Equation of state of gluon plasma from local action

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We review recent analytic and numerical results concerning the confinement scenario in Coulomb gauge. We then consider a local, renormalizable, BRST(Becchi-Rouet-Stora-Tyutin)-invariant action for QCD in Coulomb gauge that contains auxiliary bose and fermi ghost fields and sources. When the auxiliary fields are integrated out, one obtains the standard Coulomb gauge action with a cutoff at the Gribov horizon. We use the local formulation to calculate the leading correction to the Stefan-Boltzmann equation of state at high temperature due to the cutoff at the Gribov horizon. It is of order g^6, which is precisely the order at which the infrared divergence found by Lindé divergence first occurs. No such divergence arises in the present calculation because the propagator of would-be physical gluons is suppressed in the infrared due to the proximity of the Gribov horizon in infrared directions.

I. INTRODUCTION

A. Confinement scenario in Coulomb gauge

One would like to explain the presence of a long-range force that confines colored objects. At the same time one is faced by the apparently contradictory requirement that the massless gluons that are supposed to transmit this force are absent from the physical spectrum.

This paradox is addressed in a scenario in Coulomb gauge that was originally developed by Gribov [1]. In this scenario, the 3-dimensionally transverse, would-be physical gluon propagator, $D_{A_iA_j}(\mathbf{k}, k_0)$, is suppressed at small $\mathbf{k}$ because of the proximity of the Gribov horizon in infrared directions, so physical gluons are absent from the physical spectrum. On the other hand confinement of quarks, or any colored object, is explained by the long range of the instantaneous part, $V_{\text{coul}}(R)$, of the time-time component of the gluon propagator,

$$g^2 D_{A_0A_0}(\mathbf{x}, t) = V_{\text{coul}}(|\mathbf{x}|)\delta(t) + P(\mathbf{x}, t), \qquad (1)$$

that couples universally to color charge. Here $P(\mathbf{x}, t)$ is a noninstantaneous vacuum polarization term that is screening, whereas the color-Coulomb potential, $V_{\text{coul}}(R)$, is antiscreening [2]. The concentration of probability at the Gribov horizon causes $V_{\text{coul}}(R)$ to have the long-range, confining property $\lim_{R\to\infty} V_{\text{coul}}(R) = \infty$. The different behavior of space and time components of the gluon propagator is possible because manifest Lorentz invariance does not hold in Coulomb gauge.

The local, renormalizable theory which will be presented here yields results in agreement with this scenario. For example, at tree level, the 3-dimensionally transverse gluon propagator,

$$D(\mathbf{k}, k_0) = \frac{\mathbf{k}^2}{(k_0^2 + \mathbf{k}^2)\mathbf{k}^2 + m^4}, \qquad (2)$$

is strongly suppressed in the infrared, vanishing like $\mathbf{k}^2$. It has poles at

$$-k_0^2 = E^2(\mathbf{k}) \equiv \mathbf{k}^2 + \frac{m^4}{\mathbf{k}^2}, \qquad (3)$$

in agreement with the Coulomb-gauge energy obtained by Gribov.

All states are physical in Coulomb gauge, and the instantaneous part, $V_{\text{coul}}(R)$, of the gluon propagator is directly related to the energy of a quark-pair state, $|\Psi_{\bar{q}q}\rangle = \bar{q}(0)q(R)|\Psi_0\rangle$. This state in Coulomb gauge contains massive quark-antiquark sources at separation R, and Ψ_0 is the vacuum state of pure glue. $\Psi_{\bar{q}q}$ is invariant under the remnant gauge symmetry of time-dependent, space-independent gauge transformations $g(t)$ that is left unfixed by the Coulomb-gauge condition,

$$\sum_{i=1}^{3} \partial_i A_i(\mathbf{x}, t) = 0. \qquad (4)$$

Let $L(\mathbf{x}, T)$ be the timelike Wilson (not Polyakov) line[1]

$$L(\mathbf{x}, T) \equiv P \exp\left[\int_0^T dx_0 g t^b A_0^b(\mathbf{x}, x_0)\right], \qquad (5)$$

for quarks in an irreducible representation r of SU(N), of dimension d_r. Here t^a is a basis of the Lie algebra of SU(N), $[t^a, t^b] = f^{abc} t^c$ in the representation r. The correlator of two Wilson lines,

$$G(R, T) \equiv d_r^{-1}\langle\text{Tr}[L^\dagger(\mathbf{x}, T)L(\mathbf{y}, T)], \qquad (6)$$

may be expressed in terms of the Hamiltonian H and the quark-pair state,

$$G(R, T) = \langle\Psi_{\bar{q}q}|e^{-(H-E_0)T}|\Psi_{\bar{q}q}\rangle, \qquad (7)$$

where E_0 is the vacuum energy, and $R = |\mathbf{x} - \mathbf{y}|$. We have

$$G(R, T) = \sum_n c_n e^{-\epsilon_n T}, \qquad (8)$$

where $\epsilon_n = E_n - E_0$ is an excitation energy and $c_n = |\langle\Psi_n|\Psi_{\bar{q}q}\rangle|^2$ is a positive coefficient. The logarithmic de-

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[1]Here we follow the discussion of [3].

rivative,

$$\bar{\epsilon}(R, T) \equiv -\frac{\partial \ln[G(R, T)]}{\partial T} = \frac{\sum_n \epsilon_n c_n e^{-\epsilon_n T}}{\sum_n c_n e^{-\epsilon_n T}}, \qquad (9)$$

yields the mean excitation energy above the vacuum energy of the quark-pair state $e^{-(H-E_0)T/2}\Psi_{\bar{q}q}$.

For small T, the Wilson line is given by

$$L(\mathbf{x}, T) = 1 + g \int_0^T dx_0 t^b A_0^b(\mathbf{x}, x_0) + \dots \qquad (10)$$

and

$$G(R, T) = 1 - T E_{\text{se}} + (g^2/d_r)\operatorname{Tr}(t^b t^c)$$
$$\times \int_0^T \int_0^T dx_0 dy_0 \langle A_0^b(\mathbf{x}, x_0) A_0^c(\mathbf{y}, y_0) \rangle + \dots, \qquad (11)$$

where E_{se} is the regularized self-energy of the two quarks. It might be thought that the integral is of order T^2. However, with

$$\langle A_0^b(\mathbf{x}, x_0) A_0^c(\mathbf{y}, y_0) \rangle = \delta^{bc} D_{A_0 A_0}(\mathbf{x} - \mathbf{y}, x_0 - y_0), \quad (12)$$

the instantaneous term in (1), proportional to $\delta(x_0 - y_0)$, gives a contribution of order T, and we obtain, for small T

$$G(R, T) = 1 - T[E_{\text{se}} - C_r V_{\text{coul}}(R)] + \dots, \qquad (13)$$

where $C_r = -t^b t^b > 0$ is the Casimir invariant in the representation r of SU(N). This gives

$$\lim_{T \to 0} \bar{\epsilon}(R, T) = E_{\text{se}} - C_r V_{\text{coul}}(R). \qquad (14)$$

[The minus sign in front of $V_{\text{coul}}(R)$ occurs because, by (1) and (12), $V_{\text{coul}}(R)$ is the color-Coulomb potential between *like* charges.] Thus the *physical* energy of the quark-pair state $\Psi_{\bar{q}q}$ is given by the instantaneous part of the gluon propagator in Coulomb gauge. Consistent with this, the color-Coulomb potential $V_{\text{coul}}(R)$ and, more generally, gA_0 are renormalization-group invariants [4].

It may be shown that the remnant gauge invariance of time-dependent gauge transformations $g(t)$ is spontaneously broken, or not, according as $V_{\text{coul}}(R)$ is not, or is, confining [3].

With these results, it is simple to establish that there is "no confinement without Coulomb confinement" [5]. One sees from (9) that, for large T, $\bar{\epsilon}(R, T)$ converges to the minimum energy of a quark pair in the representation r

$$\lim_{T \to \infty} \bar{\epsilon}(R, T) = V_r(R) + E_{\text{se}}, \qquad (15)$$

which, for a confining representation, is the energy of the flux-tube ground state plus E_{se}. One also sees from (9) that the time derivative of $\bar{\epsilon}(R, T)$ gives the negative of the variance of the excitation energy

$$\frac{\partial \bar{\epsilon}(R, T)}{\partial T} = -(\overline{\epsilon^2} - \bar{\epsilon}^2) < 0, \qquad (16)$$

which is negative. Thus $\bar{\epsilon}(R, T)$ is a monotonically decreas-

ing function of T, and so $\bar{\epsilon}(R, 0) > \bar{\epsilon}(R, \infty)$. We conclude from (14) and (15) that if the quark-pair energy $V_r(R)$ in some representation r is confining, $\lim_{R \to \infty} V_r(R) \to \infty$, then

$$V_r(R) < -C_r V_{\text{coul}}(R) \qquad R \to \infty. \qquad (17)$$

Thus if the quark-pair energy is confining for some representation r, then also the color-Coulomb potential is confining $-V_{\text{coul}}(R) \to \infty$. This analytic result strongly supports the confinement scenario in Coulomb gauge.

