Renormalization of Lorentz violating theories

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We classify the unitary, renormalizable, Lorentz violating quantum field theories of interacting scalars and fermions, obtained improving the behavior of Feynman diagrams by means of higher space derivatives. Higher time derivatives are not generated by renormalization. Renormalizability is ensured by a "weighted power-counting" criterion. The theories contain a dimensionful parameter Λ_L, yet a set of models are classically invariant under a weighted scale transformation, which is anomalous at the quantum level. Formulas for the weighted trace anomaly are derived. The renormalization-group properties are studied.

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I. INTRODUCTION

The set of power-counting renormalizable theories is considerably restricted by the assumptions of unitarity, locality, causality and Lorentz invariance. If we relax one or some of these assumptions we can enlarge the set of renormalizable theories. However, usually the enlargement is too wide. For example, there exists an infinite set of renormalizable nonunitary theories. Improving the behavior of propagators at large momenta with the help of higher-derivative kinetic terms [1] it is possible to define a renormalizable higher-derivative version of every theory, including gravity [2]. Relaxing locality can in principle make every theory renormalizable, smoothing away the small distance singularities that originate the UV divergences [3]. Unitarity violations due to higher derivatives can in some cases be traded for causality violations [4,5].

The purpose of this paper is to investigate the issue of renormalizability in the presence of Lorentz violations, while preserving both locality and unitarity. The UV behavior of propagators is improved with the help of higher space derivatives. It is proved that, under certain conditions, renormalization does not turn on terms with higher time derivatives, thus preserving unitarity. Renormalizability follows from a modified power-counting criterion, which weights time and space differently. The set of consistent theories is still very restricted, yet considerably larger than the set of Lorentz invariant theories. Renormalizable models exist in arbitrary spacetime dimensions.

The quadratic terms that contain higher space derivatives, as well as certain vertices, are multiplied by inverse powers of a scale Λ_L. Despite the presence of the dimensionful parameter Λ_L certain models have a weighted scale invariance, which is anomalous at the quantum level. The weighted trace anomaly is worked out explicitly.

In this paper we concentrate on scalar and fermion theories, leaving the study of gauge theories and gravity to separate publications. Lorentz violating models with higher space derivatives might be useful to define the ultraviolet limit of theories that are otherwise nonrenormalizable, including quantum gravity, and allow to remove the divergences with a finite number of independent couplings. Other domains where the models of this paper might find applications are Lorentz violating extensions of the standard model [6], effective field theory [7], renormalization-group (RG) methods for the search of asymptotically safe fixed points [8], nonrelativistic quantum field theory for nuclear physics [9], condensed matter physics and the theory of critical phenomena [10]. Certain φ^4-models that fall in our class of renormalizable theories are useful to describe the critical behavior at Lifshitz points [11] and have been widely studied in that context [12], with a variety of applications to real physical systems. Effects of Lorentz and CPT violations on stability and microcausality have been studied [13], as well as the induction of Lorentz violations by the radiative corrections [14]. The renormalization of gauge theories containing Lorentz violating terms has been studied in [15]. For a recent review on astrophysical constraints on the Lorentz violation at high energy see Ref. [16].

The paper is organized as follows. In Sec. II we study the renormalizability of scalar theories, while in Sec. III we include the fermions. In Sec. IV we analyze the divergent parts of Feynman diagrams and their subtractions. We prove the locality of counterterms and study the renormalization algorithm to all orders. The one-loop divergences are computed explicitly. In Sec. V we analyze the renormalization structure and the renormalization group. In Sec. VI we study the energy-momentum tensor, the weighted scale invariance and the weighted trace anomaly. In Sec. VII we generalize our results to nonrelativistic

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theories. Section VIII contains the conclusions. In the appendices we collect more observations about the cancellation of subdivergences and the locality of counterterms, and some expressions of Euclidean propagators in coordinate space.

Preliminaries. We use the dimensional-regularization technique whenever possible. Since the analysis of divergences is the same in the Euclidean and Minkowskian frameworks, we write our formulas directly in the Euclidean framework, which is more explicit. Yet, with an abuse of language, we still speak of "Lorentz symmetry," since no confusion is expected to arise.

We first consider models where the d-dimensional spacetime manifold M_d is split into the product $M_{\hat{d}} \otimes M_{\bar{d}}$ of two submanifolds, a $\hat{d}$-dimensional submanifold $M_{\hat{d}}$, containing time and possibly some space coordinates, and a $\bar{d}$-dimensional space submanifold $M_{\bar{d}}$. Lorentz and rotational symmetries in the two submanifolds are assumed. This kind of splitting could be useful to describe specific physical situations (for example the presence of a nonisotropic medium in condensed matter physics), but here it is mainly used as a starting point to illustrate our arguments in concrete examples. Indeed, most Lorentz violating theories contain a huge number of independent vertices, so it is convenient to begin with models where unnecessary complicacies are reduced to a minimum. The extension of our construction to the most general case, which is rather simple, will be described later. In the same spirit, a number of discrete symmetries, such as parity, time reversal, $\varphi \to -\varphi$, etc., are often assumed.

To apply the dimensional-regularization technique, both submanifolds have to be continued independently. The total continued spacetime manifold M_D is therefore split into the product $M_{\hat{D}} \otimes M_{\bar{D}}$, where $\hat{D} = \hat{d} - \varepsilon_1$ and $\bar{D} = \bar{d} - \varepsilon_2$ are complex and $D = \hat{D} + \bar{D}$. Each momentum p is split into "first" components $\hat{p}$, which live in $M_{\hat{D}}$, and "second" components $\bar{p}$, which live in $M_{\bar{D}}$: $p = (\hat{p}, \bar{p})$. The spacetime index μ is split into hatted and barred indices: $\mu = (\hat{\mu}, \bar{\mu})$. Notations such as $\hat{p}_{\hat{\mu}}$, $\hat{p}_\mu$ and $p_{\hat{\mu}}$ refer to the same object, as well as $\bar{p}_{\bar{\mu}}$, $\bar{p}_\mu$, $p_{\bar{\mu}}$. Frequently, Latin letters are used for the indices of the barred components of momenta. Finally, $\bar{\triangle} \equiv \bar{\partial}_i \bar{\partial}_i$.

We say that $P_{k,n}(\hat{p}, \bar{p})$ is a weighted polynomial in $\hat{p}$ and $\bar{p}$, of degree k and weight $1/n$, where k is a multiple of $1/n$, if $P_{k,n}(\xi^n \hat{p}, \xi \bar{p})$ is a polynomial of degree kn in ξ. Clearly,

$$P_{k_1,n}(\hat{p}, \bar{p}) P_{k_2,n}(\hat{p}, \bar{p}) = P_{k_1+k_2,n}(\hat{p}, \bar{p}).$$

We say that $H_{k,n}(\hat{p}, \bar{p})$ is a homogeneous weighted polynomial in $\hat{p}$ and $\bar{p}$, of degree k and weight $1/n$, if $H_{k,n}(\lambda \hat{p}, \lambda^{1/n} \bar{p}) = \lambda^k H_{k,n}(\hat{p}, \bar{p})$. It is straightforward to prove that a weighted polynomial $P_{k,n}$ of degree k can be expressed as a linear combination of homogeneous weighted polynomials $H_{k',n}$ of degrees $k' \leq k$.

II. RENORMALIZABILITY BY WEIGHTED POWER COUNTING

In this section we classify the renormalizable Lorentz violating scalar field theories that can be constructed with the help of quadratic terms containing higher space derivatives and prove that renormalization does not generate higher time derivatives.

Consider a generic scalar field theory with a propagator defined by the quadratic terms

$$\mathcal{L}_{\text{free}} = \frac{1}{2} (\hat{\partial} \varphi)^2 + \frac{1}{2\Lambda_L^{2n-2}} (\bar{\partial}^n \varphi)^2, \qquad (2.1)$$

where Λ_L is an energy scale. Up to total derivatives it is not necessary to specify how the $2n$ derivatives $\bar{\partial}$ contract among themselves. The n of (2.1) should be understood as the highest power of $\bar{\partial}$ that appears in the quadratic terms of the total Lagrangian. Other quadratic terms of the form

$$\frac{a_m}{2\Lambda_L^{2m-2}} (\bar{\partial}^m \varphi)^2, \qquad m < n, \qquad (2.2)$$

could be present, or generated by renormalization. They are weighted monomials of degrees <2 and weight $1/n$. For the purposes of renormalization, it is convenient to consider such terms as "interactions" (two-leg vertices) and treat them perturbatively. Indeed, the counterterms depend polynomially on the parameters a_m, because when the integral associated with a graph is differentiated a sufficient number of times with respect to the a_m's it becomes overall convergent. The a_m-polynomiality of counterterms generalizes the usual polynomiality in the masses. Thus we can assume that the propagator is defined by (2.1) and treat every other term as a vertex. Then the propagator is the inverse of a weighted homogeneous polynomial of degree 2 and weight $1/n$. The coefficient of the term $(\bar{\partial}^n \varphi)^2$ must be positive, to have an action bounded from below in the Euclidean framework or, equivalently, an energy bounded from below in the Minkowskian framework.

