren}$. In order to develop these investigations we calculated the respective Euclidean scalar Green function, $G_E^{(n)}(x,x')$.

Continuing along the same line of investigation, in this paper we shall analyze the polarization effect of a fermionic vacuum induced by a pointlike three-dimensional global monopole embedded in a higher-dimensional bulk in a braneworld scenario. Specifically, we shall consider that the bulk has its geometric structure given by the line element (1). Our main objective is to calculate and analyze the renormalized vacuum expectation value of the energy-momentum tensor, $\langle T_{AB}(x)\rangle_{\rm Ren}$. Because in this model the global monopole has its core on the brane, the region of physical interest is near $r\approx 0$. Differently from the scalar case, here we obtain a simpler expression for the fermionic Green function and for the energy-momentum tensor, which is obtained in a closed form for an arbitrary parameter angle deficit.

This paper is organized as follows: In Sec. II we calculate the Euclidean fermionic propagator associated with a massless field in the background of a pointlike global monopole transverse to a flat n-dimensional brane in a braneworld scenario, considering $n = 1$, 2, and 3. In order to do this, we write the general Dirac differential operator and the equation obeyed by the propagator. In Sec. III we analyze this function in the coincidence limit and extract all divergences from it in manifest form. In this way we provide explicit expressions for the components of the renormalized energy-momentum tensor, and analyze their behavior in some limiting cases. In Sec. IV, we summarize our most important results. The Appendix contains some technical formulas. In this paper we use the signature $+2$ and the definitions $R^\alpha_{\beta\gamma\delta}=\partial_\gamma\Gamma^\alpha_{\beta\delta}-\ldots$, $R_{\alpha\beta}=R^\gamma_{\alpha\gamma\beta}$. We also use units $\hbar=c=1$.

II. SPINOR GREEN FUNCTION

Before beginning the calculation of the spinor Green function in the six-dimensional global monopole spacetime, we shall first briefly review some important properties of the Dirac equation in a flat space.

[1]Although the physically interesting case corresponds to $n=4$, we developed our formalism considering n as an arbitrary number.

In a flat six-dimensional space, the Dirac matrices, $\Gamma^{(M)}$, are 8×8 matrices, which can be constructed from the four-dimensional 4×4 ones [24] as shown below [25]:

$$\Gamma^{(\mu)}=\begin{pmatrix}0 & \gamma^\mu\\ \gamma^\mu & 0\end{pmatrix},\qquad \Gamma^{(4)}=\begin{pmatrix}0 & i\gamma_5\\ i\gamma_5 & 0\end{pmatrix},\qquad \Gamma^{(5)}=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}, \tag{2}$$

where $\gamma_5=i\gamma^o\gamma^1\gamma^2\gamma^3$ and I is the 4×4 identity matrix. It can be easily verified that these matrices obey the Clifford algebra: $\{\Gamma^{(M)},\Gamma^{(N)}\}=-2\eta^{(M)(N)}$, for M, $N=0,1,\ldots,5$.

In these representations, the $\Gamma^{(7)}$ matrix is written as

$$\Gamma^{(7)}=\Gamma^{(0)}\Gamma^{(1)}\ldots\Gamma^{(5)}=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}. \tag{3}$$

This matrix has two chiral eigenstates defined as Ψ_+ and Ψ_-. Consequently, any six-dimensional fermionic wave function, Ψ, can be decomposed in terms of its chiral components as[2]

$$\Psi=\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}. \tag{4}$$

Solutions for the Dirac equation,

$$i\Gamma^{(M)}\partial_{(M)}\Psi=M\Psi, \tag{5}$$

with defined chirality can only be possible for $M=0$. In this way, for positive chirality, Eq. (5) reduces to

$$\sigma^{(M)}\partial_{(M)}\Psi_+=0, \tag{6}$$

$\sigma^{(M)}=(\gamma^\mu, i\gamma_5, -I)$ being a set of 4×4 matrices; and for negative chirality it reduces to

$$\tilde\sigma^{(M)}\partial_{(M)}\Psi_-=0, \tag{7}$$

now with $\tilde\sigma^{(M)}=(\gamma^\mu, i\gamma_5, I)$.

In order to write the Dirac equation in the six-dimensional global monopole spacetime, we shall choose the following coordinate system and basis tetrad:

$$x^A=(t,r,\theta,\phi,x,y)=(x^\mu,y^a), \tag{8}$$

where $\mu=0$, 1, 2, 3 and $a=1$, 2, and

[2]It is worth mentioning that the six-dimensional chirality does not correspond to four-dimensional chirality. Six-dimensional chiral wave functions correspond to four-dimensional wave functions, which still contain two four-dimensional chiral components eigenstates of γ^5.

$$e^A_{(M)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha\sin\theta\cos\phi & \cos\theta\cos\phi/r & -\sin\phi/r\sin\theta & 0 & 0 \\ 0 & \alpha\sin\theta\sin\phi & \cos\theta\sin\phi/r & \cos\phi/r\sin\theta & 0 & 0 \\ 0 & \alpha\cos\theta & -\sin\theta/r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

The Dirac equation for a massive field in the above coordinate system reads

$$i\not\nabla\Psi - M\Psi = 0, \tag{10}$$

with the covariant derivative operator given by

$$\not\nabla = e^A_{(M)}\Gamma^{(M)}(\partial_A + \Pi_A), \tag{11}$$

where Π_A is the spin connection, given in terms of the flat spacetime Dirac matrices by

$$\Pi_A = -\tfrac{1}{4}\Gamma^{(M)}\Gamma^{(N)}e^C_{(M)}e_{(N)C;A}. \tag{12}$$

For the above basis tetrad, the only nonzero spin connections are

$$\Pi_\theta = \frac{i}{2}(1-\alpha)\vec{\Sigma}_{(8)}\cdot\hat{\phi}, \qquad \Pi_\phi = -\frac{i}{2}(1-\alpha)\sin\theta\vec{\Sigma}_{(8)}\cdot\hat{\theta}, \tag{13}$$

where

$$\vec{\Sigma}_{(8)} = \begin{pmatrix} \vec{\Sigma} & 0 \\ 0 & \vec{\Sigma} \end{pmatrix}, \qquad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \tag{14}$$

$\hat{\theta}$ and $\hat{\phi}$ being the standard unit vectors along the angular directions and σ^k the Pauli matrices.

The fermionic propagator obeys the following differential equation:

$$(i\not\nabla - M)S_F(x,x') = \frac{1}{\sqrt{-g}}\delta^{(6)}(x-x')I_{(8)}, \tag{15}$$

where $g = \det(g_{AB})$ and $I_{(8)}$ is the 8×8 identity matrix. This propagator is a bispinor, i.e., it transforms as Ψ at point x and as $\bar{\Psi}$ at x'.

If a bispinor $D_F(x',x)$ satisfies the differential equation below,

$$\left(\Box - M^2 - \frac{1}{4}R\right)D_F(x,x') = -\frac{1}{\sqrt{-g}}\delta^{(6)}(x-x')I_{(8)}, \tag{16}$$

with the generalized d'Alembertian operator given by

$$\Box = g^{MN}\nabla_M\nabla_M = g^{MN}(\partial_M\nabla_N + \Pi_M\nabla_N - \{^{S}_{MN}\}\nabla_S), \tag{17}$$

the spinor Feynman propagator can be written as

$$S_F(x',x) = (i\not\nabla + M)D_F(x',x). \tag{18}$$

Now, after this brief review about the calculation of the spinor Feynman propagator, let us consider the six-dimensional global monopole spacetime (1), where the scalar curvature $R = \frac{2(1-\alpha^2)}{r^2}$. Choosing the basis tetrad (9), we obtain, after some intermediate steps,

$$\begin{aligned}\mathcal{K} &= \Box - \frac{1}{4}R \\ &= -\partial_t^2 + \alpha^2\left(\partial_r^2 + \frac{2}{r}\partial_r\right) - \frac{\vec{L}^2}{r^2} + \partial_x^2 + \partial_y^2 \\ &\quad - \frac{(1-\alpha)}{r^2}(1 + \vec{\Sigma}_{(8)}\cdot\vec{L}),\end{aligned} \tag{19}$$

where $\vec{L}$ is the ordinary angular momentum operator.