B. Recent numerical results

Numerical studies provide a valuable laboratory for testing various confinement scenarios. In accord with the above scenario, it was found [6] that the equal-time would-be physical gluon propagator $D_{A_i A_j}(\mathbf{k})|_{t=0}$ is indeed suppressed at small $\mathbf{k}$, while the Fourier transform $\tilde{V}_{\text{coul}}(\mathbf{k})$ is enhanced at small $\mathbf{k}$, which corresponds to a long-range, color-Coulomb potential $V_{\text{coul}}(R)$. It was subsequently found numerically [7] that the long-distance behavior of the color-Coulomb potential is consistent with a linear increase at large R,

$$V_{\text{coul}}(R) \sim \sigma_{\text{coul}} R, \qquad (18)$$

with the Coulomb string tension given approximately by $\sigma_{\text{coul}} \sim 3\sigma$. Here σ is the physical string tension between a pair of external quarks as determined from a large Wilson loop in the fundamental representation. Other studies provided additional support for the confinement scenario in Coulomb gauge, including, in particular, the consistency of Coulomb gauge and center-vortex scenarios [3,8]. These numerical studies were certainly encouraging for the confinement scenario in Coulomb gauge.

However, as in the old tale of the sorcerer's apprentice, it was unexpectedly discovered that this "confinement scenario" works also in the *deconfined* phase. Indeed it was found numerically that the long-distance behavior of $V_{\text{coul}}(R)$ is consistent with a linear increase, $\sigma_{\text{coul}} > 0$, at temperatures T above the phase transition, $T > T_c$, where the physical string tension vanishes, $\sigma = 0$ [3]. Investigation of the temperature dependence of σ_{coul} revealed that at high T in the deconfined phase, the Coulomb string tension *increases* with T, consistent with a magnetic mass [9],

$$\sigma_{\text{coul}}^{1/2}(T) \sim c g^2(T) T. \qquad (19)$$

Thus, from the numerical evidence, what has been called the Gribov confinement scenario works as well in the deconfined phase of QCD as in the confined phase.

Although this came as a surprise, it was a surprise that could have been predicted. Recall that the temperature T determines the extent, $\beta = T^{-1}$, of Euclidean space-time in the time direction. On the other hand gauge fixing to the minimal Coulomb gauge, defined in (21) below, is done at each t, so the Coulomb-gauge condition (4) holds at each

Euclidean time t and the cutoff at the Gribov horizon applies to 3-dimensional configurations $\mathbf{A}(\mathbf{x}, t)$ at each time, t, regardless of the extent β in the time direction. Thus the arguments which lead to suppression at small $\mathbf{k}$ of the space components $D_{A_i A_j}(\mathbf{k}, k_0)$ of the gluon propagator and enhancement at small $\mathbf{k}$ of the time component $D_{A_0, A_0}(\mathbf{k}, k_0)$ hold at all temperatures T, including in the deconfined phase.

It thus appears, from both numerical and analytic evidence, that what originated as a scenario for confinement provides a more general framework for QCD that holds both in the confined and deconfined phases. Although this is surprising at first, there is no paradox because the bound (17) provides a necessary condition for confinement, but not a sufficient condition.

[To see that the converse may be false—namely, that the color-Coulomb energy may diverge even though the quark-pair energy is finite—suppose that the ground-state quark-pair energy is confining when the quarks are in the fundamental representation, $V_{\text{fund}}(R) \sim R$. In this case according to the bound (17), $-V_{\text{coul}}(R) \to \infty$. But then the color-Coulomb quark-pair energy $-C_r V_{\text{coul}}(R)$ is confining in every representation r with $C_r \neq 0$. This includes the adjoint representation which, however, is not confining.]

C. Present results

In a first attempt to account for the influence of the cutoff at the Gribov horizon on the equation of state of the gluon plasma, a simple Planck formula was taken as Ansatz ([10]), with a dispersion relation given by the Gribov energy $E = (\mathbf{k}^2 + \frac{m^4}{\mathbf{k}^2})^{1/2}$, where the Gribov mass m was assumed to be temperature-independent. In the present work we develop a systematic calculational scheme in which the cutoff at the Gribov horizon is effected within the framework of a local, BRST(Becchi-Rouet-Stora-Tyutin)-invariant, renormalizable quantum field theory, and a renormalizable gap equation [BRST transformations are defined below, Eq. (35)]. The equation of state is no longer given by a simple Planck formula, and the Gribov mass becomes a calculable quantity that is temperature-dependent, $m = m(T)$. We calculate the leading correction to the Stefan-Boltzmann equation of state at high temperature due to the cutoff at the Gribov horizon.

In 1980 Lindé showed that standard and resummed finite-temperature perturbation theory suffer from infrared divergences [11]. Since then no solution has been found, although the infrared divergences may be avoided by introducing a magnetic mass for the gluon in an *ad hoc* manner. Some time ago it was suggested that these divergences arise because standard perturbation theory neglects the suppression of infrared gluons required by the proximity of the Gribov horizon in infrared directions [12]. No infrared divergence appears in the present calculation because the tree-level propagator (2) of would-be physical gluons is strongly suppressed in the infrared. The correc-

tion obtained here is of order g^6, which is precisely the order at which the Lindé divergence would otherwise occur.

We show by integrating out auxiliary fields that the local, renormalizable formulation presented here is equivalent to the nonlocal action and gap equation that had been obtained previously from the standard Coulomb-gauge action by imposing a cutoff at the Gribov horizon [13]. Determination of the number of independent renormalization constants (there are two) will be presented elsewhere [14].

D. Other approaches

Equal-time correlators in Coulomb gauge may be calculated using the Schwinger-Dyson equations of the Hamiltonian formulation [15–19]. The gluon energy obtained by variational calculation accords at both high and low momentum with the Gribov energy (3), provided that the Faddeev-Popov determinant is properly accounted for [17,18]. Indeed the Faddev-Popov determinant dominates the low-energy dynamics. A systematic study of the infrared limits of propagators and vertices has been reported recently in [19], which incorporates earlier results [20,21]. The Schwinger-Dyson equations in Coulomb and Landau gauge should be quite reliable in the infrared limit because the weight in the infrared limit is precisely given by the Faddeev-Popov determinant, $\det(-D_i \partial_i)$ with negligible contribution from the gluon wave-functional or action [22], a phenomenon known as ghost dominance [23,24]. This is used to advantage with the horizon condition and non-renormalization of ghost-gluon vertices and, as a result, the infrared limit of the Schwinger-Dyson equations in Coulomb and Landau gauges decouples from finite-momentum correlators. Moreover errors from truncation of 3-vertices have been controlled [19,25,26]. By contrast the semiperturbative (or seminonperturbative) method described here may not give accurate results in one-loop approximation for infrared quantities. Nevertheless both methods have their advantages. In particular the present approach also gives the correlators in Coulomb gauge at unequal times and finite temperature, and does not rely on a variational Ansatz for the wave-functional that would be needed to go beyond the infrared limit.

The instantaneous color-Coulomb potential has recently been used in Dyson-Schwinger and Bethe-Salpeter equations to find solutions for pseudoscalar and vector mesons [27]. Cancellation of the energy divergences of Coulomb gauge has been demonstrated at the two-loop level [28], and renormalization and cancellation of energy divergences to all orders in perturbation theory in Coulomb gauge has been elucidated recently, [29,30].

Equivalence of a cutoff at the Gribov horizon to a modified local action with auxiliary fields was first established in Landau gauge some time ago [31]. It was then shown that the horizon condition renormalizes consistently, and that the ghost propagator in Landau gauge has

a $\frac{1}{(k^2)^2}$ or dipole singularity [32]. The symmetries of the local theory in Landau gauge were exhibited and algebraic renormalizability established [33,34]. More recently, using this local action in Landau gauge, the gap equation was determined to two-loop order, and it was verified at the two-loop level that the ghost propagator has a dipole singularity in the infrared [35]. It has also been found to one-loop order that the gluon propagator in Landau gauge vanishes like k^2, and that the renormalization-group invariant coupling $\alpha_s(k)$ appropriate to the Landau gauge is finite at $k = 0$ [36]. Thus the elements of Gribov's scenario in Landau gauge [1] have been derived from a local, renormalizable, BRST-invariant action. The Landau gauge case has been reviewed in [37].

The Coulomb gauge provides a more straightforward confinement scenario than the Landau gauge (but see [38]), and for this reason Gribov and others turned to the Coulomb gauge. The Coulomb gauge is also well suited to finite-temperature calculations because the heat-bath provides a preferred Lorentz frame, so symmetries of the Coulomb gauge, which breaks manifest Lorentz invariance, are the physical symmetries of the system. Unitarity is immediate in the nonperturbative Coulomb gauge because of the equivalence to a canonical system whereas it is problematical in the nonperturbative Euclidean Landau gauge.