Label the vertices that have N φ-legs with indices α to distinguish different derivative structures. Each vertex of type (N, α) defines a monomial in the momenta of the fields. Denote the weighted degree of such a monomial by $\delta_N^{(\alpha)}$. A vertex with p_1 derivatives $\hat{\partial}$, p_2 derivatives $\bar{\partial}$ and N φ-legs is symbolically written as

$$[\hat{\partial}^{p_1} \bar{\partial}^{p_2} \varphi^N]_\alpha$$

and its weighted degree is

$$\delta_N^{(\alpha)} = p_1 + \frac{p_2}{n}.$$

Consider a Feynman graph G made of L loops, E external legs, I internal legs and $v_N^{(\alpha)}$ vertices of type (N, α). The integral associated with G has the form

$$\mathcal{I}_G(k) = \int \frac{\mathrm{d}^{L\hat{D}}\hat{p}}{(2\pi)^{L\hat{D}}} \int \frac{\mathrm{d}^{L\bar{D}}\bar{p}}{(2\pi)^{L\bar{D}}} \prod_{i=1}^{I} \mathcal{P}^{(i)}_{-2,n}(p, k) \times \prod_{j=1}^{V} \mathcal{V}^{(j)}_{\delta_j,n}(p, k),$$

where p are the loop momenta, k are the external momenta, $\mathcal{P}^{(i)}_{-2,n}$ are the propagators, which have weighted degree -2, and $\mathcal{V}^{(j)}_{\delta_j,n}$ are the vertices, with weighted degrees δ_j. The integral measure $\mathrm{d}^{\hat{D}}\hat{p}\mathrm{d}^{\bar{D}}\bar{p}$ is a weighted measure of degree $\text{Đ} \equiv \hat{D} + \bar{D}/n$. Performing a rescaling $(\hat{k}, \bar{k}) \to (\lambda\hat{k}, \lambda^{1/n}\bar{k})$, accompanied by an analogous change of variables $(\hat{p}, \bar{p}) \to (\lambda\hat{p}, \lambda^{1/n}\bar{p})$, it is immediate to prove that $\mathcal{I}_G(k)$ is a weighted function of degree

$$L\text{Đ} - 2I + \sum_{j=1}^{V} \delta_j = L\text{Đ} - 2I + \sum_{(N,\alpha)} \delta_N^{(\alpha)} v_N^{(\alpha)}.$$

By the locality of counterterms, once the subdivergences of G have been inductively subtracted away, the overall divergent part of G is a weighted polynomial of degree

$$\omega(G) = L\text{đ} - 2I + \sum_{(N,\alpha)} \delta_N^{(\alpha)} v_N^{(\alpha)}$$

in the external momenta, where $\text{đ} \equiv \hat{d} + \bar{d}/n$. The usual relations

$$L = I - V + 1, \qquad E + 2I = \sum_{(N,\alpha)} N v_N^{(\alpha)}, \tag{2.3}$$

allow us to write

$$\omega(G) = d(E) + \sum_{(N,\alpha)} v_N^{(\alpha)}[\delta_N^{(\alpha)} - d(N)], \tag{2.4}$$

where

$$d(X) \equiv \text{đ}\left(1 - \frac{X}{2}\right) + X; \tag{2.5}$$

The theory is i) renormalizable, if it contains all vertices with $\delta_N^{(\alpha)} \leq d(N)$, and only those: $\omega(G)$ does not increase when the number of vertices increases; ii) super-renormalizable, if it contains all vertices with $\delta_N^{(\alpha)} < d(N)$, and only those: $\omega(G)$ decreases when the number of vertices increases; iii) strictly-renormalizable, if it contains all vertices with $\delta_N^{(\alpha)} = d(N)$, and only those: $\omega(G)$ does not depend on $v_N^{(\alpha)}$; iv) nonrenormalizable, if it contains some vertices with $\delta_N^{(\alpha)} > d(N)$: $\omega(G)$ increases when the number of those vertices increases.

The vertices with $\delta_N^{(\alpha)} = d(N)$ are called "weighted marginal," those with $\delta_N^{(\alpha)} < d(N)$ are called "weighted relevant" and those with $\delta_N^{(\alpha)} > d(N)$ are called "weighted irrelevant."

By locality, $\delta_N^{(\alpha)}$ cannot be negative. Moreover, polynomiality demands that there must exist a bound $N_{\max}$ on the number of legs that the vertices can contain. It is easy to show that these requirements are fulfilled if and only if

$$\text{đ} > 2 \tag{2.6}$$

and the bound is

$$N_{\max} = \left[\frac{2\text{đ}}{\text{đ} - 2}\right], \tag{2.7}$$

where $[x]$ denotes the integral part of x. The existence of nontrivial interactions ($N_{\max} \geq 3$) requires $\text{đ} \leq 6$, while the existence of nontrivial even interactions ($N_{\max} \geq 4$) requires $\text{đ} \leq 4$.

To complete the proof of renormalizability, observe that when $\delta_N^{(\alpha)} \leq d(N)$ the weighted degree of divergence $\omega(G)$ of a graph G satisfies

$$\omega(G) \leq d(E). \tag{2.8}$$

The inequality (2.6) ensures also that $\omega(G)$ decreases when the number of external legs increases. Finally, since the vertices that subtract the overall divergences of G are of type (E, α) with $\delta_E^{(\alpha)} = \omega(G)$, it is straightforward to check that the Lagrangian contains all needed vertices. Indeed, (2.8) coincides with the inequality satisfied by $\delta_E^{(\alpha)}$.

Now we prove that the renormalizable models just constructed are perturbatively unitary, in particular, that no higher time derivatives are present, both in the kinetic part and in the vertices, and no higher time derivatives are generated by renormalization. Indeed, a Lagrangian term with higher time derivatives would have $\delta_N^{(\alpha)} \geq 2$ for $N > 2$ or $\delta_2^{(\alpha)} > 2$ (terms with $N = 1$ need not be considered, since they cannot contain derivatives). This cannot happen in a renormalizable theory, because (2.6) and $\delta_N^{(\alpha)} \leq d(N)$ imply $\delta_N^{(\alpha)} \leq 2$ in general and $\delta_N^{(\alpha)} < 2$ for $N > 2$. In particular, true vertices ($N > 2$) cannot contain any $\hat{\partial}$-derivative at all, because invariance under the reduced Lorentz and rotational symmetries of $M_{\hat{D}}$ and $M_{\bar{D}}$ exclude also terms containing an odd number of $\hat{\partial}$'s or an odd number of $\bar{\partial}$'s. Similar conclusions apply to the counterterms, because of (2.8). Therefore, renormalization does not turn on higher time derivatives, as promised.

Weighted scale invariance. The strictly renormalizable models have the $\delta_N^{(\alpha)} = d(N)$. Their Lagrangian has the form

$$\mathcal{L}_{(\hat{d},\bar{d})} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^{2n-2}}(\bar{\partial}^n\varphi)^2 + \sum_{(N,\alpha)} \frac{\lambda_{(N,\alpha)}}{N!\Lambda_L^{(n-1)(N+\hat{d}-\hat{d}N/2)}}[\bar{\partial}^{nd(N)}\varphi^N]_\alpha. \tag{2.9}$$

Here $[\bar{\partial}^{nd(N)}\varphi^N]_\alpha$ denotes a basis of Lagrangian terms constructed with N fields φ and $nd(N)$ $\bar{\partial}$-derivatives acting on them, contracted in all independent ways, and $\lambda_{(N,\alpha)}$ are dimensionless couplings.

In the physical spacetime dimension $d = \hat{d} + \bar{d}$ (the continuation to complex dimensions will be discussed later) the classical theories with Lagrangians $\mathcal{L}_{(\hat{d},\bar{d})}$ are invariant under the weighted dilatation

$$\hat{x} \to \hat{x}e^{-\Omega}, \qquad \bar{x} \to \bar{x}e^{-\Omega/n}, \qquad \varphi \to \varphi e^{\Omega(\text{đ}/2-1)}, \tag{2.10}$$

where Ω is a constant parameter. Each Lagrangian term scales with the factor đ, compensated by the scaling factor of the integration measure $d^d x$ of the action.