The system that we shall consider consists of a massless positive chiral field. In this way, Eq. (10) can be written in terms of a 4×4 matrix differential equation,

$$\not D\Psi_+ = 0, \tag{20}$$

with

$$\not D = \gamma^0\partial_t + \alpha\gamma_r\partial_r - \frac{1}{r}\gamma_r(\vec{\Sigma}\cdot\vec{L} + 1) + i\gamma_5\partial_x - I\partial_y + \frac{\alpha}{r}\gamma_r, \tag{21}$$

where $\gamma_r = \hat{r}\cdot\vec{\gamma}$.

The Feynman four-component propagator obeys the equation

$$i\not D S_F(x,x') = \frac{1}{\sqrt{-g}}\delta^{(6)}(x-x')I, \tag{22}$$

and can be written in terms of the bispinor $\mathcal{G}_{\mathcal{F}}$ by

$$S_F(x,x') = i\not D\mathcal{G}_F(x',x), \tag{23}$$

where now $\mathcal{G}_F(x',x)$ obeys the 4×4 differential equation

$$\bar{\mathcal{K}}\mathcal{G}_F(x,x') = -\frac{1}{\sqrt{-g}}\delta^{(6)}(x-x')I, \tag{24}$$

with

$$\bar{\mathcal{K}} = -\partial_t^2 + \alpha^2\left(\partial_r^2 + \frac{2}{r}\partial_r\right) - \frac{\vec{L}^2}{r^2} + \partial_x^2 + \partial_y^2 - \frac{(1-\alpha)}{r^2}(1 + \vec{\Sigma}\cdot\vec{L}). \tag{25}$$

The vacuum average for the energy-momentum tensor can be expressed in terms of the Euclidean Green function.

It is related to the ordinary Feynman Green function [26] by the relation $\mathcal{G}_E(\tau, \vec{r}; \tau', \vec{r'}) = -i\mathcal{G}_F(x, x')$, where $t = i\tau$. In the following we shall consider the Euclidean Green function.

In order to find a solution for the bispinor $\mathcal{G}_E(x, x')$, we shall obtain the solution for the eigenvalue equation

$$\bar{\mathcal{K}}_E \Phi_\lambda(x) = -\lambda^2 \Phi_\lambda(x), \tag{26}$$

with $\lambda^2 \geq 0$, so we can write

$$\mathcal{G}_E(x, x') = \sum_\lambda \frac{\Phi_\lambda(x)\Phi_\lambda^\dagger(x')}{\lambda^2}. \tag{27}$$

Because of the fact that our operator (19) is self-adjoint, the set of its eigenfunctions constitutes a basis for the Hilbert space associated with four-component spinors. Moreover, because operator $\bar{\mathcal{K}}_E$ is a parity-even operator, its eigenfunctions present a defined parity, so the normalized eigenfunctions can be written as

$$\Phi_\lambda^{(\sigma)}(x) = \frac{e^{-ikx}}{(2\pi)^{3/2}} \times \sqrt{\frac{\alpha p}{r}} \begin{pmatrix} J_{\nu_\sigma}(pr)\varphi_{j,m_j}^{(\sigma)}(\theta, \phi) \\ in_\sigma J_{\nu_\sigma + n_\sigma}(pr)\hat{r}\cdot\vec{\sigma}\varphi_{j,m_j}^{(\sigma)}(\theta, \phi) \end{pmatrix}, \tag{28}$$

$$\lambda^2 = k^2 + \alpha^2 p^2, \tag{29}$$

where $kx = \bar{\eta}_{ab}k^a x^b = k_0\tau + k_x x + k_y y$, and J_ν represents the cylindrical Bessel function of order

$$\nu_\sigma = \frac{j + 1/2}{\alpha} - \frac{n_\sigma}{2}, \quad \text{with } n_\sigma = (-1)^\sigma, \quad \sigma = 0, 1. \tag{30}$$

These functions are specified by the set of quantum numbers (σ, k^a, p, j, m_j), where $k^a \in (-\infty, \infty)$, $p \in [0, \infty)$, $j = 1/2, 3/2, \ldots$ denotes the value of the total angular quantum number and $m_j = -j, \ldots, j$ determines its projection. σ specifies two types of eigenfunctions with different parities corresponding to $l = j - n_\sigma/2$, where l is the orbital quantum number. In (28), $\varphi_{j,m_j}^{(\sigma)}$ are the spinor spherical harmonics which are eigenfunctions of the operators $\vec{L}^2$ and $\vec{\sigma}\cdot\vec{L}$ as shown below:

$$\vec{L}^2\varphi_{j,m_j}^{(\sigma)} = l(l+1)\varphi_{j,m_j}^{(\sigma)}, \tag{31}$$

$$\vec{\sigma}\cdot\vec{L}\varphi_{j,m_j}^{(\sigma)} = -(1 + \kappa^{(\sigma)})\varphi_{j,m_j}^{(\sigma)}, \tag{32}$$

with $\kappa^{(0)} = -(l+1) = -(j + 1/2)$ and $\kappa^{(1)} = l = j + 1/2$. Explicit forms of the above standard functions are given in Ref. [24], for example.

Although we have developed this formalism for a six-dimensional spacetime, it can be adapted to four- and five-dimensional spaces. The reason for this resides in the representations for the Dirac matrices in these dimensions. For four dimensions an irreducible representation for the flat Dirac matrices is the well-known 4×4 one, so we may use $\sigma^M \equiv \gamma^\mu$. As a consequence, in the corresponding analysis we have to discard the derivatives with respect to the coordinates x and y in the operator $\mathcal{K}$. For five dimensions a possible representation for the flat Dirac matrices is also the 4×4 one, which can be given by $\sigma^M \equiv (\gamma^\mu, i\gamma_5)$. In this case the coordinate y should be discarded. So on the basis of these arguments it is possible to generalize the eigenfunctions of the operator $\bar{\mathcal{K}}_E$ as

$$\Phi_\lambda^{(\sigma)}(x) = \frac{e^{-ikx}}{(2\pi)^{n/2}} \times \sqrt{\frac{\alpha p}{r}} \begin{pmatrix} J_{\nu_\sigma}(pr)\varphi_{j,m_j}^{(\sigma)}(\theta, \phi) \\ in_\sigma J_{\nu_\sigma + n_\sigma}(pr)\hat{r}\cdot\vec{\sigma}\varphi_{j,m_j}^{(\sigma)}(\theta, \phi) \end{pmatrix}, \tag{33}$$

with $n = 1, 2, 3$. From now on, we shall use the above eigenfunctions and make applications for specific values attributed to n.