E. Organization of paper

In Sec. II we briefly review previous work on which this article relies, which is scattered in a number of different places. A local, renormalizable, BRST-invariant action with auxiliary ghosts and sources is introduced in Sec. III. In Sec. IV the nonperturbative Coulomb gauge is defined by a gap equation that determines a nonzero "physical value" for sources. Unitarity and confining properties of the nonperturbative Coulomb gauge are established in Sec. V. Free propagators are calculated in Sec. VI. The gap equation is evaluated in one-loop approximation in Sec. VII and solved in Sec. VIII. The free energy is calculated in Sec. IX, and in Sec. X the leading high-temperature correction to the equation of state from the cutoff at the Gribov horizon is obtained. Our conclusions are presented in Sec. XI. The auxiliary fields are integrated out in Appendix A, which results in the standard Coulomb gauge with a cutoff at the Gribov horizon. In Appendix B the partition function in Coulomb gauge with a cutoff at the Gribov horizon is expressed in canonical Hamiltonian form.

II. HOW TO CUT OFF A FUNCTIONAL INTEGRAL (OLD STUFF)

In Euclidean quantum field theory the functional integral

$$Z = \int_\Lambda dA\, \delta(\partial_i A_i)\, \det[D_i(A)\partial_i]\exp(-S_{\text{YM}}), \quad (20)$$

must be cut off at the boundary $\partial \Lambda$ of the "fundamental modular region" Λ, this being a region on the gauge-fixing surface that intersects each gauge orbit once and only once. The integrand is the familiar Faddeev-Popov weight Coulomb gauge.

In Coulomb gauge we may take Λ to be the set of configurations $A_i(\mathbf{x})$ each of which is the absolute minimum on its gauge orbit of the Hilbert norm,

$$\Lambda \equiv \{A_i(\mathbf{x}) : \|A\| \leq \|{}^g A\|\}. \quad (21)$$

Here $A_i(\mathbf{x})$ is a 3-dimensional configuration, ${}^g A_i = g^{-1}A_i g + g^{-1}\partial_i g$ is its gauge transform by a 3-dimensional local gauge transformation $g(\mathbf{x})$, and $\|A\|^2 \equiv \int d^3x |A_i(\mathbf{x})|^2$ is the Hilbert square norm. This gauge choice is done for each time t.

At a local or global minimum, the functional $F_A[g] = \|{}^g A\|^2$ is stationary at $g = 1$, and the second variation of $F_A[g]$ is a positive matrix (all its eigenvalues are positive). This translates into the Coulomb-gauge condition $\partial_i A_i = 0$, and the positivity of the 3-dimensional Faddeev-Popov operator,

$$-D_i(A)\partial_i \geq 0. \quad (22)$$

These two properties characterize a set known as the Gribov region Ω,

$$\Omega \equiv \{A: \partial_i A_i = 0, \quad \text{and} \quad -D_i(A)\partial_i \geq 0\}, \quad (23)$$

which is the set of configurations each of which is a local minimum on its gauge orbit of the Hilbert norm. The set of local minima is larger than the set Λ of global minima, $\Lambda \subset \Omega$.

We do not possess an explicit description of the set Λ of global minima. However, it has been argued from stochastic quantization [39], a geometrically correct but unwieldy quantization of gauge theory, that integration over Ω or over Λ gives the same expectation value for all n-point functions with finite n. This is consistent with Abelian or center-vortex dominance scenarios, because Abelian and center-vortex configurations lie on the common boundary of the Gribov region and the fundamental modular region [8]. Either as exact truth, or to provide a model worthy of investigation, we replace the fundamental region Λ by the Gribov region Ω in the functional integral,

$$Z = \int_\Omega dA\, \delta(\partial_i A_i)\, \det[-D_i(A)\partial_i]\exp(-S_{\text{YM}}). \quad (24)$$

The boundary $\partial\Omega$ of the Gribov region, known as the "Gribov horizon," is characterized in Euclidean Landau gauge or Coulomb gauge at fixed time (with $D \to D - 1$) by "horizon function" [31],

$$G(A) \equiv \int d^{D-1}x [gf^{abc}A_i^b(M^{-1})^{ad}gf^{dec}A_i^e$$
$$- (N^2 - 1)(D - 1)], \quad (25)$$

where $M^{ab} = -D_i^{ab}(A)\partial_i$ is the $D - 1$-dimensional

Faddeev-Popov operator, and $M^{ab}(M^{-1})^{bd}f^{\text{dec}}A_i^e = f^{aec}A_i^e$. The horizon function vanishes on the boundary, $G(A) = 0$ for $A \in \partial\Omega$, and is negative inside, $G(A) < 0$ for $A \in \Omega$. Thus the partition function may be written

$$Z = \int dA\theta[-G(A)]\delta(\partial_i A_i)\det[-D_i(A)\partial_i]\exp(-S_{\text{YM}}),$$
(26)

where $\theta(-G)$ is the step function, and a product over all times t is understood. Configuration space with, for example, a lattice cutoff is a high-dimensional space, and classical statistical mechanical arguments apply. Entropy favors population at the boundary $\partial\Omega$ where $G(A) = 0$, and in the last integral we may make the replacement $\theta[-G(A)] \to \delta[G(A)]$.[2] The horizon function $G(A)$ is a bulk quantity, like the Hamiltonian $H(A)$ in classical statistical physics. Just as the microcanonical ensemble is equivalent to the canonical ensemble, $\delta[H(A) - E] \to \exp[-\beta H(A)]$, where β is determined by the condition $\langle H\rangle = E$, likewise the microcanonical weight $\delta[G(A)]$ is equivalent to the Boltzmann weight $\exp[-\gamma G(A)]$, so partition function is given by [1,13,31]

$$Z = \int dA\delta(\partial_i A_i)\det(-D_i(A)\partial_i)\exp(-S_{\text{YM}} - \gamma S_h),$$
(27)

where γ is determined by the horizon condition

$$\langle S_h\rangle = 0.$$
(28)

Here, as a result of the product over all t, S_h is the integrated horizon function

$$S_h = \int dt G[A(t)],$$
(29)

where $G(A)$ is given in (25). Because $\langle G[A(t)]\rangle$ is independent of t, (28) is equivalent to $\langle G[A(t)]\rangle = 0$. In terms of the free energy, $W = \ln Z$, the horizon condition may be written

$$\frac{\partial W}{\partial\gamma} = 0.$$
(30)

The action S_h is nonlocal in space, but local in time. It may be expressed in terms of a local, renormalizable action by integration over auxiliary fields, as has been known for some time in the Landau gauge [31]. Precisely the same argument holds for the Coulomb gauge. It is written out in Appendix A, where we start with the local, renormalizable action that is given below and integrate out the auxiliary fields to obtain (27) and (30).

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[2]As an example of this, the radial probability distribution inside an N-dimensional sphere of radius R is given by $r^{N-1}dr$. This gets concentrated on the surface of the sphere, at $r = R$, for $N \to \infty$, thus $\int_0^R dr r^{N-1} \to \int dr\delta(R - r)$.

III. PERTURBATIVE COULOMB GAUGE

A. Faddeev-Popov action

For simplicity we shall be interested in pure SU(N) gauge theory at temperature T, but there is no obstacle to extending our considerations to include quarks. It is described by a Euclidean action which, for pure SU(N) gauge theory, is of the form

$$S = S_{\text{YM}} + s\Xi,$$
(31)

where $S_{\text{YM}} = \int d^D x\mathcal{L}_{\text{YM}}$ is the Yang-Mills action, with

$$\mathcal{L}_{\text{YM}} = \tfrac{1}{4}F_{\mu\nu}^2,$$
(32)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + gA_\mu \times A_\nu.$$
(33)

The s-exact term $s\Xi$ is defined below. Here g is the coupling constant, and we use the notation for the Lie bracket $(A \times B)^a \equiv f^{abc}A^b B^c$, where f^{abc} are the fully antisymmetric structure constants of the SU(N) group. The color indices a, b, c are taken in the adjoint representation, $a = 1, \ldots, N^2 - 1$. We shall generally suppress the color index, and leave summation over it implicit. Configurations are periodic in x_0,

$$A_\mu(x_i, x_0) = A_\mu(x_i, x_0 + \beta),$$
(34)

with period $\beta = 1/T$, where T is the temperature. The integral over x_0 always extends over one cycle, $\int dx_0 \equiv \int_0^\beta dx_0$. We are in D Euclidean dimensions. Lower case Latin indices take values, $i = 1, 2, \ldots, D - 1$, while Greek indices take values $\mu = 0, 1, \ldots, D - 1$.

In the BRST formulation there are, in addition to A_μ, a pair of Faddeev-Popov ghost fields c and $\bar{c}$ and a Lagrange-multiplier field, b, on which the BRST operator acts according to

$$sA_\mu = D_\mu c; \qquad sc = -(g/2)c \times c$$
$$s\bar{c} = ib; \qquad sb = 0.$$
(35)

It is nil-potent, $s^2 = 0$. Here D_μ is the gauge-covariant derivative in the adjoint representation, $D_\mu c \equiv \partial_\mu c + gA_\mu \times c$.