We call the models (2.9) homogeneous. Homogeneity is preserved by renormalization, namely, there exists a subtraction scheme in which no Lagrangian terms of weighted degrees smaller than $d(N)$ are turned on by renormalization. This fact is evident using the dimensional-regularization technique. Indeed, when $\delta_N^{(\alpha)} = d(N)$, the equality in (2.8) holds, so $\omega(G) = d(E) = \delta_E^{(\alpha)}$.

The weighted scale invariance (2.10) is anomalous at the quantum level. The weighted trace anomaly and its relation with the renormalization group are studied in Sec. VI.

Nonhomogeneous theories are those that contain both weighted marginal and weighted relevant vertices. In these cases the weighted dilatation (2.10) is explicitly broken by the super-renormalizable vertices, and dynamically broken by the anomaly.

Let us analyze some explicit examples, starting from the homogeneous models.

Homogeneous models. We begin with the φ^4-theories. Setting $N_{\max} = 4$ in (2.7) we get

$$\frac{10}{3} < \text{đ} \leq 4. \tag{2.11}$$

One solution with đ $= 4$ is the usual Lorentz invariant φ^4-theory in four dimensions ($\hat{d} = \bar{d} = 2$, $n = 1$). A simple Lorentz violating solution is the model with $n = 2$ described by the Lagrangian

$$\mathcal{L}_{(2,4)} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^2}(\bar{\Delta}\varphi)^2 + \frac{\lambda}{4!\Lambda_L^2}\varphi^4. \tag{2.12}$$

in six dimensions, with $\hat{d} = 2$, $\bar{d} = 4$. The φ^4-theories with $n = 2$ are used to describe the critical behavior at Lifshitz points [11,12].

It is clear that (2.11) admits infinitely many solutions for each value of đ. For example, given a solution, such as (2.12), infinitely many others are obtained multiplying $\bar{d}$ and n by a common integer factor. For đ $= 4$ we have the family of $2(n+1)$-dimensional theories

$$\mathcal{L}_{(2,2n)} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^{2(n-1)}}(\bar{\partial}^n\varphi)^2 + \frac{\lambda}{4!\Lambda_L^{2(n-1)}}\varphi^4. \tag{2.13}$$

In general, for every Lorentz invariant renormalizable theory there exists an infinite family of Lorentz violating renormalizable theories.

Let us now concentrate on four dimensions. The spacetime manifold can be split as $(\hat{d}, \bar{d}) = (0, 4), (1, 3), (2, 2), (3, 1), (4, 0)$. There is no nontrivial solution with $\hat{d} = 0$. Indeed, (2.7) implies

$$N_{\max} = \left[\frac{4}{2-n}\right],$$

so n can only be 1, which gives back the Lorentz invariant φ^4-theory. For $\hat{d} = 1$ we get

$$N_{\max} = \left[\frac{2(n+3)}{3-n}\right].$$

The only nontrivial solution is $n = 2$, which implies $N_{\max} = 10$ and

$$\mathcal{L}_{(1,3)} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^2}(\bar{\Delta}\varphi)^2 + \frac{\lambda_6}{6!\Lambda_L^4}\varphi^4(\bar{\partial}\varphi)^2 + \frac{\lambda_{10}}{10!\Lambda_L^6}\varphi^{10}. \tag{2.14}$$

For $\hat{d} = 2$ we get $N_{\max} = 2(n+1)$: every integer $n > 1$ defines a nontrivial solution in this case. The simplest example is $(\hat{d}, \bar{d}) = (2, 2)$, $n = 2$. Listing all allowed vertices we get the theory

$$\mathcal{L}_{(2,2)} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^2}(\bar{\Delta}\varphi)^2 + \frac{\lambda_4}{4!\Lambda_L^2}\varphi^2(\bar{\partial}\varphi)^2 + \frac{\lambda_6}{6!\Lambda_L^2}\varphi^6. \tag{2.15}$$

This model belongs to a family of đ $= 3$, $(2+n)$-dimensional φ^6-theories, whose Lagrangian is

$$\mathcal{L}_{(2,n)} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^{2(n-1)}}(\bar{\partial}^n\varphi)^2 + \frac{\lambda_6}{6!\Lambda_L^{2(n-1)}}\varphi^6, \tag{2.16}$$

when n is odd, and

$$\mathcal{L}_{(2,n)} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^{2(n-1)}}(\bar{\partial}^n\varphi)^2 + \frac{1}{4!\Lambda_L^{2(n-1)}}\sum_\alpha \lambda_\alpha[\bar{\partial}^n\varphi^4]_\alpha + \frac{\lambda_6}{6!\Lambda_L^{2(n-1)}}\varphi^6, \tag{2.17}$$

when n is even. Observe that (2.16) includes the Lorentz invariant φ^6-theory in three spacetime dimensions, which is the case $n = 1$.

For $\hat{d} = 3$ we get

$$N_{\max} = \left[\frac{2(3n+1)}{n+1}\right].$$

The solution with $n = 2$ has $N_{\max} = 4$. However, this solution is trivial, since its unique vertex would have just one $\bar{\partial}$-derivative. Instead, for every $n \geq 3$, $N_{\max}$ is equal to 5. For example, the theory with $n = 3$ is

$$\mathcal{L}_{(3,1)} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^4}(\bar{\partial}\bar{\Delta}\varphi)^2 + \frac{\lambda_3'}{3!\Lambda_L^3}\varphi^2\bar{\Delta}^2\varphi + \frac{\lambda_3}{3!\Lambda_L^3}\varphi(\bar{\Delta}\varphi)^2 + \frac{\lambda_4}{4!\Lambda_L^2}\varphi^2(\bar{\partial}\varphi)^2 + \frac{\lambda_5}{5!\Lambda_L}\varphi^5,$$

which is clearly unstable. Imposing the symmetry $\varphi \to -\varphi$ we have the modified φ^4-theory

$$\mathcal{L}_{(3,1)}^{\text{even}} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^4}(\bar{\partial}\bar{\Delta}\varphi)^2 + \frac{\lambda_4}{4!\Lambda_L^2}\varphi^2(\bar{\partial}\varphi)^2,$$

which is stable for $\lambda_4 > 0$. Finally, for $\hat{d} = 4$ we get again the Lorentz invariant φ^4-theory.

Nonhomogeneous models. Nonhomogeneous theories can be obtained from the homogeneous ones adding all super-renormalizable terms, which are those that satisfy the strict inequality $\delta_N^{(\alpha)} < d(N)$. For example, keeping the symmetry $\varphi \to -\varphi$, the nonhomogeneous extension of (2.12) is just

$$\mathcal{L}_{(2,4)}^{\text{nh}} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{a}{2}(\bar{\partial}\varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{1}{2\Lambda_L^2}(\bar{\Delta}\varphi)^2 + \frac{\lambda}{4!\Lambda_L^2}\varphi^4$$

and the one of (2.15) is

$$\mathcal{L}_{(2,2)}^{\text{nh}} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{a}{2}(\bar{\partial}\varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{1}{2\Lambda_L^2}(\bar{\Delta}\varphi)^2 + \frac{\lambda_4}{4!\Lambda_L^2}\varphi^2(\bar{\partial}\varphi)^2 + \frac{\lambda_4'}{4!}\varphi^4 + \frac{\lambda_6}{6!\Lambda_L^2}\varphi^6.$$

Splitting the spacetime manifold into the product of more submanifolds. Instead of splitting the spacetime manifold into two submanifolds, we can split it into the product of more submanifolds, eventually one for each coordinate. This analysis covers the most general case. We still need to distinguish a $\hat{d}$-dimensional submanifold $M_{\hat{d}}$ containing time from the $\bar{d}_i$-dimensional space submanifolds $M_{\bar{d}_i}$, $i = 1, \ldots \ell$, so we write

$$M_d = M_{\hat{d}} \otimes \prod_{i=1}^{\ell} M_{\bar{d}_i}.$$

Denote the space derivatives in the ith space subsector with $\bar{\partial}_i$ and assume that they have weights $1/n_i$. Then the kinetic term of the Lagrangian reads

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2}\varphi P_2(\bar{\partial}_i, \Lambda_L)\varphi,$$

where $P_2(\bar{\partial}_i, \Lambda_L)$ is the most general weighted homogeneous polynomial of degree 2 in the spatial derivatives, $P_2(\lambda^{1/n_i}\bar{\partial}_i, \Lambda_L) = \lambda^2 P_2(\bar{\partial}_i, \Lambda_L)$, invariant under rotations in the subspaces $M_{\bar{d}_i}$. The Λ_L-dependence is arranged so that P_2 has dimensionality 2. The previous analysis can be repeated straightforwardly. It is easy to verify that the weighted power-counting criterion works as before with

$$\text{đ} = \hat{d} + \sum_{i=1}^{\ell}\frac{\bar{d}_i}{n_i}.$$

Edge renormalizability. By edge renormalizable theories we mean theories where renormalization preserves the derivative structure of the Lagrangian, but the powers of the fields are unrestricted. With scalars and fermions, such theories contain arbitrary functions of the fields and therefore infinitely many independent couplings. The notion of edge renormalizability is interesting in the perspective to study gravity. Indeed, Einstein gravity is an example of theory where all vertices have the same number of derivatives, but are nonpolynomial in the fluctuation around flat space. Yet, diffeomorphism invariance ensures that the number of invariants with a given dimensionality in units of mass is finite. Therefore, in quantum gravity a polynomial derivative structure is sufficient to reduce the arbitrariness to a finite set of independent couplings.