Now we are in a position to obtain the bispinor $\mathcal{G}_E$, which is given by

$$\mathcal{G}_E(x, x') = \int_0^\infty ds \int d^n k \int_0^\infty dp \sum_{\sigma,j,m_j} \Phi_\lambda^{(\sigma)}(x) \times \Phi_\lambda^{(\sigma)\dagger}(x') e^{-s\lambda^2}. \tag{34}$$

Finally, substituting (33) into (34) with $\lambda^2 = k^2 + \alpha^2 p^2$, we obtain, with the help of [27], an explicit expression for the Euclidean Green function:

$$\mathcal{G}_E(x', x) = \frac{1}{(2\pi r r')^{(n+1)/2}}\left(-\frac{\alpha^2}{\sinh u}\right)^{(n-1)/2} \sum_{j,m_j} \begin{pmatrix} F_{j,m_j}(\cosh u, \Omega, \Omega') & 0 \\ 0 & F_{j,m_j}(\cosh u, \Omega, \Omega') \end{pmatrix}, \tag{35}$$

where

$$F_{j,m_j}(\cosh u, \Omega, \Omega') = Q_{\nu_0 - 1/2}^{(n-1)/2}(\cosh u) C_{j,m_j}^{(0)}(\Omega, \Omega') + Q_{\nu_1 - 1/2}^{(n-1)/2}(\cosh u) C_{j,m_j}^{(1)}(\Omega, \Omega'), \tag{36}$$

with

$$\cosh u = \frac{r^2 + r'^2 + \alpha^2(\Delta x)^2}{2rr'} \geq 1, \qquad (\Delta x)^2 = \bar{\eta}_{ab}(x - x')^a(x - x')^b, \tag{37}$$

and $C^{(\sigma)}_{j,m_j}(\Omega, \Omega') = \varphi^{(\sigma)}_{j,m_j}(\theta, \phi)\varphi^{(\sigma)\dagger}_{j,m_j}(\theta', \phi')$ a 2×2 matrix. Q^λ_ν is the associated Legendre function. We can express this function in terms of a hypergeometric function [27] by

$$Q^\lambda_\nu(\cosh u) = e^{i\lambda\pi}2^\lambda\sqrt{\pi}\frac{\Gamma(\nu + \lambda + 1)}{\Gamma(\nu + 3/2)}\frac{e^{-(\nu+\lambda+1)u}}{(1 - e^{-2u})^{\lambda+1/2}}(\sinh u)^\lambda F\left(\lambda + 1/2, -\lambda + 1/2; \nu + 3/2; \frac{1}{1 - e^{2u}}\right). \tag{38}$$

In this analysis we identify the parameter ν with $\nu_\sigma - 1/2$ and take $\lambda = (n - 1)/2$. So the relevant hypergeometric function is

$$F\left(\frac{n}{2}, -\frac{n-2}{2}; \nu_\sigma + 1; \frac{1}{1 - e^{2u}}\right).$$

If n is an even number, this function becomes a polynomial of degree $\frac{n-2}{2}$; however, n being an odd number, this function is an infinite series.

Taking $\alpha = 1$, $\nu_0 = j = l + 1/2$ and $\nu_1 = j + 1 = l + 1/2$; however, for the first index the angular quantum number l assumes increasing integer numbers starting from zero, i.e., $l = 0, 1, 2, \ldots$, while for the second index $l = 1, 2, \ldots$. For $l = 0$ the only term that contributes to (35) in the summation is $Q_0^{(n-1)/2}(C^{(0)}_{1/2,1/2} + C^{(0)}_{1/2,-1/2})$. For $l \geq 1$ both associated Legendre functions contribute, $\sum_{l\geq1}Q_l^{(n-1)/2}\sum_{m_l}(C^{(0)}_{l,m_l} + C^{(1)}_{l,m_l})$. Using the explicit expressions for the spinor spherical harmonics, and using the addition theorem for spherical harmonics, it is possible to express (35) by

$$\mathcal{G}_E(x', x) = \frac{1}{(2\pi rr')^{(n+1)/2}}\frac{1}{4\pi}\left(-\frac{1}{\sinh u}\right)^{(n-1)/2} \times \sum_{l\geq0}(2l + 1)Q_l^{(n-1)/2}(\cosh u)P_l(\cos\gamma)I. \tag{39}$$

Now taking $n = 1$, 2, 3 and using the sum of Legendre functions and polynomials [27], we obtain standard expressions for the Green function. So,

(i) For $n = 1$, we have

$$\mathcal{G}_E(x', x) = \frac{1}{4\pi^2}\frac{1}{(x' - x)^2}I. \tag{40}$$

(ii) For $n = 2$, we have

$$\mathcal{G}_E(x', x) = \frac{1}{8\pi^2}\frac{1}{(x' - x)^3}I. \tag{41}$$

(iii) For $n = 3$, we have

$$\mathcal{G}_E(x', x) = \frac{1}{4\pi^3}\frac{1}{(x' - x)^4}I. \tag{42}$$

After this application of our formalism, let us return to higher-dimensional global monopole spacetime. In this space the fermionic Green function, $\mathcal{S}_\mathcal{F}$, can be given by applying the Dirac operator on the bispinor, given by (35), according to (23).

III. VACUUM AVERAGE OF THE ENERGY-MOMENTUM TENSOR

In this section we shall calculate, in an explicit way, the renormalized VEV of the energy-momentum tensor, $\langle T^A_B\rangle_{\rm Ren}$. Because the metric tensor does not depend of any dimensional parameter, and also because we are working with a natural unit system, we can infer that the VEV of the energy-momentum tensor depends only on the radial coordinate r. Moreover, due to the nonvanishing of the Riemann and Ricci tensors, and the scalar curvature of the spacetime, this VEV also depends on the arbitrary mass scale μ introduced by the renormalization prescription. So, by dimensional analysis it is expected that

$$\langle T^A_B\rangle_{\rm Ren} = \frac{1}{(\sqrt{4\pi}r)^{n+3}}(F^A_B + G^A_B\ln(\mu r/\alpha)), \tag{43}$$

where the tensors F^A_B and G^A_B depend only on the parameter α. Because of the presence of the arbitrary cutoff scale μ, there is an ambiguity in the definition of (43). Finally for a spacetime of odd dimension, i.e., for an even value of n, $G^A_B = 0$. Obviously the tensors are diagonal and, due to the spherical symmetry of the problem, we should have $F^\theta_\theta = F^\phi_\phi$ and $G^\theta_\theta = G^\phi_\phi$.

The renormalized VEV of the energy-momentum tensor must be conserved,

$$\nabla_A\langle T^A_B\rangle_{\rm Ren} = 0, \tag{44}$$

and provide the correct trace anomaly, which for massless fields and spacetimes of even dimension reads [28]

$$\langle T^A_A\rangle_{\rm Ren} = \frac{1}{(4\pi)^{(n+3)/2}}\,{\rm Tr}(a_{(n+3)/2}) = \frac{4T}{(\sqrt{4\pi}r)^{n+3}}. \tag{45}$$

As we shall see, taking into account this information, it is possible to express all components F^A_B and G^A_B in terms of the zero-zero ones, F^0_0 and G^0_0, and the trace T.

Using the point-splitting procedure [26], the VEV of the energy-momentum tensor for this four-component spinor Feynman propagator, compatible with the eight-component one, has the following form:

$$\langle T_{AB}(x)\rangle = \frac{1}{4}\lim_{x'\to x}\mathrm{Tr}[\tilde{\sigma}_A(\nabla_B - \nabla_{B'}) + \tilde{\sigma}_B(\nabla_A - \nabla_{A'})]S_F(x, x'). \quad (46)$$

Because of the dependence of the fermionic Green function on the time variable, the zero-zero component of the energy-momentum tensor reads

$$\langle T_{00}(x)\rangle = \lim_{x'\to x}\mathrm{Tr}\gamma_0\partial_0 S_F(x, x'), \quad (47)$$

which can be expressed by

$$\langle T_{00}(x)\rangle = -i\lim_{x'\to x}\partial_t^2\,\mathrm{Tr}\mathcal{G}_F(x', x) = -\lim_{x'\to x}\partial_\tau^2\,\mathrm{Tr}\mathcal{G}_E(x, x'). \quad (48)$$

In obtaining the above expression, we have first taken in the Green function, (35) and (36), the coincidence limit of the angular variables, $\Omega = \Omega'$, and the sum over m_j. This makes the function proportional to the unit matrix I. So, only the term with a time derivative in (21) provides a nonvanishing contribution for the zero-zero component of the energy-momentum tensor.