The choice of Ξ is the choice of gauge. Physics is independent of Ξ, provided that it provides a well-defined calculational scheme. The standard Coulomb gauge is defined by the choice $S_{\text{coul}} = \int d^D x\mathcal{L}_{\text{coul}}$

$$\mathcal{L}_{\text{coul}} = s\xi_{\text{coul}} = s\partial_i\bar{c}A_i = i\partial_i bA_i - \partial_i\bar{c}D_i c.$$
(36)

The Lagrange-multiplier field b imposes the Coulomb-gauge condition $\partial_i A_i = 0$.

B. Auxiliary ghosts

We first introduce the auxiliary ghosts and related sources in the perturbative Coulomb gauge where their role is trivial and superfluous. We shall then use them to

write a local theory that is equivalent to the standard Coulomb gauge with a cutoff at the Gribov horizon.

Recall that observables O are in the cohomology of s (namely, s-invariant operators $sO = 0$, modulo s-exact operators, $O \sim O + sX$.) We may freely introduce additional quartets of auxiliary ghost fields on which s acts trivially, because such fields cannot appear in the cohomology of s,

$$s\phi_B^a = \omega_B^a; \qquad s\omega_B^a = 0 \qquad s\bar{\omega}_B^a = \bar{\phi}_B^a; \qquad s\bar{\phi}_B^a = 0. \quad (37)$$

Here a labels components in the adjoint representation of the global gauge group $a = 1, \ldots, N^2 - 1$, and B is a mute index that is arbitrary for the moment. The fields ϕ_B^a and $\bar{\phi}_B^a$ are a pair of bose ghosts, while ω_B^a and $\bar{\omega}_B^a$ are fermi ghost and antighost. The BRST method [40] insures that physics is unchanged if we add to the action an s-exact term, $S_{\text{aux}} = \int d^D x \mathcal{L}_{\text{aux}} = \int d^D x s \xi_{\text{aux}}$ that depends on the auxiliary ghost fields. The auxiliary Lagrangian, adapted to the Coulomb gauge, is taken to be[3]

$$\begin{aligned} \mathcal{L}_{\text{aux}} &= s\partial_i \bar{\omega}_B^a (D_i \phi_B)^a \\ &= \partial_i \bar{\phi}_B^a (D_i \phi_B)^a - \partial_i \bar{\omega}_B^a [(D_i \omega_B)^a \\ &\quad + (gD_i c \times \phi_B)^a]. \end{aligned} \quad (38)$$

The action and Lagrangian density are now

$$S = \int d^D x \mathcal{L}, \qquad \mathcal{L} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{coul}} + \mathcal{L}_{\text{aux}}, \quad (39)$$

where $\mathcal{L}_{\text{coul}}$ and $\mathcal{L}_{\text{aux}}$ are given in (36) and (38).

This action is renormalizable by power counting. Symmetries and renormalizability of the analogous action in Landau gauge were established in [32,33], and similar considerations hold also in Coulomb gauge [14].

C. Auxiliary sources

To derive the Slavnov-Taylor identities it is standard to introduce sources K_μ and L for the BRST-transforms that are nonlinear $(K_\mu, sA_\mu) = (K_\mu, D_\mu c)$ and $(L, sc) = (L, (-g/2)c \times c)$. We also introduce auxiliary sources, namely, pairs of bosonic sources, $\bar{V}_{iB}^a$ and V_{iB}^a and pairs of fermionic sources N_{iB}^a and $\bar{N}_{iB}^a$ [32]. The extended action with all sources is defined by,

$$\begin{aligned} \Sigma &= \int d^D x \Lambda, \\ \Lambda &\equiv \mathcal{L} + K_\mu sA_\mu + Lsc + \bar{V}_i D_i \phi + sD_i \bar{\omega} V_i + D_i \bar{\omega} N_i \\ &\quad + \bar{N}_i sD_i \phi + \bar{V}_i V_i - \bar{N}_i N_i. \end{aligned} \quad (40)$$

Here the color index a and the mute index B are summed

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[3]More generally we may take the gauge-fixing term to be $s \int d^D x \partial_\kappa \bar{c} \alpha_{\kappa\lambda} A_\lambda$ and the auxiliary action to be $s \int d^D x \partial_\kappa \bar{\omega}_B^a \alpha_{\kappa\lambda} (D_\lambda \phi_B)^a$, where $\alpha_{\kappa\lambda}$ is a positive symmetric matrix that may be diagonal [41].

over and suppressed, $\bar{V}_i D_i \phi \equiv \bar{V}_{iB} D_i \phi_B$ etc. We have also added the purely source term $\bar{V}_i V_i - \bar{N}_i N_i$, which will allow us to write the horizon condition in a multiplicatively renormalizable form.

D. Quantum effective action

The partition function is given in terms of the extended action by

$$Z(J, K, L, V, \bar{V}, N, \bar{N}) = \int d\Phi \exp[-\Sigma + (J, \Phi)], \quad (41)$$

where $\Phi \equiv (A, c, \bar{c}, \varphi, \omega, \bar{\omega}, \bar{\varphi})$ represents the set of all fields, and J the corresponding sources. The free energy is defined by $W(J, K, L, V, \bar{V}, N, \bar{N}) \equiv \ln Z$, from which the quantum effective action $\Gamma(\Phi, K, L, V, \bar{V}, N, \bar{N})$ is obtained by Legendre transformation.

The quantum effective action satisfies the same Slavnov-Taylor identity in Coulomb gauge as in Landau gauge and the proof is the same as in Landau gauge [32],

$$\int d^D x \left(\frac{\delta\Gamma}{\delta K} \frac{\delta\Gamma}{\delta A} + \frac{\delta\Gamma}{\delta L} \frac{\delta\Gamma}{\delta c} + ib \frac{\delta\Gamma}{\delta \bar{c}} + \omega \frac{\delta\Gamma}{\delta \phi} \right. \\ \left. + \bar{\phi} \frac{\delta\Gamma}{\delta \bar{\omega}} - N \frac{\delta\Gamma}{\delta V} - \bar{V} \frac{\delta\Gamma}{\delta \bar{N}} \right) = 0. \quad (42)$$

We will shortly assign nonzero "physical" values to the sources V and $\bar{V}$. This gives an additional contribution $-\bar{V}_{\text{ph}} \frac{\delta\Gamma}{\delta \bar{N}}$ to the usual Slavnov-Taylor identity.

IV. NONPERTURBATIVE COULOMB GAUGE

In this section we describe a local theory which is shown in Appendix A to be equivalent to the standard Coulomb gauge with a cutoff at the Gribov horizon.

A. Modified local action

The mute index B on the auxiliary ghosts is now defined to be the pair of indices $B \equiv (c, \mu)$, where c labels components in the adjoint representation of the global gauge group, $c = 1, \ldots, N^2 - 1$, and μ is a Lorentz index. With this assignment, the BRST operator acts according to

$$\begin{aligned} s\phi_\mu^{ac} &= \omega_\mu^{ac}; \qquad s\omega_\mu^{ac} = 0 \\ s\bar{\omega}_\mu^{ac} &= \bar{\phi}_\mu^{ac}; \qquad s\bar{\phi}_\mu^{ac} = 0, \end{aligned} \quad (43)$$

and the auxiliary Lagrangian (38) reads

$$\begin{aligned} \mathcal{L}_{\text{aux}} &= \partial_i \bar{\phi}_\mu^{ab} (D_i \phi_\mu)^{ab} - \partial_i \bar{\omega}_\mu^{ab} [(D_i \omega_\mu)^{ab} \\ &\quad + (gD_i c \times \phi_\mu)^{ab}]. \end{aligned} \quad (44)$$

[The gauge-covariant derivative and the Lie commutator act on the first color index only, because the second color index is mute, thus $(D_i \phi_\mu)^{ac} = \partial_i \phi_\mu^{ac} + g(A_i \times \phi_\mu)^{ac}$ where $(A_i \times \phi_\mu)^{ab} \equiv f^{acd} A_i^c \phi^{db}$]. We further stipulate that the "physical value" of the sources V and $\bar{V}$ is not zero but rather at

$$V_{i\mu}^{ab} = V_{\text{ph},i\mu}^{ab} \equiv -\gamma^{1/2}\delta_{i\mu}\delta^{ab},$$
$$\bar{V}_{i\mu}^{ab} = \bar{V}_{\text{ph},i\mu}^{ab} \equiv \gamma^{1/2}\delta_{i\mu}\delta^{ab}, \tag{45}$$

where γ is a constant with engineering dimension m^4 that will be determined shortly. All other sources are set to 0, in particular $N_{\text{ph}} = \bar{N}_{\text{ph}} = 0$. For purposes of a semiperturbative expansion in g, described below, we write

$$\gamma^{1/2} = \frac{m^2}{(2N)^{1/2}g}, \tag{46}$$

and we take m to be of order g^0. This yields the physical local action

$$S_{\text{ph}} = \Sigma|_{V=V_{\text{ph}},\bar{V}=\bar{V}_{\text{ph}},K=L=N=\bar{N}=0} = \int d^D x \mathcal{L}_{\text{ph}}, \tag{47}$$

where

$$\mathcal{L}_{\text{ph}} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{coul}} + \mathcal{L}_{\text{aux}} + \mathcal{L}_m, \tag{48}$$

$$\mathcal{L}_m = -\frac{m^4}{2Ng^2}(D-1)(N^2-1)$$
$$+ \frac{m^2}{(2N)^{1/2}g}[D_i\phi_i - s(D_i\bar{\omega}_i)]^{aa}, \tag{49}$$

and $s(D_i\bar{\omega}_i) = D_i\bar{\phi}_i + g(D_i c \times \bar{\omega}_i)$. The vacuum free energy is the free energy $W = \ln Z$, with sources set to their physical value,

$$W_{\text{ph}}(m) = -\Gamma_{\text{ph}}(m) \equiv -\Gamma|_{V=V_{\text{ph}},\bar{V}=\bar{V}_{\text{ph}}}. \tag{50}$$