Edge renormalizable theories are those where $\omega(G)$ does not decrease when E increases, rather it is independent of E. By formula (2.8) this means $\text{đ} = 2$ ($N_{\max} = \infty$), in which case $\omega(G)$ is always equal to 2. Since $\text{đ} = 2$, $\hat{d}$ can be either 0 or 1. The theories with $\hat{d} = 0$ contain higher time derivatives, so they are not unitary. Thus we must take $\hat{d} = 1$. The homogeneous theory in four dimensions has Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{I}}, \tag{2.18}$$

where

$$\mathcal{L}_{\text{free}} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^4}(\bar{\partial}\bar{\Delta}\varphi)^2$$

and

$$\begin{aligned}\mathcal{L}_{\text{I}} = {} & V_1(\varphi)(\hat{\partial}\varphi)^2 + V_2(\varphi)[(\partial_i\varphi)^2]^3 + V_3(\varphi)\bar{\Delta}\varphi(\partial_i\varphi)^2 \\ & \times(\partial_j\varphi)^2 + V_4(\varphi)(\partial_i\partial_j\varphi)(\partial_i\partial_j\bar{\Delta}\varphi) \\ & + V_5(\varphi)\bar{\Delta}^2\varphi(\partial_i\varphi)^2 + V_6(\varphi)(\bar{\Delta}\varphi)^3 + V_7(\varphi) \\ & \times(\partial_i\bar{\Delta}\varphi)^2 + V_8(\varphi)(\partial_i\partial_j\partial_k\varphi)^2 + V_9(\varphi)\bar{\Delta}^3\varphi,\end{aligned} \tag{2.19}$$

where the V_i's are unspecified functions of φ with $V_1(\varphi) = \mathcal{O}(\varphi)$, $V_4(\varphi), V_7(\varphi), V_8(\varphi), V_9(\varphi) = \mathcal{O}(\varphi^2)$.

The Lagrangian of the most general nonhomogeneous theory is (2.18) with

$$\mathcal{L}_{\text{free}} = \frac{1}{2}(\hat{\partial}\varphi)^2 - \frac{1}{2}\varphi\left(a\bar{\Delta} + b\frac{\bar{\Delta}^2}{\Lambda_L^2} + \frac{\bar{\Delta}^3}{\Lambda_L^4}\right)\varphi$$

and $\mathcal{L}_{\text{I}}$ equal to (2.19) plus

$$V_{10}(\varphi) + V_{11}(\varphi)\bar{\Delta}\varphi + V_{12}(\varphi)\bar{\Delta}^2\varphi + V_{13}(\varphi)(\bar{\Delta}\varphi)^2 + V_{14}(\varphi)[(\partial_i\varphi)^2]^2,$$

with $V_{11}(\varphi), V_{12}(\varphi) = \mathcal{O}(\varphi^2)$, $V_{13}(\varphi) = \mathcal{O}(\varphi)$.

III. INCLUSION OF FERMIONS

In this section we classify the models of interacting fermions and scalars. We start from pure fermionic theories, with quadratic Lagrangian

$$\mathcal{L}_{\text{free}} = \bar{\psi}\hat{\partial\!\!\!/}\psi + \frac{1}{\Lambda_L^{n-1}}\bar{\psi}\bar{\partial\!\!\!/}^n\psi,$$

where n is the maximal number of $\bar{\partial}$-derivatives. The propagator

$$\frac{-i\hat{p\!\!\!/} + (-i)^n\frac{\bar{p\!\!\!/}^n}{\Lambda_L^{n-1}}}{\hat{p}^2 + \frac{(\bar{p}^2)^n}{\Lambda_L^{2n-2}}},$$

is, in momentum space, a weighted function of degree -1. The loop-integral measure is, as usual, a weighted measure of degree đ. For the purposes of renormalization, the kinetic terms with fewer than n $\bar{\partial}$-derivatives can be treated as vertices.

Label the vertices that have $2N$ ψ-$\bar{\psi}$-legs by means of indices α and denote their weighted degree with $\delta_N^{(\alpha)}$. Consider a diagram G with $2E$ external ψ-$\bar{\psi}$-legs, constructed with $v_N^{(\alpha)}$ vertices of type (N, α). Once the subdivergences have been subtracted away, its overall divergence is a weighted polynomial of degree

$$\omega(G) = \text{đ} - E(\text{đ} - 1) + \sum_{(N,\alpha)} v_N^{(\alpha)}[\delta_N^{(\alpha)} - \text{đ}(1 - N) - N]$$

in the external momenta. Renormalizability demands

$$\delta_N^{(\alpha)} \leq \text{đ}(1 - N) + N \equiv d_F(N). \tag{3.1}$$

Polynomiality demands

$$\text{đ} > 1,$$

in which case the maximal number of external ψ-$\bar{\psi}$-legs is

$$N_{\max} = \left[\frac{\text{đ}}{\text{đ} - 1}\right].$$

Pure fermionic homogeneous models have strictly renormalizable vertices, namely, those with $\delta_N^{(\alpha)} = d_F(N)$. Their Lagrangian has the form

$$\mathcal{L} = \bar{\psi}\hat{\partial\!\!\!/}\psi + \frac{1}{\Lambda_L^{n-1}}\bar{\psi}\bar{\partial\!\!\!/}^n\psi + \sum_{(N,\alpha)}\frac{\lambda_{(N,\alpha)}}{(N!)^2\Lambda_L^{(n-1)(N-\hat{d}-N\hat{d})}} \times [\bar{\partial}^{nd_F(N)}\bar{\psi}^N\psi^N]_\alpha.$$

Here $[\bar{\partial}^{nd_F(N)}\bar{\psi}^N\psi^N]_\alpha$ denotes a basis of Lagrangian terms constructed with N fields ψ, N fields $\bar{\psi}$ and $nd_F(N)$ $\bar{\partial}$-derivatives, invariant under the reduced Lorentz symmetry. For simplicity, we can assume also invariance under parities in both portions of spacetime.

Let us concentrate on four spacetime dimensions. The Lorentz split (1, 3) gives $N_{\max} = 1 + [n/3]$, which admits infinitely many nontrivial solutions, beginning from $n = 3$. For example, the $n = 3$ and $n = 6$ theories read

$$\mathcal{L}_{(1,3)} = \bar{\psi}\hat{\partial\!\!\!/}\psi + \frac{1}{\Lambda_L^2}\bar{\psi}\bar{\Delta}\bar{\partial\!\!\!/}\psi + \sum_\alpha\frac{\lambda_\alpha}{\Lambda_L^2}[\bar{\psi}^2\psi^2]_\alpha,$$

$$\mathcal{L}'_{(1,3)} = \bar{\psi}\hat{\partial\!\!\!/}\psi + \frac{1}{\Lambda_L^5}\bar{\psi}\bar{\Delta}^3\psi + \sum_\alpha\frac{\lambda_\alpha}{\Lambda_L^5}[\bar{\partial}^3\bar{\psi}^2\psi^2]_\alpha + \sum_\alpha\frac{\lambda'_\alpha}{\Lambda_L^5}[\bar{\psi}^3\psi^3]_\alpha,$$

respectively. The Lorentz splits (2, 2) and (3, 1) do not admit nontrivial solutions, since $N_{\max} = 1$ in those cases.