Now after these brief comments about general properties of the VEV of the energy-momentum tensor, we shall start explicit calculations of this quantity, specifying the values for n.

A. Case $n = 1$

For $n = 1$, the bulk spacetime corresponds to the four-dimensional global monopole spacetime [29]. For this case the calculation of the vacuum polarization effect associated with the massless two-component spinor field on this spacetime was developed in [30] a long time ago. More recently, some Casimir densities associated with the four-component massive fermionic field obeying the MIT bag boundary condition on one spherical shell and two concentric spherical shells in the global monopole spacetime have been analyzed in [31,32], respectively.

B. Case $n = 2$

The case $n = 2$ is a new one; it corresponds to a two-dimensional Minkowski brane with a global monopole on the transverse submanifold. For this case the associated Legendre function in (36) assumes a very simple expression,

$$Q^{1/2}_{\nu_\sigma - 1/2}(\cosh u) = i\sqrt{\frac{\pi}{2}}\frac{e^{-\nu_\sigma}}{\sqrt{\sinh u}}. \quad (49)$$

Taking $r = r'$ and $\Omega = \Omega'$ into (35), summing over m_j, and using the above expression for the Legendre function, it is possible to develop the sum over j, which is a geometric series, and obtain a closed formula for the Euclidean Green function:

$$\mathcal{G}_E(x', x) = \frac{\alpha}{64\pi^2 r^3}\frac{1}{\sinh(u/2)}\frac{1}{\sinh^2(u/2\alpha)}I, \quad (50)$$

with $u = 2\mathrm{arcsinh}(\alpha\Delta x/2r)$.

The above Green function is divergent in the coincidence limit, $\Delta x \to 0$. We can verify its singular behavior by expanding (50) in a power series of Δx:

$$\mathcal{G}_E(x', x) = \left[\frac{1}{8\pi^2}\frac{1}{(\Delta x)^3} - \frac{1-\alpha^2}{96\pi^2 r^2}\frac{1}{(\Delta x)} + \frac{1-\alpha^4}{1920\pi^2 r^4}(\Delta x) + \frac{31\alpha^6 - 21\alpha^2 - 10}{483\,840\pi^2 r^6}(\Delta x)^3 + O((\Delta x)^5)\right]I. \quad (51)$$

In order to obtain a finite and well-defined result for the VEV of the zero-zero component of the energy-momentum tensor, we must extract all divergent terms in the coincidence limit after applying the second time derivative. In order to do that in a manifest form, we subtract from the Green function the Hadamard one. In [28], the general formal expression for the Hadamard function is given for any dimensional spacetime. For a five-dimensional spacetime it reads

$$G_H(x', x) = \frac{\Delta^{1/2}(x', x)}{16\pi^2\sqrt{2}}\frac{1}{\sigma^{3/2}(x', x)}[a_0(x,', x) + a_1(x', x)\sigma(x', x) - a_2(x', x)\sigma^2(x', x)]. \quad (52)$$

For this manifold, $\Delta(x', x)$, the Van Vleck-Morette determinant, and the coefficients $a_i(x)$ are given below:

$$\Delta = 1, \quad a_0 = I, \quad a_1 = -\frac{1-\alpha^2}{6r^2}I, \quad \text{and} \quad a_2 = -\frac{1-\alpha^4}{60r^4}I. \quad (53)$$

One-half of the geodesic distance for $r' = r$ and $\Omega' = \Omega$ is $\sigma(x', x) = \frac{(\Delta x)^2}{2}$. So, we can see that the singular behavior for the Green function, Eq. (51), after taking its second time derivative, has the same structure as is given by the Hadamard function,

$$G_H(x', x) = \left[\frac{1}{8\pi^2}\frac{1}{(\Delta x)^3} - \frac{1-\alpha^2}{96\pi^2 r^2}\frac{1}{(\Delta x)} + \frac{1-\alpha^4}{1920\pi^2 r^4}(\Delta x)\right]I. \quad (54)$$

So, on the basis of this fact we can see that the renormalized VEV of the zero-zero component of the energy-momentum tensor vanishes, i.e.,

$$\langle T_0^0(x)\rangle_{\mathrm{Ren}} = \lim_{x'\to x}\partial_\tau^2\,\mathrm{Tr}[\mathcal{G}_E(x', x) - G_H(x', x)] = 0. \quad (55)$$

For an odd dimensional spacetime there is no trace anomaly, i.e., $\langle T_A^A\rangle_{\mathrm{Ren}} = 0$. Writing $F_B^A = (F_0^0, F_r^r, F_\theta^\theta,$

F^ϕ_ϕ, F^x_x), we have that the sum of all these components vanishes; moreover, the geometric structure of the brane section of this five-dimensional spacetime is Minkowski-type; consequently, the Green function and the Hadamard one depend on the variables on the two-dimensional brane through $\Delta x^2 = -(\Delta x^0)^2 + (\Delta x^4)^2 = (\Delta\tau)^2 + (\Delta x^4)^2$. By the definition of the VEV of the energy-momentum tensor, Eq. (46), we have

$$\langle T^4_4(x)\rangle = \lim_{x'\to x}\mathrm{Tr}\,\partial^2_x \mathcal{G}_E(x,x'). \tag{56}$$

So, we can infer that $\langle T^0_0(x)\rangle_{\rm Ren} = \langle T^4_4(x)\rangle_{\rm Ren}$. In this way we have $F^0_0 = F^x_x$, being both zero. The conservation condition, $\nabla_A\langle T^A_r\rangle_{\rm Ren} = 0$, provides $3F^r_r = -F^\theta_\theta - F^\phi_\phi$, and $\nabla_A\langle T^A_\theta\rangle_{\rm Ren} = 0$, $F^\theta_\theta = F^\phi_\phi$. So on the basis of all this information, we conclude that all components of the tensor F^A_B are zero, and consequently,

$$\langle T^A_A\rangle_{\rm Ren} = 0. \tag{57}$$

C. Case $n = 3$

The case $n = 3$ corresponds to a flat three-dimensional brane transverse to a three-dimensional global monopole space. The respective Euclidean Green function is given in terms of an infinite sum of the associated Legendre function $Q^1_{\nu_\sigma-1/2}$ with $C^{(\sigma)}_{j,m_j}$. Taking the angular coincidence limit, $\Omega = \Omega'$, the sum S_0, given in (35), can be expressed as

$$S_0 = \sum_{j,m_j} Q^1_{\nu_0-1/2}(\cosh u)C^{(0)}_{j,m_j}(\Omega,\Omega) = \frac{1}{4\pi}\sum_{l\geq 0}(l+1)Q^1_{((l+1)/\alpha)-1}(\cosh u)I_{(2)}, \tag{58}$$

and for S_1,

$$S_1 = \sum_{j,m_j} Q^1_{\nu_1-1/2}(\cosh u)C^{(1)}_{j,m_j}(\Omega,\Omega) = \frac{1}{4\pi}\sum_{l\geq 1} l Q^1_{l/\alpha}(\cosh u)I_{(2)}. \tag{59}$$