B. Renormalizable horizon condition

The specification of the nonperturbative Coulomb gauge is completed by requiring that m be a stationary point of the vacuum free energy

$$\frac{\partial W_{\text{ph}}}{\partial m} = -\frac{\partial \Gamma_{\text{ph}}}{\partial m} = 0. \tag{51}$$

This is a nonperturbative gap equation that determines m as a function of Λ_{QCD}, and eventually also of the temperature T. As shown in Appendix A, this equation expresses the "horizon condition" obtained previously [13]. It is compatible with multiplicative renormalization because m renormalizes multiplicatively, as in Landau gauge [32].

From $W_{\text{ph}} = \ln Z_{\text{ph}}$, and $Z_{\text{ph}} = \int d\Phi \exp(-S_{\text{ph}})$, the horizon condition may be written

$$\left\langle \frac{\partial S_{\text{ph}}}{\partial m} \right\rangle = 0, \tag{52}$$

where the expectation value is defined by the action S_{ph}. By (49), this reads

$$\langle D_i(\phi_i - \bar{\phi}_i)^{aa} - (gD_i c \times \bar{\omega}_i)^{aa} \rangle$$
$$= \frac{2^{1/2}m^2}{N^{1/2}g}(N^2-1)(D-1). \tag{53}$$

The second term vanishes,

$$\langle (gD_i c \times \bar{\omega}_i)^{aa} \rangle = 0, \tag{54}$$

because there is no $\bar{c}\omega$ term in the action (39) and (48), and translation invariance implies that the terms $\partial_i\phi$ and $\partial_i\bar{\phi}$ do not contribute to (53). The horizon condition reads

$$\langle f^{abc}A_i^b(\phi - \bar{\phi})_i^{ca} \rangle = \frac{2^{1/2}m^2}{N^{1/2}g^2}(D-1)(N^2-1). \tag{55}$$

V. PROPERTIES OF THE NONPERTURBATIVE COULOMB GAUGE

A. Unitarity

In Appendix B, the equivalence to a canonical system is established, and with it, unitarity. Unitarity is already manifest at tree level. Indeed the only poles in k_0 of the tree-level propagators (given below) occur at $k_0^2 + \mathbf{k}^2 + \frac{m^4}{\mathbf{k}^2} = 0$, corresponding to the Gribov [1] energy $E = (\mathbf{k}^2 + \frac{m^4}{\mathbf{k}^2})^{1/2}$. This real energy is consistent with a Hermitian Hamiltonian. By contrast, the tree-level gluon propagator of the nonperturbative Landau gauge [31] has poles at $k^2 + \frac{m^4}{k^2} = 0$, where $k^2 = k_0^2 + \mathbf{k}^2$. This corresponds to complex energy $E = (\mathbf{k}^2 \pm im^2)^{1/2}$.

B. A confining property

Integration over Lagrange-multiplier field b produces a functional δ-function that imposes the gauge constraint $\partial_i A_i = 0$. Suppose this integration is done, so A_i is identically transverse. The auxiliary field $\bar{\phi}_\mu^{ac}$ is also a Lagrange multiplier, and integration over it yields a functional δ-function that imposes the constraint

$$-\partial_i D_i^{ab}\phi_j^{bc} - \gamma^{1/2}gf^{cba}A_j^b = 0. \tag{56}$$

Quantization is done in a periodic box of finite spatial volume V. Integration of the last equation over V at fixed time t kills the first term, which is a spatial divergence, and yields a new constraint satisfied by A_i,

$$\int_V d^3x A_j^b = 0. \tag{57}$$

This states that the zero-momentum component of A_i vanishes. This is an additional gauge condition on A_i that expresses a confining property: a gluon with 3-momentum $\mathbf{k} = 0$ is forbidden. In accordance with this condition, the propagator $D_{A_iA_j}(\mathbf{k}, k_0)$, calculated below, vanishes with $\mathbf{k}$.

C. Automatic cutoff at Gribov horizon

In Appendix A it is shown that after integrating out the auxiliary fields the action acquires a term proportional to the "horizon function"

$$S_h \equiv \int d^D x [g f^{abc} A_i^b (M^{-1})^{ad} g f^{\text{dec}} A_i^e$$
$$- (N^2 - 1)(D - 1)]. \tag{58}$$

If one expands in eigenfunctions of the Faddeev-Popov operator $M(A)$, the functional weight acquires the factor $\exp(-c^2/\lambda_0)$, where $\lambda_0(\mathbf{A})$ is the lowest nontrivial eigenvalue of $M(A)$. By definition of the Gribov region Ω, $\lambda_0(\mathbf{A})$ vanishes as $\mathbf{A}$ approaches the boundary $\partial\Omega$ of the Gribov region from within. This constitutes an "automatic," non-analytic cutoff of the functional integral at the Gribov horizon.

VI. FREE PROPAGATORS

We now develop a semipertubative calculational scheme. For this purpose we treat m as an independent parameter of order g^0, and calculate all quantities perturbatively, including the gap equation, to a given order in g. Then $m = m(g, T)$ is determined by solving the gap Eq. (55) nonperturbatively.

We expand the action (48) in powers of g,

$$S = S_{-2} + S_0 + \dots. \tag{59}$$

The leading term is of order g^{-2},

$$S_{-2} \equiv -\frac{m^4}{2Ng^2}(N^2 - 1)(D - 1)L^3\beta, \tag{60}$$

where the spatial quantization volume is $V = L^3$, and the time extent $\beta = T^{-1}$. Although this term is independent of the fields, it should not be ignored because, when the gap equation is solved for $m = m(T)$, it gives a T-dependent contribution to the free energy. The terms in the action of order g^0,

$$S_0 = S_{0,\text{YM}} + S_{0,1} + S_{0,2} + S_{0,3}, \tag{61}$$

are quadratic in the fields and determine the "free" propagators,

$$S_{0,\text{YM}} \equiv \int d^D x \frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2, \tag{62}$$

$$S_{0,1} \equiv \int d^D x (i\partial_i b^a A_i^a - \partial_i \bar{c}^a \partial_i c^a), \tag{63}$$

$$S_{0,2} \equiv \int d^D x (\partial_i \bar{\varphi}_\mu^{ab} \partial_i \varphi_\mu^{ab} - \partial_i \bar{\omega}_\mu^{ab} \partial_i \omega_\mu^{ab}), \tag{64}$$

$$S_{0,3} \equiv \int d^D x \frac{m^2}{(2N)^{1/2}} f^{abc} A_i^b (\varphi_i - \bar{\varphi}_i)^{ca}. \tag{65}$$

The term $S_{0,3}$ causes a mixing of the zero-order transverse gluon and bose-ghost propagators.