Now we study the models containing coupled scalars and fermions. It is important to note that when different types of fields are involved, they must have the same n. We classify the vertices with labels $(N_\psi, N_\varphi, \alpha)$, where $2N_\psi$ is the number of ψ-$\bar{\psi}$-legs, N_φ is the number of φ-legs and α is an extra label that distinguishes vertices with different structures. Call $\delta_{(N_\psi,N_\varphi)}^{(\alpha)}$ the weighted degree of the α-th vertex. Consider a diagram G with $2E_\psi$ external ψ-$\bar{\psi}$-legs, E_φ external φ-legs and $v_{(N_\psi,N_\varphi)}^{(\alpha)}$ vertices of type $(N_\psi, N_\varphi, \alpha)$. Once the subdivergences have been subtracted away, the overall divergent part of G a is a weighted polynomial of degree

$$\omega(G) = \text{đ} - E_\psi(\text{đ} - 1) - \frac{E_\varphi}{2}(\text{đ} - 2) + \sum_{(N_\psi,N_\varphi,\alpha)} v_{(N_\psi,N_\varphi)}^{(\alpha)} \times \left[\delta_{(N_\psi,N_\varphi)}^{(\alpha)} - \text{đ}\left(1 - N_\psi - \frac{N_\varphi}{2}\right) - N_\psi - N_\varphi\right]$$

in the external momenta. Renormalizability demands

$$\delta_{(N_\psi,N_\varphi)}^{(\alpha)} \leq \text{đ}\left(1 - N_\psi - \frac{N_\varphi}{2}\right) + N_\psi + N_\varphi \equiv d(N_\psi, N_\varphi).$$

Because $\delta_{(N_\psi,N_\varphi)}^{(\alpha)}$ is non-negative, the numbers of fermionic and bosonic legs are bound by the inequality

$$N_\psi(\text{đ} - 1) + \frac{N_\varphi}{2}(\text{đ} - 2) \leq \text{đ}.$$

Polynomiality demands $\text{đ} > 2$.

The homogeneous models have a Lagrangian of the form

$$\mathcal{L} = \bar{\psi}\hat{\partial\!\!\!/}\psi + \frac{\eta}{\Lambda_L^{n-1}}\bar{\psi}\bar{\partial\!\!\!/}^n\psi + \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^{2n-2}}(\bar{\partial}^n\varphi)^2 + \sum_{(N_\psi,N_\varphi,\alpha)}\frac{\lambda_{(N_\psi,N_\varphi,\alpha)}}{N_\varphi!(N_\psi!)^2\Lambda_L^{(n-1)(N_\varphi+N_\psi+\hat{d}-\hat{d}N_\psi-\hat{d}N_\varphi/2)}} \times [\bar{\partial}^{nd(N_\psi,N_\varphi)}\bar{\psi}^{N_\psi}\psi^{N_\psi}\varphi^{N_\varphi}]_\alpha.$$

In four dimensions the splitting (1, 3) has a unique nontrivial solution, which is the model (2.14) coupled to fermions. It has $n = 2$ and its Lagrangian reads

$$\mathcal{L}_{(1,3)} = \bar{\psi}\hat{\not\partial}\psi + \frac{\eta}{\Lambda_L}\bar{\psi}\bar{\Delta}\psi + \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^2}(\bar{\Delta}\varphi)^2 + \frac{\lambda_2}{2\Lambda_L^2}\varphi^2(\bar{\psi}\overleftrightarrow{\bar{\not\partial}}\psi) + \frac{\lambda_2'}{2\Lambda_L^2}\varphi^2\bar{\partial}\cdot(\bar{\psi}\bar{\gamma}\psi) + \frac{\lambda_4}{4!\Lambda_L^3}\varphi^4\bar{\psi}\psi + \frac{\lambda_6}{6!\Lambda_L^4}\varphi^4(\bar{\partial}\varphi)^2 + \frac{\lambda_{10}}{10!\Lambda_L^6}\varphi^{10}.$$

The splitting (2, 2) admits infinitely many solutions. The simplest one is the theory with $n = 2$, symmetric under $\varphi \leftrightarrow -\varphi$, that couples (2.15) to fermions:

$$\mathcal{L}_{(2,2)} = \bar{\psi}\hat{\not\partial}\psi + \frac{\eta}{\Lambda_L}\bar{\psi}\bar{\Delta}\psi + \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^2}(\bar{\Delta}\varphi)^2 + \frac{\lambda_2}{2\Lambda_L}\varphi^2\bar{\psi}\psi + \frac{\lambda_4}{4!\Lambda_L^2}\varphi^2(\bar{\partial}\varphi)^2 + \frac{\lambda_6}{6!\Lambda_L^2}\varphi^6.$$

The splitting (3, 1) admits, again, infinitely many solutions.

IV. RENORMALIZATION

In this section we study the structure of Feynman diagrams, their divergences and subdivergences and the locality of counterterms. For definiteness, we work with scalar fields, but the conclusions are general.

One-loop. Consider the most general one-loop Feynman diagram G, with E external legs, I internal legs and $v_N^{(\alpha)}$ vertices of type (N, α) and weighted degree $\delta_N^{(\alpha)}$. Collectively denote the external momenta by k. The divergent part of G can be calculated expanding the integral in powers of k. We obtain a linear combination of contributions of the form

$$\mathcal{I}^{(I,n)}_{\mu_1\cdots\mu_{2r}|j_1\cdots j_{2s}}\hat{k}_{\nu_1}\cdots\hat{k}_{\nu_u}\bar{k}_{i_1}\cdots\bar{k}_{i_v}, \tag{4.1}$$

where

$$\mathcal{I}^{(I,n)}_{\mu_1\cdots\mu_{2r}|j_1\cdots j_{2s}} = \int\frac{d^{\hat{D}}\hat{p}}{(2\pi)^{\hat{D}}}\int\frac{d^{\bar{D}}\bar{p}}{(2\pi)^{\bar{D}}} \times \frac{\hat{p}_{\mu_1}\cdots\hat{p}_{\mu_{2r}}\bar{p}_{j_1}\cdots\bar{p}_{j_{2s}}}{(\hat{p}^2 + (\bar{p}^2)/\Lambda_L^{2(n-1)} + m^2)^I}.$$

To avoid infrared problems we insert a mass m in the denominators. For the purposes of renormalization, it is not necessary to think of m as the real mass. It can be considered as a fictitious parameter, introduced to calculate the divergent part of the integral and set to zero afterwards. The real mass, as well as the other parameters a_m of (2.2), can be treated perturbatively, so they are included in the set of "vertices."

From the weighted power-counting analysis of Sec. II we know that the numerator of (4.1), namely

$$\hat{p}_{\mu_1}\cdots\hat{p}_{\mu_{2r}}\bar{p}_{j_1}\cdots\bar{p}_{j_{2s}}\hat{k}_{\nu_1}\cdots\hat{k}_{\nu_u}\bar{k}_{i_1}\cdots\bar{k}_{i_v},$$

is a weighted monomial $P_{q,n}(\hat{p},\hat{k};\bar{p},\bar{k})$ of weight $1/n$ and degree

$$q = u + 2r + \frac{v}{n} + \frac{2s}{n} = \sum_{(N,\alpha)}\delta_N^{(\alpha)}v_N^{(\alpha)}.$$

At one-loop the number of vertices equals the number of propagators. Using (2.3) and $\delta_N^{(\alpha)} \leq d(N)$ we get

$$u + \frac{v}{n} \leq 2\left(I - r - \frac{s}{n}\right) + E\left(1 - \frac{\text{đ}}{2}\right). \tag{4.2}$$