Using $Q^1_\nu(z) = (z^2-1)^{1/2}\frac{dQ_\nu(z)}{dz}$, and the integral representation below for the Legendre function Q_ν,

$$Q_\nu(\cosh u) = \frac{1}{\sqrt{2}}\int_u^\infty dt\frac{e^{-(\nu+1/2)t}}{\sqrt{\cosh t - \cosh u}}, \tag{60}$$

it is possible to develop the sums over l in (58) and (59), and obtain

$$S_0(\cosh u) = \frac{I_{(2)}\sinh u}{16\pi\sqrt{2}}\frac{d}{dz}\bigg[\int_{\mathrm{arccosh} z}^\infty \frac{dt e^{t/2}}{\sqrt{\cosh t - z}} \times \frac{1}{\sinh^2(t/2\alpha)}\bigg]\bigg|_{z=\cosh u} \tag{61}$$

and

$$S_1(\cosh u) = \frac{I_{(2)}\sinh u}{16\pi\sqrt{2}}\frac{d}{dz}\bigg[\int_{\mathrm{arccosh} z}^\infty \frac{dt e^{-t/2}}{\sqrt{\cosh t - z}} \times \frac{1}{\sinh^2(t/2\alpha)}\bigg]\bigg|_{z=\cosh u}. \tag{62}$$

Substituting the sum $S_0 + S_1$ into (35), we can express the Euclidean Green function by

$$\mathcal{G}_E(x',x) = -\frac{I}{32\pi^3}\frac{\alpha^2}{(r'r)^2}\frac{d}{dz}\bigg[\int_{\sqrt{z-1}}^\infty \frac{dy}{\sqrt{y^2+1-z}} \times \frac{1}{\sinh^2(\frac{\mathrm{arcsinh}(y/\sqrt{2})}{\alpha})}\bigg]\bigg|_{z=\cosh u}, \tag{63}$$

where we have introduced a new variable $t := 2\,\mathrm{arcsinh}(y/\sqrt{2})$.

The VEV of the zero-zero component of the energy-momentum tensor can be formally given by substituting (63) into (48). Because this procedure provides a divergent result, we must renormalize it by extracting all the divergent terms. In what follows we shall develop a procedure to extract the divergences in a manifest form. Let us consider the integral inside the bracket of (63) and write it as

$$I_\alpha(z) = I_1(z) + I_2(z) = \int_{\sqrt{z-1}}^1 \frac{dy}{\sqrt{y^2+1-z}}\frac{1}{\sinh^2(\frac{\mathrm{arcsinh}(y/\sqrt{2})}{\alpha})} + \int_1^\infty \frac{dy}{\sqrt{y^2+1-z}}\frac{1}{\sinh^2(\frac{\mathrm{arcsinh}(y/\sqrt{2})}{\alpha})}. \tag{64}$$

Subtracting and adding into the integrand of I_1 the first four terms of the power series of the function

$$\frac{1}{\sinh^2(\frac{\mathrm{arcsinh}(y/\sqrt{2})}{\alpha})} = \frac{2\alpha^2}{y^2} + \frac{(\alpha^2-1)}{3} - \frac{(\alpha^4-1)}{30\alpha^2}y^2 + \frac{(31\alpha^6-21\alpha^2-10)}{3780\alpha^4}y^4 + \ldots, \tag{65}$$

we get

$$I_1(z) = I_1^{\rm fin}(z) + I_1^{\rm sing}(z), \tag{66}$$

with

$$I_1^{\rm fin}(z) = \int_{\sqrt{z-1}}^1 \frac{dy}{\sqrt{y^2+1-z}}\bigg[\frac{1}{\sinh^2(\frac{\mathrm{arcsinh}(y/\sqrt{2})}{\alpha})} - \frac{2\alpha^2}{y^2} - \frac{(\alpha^2-1)}{3} + \frac{(\alpha^4-1)}{30\alpha^2}y^2 - \frac{(31\alpha^6-21\alpha^2-10)}{3780\alpha^4}y^4\bigg] \tag{67}$$

and

$$I_1^{\rm sing}(z) = \int_{\sqrt{z-1}}^1 \frac{dy}{\sqrt{y^2+1-z}}\bigg[\frac{2\alpha^2}{y^2} + \frac{(\alpha^2-1)}{3} - \frac{(\alpha^4-1)}{30\alpha^2}y^2 + \frac{(31\alpha^6-21\alpha^2-10)}{3780\alpha^4}y^4\bigg]. \quad (68)$$

As we shall see, $I_1^{\rm fin}$, together with I_2, provides a finite contribution to the VEV. All the divergences are contained in $I_1^{\rm sing}$.

Taking the derivative of $I_1^{\rm sing}$ with respect to z and substituting $z = \cosh u = 1 + \frac{\alpha^2\sigma}{r^2}$, with $\sigma = \frac{(\Delta x)^2}{2}$, we expand the result in powers of σ, which survives after taking the second derivative and the coincidence limit. Performing this procedure we have

$$\frac{d_1^{\rm sing}(\sigma)}{dz} = -\frac{2r^4}{\alpha^2\sigma^2} - \frac{\bar{a}_1 r^2}{2\alpha^2\sigma} + \frac{\bar{a}_2}{4}\ln\bigg(\frac{\sigma\alpha^2}{4r^2}\bigg) + \frac{(-4\alpha^2 - 10\bar{a}_3 + 6\bar{a}_2 - 3\bar{a}_1 - 6\bar{a}_3\ln(\frac{\sigma\alpha^2}{4r^2}))\sigma\alpha^2}{16r^2} + \ldots, \quad (69)$$

with

$$\bar{a}_1 = \frac{(\alpha^2-1)}{3}, \quad \bar{a}_2 = \frac{(\alpha^4-1)}{30\alpha^2}, \quad \text{and} \quad \bar{a}_3 = \frac{(31\alpha^6-21\alpha^2-10)}{3780\alpha^4}, \quad (70)$$

being the last three coefficients of the expansion (65).

We can see that the contribution to the Euclidean Green function given by $I_1^{\rm sing}$ has the same structure as the Hadamard function for a six-dimensional spacetime,

$$G_H(x',x) = \frac{\Delta^{1/2}}{16\pi^3}\bigg[\frac{a_0}{\sigma^2} + \frac{a_1}{2\sigma} - \frac{1}{4}\bigg(a_2 - \frac{a_3}{2}\sigma\bigg)\ln\bigg(\frac{\mu^2\sigma}{4}\bigg)\bigg], \quad (71)$$

where μ is an arbitrary energy scale [28]. Δ, a_0, a_1, and a_2 are the same as given in (53) for this six-dimensional global monopole spacetime.