To calculate the free propagators, we define the field that mixes with A_i^b,

$$\psi_j^b \equiv \frac{i}{(2N)^{1/2}} f^{abc} (\varphi_j^{ca} - \bar{\varphi}_j^{ca}). \tag{66}$$

The orthogonal component $\frac{1}{(2N)^{1/2}} f^{abc}(\varphi_j^{ca} + \bar{\varphi}_j^{ca})$ and other components of φ and $\bar{\varphi}$ do not mix with A_i. Moreover because of the Lagrange-multiplier term $i\partial_j b A_j$, only the 3-dimensionally transverse part A_j^T of A_j contributes to propagators, and it mixes only with the transverse part ψ_j^T of ψ_j. Consequently, the free propagators of the fields A_j and ψ_j are determined by the mixed action

$$S_0(A, \psi) \equiv \int d^D x ((1/2)[(\dot{A}_j^T)^2 + (\partial_i A_j^T)^2 + (\partial_i \psi_j^T)^2]$$
$$- im^2 A_j^T \psi_j^T), \tag{67}$$

which corresponds to the matrix in momentum space,

$$\begin{pmatrix} \mathbf{k}^2 + k_0^2 & -im^2 \\ -im^2 & \mathbf{k}^2 \end{pmatrix},$$

with determinant

$$\Delta_1 = (\mathbf{k}^2 + k_0^2)\mathbf{k}^2 + m^4. \tag{68}$$

The free propagators are given by

$$D_{A_i A_j}(x - y) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} T \sum_{k_0} \exp[ik \cdot (x - y)] P_{ij}(\mathbf{k})$$
$$\times \frac{\mathbf{k}^2}{(\mathbf{k}^2 + k_0^2)\mathbf{k}^2 + m^4}, \tag{69}$$

$$D_{A_i \psi_j}(x - y) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} T \sum_{k_0} \exp[ik \cdot (x - y)] P_{ij}(\mathbf{k})$$
$$\times \frac{im^2}{(\mathbf{k}^2 + k_0^2)\mathbf{k}^2 + m^4}, \tag{70}$$

$$D_{\psi_i \psi_j}(x - y)\rangle = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} T \sum_{k_0} \exp[ik \cdot (x - y)] P_{ij}(\mathbf{k})$$
$$\times \frac{\mathbf{k}^2 + k_0^2}{(\mathbf{k}^2 + k_0^2)\mathbf{k}^2 + m^4}. \tag{71}$$

Here $k_0 = 2\pi n/\beta$ are the Matsubara frequencies, where n is any integer, $T = 1/\beta$, and $P_{ij}(\mathbf{k}) = \delta_{ij} - \hat{k}_i \hat{k}_j$ is the transverse projector. We have suppressed the trivial color factor δ^{bc}. The gap Eq. (55) reads

$$-i\langle A_j^c(0)\psi_j^c(0)\rangle = \frac{m^2}{Ng^2}(D - 1)(N^2 - 1). \tag{72}$$

VII. GAP EQUATION IN ONE-LOOP APPROXIMATION

When the left-hand side of the gap equation is evaluated to zeroth order in g, using the mixed propagator (70), it reads

$$\int \frac{d^{D-1}k}{(2\pi)^{D-1}} T\sum_{k_0} \frac{D-2}{(\mathbf{k}^2 + k_0^2)\mathbf{k}^2 + m^4} = \frac{D-1}{Ng^2}, \quad (73)$$

where we used $P_{ii}(\mathbf{k}) = D - 2$.

To evaluate the sum over Matsubara frequencies,

$$Q \equiv T\sum_{k_0} \frac{1}{(\mathbf{k}^2 + k_0^2)\mathbf{k}^2 + m^4}$$
$$= \frac{1}{E^2\mathbf{k}^2\beta} \sum_{n=0,\pm 1...} \frac{1}{1 + (2\pi n/\beta E)^2}, \quad (74)$$

where $E \equiv (\mathbf{k}^2 + \frac{m^4}{\mathbf{k}^2})^{1/2}$, we use the identity

$$\sinh\theta = \theta \prod_{n=1}^{\infty} [1 + (\theta/n\pi)^2] \quad (75)$$

or

$$\ln\sinh\theta = \ln\theta + \sum_{n=1}^{\infty} \ln[1 + (\theta/n\pi)^2]. \quad (76)$$

This gives upon differentiation

$$\frac{\cosh\theta}{\sinh\theta} = \frac{1}{\theta} + \frac{2}{\theta} \sum_{n=1}^{\infty} \frac{1}{1 + (n\pi/\theta)^2}, \quad (77)$$

and we obtain for the sum over Matsubara frequencies

$$Q = \frac{1}{2\mathbf{k}^2 E} \frac{\cosh(\beta E/2)}{\sinh(\beta E/2)} = \frac{1}{2\mathbf{k}^2 E}\left(1 + \frac{2}{\exp(\beta E) - 1}\right). \quad (78)$$

The first term is the value of Q at $T = 0$ and the second is a Planck-type finite-temperature correction. The gap equation reads

$$\mu^{4-D} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{(D-2)}{2\mathbf{k}^2 E}\left(1 + \frac{2}{\exp(\beta E) - 1}\right) = \frac{D-1}{Ng^2}, \quad (79)$$

where we have made the substitution $g^2 \rightarrow g^2 \mu^{4-D}$.

The first term on the left-hand side,

$$I \equiv \mu^{4-D} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{(D-2)}{2\mathbf{k}^2 E}, \quad (80)$$

converges for $D < 4$. The integral is readily evaluated by taking $y = |\mathbf{k}|^4$ as integration variable, with the result

$$I = \left(\frac{m}{\mu}\right)^{D-4} \frac{D-2}{2} \frac{1}{(4\pi)^{D/2}} \frac{\Gamma((D-2)/4)\Gamma((4-D)/4)}{\Gamma((D-1)/2)}. \quad (81)$$

In terms of $\epsilon \equiv (4 - D)/2$, this gives

$$I = \frac{1}{4\pi^2}\left[\frac{1}{\epsilon} - \ln\left(\frac{m^2}{\mu^2}\right) + \ln(4\pi)\right.$$
$$\left. + \frac{1}{2}\left(\Gamma'(1) + \frac{\Gamma'(1/2)}{\Gamma(1/2)}\right) + 1\right], \quad (82)$$

with neglect of terms that vanish with ϵ. From the identity [42],

$$\frac{\Gamma'(1/2)}{\Gamma(1/2)} = \frac{\Gamma'(1)}{\Gamma(1)} - 2\ln 2, \quad (83)$$

one obtains

$$I = \frac{1}{4\pi^2}\left[\frac{1}{\epsilon} - \ln\left(\frac{m^2}{\mu^2}\right) + \ln(4\pi) - \gamma - \ln 2 + 1\right], \quad (84)$$

where $\gamma \equiv -\Gamma'(1) = 0.577215\ldots$. Upon making the standard $\overline{MS}$ subtraction, one obtains

$$I \rightarrow \frac{1}{4\pi^2} \ln\left(\frac{e}{2}\frac{\mu^2}{m^2}\right). \quad (85)$$

After this substitution, the gap Eq. (79) reads

$$\frac{1}{4}\ln\left(\frac{e}{2}\frac{\mu^2}{m^2}\right) + \int_0^\infty \frac{dx}{u} \frac{1}{\exp(m\beta u) - 1} = \frac{3\pi^2}{Ng^2(T)}, \quad (86)$$

where $u \equiv (x^2 + \frac{1}{x^2})^{1/2}$.

VIII. SOLUTION OF ONE-LOOP GAP EQUATION

To solve the gap equation for m, we change unknown from m to $m^* \equiv m/T$ so it reads

$$f(m^*) \equiv \frac{1}{2}\ln\left(\frac{1}{m^*}\right) + \int_0^\infty \frac{dx}{u} \frac{1}{\exp(m^*u) - 1} = y, \quad (87)$$

where

$$y \equiv \frac{3\pi^2}{Ng^2(T)} - \frac{1}{4}\ln\left(\frac{e\mu^2\beta^2}{2}\right). \quad (88)$$

The function $f(m^*)$ decreases monotonically, $f'(m^*) < 0$, with $f(0) = \infty$ and $f(\infty) = -\infty$, so this equation always has a unique solution

$$m^* = \frac{m}{T} = h(y). \quad (89)$$

Moreover by the inverse function theorem, $h(y)$ is analytic, so in this approximation there is no phase transition.

Asymptotically at high T, the running coupling $g(T)$ in the $\overline{MS}$-scheme is given by

$$\frac{1}{g^2(T)} = \frac{11N}{24\pi^2} \ln\left(\frac{2\pi T}{\Lambda_{\overline{MS}}}\right), \quad (90)$$

where $\Lambda_{\overline{MS}}$ is a physical QCD mass scale. We specialize to high-temperature T where $g(T)$ is small, so the one-loop gap equation should be a good approximation. We also

suppose that at high-temperature $\mu = \mu(T) \approx 2\pi T$, so

$$y(T) \approx \frac{3\pi^2}{Ng^2(T)} - \frac{1}{4}\ln(2e\pi^2). \qquad (91)$$

According to (87) and (88), large T or small g implies small m^*, and for small m^* we have

$$\int_0^\infty \frac{dx}{u}\frac{1}{\exp(m^*u) - 1} \to \frac{1}{m^*}\int_0^\infty dx\frac{x^2}{x^4 + 1} = \frac{\pi}{2^{3/2}m^*}, \qquad (92)$$

so the gap equation at high T simplifies to

$$\frac{1}{2}\ln\left(\frac{1}{m^*}\right) + \frac{\pi}{2^{3/2}m^*} = y(T). \qquad (93)$$

To leading order at high T or small m^*, the first term on the left is negligible compared to the second, and we obtain, with $m^* = \frac{m}{T}$ and, with neglect of the second term in (91),

$$m(T) = \frac{N}{2^{3/2}3\pi}g^2(T)T \qquad T \to \infty. \qquad (94)$$

Thus, in the high-temperature limit, $m(T)$ approaches a standard magnetic mass, $m \sim g^2(T)T$.