By symmetric integration, we can write

$$\mathcal{I}^{(I,n)}_{\mu_1\cdots\mu_{2r}|j_1\cdots j_{2s}} = \delta^{(1)}_{\mu_1\cdots\mu_{2r}}\delta^{(2)}_{j_1\cdots j_{2s}}\mathcal{I}^{(I,n)}_{r,s},$$

$$\mathcal{I}^{(I,n)}_{r,s} = \int\frac{d^{\hat{D}}\hat{p}}{(2\pi)^{\hat{D}}}\int\frac{d^{\bar{D}}\bar{p}}{(2\pi)^{\bar{D}}} \times \frac{(\hat{p}^2)^r(\bar{p}^2)^s}{(\hat{p}^2 + (\bar{p}^2)^n/\Lambda_L^{2(n-1)} + m^2)^I},$$

where $\delta^{(1)}_{\mu_1\cdots\mu_{2r}}$ and $\delta^{(2)}_{j_1\cdots j_{2s}}$ are appropriately normalized completely symmetric tensors constructed with the Kronecker tensors of $M^{\hat{D}}$ and $M^{\bar{D}}$, respectively. Performing the change of variables

$$\bar{p}_i = \bar{p}'_i\left(\frac{\Lambda_L^2}{\bar{p}'^2}\right)^{(n-1)/(2n)}, \tag{4.3}$$

the integral $\mathcal{I}^{(I,n)}_{r,s}$ can be calculated using the standard formulas of the dimensional-regularization technique. We obtain

$$\mathcal{I}^{(I,n)}_{r,s} = \frac{1}{n}\Lambda_L^{(2s+\bar{D})(n-1)/n}\int\frac{d^{\hat{D}}\hat{p}}{(2\pi)^{\hat{D}}}\int\frac{d^{\bar{D}}\bar{p}'}{(2\pi)^{\bar{D}}}\frac{(\hat{p}^2)^r(\bar{p}'^2)^{(2s+\bar{D}-n\bar{D})/(2n)}}{(\hat{p}^2 + \bar{p}'^2 + m^2)^I} = \frac{\Lambda_L^{(2s+\bar{D})(n-1)/n}(m^2)^{r-I+s/n+\text{Đ}/2}\Gamma(\frac{2s+\bar{D}}{2n})\Gamma(\frac{2r+\hat{D}}{2})\Gamma(I - r - \frac{s}{n} - \frac{\text{Đ}}{2})}{n(4\pi)^{D/2}\Gamma(\hat{D}/2)\Gamma(\bar{D}/2)\Gamma(I)}.$$

The factor $1/n$ is due to the Jacobian determinant of the transformation (4.3). The singularities occur for

$$I \leq r + \frac{s}{n} + \frac{\text{đ}}{2}. \tag{4.4}$$

Combining this inequality with (4.2) we find that the divergent contributions satisfy

$$u + \frac{v}{n} \leq đ + E\left(1 - \frac{đ}{2}\right) = d(E). \qquad (4.5)$$

The counterterms are a $P_{u+v/n,n}(\hat{k}, \bar{k})$:

$$\frac{1}{\varepsilon}\hat{k}_{\nu_1} \cdots \hat{k}_{\nu_u}\bar{k}_{i_1} \cdots \bar{k}_{i_v}, \quad \text{where } \varepsilon = đ - Đ = \varepsilon_1 + \frac{\varepsilon_2}{n}.$$

Thus (4.5) ensures that the divergent terms can be subtracted away renormalizing the fields and couplings of the initial Lagrangian. Observe that while the poles are proportional to $1/\varepsilon$, the residues of the poles can depend on ε_1 and ε_2 separately. We know that taking a sufficient number of derivatives with respect to the masses, the external momenta and the parameters a_m of (2.2), the integral becomes convergent. Therefore, the finite parts are regular in the limits $\varepsilon_1, \varepsilon_2 \to 0$, which can be safely taken in any preferred order. Objects such as $\varepsilon_1/\varepsilon$ and $\varepsilon_2/\varepsilon$ multiply only local terms, so they parametrize different scheme choices and never enter the physical quantities. These observations generalize immediately to all orders. We define the minimal subtraction schemes as the schemes where

$$\varepsilon_1 = \alpha\varepsilon, \qquad \varepsilon_2 = n(1 - \alpha)\varepsilon,$$

with $\alpha =$ constant, and only the pure poles in ε are subtracted away, with no finite contributions.

Overall divergences and subdivergences. Before considering Lorentz violating theories to all orders in the loop expansion it is convenient to briefly review the usual classification of divergences and the proof of locality of counterterms [17] in Lorentz symmetric theories. Consider the L-loop integral

$$\mathcal{I}(k) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D p^{(i)}}{(2\pi)^D} Q(p^{(1)}, \ldots, p^{(L)}; k)$$

with Lorentz invariant propagators $1/(p^2 + m^2)$, where k denotes the external momenta. The ultraviolet behavior of $\mathcal{I}(k)$ is studied letting any (sub)set of the momenta $p^{(1)}, \ldots, p^{(L)}$ tend to infinity with the same velocity. Proper subsets of the momenta test the presence of subdivergences, while the whole set tests the presence of overall divergences. i) When any subconvergence fails, counterterms corresponding to the divergent subdiagrams have to be included to subtract the subdivergences. ii) Once all subdivergences are removed, the subtracted integral $\mathcal{I}_{\text{sub}}(k)$ can still be overall divergent. Taking an appropriate number M of derivatives with respect to the external momenta k the integral $\partial_k^M \mathcal{I}_{\text{sub}}(k)$ becomes overall convergent. This proves the locality of counterterms.

The overlapping divergences can be tested sending momenta to infinity with different velocities. For example, rescale $p_1, \ldots, p_L$ as $\lambda p_1, \ldots, \lambda p_l, \lambda^2 p_{l+1}, \ldots, \lambda^2 p_L$. This test, however, is already covered by the previous ones, since there is always a (sub)set s_{fast} of momenta tending to infinity with maximal velocity. In the example just given, $s_{\text{fast}} = (p_{l+1}, \ldots, p_L)$. The other momenta s_{slow} grow slower, so they can be considered fixed in the first analysis and taken to infinity at a second stage. Weinberg's theorem [18] ensures that when s_{fast} tends to infinity the behavior of the relevant subintegral is governed by power counting and can generate logarithmic corrections depending on the momenta of s_{slow}. Then, when s_{slow} tends to infinity the behavior of the integral over s_{slow} is still governed by power counting, because the corrections due to the integrals over s_{fast} do not affect the powers of the momenta s_{slow}. Thus the power-counting analysis done in steps i) and ii) suffices.

Now we generalize the analysis to Lorentz violating theories. We say that the components $\hat{p}$ and $\bar{p}$ of each momentum are rescaled with the same "weighted velocity" when

$$\hat{p} \to \lambda\hat{p}, \qquad \bar{p} \to \lambda^{1/n}\bar{p}.$$

Step i) is modified studying the convergence when any subset of momenta tends to infinity with the same weighted velocity. Whenever a subconvergence fails the counterterms associated with the divergent subdiagrams have to be included. Once the subdivergences are subtracted away, step ii) consists of taking an appropriate number of "weighted derivatives" (see below) with respect to the external momenta, to eliminate the overall divergences. It is easy to check that this procedure automatically takes care of the overlapping divergences.

Weighted Taylor expansion. Every Taylor expansion

$$f(\hat{k}, \bar{k}) = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{f_{\nu_1\cdots\nu_u, i_1\cdots i_v}}{u!v!}\hat{k}_{\nu_1} \cdots \hat{k}_{\nu_u}\bar{k}_{i_1} \cdots \bar{k}_{i_v}$$

can be rearranged into a "weighted Taylor expansion"

$$f(\hat{k}, \bar{k}) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} f^{(\ell)}(\hat{k}, \bar{k}),$$

where

$$f^{(\ell)}(\hat{k}, \bar{k}) = \sum_{u=0}^{[\ell/n]} \frac{\ell!}{u!(\ell - nu)!} \times f_{\nu_1\cdots\nu_u, i_1\cdots i_{\ell-nu}}\hat{k}_{\nu_1} \cdots \hat{k}_{\nu_u}\bar{k}_{i_1} \cdots \bar{k}_{i_{\ell-nu}}$$

is a weighted homogeneous polynomial of degree ℓ/n:

$$f^{(\ell)}(\lambda\hat{k}, \lambda^{1/n}\bar{k}) = \lambda^{\ell/n} f^{(\ell)}(\hat{k}, \bar{k}).$$

The ℓ-th weighted derivatives with weight $1/n$ are the coefficients $f_{\nu_1\cdots\nu_u, i_1\cdots i_{\ell-nu}}$.

The weighted Taylor expansion is useful to subtract the overall divergences. The overall-subtracted version of an integral whose weighted degree of divergence is ω reads

$$\int \frac{\mathrm{d}^{L\hat{D}}\hat{p}}{(2\pi)^{L\hat{D}}}\frac{\mathrm{d}^{L\bar{D}}\bar{p}}{(2\pi)^{L\bar{D}}}[Q(\hat{p},\bar{p};\hat{k},\bar{k}) - \sum_{\ell=0}^{n\omega}\frac{1}{\ell!}Q^{(\ell)}(\hat{p},\bar{p};\hat{k},\bar{k})],$$

where $Q^{(\ell)}$ denotes the ℓ-th homogeneous polynomial of the weighted Taylor expansion of Q in $\hat{k}, \bar{k}$.