In [33,34], an explicit expression for the coefficient a_3 is provided for a general second order differential operator $D^2 + X$, D_M being the covariant derivative including the gauge field and X an arbitrary scalar function. For this six-dimensional global monopole spacetime, by using the computer program GRTENSORII, we found, considering $X = -\frac{1}{4}RI$,

$$a_3 = \frac{31\alpha^6 - 21\alpha^2 - 10}{2520r^6}I. \quad (72)$$

The renormalized bispinor given by

$$\mathcal{G}_{\rm Ren}(x',x) = \mathcal{G}_E(x',x) - G_H(x',x), \quad (73)$$

with

$$\mathcal{G}_E(x',x) = -I\frac{\alpha^2}{32\pi^3r^4}\bigg[\frac{dI_1^{\rm sing}(\sigma)}{dz} + \frac{dI_1^{\rm fin}(\sigma)}{dz} + \frac{dI_2(\sigma)}{dz}\bigg] \quad (74)$$

and $G_H(x',x)$ given by (71), provides

$$\mathcal{G}_{\rm Ren}(x',x) = I\frac{\alpha^2}{64\pi^3r^4}\bigg[\bar{a}_2 - \frac{3\bar{a}_3\alpha^2\sigma}{2r^2}\bigg]\ln\bigg(\frac{\mu r}{\alpha}\bigg) - I\frac{[-4\alpha^2 - 10\bar{a}_3 + 6\bar{a}_2 - 3\bar{a}_1]\alpha^4\sigma}{512\pi^3r^6} - I\frac{\alpha^2}{32\pi^3r^4}\bigg[\frac{dI_1^{\rm fin}(\sigma)}{dz} + \frac{dI_2(\sigma)}{dz}\bigg]. \quad (75)$$

Now we are in a position to obtain the renormalized VEV of the zero-zero component of the energy-momentum tensor. It is given by

$$\langle T_0^0(x)\rangle_{\rm Ren} = \lim_{x'\to x}\partial_\tau^2\,{\rm Tr}\mathcal{G}_{\rm Ren}(x',x). \quad (76)$$

In [35], Wald proved that, in order to obtain an energy-momentum tensor which obeys the conservation condition law (44) and provides the correct trace anomaly (45), an additional contribution must be considered. In a six-dimensional spacetime, this terms reads

$$\frac{1}{384\pi^3}\delta_A^B\,{\rm Tr}a_3. \quad (77)$$

So, based on this fact we explicitly present in the Appendix the steps needed to obtain F_0^0 and G_0^0. With respect to F_0^0 we provide an integral expression for it, and for G_0^0 a closed expression. Briefly speaking, they are given by substituting (75) into (76) and taking into account (77). These components are

$$F_0^0 = -6\alpha^4\int_0^1\frac{dx}{x^5}f_1(x) - 6\alpha^4\int_1^\infty\frac{dx}{x^5}f_2(x) + 6\alpha^4\bar{a}_3 \quad (78)$$

with

$$f_1(x) = \frac{1}{\sinh^2(\frac{{\rm arcsinh}(x/\sqrt{2})}{\alpha})} - \frac{2\alpha^2}{x^2} - \frac{(\alpha^2-1)}{3} + \frac{(\alpha^4-1)}{30\alpha^2}x^2 - \frac{(31\alpha^6-21\alpha^2-10)}{3780\alpha^4}x^4, \quad (79)$$

$$f_2(x) = \frac{1}{\sinh^2(\frac{\operatorname{arcsinh}(x/\sqrt{2})}{\alpha})} - \frac{2\alpha^2}{x^2} - \frac{(\alpha^2-1)}{3} + \frac{(\alpha^4-1)}{30\alpha^2}x^2, \tag{80}$$

and

$$G_0^0 = -r^6\,\mathrm{Tr}a_3 = -\frac{31\alpha^6 - 21\alpha^2 - 10}{630}. \tag{81}$$

In (78) we have used the fact that $(r^6/6)\,\mathrm{Tr}a_3 = \alpha^4\bar{a}_3$.

Unfortunately, it is not possible to provide analytical results for the integrals above for a general value of α. Their dependence on the parameter α can only be provided numerically. Our numerical results for these integrals are exhibited in Fig. 1. Also, it is possible to provide the approximate behavior for the integrals, $I_1 = \int_0^1 \frac{dx}{x^5} f_1$ and $I_2 = \int_1^\infty \frac{dx}{x^5} f_2$, for a specific limit of this parameter:

(i) For a large solid angle deficit ($\alpha \ll 1$),

$$I_1(\alpha) \approx -\frac{1}{378}\frac{\ln(\alpha)}{\alpha^4}, \tag{82}$$

$$I_2(\alpha) \approx -\frac{1}{60\alpha^2}. \tag{83}$$

(ii) For a small solid angle deficit ($|\alpha - 1| \ll 1$),

$$I_1(\alpha) \approx -0.0052(\alpha - 1), \tag{84}$$

$$I_2(\alpha) \approx 0.0275(\alpha - 1). \tag{85}$$

(iii) For a large solid angle excess ($\alpha \gg 1$),

$$I_1(\alpha) \approx -0.0025\alpha^2, \tag{86}$$

$$I_2(\alpha) \approx \frac{\alpha^2}{60}. \tag{87}$$

Having found F_0^0 and G_0^0, the other components of the renormalized VEV of the energy-momentum tensor can be expressed in terms of them. Because of the Minkowski structure of the brane section of this spacetime, the Green function and also the Hadamard one depend on the variables on the two-dimensional brane through $(\Delta x)^2 = \bar{\eta}_{ab}(x - x')^a(x - x')^b$; on the other hand, by using (46) we can verify that

$$\langle T_a^a(x)\rangle = \lim_{x'\to x}\mathrm{Tr}\partial_a^2\mathcal{G}_E(x', x), \quad \text{for } a = 4, 5. \tag{88}$$

So we may conclude that the renormalized VEVs of these components are equal to the zero-zero one. Consequently, $F_0^0 = F_4^4 = F_5^5$ and $G_0^0 = G_4^4 = G_5^5$. Besides, the trace anomaly,

$$\langle T_A^A(x)\rangle_{\mathrm{Ren}} = \frac{1}{64\pi^3}\,\mathrm{Tr}a_3 = \frac{T}{16\pi^3 r^6}, \tag{89}$$

with

$$T = \frac{31\alpha^6 - 21\alpha^2 - 10}{2520} \tag{90}$$

give us $F_A^A = 4T$ and $G_A^A = 0$. The conservation conditions, $\nabla_A\langle T_\theta^A\rangle_{\mathrm{Ren}} = 0$ and $\nabla_A\langle T_r^A\rangle_{\mathrm{Ren}} = 0$, provide, respectively, $T_\theta^\theta = T_\phi^\phi$ and $4F_r^r - G_r^r + 2F_\theta^\theta = 0$. So, after a simple calculation we obtain

$$F_B^A = \mathrm{diag}(F_0^0, F_0^0 + G_0^0/3 - 4T/3, 8T/3 - 2F_0^0 - G_0^0/6, 8T/3 - 2F_0^0 - G_0^0/6, F_0^0, F_0^0) \tag{91}$$

and

$$G_B^A = G_0^0\mathrm{diag}(1, 1, -2, -2, 1, 1). \tag{92}$$

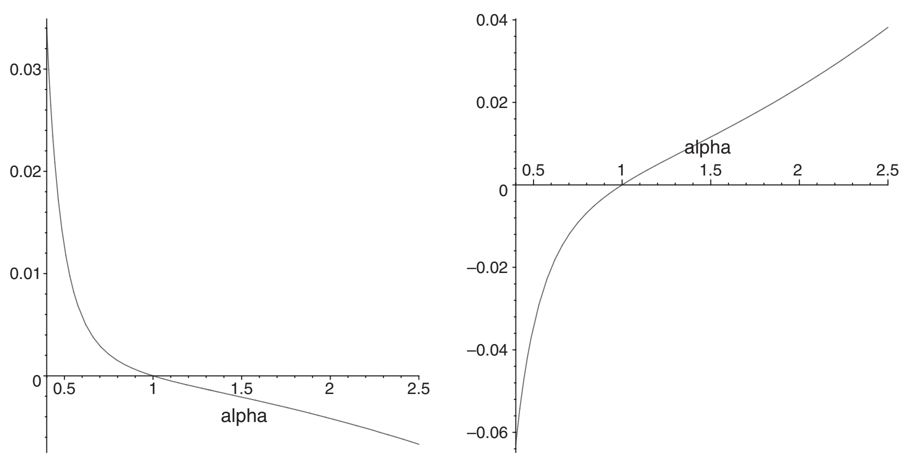

FIG. 1. These graphs represent the dependence of the two integrals in (78) as a function of the parameter α in a specific interval. The left panel corresponds to the integral associated with the function $f_1(x)/x^5$, and the right panel corresponds to the integral associated with the function $f_2(x)/x^5$.