IX. FREE ENERGY

To order g^0, the free energy $W = \ln Z$ is given by

$$\exp W = \int d\Phi \exp(-S_{-2} - S_0), \qquad (95)$$

where S_{-2}, Eq. (60), is field-independent and of order g^{-2}, while S_0, Eq. (61), is quadratic in the fields. We obtain

$$W = W_{-2} + W_0 \qquad (96)$$

where, for $D - 1 = 3$

$$W_{-2} = -S_{-2} = \frac{3m^4}{2Ng^2}(N^2 - 1)L^3\beta \qquad (97)$$

and

$$\exp W_0 = \int d\Phi \exp(-S_0). \qquad (98)$$

To evaluate W_0 we observe first that the Faddeev-Popov ghost pair contributes a factor $\mathbf{k}^2$ which is cancelled by the two factors of $|\mathbf{k}|^{-1}$ that come from the A_0 and b integrations. Moreover all auxiliary bose and fermi ghost pairs with the same action give contributions to W^0 that cancel. Each single 4-momentum mode of the action $S^0(A, \psi)$, Eq. (67), contributes $\Delta_1^{-1/2}$ to $\exp W_0$, where, by (68), $\Delta_1 = (\mathbf{k}^2 + k_0^2)\mathbf{k}^2 + m^4$. Corresponding to the bose-ghost ψ_j^b that mixes with A_j^b is an otherwise unpaired fermi ghost mode that contributes $\Delta_2^{1/2}$, where $\Delta_2 = \mathbf{k}^2$. The net result is that for each 4-momentum mode of the A_i^b field we obtain a contribution $(\Delta_1/\Delta_2)^{-1/2}$, where $\Delta_1/\Delta_2 = k_0^2 + E^2$, and $E = (\mathbf{k}^2 + \frac{m^4}{\mathbf{k}^2})^{1/2}$. There is an infinite product over

all frequencies k_0 for each 3-momentum mode of the A_i field. For each 3-momentum $\mathbf{k}$ of the A_i^b field the result of this infinite product is the Planck partition function for a single mode

$$\exp[W_0(\mathbf{k})] = \sum_{n=0}^\infty \exp(-nE\beta) = [1 - \exp(-E\beta)]^{-1}, \qquad (99)$$

$$W_0(\mathbf{k}) = -\ln[1 - \exp(-E\beta)]. \qquad (100)$$

The sum over the $N^2 - 1$ color modes, the 2 degrees of transverse polarization, and over all 3-momentum modes $\mathbf{k}$ yields, with $\sum_{\mathbf{k}} \to V\int\frac{d^3k}{(2\pi)^3}$,

$$\begin{aligned} W_0 &= -2(N^2 - 1)V\int\frac{d^3k}{(2\pi)^3}\ln[1 - \exp(-E\beta)] \\ &= -\frac{(N^2 - 1)V}{\pi^2}\int_0^\infty dk k^2\ln[1 - \exp(-E\beta)] \\ &= \frac{(N^2 - 1)V\beta}{3\pi^2}\int_0^\infty \frac{dk(k^4 - m^4)}{E[\exp(E\beta) - 1]}, \end{aligned} \qquad (101)$$

where $E = E(k) = (k^2 + \frac{m^4}{k^2})^{1/2}$. We add the term W_{-2} and obtain to order g^0 the free energy per unit volume, $w = W/V$,

$$w = (N^2 - 1)\beta\left(\frac{3m^4}{2Ng^2} + \frac{1}{3\pi^2}\int_0^\infty \frac{dk(k^4 - m^4)}{E[\exp(E\beta) - 1]}\right). \qquad (102)$$

Here $m = m(T)$ is the solution (94) of the gap equation.

X. EQUATION OF STATE AT HIGH TEMPERATURE

We now evaluate w in the high-temperature limit, where $g(T)$ is small, and our expansion should be reliable. From (94) we obtain

$$w = (N^2 - 1)\left(\frac{N^3 g^6(T)}{2^7 3^3 \pi^4} + \frac{1}{3\pi^2}K(\eta)\right)T^3, \qquad (103)$$

where

$$K(\eta) \equiv \int_0^\infty \frac{dy(y^4 - \eta)}{u(\exp u - 1)}, \qquad (104)$$

$$u \equiv \left(y^2 + \frac{\eta}{y^2}\right)^{1/2}, \qquad (105)$$

and

$$\eta \equiv \left(\frac{m}{T}\right)^4 = \left(\frac{Ng^2(T)}{2^{3/2}3\pi}\right)^4 \qquad (106)$$

is a small parameter.

The lowest-order expression for $K(\eta)$ at small η is obtained by setting $\eta = 0$,

$$K(0) = \int_0^\infty \frac{dy\, y^3}{\exp y - 1} = \frac{\pi^4}{15}, \qquad (107)$$

which gives the Stefan-Boltzmann free energy. To evaluate the leading correction to $K(0)$, we use

$$\frac{\partial K}{\partial \eta} = - \int_0^\infty dy \left(\frac{1}{u(\exp u - 1)} + \frac{y^4 - \eta}{2u^3 y^2 (\exp u - 1)} \right.$$
$$\left. + \frac{(y^4 - \eta)\exp u}{2u^2 y^2 (\exp u - 1)^2} \right). \qquad (108)$$

We pose $y = \eta^{1/4} x$ and obtain

$$\frac{\partial K}{\partial \eta} = - \int_0^\infty dx \left(\frac{1}{v[\exp(\eta^{1/4} v) - 1]} \right.$$
$$+ \frac{x^4 - 1}{2v^3 x^2 [\exp(\eta^{1/4} v) - 1]}$$
$$\left. + \frac{\eta^{1/4}(x^4 - 1)\exp(\eta^{1/4} v)}{2v^2 x^2 [\exp(\eta^{1/4} v) - 1]^2} \right), \qquad (109)$$

where $v \equiv (x^2 + \frac{1}{x^2})^{1/2}$. To leading order in η this gives

$$\frac{\partial K}{\partial \eta} = \frac{-1}{\eta^{1/4}} \int_0^\infty dx \left(\frac{1}{v^2} + \frac{x^4 - 1}{v^4 x^2} \right) = \frac{-3\pi}{4 \cdot 2^{1/2} \cdot \eta^{1/4}}. \qquad (110)$$

This identity, $K = K(0) + \int_0^\eta \frac{\partial K}{\partial \eta}$, yields

$$K(\eta) = \frac{\pi^4}{15} - \frac{\pi \eta^{3/4}}{2^{1/2}}. \qquad (111)$$

We obtain for the free energy

$$w = (N^2 - 1) \left(\frac{N^3 g^6}{2^7 3^3 \pi^4} + \frac{\pi^2}{45} - \frac{N^3 g^6}{2^5 3^4 \pi^4} \right) T^3$$
$$= (N^2 - 1) \left(\frac{\pi^2}{45} - \frac{N^3}{10\,368 \pi^4} g^6(T) \right) T^3, \qquad (112)$$

which gives the leading correction to the Stefan-Boltzmann limit from the cutoff at the Gribov horizon.

The equation of state of the gluon plasma follows from the thermodynamic formulas for the energy per unit volume and pressure,

$$e = - \frac{\partial w}{\partial \beta}; \qquad p = \frac{w}{\beta}, \qquad (113)$$

and entropy per unit volume, $s = \frac{e+p}{T}$. To calculate the energy density, we use $-\beta \frac{\partial g}{\partial \beta} = T \frac{\partial g}{\partial T} = \beta\text{-function} = O(g^3)$, which is of higher order. We thus obtain for the energy density and pressure at high temperature,

$$e = 3p = (N^2 - 1) \left(\frac{\pi^2}{15} - \frac{N^3}{3456 \pi^4} g^6(T) \right) T^4, \qquad (114)$$

with $s = \frac{4e}{3T}$.

This provides the leading correction to the equation of state of the gluon plasma at high T that comes from the cutoff at the Gribov horizon. Numerically it is small, whereas the correction of order g^6 is divergent when calculated perturbatively [11]. To this result must be added the perturbative contributions, including resummations [43]. Since standard, resummed perturbation theory diverges at order g^6, which is precisely the order of the correction we have found, the result obtained here is consistent with standard perturbative calculations.

XI. CONCLUSION

We have developed a semiperturbative method of calculation. It uses the systematic, perturbative expansion of local, renormalizable quantum field theory to calculate all quantities, including the gap equation. The gap equation is then solved nonperturbatively for the Gribov mass $m(T)$.

We have applied this method to calculate the leading correction to the Stefan-Boltzmann equation of state of the gluon plasma at high T that comes from the cutoff at the Gribov horizon. It is of order g^6. However we have not attempted to calculate the additional perturbative corrections to the Stefan-Boltzmann law to this order [43]. Significantly, $g^6(T)$ is precisely the order at which resummation of ordinary perturbation theory fails due to infrared divergences, as was first shown by Lindé [11]. Thus our calculation does not contradict the finite results from ordinary perturbation theory and its resummation, but instead gives a finite result precisely where ordinary perturbation theory breaks down.

APPENDIX A: ACTION IN TERMS OF HORIZON FUNCTION

In this appendix we integrate out auxiliary fields to express the action in terms of the nonlocal "horizon function" that effects the cutoff at the Gribov horizon. It is nonlocal in space, but local in time.