Subtraction algorithm. Consider an L-loop diagram with V vertices and I propagators. The integrand, which we denote with Q_G, is a ratio of weighted polynomials and has degree equal to $d_Q \equiv \sum_{(N,\alpha)}\delta_N^{(\alpha)}v_N^{(\alpha)} - 2I$. The integral $\mathcal{I}$ is a weighted function of degree $d_{\mathcal{I}} = d_Q + ĐL$. It has the form

$$\mathcal{I} = \int \frac{\mathrm{d}^{L\hat{D}}\hat{p}}{(2\pi)^{L\hat{D}}}\int \frac{\mathrm{d}^{L\bar{D}}\bar{p}}{(2\pi)^{L\bar{D}}} Q_G(\hat{p},\bar{p},k), \tag{4.6}$$

where $\hat{p}$ and $\bar{p}$ collectively denote the components of the momenta circulating in the loops, while $k = (\hat{k}, \bar{k})$ collectively denotes the external momenta. The overall degree of divergence of $\mathcal{I}$ is $\omega(G) = d_Q + đL$.

The subtraction of divergences can be arranged according to the following table:

$$\begin{array}{cc} Q_G(\hat{p},\bar{p};\hat{k},\bar{k}) & -\sum_{\gamma\in\Gamma}\bar{Q}_\gamma(\hat{p},\bar{p};\hat{k},\bar{k}) \\ -\sum_{\ell=0}^{n\omega(G)}\frac{1}{\ell!}Q_G^{(\ell)}(\hat{p},\bar{p};\hat{k},\bar{k}) & \sum_{\gamma\in\Gamma}\sum_{\ell=0}^{n\omega(G)}\frac{1}{\ell!}\bar{Q}_\gamma^{(\ell)}(\hat{p},\bar{p};\hat{k},\bar{k}) \end{array} \tag{4.7}$$

Here Γ denotes the set of divergent subdiagrams γ of the diagram G. The rational function $\bar{Q}_\gamma$ is obtained replacing the subintegrand with the appropriate, truncated, weighted Taylor expansion in the external momenta of γ. In the arrangement of (4.7) subdivergences are subtracted row-wise. Overall divergences are subtracted columnwise.

A potential caveat comes from certain "extra subdivergences," those that occur when a subdiagram γ' is convergent in Q_G, but becomes divergent in one of the $\bar{Q}_\gamma$'s. Then γ' does not belong to Γ, so its subdivergence is not subtracted row-wise. Nevertheless, it is easy to show that the extra subdivergences are automatically subtracted columnwise in (4.7). Details and an explicit example are given in Appendix A.

Thus, once the subdivergences have been subtracted away, the divergent part of every Feynman diagram is a weighted polynomial of degree $\omega(G)$ (second row of (4.7)) and can be removed renormalizing the Lagrangian (2.18).

V. RENORMALIZATION STRUCTURE AND RENORMALIZATION GROUP

In this section we study the renormalization group. We illustrate it first in the $đ = 4$ models (2.13). For the reasons that we explain below, it is convenient to parametrize the bare Lagrangian as

$$\mathcal{L}_{(2,2n)\mathrm{B}} = \frac{1}{2}(\hat{\partial}\varphi_\mathrm{B})^2 + \frac{1}{2\Lambda_{L\mathrm{B}}^{2(n-1)}}(\bar{\partial}^n\varphi_\mathrm{B})^2 + \frac{\lambda_\mathrm{B}}{4!\Lambda_{L\mathrm{B}}^{(n-1)(2-\varepsilon_2/n)}}\varphi_\mathrm{B}^4 \tag{5.1}$$

with

$$\varphi_\mathrm{B} = Z_\varphi^{1/2}\varphi, \qquad \Lambda_{L\mathrm{B}} = Z_\Lambda\Lambda_L, \qquad \lambda_\mathrm{B} = \lambda\mu^\varepsilon Z_\lambda, \qquad \varepsilon \equiv \varepsilon_1 + \frac{\varepsilon_2}{n}. \tag{5.2}$$

Observe that $Đ = 4 - \varepsilon$. The weighted scale invariance (2.10) can be extended to a transformation that rescales also μ:

$$\hat{x} \to \hat{x}\mathrm{e}^{-\Omega}, \qquad \bar{x} \to \bar{x}\mathrm{e}^{-\Omega/n}, \qquad \varphi \to \varphi\mathrm{e}^{\Omega(Đ-2)/2}, \qquad \mu \to \mu\mathrm{e}^{\Omega}. \tag{5.3}$$

The invariance under this transformation is not a symmetry. It just tells us that at the quantum level the weighted scale invariance (2.10) is equivalent to a μ-rescaling. What is important in (2.10) and (5.3) is that Λ_L is unmodified. Because of (5.3), every renormalization constant in (5.2) is just a function of λ (otherwise it could also depend on evanescent powers of the ratio μ/Λ_L). Thus, in the minimal subtraction scheme the λ-beta function has the usual form

$$\mu\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = \hat{\beta}_\lambda = -\varepsilon\lambda + \beta(\lambda).$$

The finiteness of $\hat{\beta}_\lambda$ proves that all poles contained in Z_λ are inverse powers of ε.

In more detail, let us consider the contribution of a graph G with E external legs, I propagators and V vertices to the generating functional of one-particle irreducible diagrams. Such a contribution has the schematic form

$$\mathcal{I} = \int \mathrm{d}^D x \frac{\lambda^V\mu^{V\varepsilon}}{\Lambda_L^{V(n-1)(2-\varepsilon_2/n)}} G\varphi^E,$$

where G denotes the value of the Green function. The dimensionality of G in units of mass is

$$[G] = D\left(V - \frac{E}{2} + 1\right) + E - 4V,$$

while its weighted degree is

$$\omega(G) = [G] - \delta[G] = 4 - E + \Delta\omega(G),$$

where

$$\Delta\omega(G) = -\varepsilon\left(V - \frac{E}{2} + 1\right), \qquad \delta[G] = \left(2 - \frac{\varepsilon_2}{n}\right)(n-1)\left(V - \frac{E}{2} + 1\right).$$

Recalling that $\mathcal{I}$ is invariant under the weighted scale transformation (5.3), we find that G transforms as

$$G \to e^{\Omega\omega(G)}G. \tag{5.4}$$

Once the subdivergences have been inductively subtracted away, the divergent part $G_{\rm div}$ is a weighted polynomial of degree $4-E$ in the external momenta. Matching the dimensionality and the weighted rescaling (5.4) we find

$$G_{\rm div} = P_{4-E,n}(\hat{\partial}, \bar{\partial}; \Lambda_L)\Lambda_L^{\delta[G]}\mu^{\Delta\omega(G)},$$

where $P_{4-E,n}(\hat{\partial}, \bar{\partial}; \Lambda_L)$ is a homogeneous weighted polynomial of degree $4-E$ and dimensionality equal to its degree. The corresponding Lagrangian counterterm reads

$$I_{\rm div} = -\int d^D x\left(\frac{\lambda\mu^{\varepsilon}}{\Lambda_L^{(n-1)(2-\varepsilon_2/n)}}\right)^V \mu^{\Delta\omega(G)}\Lambda_L^{\delta[G]} \times [P_{4-E,n}(\hat{\partial}, \bar{\partial}; \Lambda_L)]\varphi^E,$$

where $[P]$ means that the derivatives contained in P act on the scalar legs φ^E as appropriate. In particular, summing up all contributions for $E=4$, we get

$$-\int d^D x \frac{\lambda\mu^{\varepsilon}}{\Lambda_L^{(n-1)(2-\varepsilon_2/n)}}\varphi^4\sum_{L=1}^{\infty}c_L\lambda^L,$$

where c_L are divergent constants. Thus the renormalization constant of λ is a power series in λ,

$$Z_\lambda = 1 - \sum_{L=1}^{\infty}c_L\lambda^L,$$

with no spurious dependence on μ/Λ_L. The same conclusion holds for the other renormalization constants. We have

$$\mu\frac{d\Lambda_L}{d\mu} = \eta_L\Lambda_L, \qquad \eta_L(\lambda) = -\frac{d\ln Z_\Lambda}{d\ln\mu}.$$