IV. CONCLUDING REMARKS

In this paper we have considered the vacuum polarization effect associated with the fermionic field induced by a global monopole in the braneworld context. In this scenario our Universe is described by an $(n-1)$-flat brane transverse to a pointlike three-dimensional global monopole manifold. Two specific spacetimes have been explicitly considered, the cases with $n = 2$ and 3. In fact, our first motivation was to consider a six-dimensional bulk. However, because we have considered a defined positive chiral field, the fermionic field can be described by a four-component representation; so, in this way, we could extend the formalism to include a five-dimensional bulk. Because of the fact that the global monopole lives in a three-dimensional manifold, it was possible to express the fermionic eigenfunctions in terms of spinor spherical harmonics, $\varphi^{(\sigma)}_{j,m_j}$. Differently from the scalar case analyzed in [23], the effective total angular momentum has simple expressions: $\nu_0 = (l+1)/\alpha - 1/2$ and $\nu_1 = l/\alpha + 1/2$.[3]

The main objective of this paper was to obtain the renormalized vacuum expectation value of the energy-momentum tensor, $\langle T_A^B\rangle$. In order to do that, we have explicitly constructed the fermionic Green function, which was expressed in terms of a linear differential operator acting on a bispinor. Because all components of this tensor can be related by the conservation condition and the correct trace anomaly, we needed only to calculate the zero-zero component of this tensor. From our results we could verify that, for a five-dimensional bulk, $\langle T_A^B\rangle_{\rm Ren} = 0$.[4] However, for a six-dimensional bulk, a nonvanishing result has been obtained. This result depends on the inverse of the sixth order power of the distance from the point to the monopole's core considered, and also on the arbitrary energy scale μ:

$$\langle T_0^0(x)\rangle_{\rm Ren} = \frac{1}{64\pi^3 r^6}(F_0^0(\alpha) + G_0^0(\alpha)\ln(\mu r/\alpha)). \quad (93)$$

The expression found for F_0^0 involves a long calculation, and some details related to this calculation are presented in the Appendix. From our result this component is given in terms of a closed expression and two more integrals. For these integrals we provided numerically, in Fig. 1, their behavior as a function of α. Fortunately, we were able to provide a closed expression for G_0^0, which, in turn, depends on the coefficient a_3 of the Hadamard function. The general expression for this coefficient, for the fermionic case, presents 43 terms, so its final result requires a long and careful calculation, even using the specific computer program GRTENSORII. Because (93) is given in terms of two distinct contributions, there exists a radius $\bar{r}$ where $\langle T_0^0(x)\rangle_{\rm Ren}$ vanishes. In Fig. 2 we present the dependence of the dimensionless distance $\mu\bar{r}/\alpha$ as a function of α.

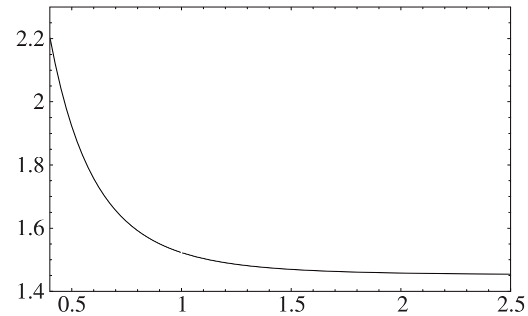

FIG. 2. This graph represents the dependence of the physical distance $\mu\bar{r}/\alpha$, where $\langle T_0^0(x)\rangle_{\rm Ren}$ vanishes, with α.

Finally, we would like to mention that the model analyzed here presents the monopole as a pointlike object having its core on the flat brane; so, the influence due to the fermionic quantum field on the brane can be evaluated in the region near the monopole's core. However, from this model the renormalized VEV of the energy-momentum tensor is divergent at the monopole's core. The problem of the singularity of vacuum polarization effects involving a global monopole, and topological defects in general, can be avoided by considering a more realistic model to the monopole, i.e., considering an inner structure to its core. A simplified model for the monopole core, where the region inside it is described by de Sitter geometry, has been presented in [36]. The vacuum polarization effects due to a massless scalar field in the region outside this model have been investigated in [37]. More recently, the analyses of vacuum polarization effects associated with quantum bosonic and fermionic fields, in the global monopole spacetime with a general spherically symmetric inner structure, have been developed in [38,39], respectively. In these analyses the asymptotic behavior of the core-induced vacuum densities is investigated at large and small distance from the core, and for small and large solid angle deficit.

ACKNOWLEDGMENTS

The author would like to thank J. Batista Fonseca for his help with the computer program GRTENSORII, as well as Aram A. Saharian, also to Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for partial financial support, FAPESQ-PB/CNPq (PRONEX) and FAPES-ES/CNPq (PRONEX).

APPENDIX: EXPLICIT CALCULATION OF F_0^0 AND G_0^0

In order to find the complete expression for (75), it is necessary to obtain the derivative of $I_1^{\rm fin}$ and I_2 with respect to z. Our first step in this direction is to introduce a new

[3]For the scalar case the effective orbital angular quantum number assumes a simple form only for a curvature coupling parameter $\xi = 1/8$.

[4]A vanishing result for the renormalized vacuum expectation value of the square of the field, $\langle\Phi^2\rangle$, has also been obtained for a scalar field in this five-dimensional bulk in a specific approximated result [23].

variable, $b=\sqrt{z-1}$. So, we may write

$$I_1^{\rm fin}(z)=\int_b^1\frac{dx}{\sqrt{x^2-b^2}}f_1(x), \tag{A1}$$

with

$$f_1(x)=\frac{1}{\sinh^2(\frac{\operatorname{arcsinh}(x/\sqrt{2})}{\alpha})}-\frac{2\alpha^2}{x^2}-\frac{(\alpha^2-1)}{3}+\frac{(\alpha^4-1)}{30\alpha^2}x^2-\frac{(31\alpha^6-21\alpha^2-10)}{3780\alpha^4}x^4. \tag{A2}$$

Consequently,

$$\frac{I_1^{\rm fin}(\sigma)}{dz}=\frac{1}{2b}\frac{d}{db}\int_b^1\frac{dx}{\sqrt{x^2-b^2}}f_1(x). \tag{A3}$$

Because the integrand of $I_1^{\rm fin}$ is divergent at the point $x=b$, we have to change the variable $x\to bx$ before applying the Leibnitz formula for the derivative of an integral. So, this derivative can be written as

$$\frac{I_1^{\rm fin}(\sigma)}{dz}=\frac{1}{2}\int_b^1\frac{dx}{\sqrt{x^2-b^2}}g(x)-\frac{f_1(1)}{2\sqrt{1-b^2}}, \tag{A4}$$

where $g(x)=\frac{d}{dx}(\frac{f_1(x)}{x})$ and $b=\frac{\alpha\sqrt{\sigma}}{r}$. The derivative of I_2 is simple and reads

$$\frac{I_2(\sigma)}{dz}=\frac{1}{2}\int_1^\infty\frac{dx}{(x^2-b^2)^{3/2}}\frac{1}{\sinh^2(\frac{\operatorname{arcsinh}(x/\sqrt{2})}{\alpha})}. \tag{A5}$$