We start with the action (48). It contains cross terms that involve $\bar{\omega}$ but none with ω. These cross terms are cancelled by an appropriate shift of ω. We then integrate out the quartet ω_0^{ab}, $\bar{\omega}_0^{ab}$, φ_0^{ab}, $\bar{\varphi}_0^{ab}$ that appears in the action in the expression

$$I_0 = \int d^D x (\bar{\varphi}_0^{ac} M^{ab} \varphi_0^{bc} - \bar{\omega}_0^{ac} M^{ab} \omega_0^{bc}), \qquad (A1)$$

where $M^{ab} = -\partial_i D_i^{ab}$ is the Faddeev-Popov operator. This

gives

$$\int d\omega_0 d\bar{\omega}_0 d\varphi_0 d\bar{\varphi}_0 \exp(-I_0) = 1, \qquad \text{(A2)}$$

because from the fermi ghost pairs ω_0^{ab}, $\bar{\omega}_0^{ab}$, we get $(\det M)^{N^2-1}$ and from the bose-ghost pairs φ_0^{ab}, $\bar{\varphi}_0^{ab}$ we get $1/(\det M)^{N^2-1}$, which cancel. (This shows that the auxiliary quartet φ_0, ω_0, $\bar{\omega}_0$, $\bar{\varphi}_0$ plays no dynamical role but it was included because it may be useful to control Lorentz invariance.)

We now integrate over the remaining auxiliary fermi ghost pairs and obtain

$$\int d\omega_j^{ab} d\bar{\omega}_j^{ab} \exp(\bar{\omega}_j^{ac}, M^{ab}\omega_j^{bc}) = (\det M)^{(N^2-1)(D-1)}. \quad \text{(A3)}$$

The remaining auxiliary bose-ghost pairs appear only in the action

$$S_{\bar{\varphi},\varphi} = \int d^D x[\bar{\varphi}_j^{ac} M^{ab}\varphi_j^{bc} + \gamma^{1/2}gf^{abc}A_j^b(\varphi_j^{ca} - \bar{\varphi}_j^{ca})],$$
$$\text{(A4)}$$

where we have integrated by parts. The integral over these ghosts is done by completing the square with the result

$$\int d\varphi_j^{ab} d\bar{\varphi}_j^{ab} \exp(-S_{\bar{\varphi},\varphi}) = (\det M)^{-(N^2-1)(D-1)}$$
$$\times \exp(-\gamma S_{h1}), \qquad \text{(A5)}$$

where

$$S_{h1} \equiv \int d^D x g f^{abc} A_i^b (M^{-1})^{ad} g f^{dec} A_i^e. \qquad \text{(A6)}$$

The powers of $\det M$ from the fermi- and bose-ghost integrations cancel, so the net result from integrating over all the auxiliary ghost pairs is simply $\exp(-\gamma S_{h1})$.

The only remaining dependence on γ occurs in the constant term in $\mathcal{L}_m$, Eq. (49). We combine S_{h1} with the constant term and obtain the nonlocal horizon function

$$S_h \equiv \int d^D x[gf^{abc}A_i^b(M^{-1})^{ad}gf^{dec}A_i^e$$
$$- (N^2 - 1)(D - 1)]. \qquad \text{(A7)}$$

The integral over the Faddeev-Popov ghosts, c and $\bar{c}$, gives $\det M$ as usual, and the integral over the Nakanishi-Lautrup field gives Coulomb-gauge condition, $\delta(\partial_i A_i)$. The partition function now coincides with (27). Moreover, the condition (51) that m, or γ, be a stationary point of the free energy $W = \ln Z$ now reads

$$\frac{\partial W}{\partial \gamma} = -\langle S_h \rangle = 0, \qquad \text{(A8)}$$

which coincides with (28) and (30). This establishes that the theory described by local, renormalizable Lagrangian (48) supplemented by the gap Eq. (51) is equivalent to the standard Coulomb gauge with a cutoff at the Gribov horizon, as asserted in Sec. II.

APPENDIX B: CANONICAL COULOMB-GAUGE HAMILTONIAN

In this appendix we express the action in terms of the 3-dimensionally transverse field A_i^T and its canonical momentum π_i^T.

Although S_h is nonlocal in space, it is local in time, because the 3-dimensional Faddeev-Popov operator M, and thus also its inverse M^{-1}, is local in time. Moreover S_h depends only on A_i but not on its time derivative $\dot{A}_i$. The latter appears only in $S_{\text{YM}} = \int d^D x \frac{1}{4} F_{\mu\nu}^2$, where $F_{\mu\nu}^2 = 2F_{0i}^2 + F_{ij}^2$ and $F_{0i} = \dot{A}_i - D_i A_0$. This allows us to introduce canonical momenta, which are the color-electric fields π_i^a, by the Gaussian identity

$$\exp(-S_{\text{YM}}) = \int d\pi_i \exp\left(-\int d^D x\left[\frac{1}{2}\pi_i^2\right.\right.$$
$$\left.\left. + i\pi_i(\dot{A}_i - D_i A_0) + \frac{1}{4}F_{ij}^2\right]\right). \quad \text{(B1)}$$

Integration over A_0 imposes Gauss's law in the form $\delta(D_i\pi_i)$, which gives

$$Z = \int dA_i d\pi_i \det M \delta(D_i\pi_i)\delta(\partial_i A_i)$$
$$\times \exp\left(-\int d^D x\left[\frac{1}{2}\pi_i^2 + i\pi_i\dot{A}_i + \frac{1}{4}F_{ij}^2\right] - \gamma S_h\right). \quad \text{(B2)}$$

We now separate transverse and longitudinal parts, $A_i = A_i^T - \partial_i\sigma$ and $\pi_i = \pi_i^T - \partial_i U$, where $\partial_i A_i^T = \partial_i\pi_i^T = 0$, and U is the color-Coulomb field, and obtain

$$\int dA_i d\pi_i \det M \delta(D_i\pi_i)\delta(\partial_i A_i)$$
$$= \int dA^T d\pi^T dU \det M \delta(D_i\pi_i), \qquad \text{(B3)}$$

where we have used $\int d\sigma \delta(\partial_i^2\sigma) = \text{const}$. To solve Gauss's law we write

$$D_i\pi_i = D_i(A^T)(\pi_i^T - \partial_i U) = gA_i^T \times \pi_i^T + M(A^T)U,$$
$$\text{(B4)}$$

where $M(A^T) = -D_i(A^T)\partial_i$ is the 3-dimensional Faddeev-Popov operator. The Faddeev-Popov determinant gets absorbed by the identity

$$\det M \int dU \delta(gA_i^T \times \pi_i^T + MU) = 1. \qquad \text{(B5)}$$

The last δ-function expresses Gauss's law, and fixes the color-Coulomb potential to its physical value,

$$U = U_{\text{phys}} \equiv M^{-1}\rho, \qquad \text{(B6)}$$

where $\rho \equiv -gA_i^T \times \pi_i^T$ is the color-charge density of the dynamical degrees of freedom. This gives the partition function in canonical form

$$Z = \int dA^T d\pi^T \exp\left(-\int d^D x\left[\frac{1}{2}\pi_i^2 + i\pi_i^T \dot{A}_i^T\right.\right.$$
$$\left.\left. + \frac{1}{4}F_{ij}^2(A^T)\right] - \gamma S_h(A^T)\right), \tag{B7}$$

where $\pi_i \equiv \pi_i^T - \partial_i U_{\text{phys}}$. This is a canonical system, and thus unitary, for all real γ, and stable for $\gamma > 0$. The canonical Coulomb-gauge Hamiltonian of Christ and Lee [44] is modified here by the addition of the term $\gamma S_h(A^T)$.

The functional measure $\int dA^T \exp(-\gamma S_h)$, supplemented by the horizon condition (A8) that fixes $\gamma = \gamma_{\text{phys}}$, were derived in [31] as a representation of the functional integral restricted to the Gribov region,

$$\int dA^T \exp(-\gamma_{\text{phys}} S_h) = \int_\Omega dA^T. \tag{B8}$$

This yields the partition function in terms of the standard Coulomb-gauge first-order action

$$Z = \int_\Omega dA^T d\pi^T \exp\left(-\int d^D x\left[\frac{1}{2}\pi_i^2 + i\pi_i^T \dot{A}_i^T\right.\right.$$
$$\left.\left. + \frac{1}{4}F_{ij}^2(A^T)\right]\right), \tag{B9}$$

where $\pi_i \equiv \pi_i^T - \partial_i U_{\text{phys}}$, and the integral over A^T is restricted to the Gribov region Ω, defined in Sec. II.

Whereas unitarity in Coulomb gauge holds in each order in the perturbative expansion, Lorentz invariance is violated at tree level. According to the equivalence with the operator formalism, Lorentz invariance is restored only when the nonperturbative gap equation is satisfied, but it is not attempted to verify this by explicit calculation in the present article.

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