Using these results we can express the renormalized Green function, Eq. (75), as

$$\mathcal{G}_{\rm Ren}(x',x)=I\frac{\alpha^2}{64\pi^3r^4}\left[\bar a_2-\frac{3\bar a_3\alpha^2\sigma}{2r^2}\right]\ln\left(\frac{\mu r}{\alpha}\right)-I\frac{[-4\alpha^2-10\bar a_3+6\bar a_2-3\bar a_1]\alpha^4\sigma}{512\pi^3r^6}-I\frac{\alpha^2}{64\pi^3r^4}\left[\int_b^1\frac{dx}{\sqrt{x^2-b^2}}g(x)-\frac{f_1(1)}{\sqrt{1-b^2}}\right]-I\frac{\alpha^2}{64\pi^3r^4}\int_1^\infty\frac{dx}{(x^2-b^2)^{3/2}}\frac{1}{\sinh^2(\frac{\operatorname{arcsinh}(x/\sqrt{2})}{\alpha})}. \tag{A6}$$

The next step, according to (76), is to obtain the second derivative with respect to the Euclidean time. Taking the coincidence limits $x'=x$ and $y'=y$, using the fact that $\partial_\tau^2=(\alpha^2/2r^2)\partial_b^2$, and adopting the same procedure as explained above, we can write

$$\frac{d^2}{db^2}\int_b^1\frac{dx}{\sqrt{x^2-b^2}}g(x)=\int_b^1\frac{dx}{\sqrt{x^2-b^2}}\left(g''(x)-\frac{g'(x)}{x}+\frac{g(x)}{x^2}\right)-\frac{g'(1)}{\sqrt{1-b^2}}-\frac{g(1)b^2}{(1-b^2)^{3/2}}. \tag{A7}$$

Taking the limit $b\to 0$ we obtain

$$\frac{d^2}{db^2}\int_b^1\frac{dx}{\sqrt{x^2-b^2}}g(x)\to\int_0^1dx\frac{g(x)}{x^3}. \tag{A8}$$

We can see that the integral above is finite because, for small values of x, $g(x)\to-\frac{(289\alpha^8-168\alpha^4-100\alpha^2-21)}{22.680\alpha^6}x^4+O(x^6)$. In the limit $b\to 0$, the other relevant terms in the renormalized Green function above provide

$$\frac{d^2}{db^2}\frac{1}{(x^2-b^2)^{3/2}}\to\frac{3}{x^5}, \tag{A9}$$

$$\frac{d^2}{db^2}\frac{1}{\sqrt{1-b^2}}\to 1. \tag{A10}$$

So, taking into account all these results and including the additional term (77), we have

$$\langle T_{00}(x)\rangle_{\rm Ren}=-\frac{\alpha^6}{32\pi^3r^6}+\frac{[6\bar a_2-3\bar a_1-10\bar a_3]\alpha^4}{128\pi^3r^6}+\frac{3\alpha^4}{32\pi^3r^6}\int_1^\infty\frac{dx}{x^5}\frac{1}{\sinh^2(\frac{\operatorname{arcsinh}(x/\sqrt{2})}{\alpha})}+\frac{3\alpha^4}{32\pi^3r^6}\int_0^1\frac{dx}{x^5}f_1(x)+\frac{3\bar a_3\alpha^4}{32\pi^6r^6}\ln\left(\frac{\mu r}{\alpha}\right)-\frac{1}{64\pi^3r^6}\alpha^4\bar a_3, \tag{A11}$$

which can be written as shown below:

$$\langle T_{00}(x)\rangle_{\rm Ren}=\frac{3\alpha^4}{32\pi^3r^6}\int_0^1\frac{dx}{x^5}f_1(x)+\frac{3\alpha^4}{32\pi^3r^6}\int_1^\infty\frac{dx}{x^5}f_2(x)-\frac{3\alpha^4}{32\pi^3r^6}\bar a_3+\frac{3\bar a_3\alpha^4}{32\pi^3r^6}\ln\left(\frac{\mu r}{\alpha}\right). \tag{A12}$$

[1] K. Akama, Lect. Notes Phys., **176**, 267 (1982).
[2] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).
[3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
[4] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **125B**, 136 (1983); **125B**, 139 (1983).
[5] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989); V. Sahni and A. Starobinsky, Int. J. Mod. Phys. D **9**, 373 (2000).
[6] S. Nojiri and S. D. Odintsov, Phys. Lett. B **484**, 119 (2000); D. J. Toms, Phys. Lett. B **484**, 149 (2000); A. Knapman and D. J. Toms, Phys. Rev. D **69**, 044023 (2004).
[7] A. A. Sararian, Phys. Rev. D **70**, 064026 (2004).
[8] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, England, 1994).
[9] A. G. Cohen and D. B. Kaplan, Phys. Lett. B **470**, 52 (1999).
[10] R. Gregory, Phys. Rev. Lett. **84**, 2564 (2000).
[11] E. Roessl and M. Shaposnikov, Phys. Rev. D **66**, 084008 (2002).
[12] I. Cho and A. Vilenkin, Phys. Rev. D **69**, 045005 (2004).
[13] I. Olasagasti and A. Vilenkin, Phys. Rev. D **62**, 044014 (2000).
[14] T. Gherghetta, E. Roessl, and M. Shaposnikov, Phys. Lett. B **491**, 353 (2000).
[15] I. Olasagasti, Phys. Rev. D **63**, 124016 (2001).
[16] K. Benson and I. Cho, Phys. Rev. D **64**, 065026 (2001).
[17] I. Cho and A. Vilenkin, Phys. Rev. D **68**, 025013 (2003).
[18] J. Spinelly, U. de Freitas, and E. R. Bezerra de Mello, Phys. Rev. D **66**, 024018 (2002).
[19] Y. Brihaye and B. Hartmann, Phys. Rev. D **66**, 064018 (2002).
[20] E. R. Bezerra de Mello, Y. Brihaye, and B. Hartmann, Phys. Rev. D **67**, 045015 (2003).
[21] E. R. Bezerra de Mello, Y. Brihaye, and B. Hartmann, Phys. Rev. D **67**, 124008 (2003).
[22] E. R. Bezerra de Mello and B. Hartmann, Phys. Lett. B **639**, 546 (2006).
[23] E. R. Bezerra de Mello, Phys. Rev. D **73**, 105015 (2006).
[24] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[25] R. N. Mohapatra and A. Pérez-Lorenzana, Phys. Rev. D **67**, 075015 (2003).
[26] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[27] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1980).
[28] S. M. Christensen, Phys. Rev. D **17**, 946 (1978).
[29] M. Barriola and A. Vilenkin, Phys. Rev. Lett. **63**, 341 (1989).
[30] E. R. Bezerra de Mello, V. B. Bezerra, and N. R. Khusnutdinov, Phys. Rev. D **60**, 063506 (1999).
[31] A. A. Saharian and E. R. Bezerra de Mello, J. Phys. A **37**, 3543 (2004).
[32] A. A. Saharian and E. R. Bezerra de Mello, Classical Quantum Gravity **23**, 4673 (2006).
[33] P. B. Gilkey, J. Diff. Geom. **10**, 601 (1975).
[34] I. Jack and L. Parker, Phys. Rev. D **31**, 2439 (1985).
[35] R. M. Wald, Phys. Rev. D **17**, 1477 (1978).
[36] D. Harari and C. Lousto, Phys. Rev. D **42**, 2626 (1990).
[37] J. Spinelly and E. R. Bezerra de Mello, Classical Quantum Gravity **22**, 3247 (2005).
[38] E. R. Bezerra de Mello and A. A. Saharian, J. High Energy Phys. 10 (2006) 049.
[39] E. R. Bezerra de Mello and A. A. Saharian, Phys. Rev. D **75**, 065019 (2007).