The Callan-Symanzik equation has the same form as usual. Calling

$$G_k(\hat{x}_1, \cdots, \hat{x}_k; \bar{x}_1, \cdots, \bar{x}_k; \lambda, \Lambda_L, \mu) = \langle\varphi(x_1)\cdots\varphi(x_k)\rangle,$$

we have

$$\left(\mu\frac{\partial}{\partial\mu} + \hat{\beta}_\lambda\frac{\partial}{\partial\lambda} + \eta_L\Lambda_L\frac{\partial}{\partial\Lambda_L} + k\gamma_\varphi\right) \times G_k(\hat{x}_1, \cdots, \hat{x}_k; \bar{x}_1, \cdots, \bar{x}_k; \lambda, \Lambda_L, \mu) = 0. \tag{5.5}$$

The equation can be immediately integrated to give

$$G_k(\hat{x}_1, \cdots, \hat{x}_k; \bar{x}_1, \cdots, \bar{x}_k; \lambda, \Lambda_L, \xi\mu) = z^{-k}(t)G_k(\hat{x}_1, \cdots, \hat{x}_k; \bar{x}_1, \cdots, \bar{x}_k; \lambda(t), \Lambda_L(t), \mu),$$

where $t = \ln\xi$ and

$$z(t) = \exp\left(\int_0^t \gamma_\varphi(\lambda(t'))dt'\right), \qquad \frac{d\lambda(t)}{dt} = -\hat{\beta}_\lambda(\lambda(t)),$$
$$\Lambda_L(t) = \Lambda_L\exp\left(-\int_0^t \eta_L(\lambda(t'))dt'\right),$$

with $\lambda(0)=\lambda$. Now the renormalization-group flow specifies how the correlation function changes under a weighted overall rescaling. Indeed, the weighted scale invariance (5.3) and (5.4) tells us that

$$G_k(\hat{x}_1, \cdots, \hat{x}_k; \bar{x}_1, \cdots, \bar{x}_k; \lambda, \Lambda_L, \xi\mu) = \xi^{k(\text{Ð}-2)/2}G_k(\xi\hat{x}_1, \cdots, \xi\hat{x}_k; \xi^{1/n}\bar{x}_1, \cdots, \xi^{1/n}\bar{x}_k; \lambda, \Lambda_L, \mu).$$

A one-loop calculation for the models (2.13) gives

$$\hat{\beta}_\lambda = -\varepsilon\lambda + \frac{3\lambda^2}{(4\pi)^{n+1}n!} + \mathcal{O}(\lambda^3), \qquad \gamma_\varphi = \mathcal{O}(\lambda^2),$$
$$\eta_L = \mathcal{O}(\lambda^2),$$

so these models are IR free. Only the beta function has a nonvanishing one-loop contribution. Indeed, using the dimensional-regularization technique tadpoles vanish in homogeneous models, so γ_φ and η_L start from two loops.

Let us now consider the model (2.15). The bare Lagrangian reads

$$\mathcal{L}_{(2,2)B} = \frac{1}{2}(\hat{\partial}\varphi_B)^2 + \frac{1}{2\Lambda_{LB}^2}(\bar{\triangle}\varphi_B)^2 + \frac{\lambda_{4B}}{4!\Lambda_{LB}^{2-\varepsilon_2/2}}\varphi_B^2(\bar{\partial}\varphi_B)^2 + \frac{\lambda_{6B}}{6!\Lambda_{LB}^{2-\varepsilon_2}}\varphi_B^6,$$

where

$$\varphi_B = Z_\varphi^{1/2}\varphi, \quad \Lambda_{LB} = Z_\Lambda\Lambda_L, \quad \lambda_{4B} = \mu^{\varepsilon}(\lambda_4 + \Delta_4),$$
$$\lambda_{6B} = \mu^{2\varepsilon}(\lambda_6 + \Delta_6), \quad \varepsilon \equiv \varepsilon_1 + \frac{\varepsilon_2}{2}.$$

The theory is invariant under the scale transformation (5.3) with $n=2$. At one-loop we find $Z_\varphi = 1$, $Z_\Lambda = 1$ and

$$\Delta_4 = \frac{5\lambda_4^2}{2(12\pi)^2\varepsilon}, \qquad \Delta_6 = \frac{5\lambda_4\lambda_6}{(8\pi)^2\varepsilon} - \frac{5\lambda_4^3}{(48\pi)^2\varepsilon},$$

so the beta functions read

$$\hat{\beta}_4 = -\varepsilon\lambda_4 + \frac{5\lambda_4^2}{2(12\pi)^2},$$
$$\hat{\beta}_6 = -2\varepsilon\lambda_6 + \frac{5\lambda_4\lambda_6}{(8\pi)^2} - \frac{5\lambda_4^3}{(48\pi)^2}.$$

The asymptotic solutions of the RG flow equations are

$$\lambda_4 \sim \frac{2(12\pi)^2}{5t}, \qquad \lambda_6 \sim \frac{1}{20}\lambda_4^2,$$

where $t = \ln|x|\mu$ and $|x|$ is a typical weighted scale of the process. Since λ_4 and λ_6 must be non-negative, the theory is IR free.

VI. WEIGHTED TRACE ANOMALY

The weighted scale invariance (2.10) of the homogeneous models can be anomalous due to the radiative corrections. In this section we calculate the weighted trace anomaly, following [19]. For definiteness, we work with the model (2.12), but the discussion generalizes immediately to the other models.

Weighted dilatation. In the case of the model (2.12), write the Lagrangian as $\mathcal{L}(\varphi, \hat{\partial}_\mu\varphi, \bar{\triangle}\varphi)$. The infinitesimal version of the transformation (2.10) reads

$$\delta\varphi = \Omega\left(1 + \hat{x}\cdot\hat{\partial} + \frac{1}{2}\bar{x}\cdot\bar{\partial}\right)\varphi \equiv \Omega\check{D}\varphi,$$

with $\Omega \ll 1$. The conserved Noether current $J^\mu = (\hat{J}^\mu, \bar{J}^\mu)$ is given by

$$\hat{J}^\mu = -\hat{x}^\mu\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\hat{\partial}_\mu\varphi)}\check{D}\varphi,$$
$$\bar{J}^\mu = -\frac{1}{2}\bar{x}^\mu\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\bar{\triangle}\varphi)}\overleftrightarrow{\bar{\partial}}^\mu\check{D}\varphi.$$

We continue the spacetime dimensions to complex values as explained in Sec. I. The continued transformation $\delta\varphi'$ and the continued current J'^μ are obtained replacing $\check{D}\varphi$ in $\delta\varphi$ and J^μ with

$$\check{D}'\varphi = \left(\frac{Đ}{2} - 1 + \hat{x}\cdot\hat{\partial} + \frac{1}{2}\bar{x}\cdot\bar{\partial}\right)\varphi \qquad (6.1)$$

(see (5.3)), where $Đ = 4 - \varepsilon$. At the bare level, the anomaly of (6.1) is expressed by the divergence of J'^μ. We find

$$\partial_\mu J'^\mu = -\varepsilon\frac{\lambda_B\varphi_B^4}{4!\Lambda_{BL}^2}. \qquad (6.2)$$

Improved energy-momentum tensor and its weighted trace. The anomaly of the weighted dilatation is encoded also in the energy-momentum tensor, precisely in its "weighted trace." Let us start from the energy-momentum tensor given by the Noether method. For the model (2.12) we have

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\hat{\partial}_\mu\varphi)}\partial_\nu\varphi + \frac{\partial\mathcal{L}}{\partial(\bar{\triangle}\varphi)}\overleftrightarrow{\bar{\partial}}_\mu\partial_\nu\varphi - \delta_{\mu\nu}\mathcal{L}. \qquad (6.3)$$

This tensor is not symmetric, but conserved: it is easy to check that $\partial_\mu T_{\mu\nu} = 0$, using the field equations. Next, define the improved energy-momentum tensor

$$\begin{aligned}\tilde{T}_{\mu\nu} = {}& \hat{\partial}_\mu\varphi\partial_\nu\varphi - \frac{1}{\Lambda_L^2}\partial_\nu\varphi\overleftrightarrow{\bar{\partial}}_\mu\bar{\triangle}\varphi - \delta_{\mu\nu}\mathcal{L} \\ & - \frac{Đ-2}{4(\hat{D}-1)}\hat{\pi}_{\mu\nu}\varphi^2 \\ & + \frac{3Đ - 2\bar{D}Đ + 3\bar{D} - 5}{(\bar{D}-1)\Lambda_L^2}\bar{\pi}_{\mu\nu}(\varphi\bar{\triangle}\varphi) \\ & + \frac{3-2Đ}{2(\bar{D}-1)\Lambda_L^2}\bar{\pi}_{\mu\nu}(\bar{\partial}_\alpha\varphi)^2 \\ & + \frac{3-2Đ}{\Lambda_L^2}\bar{\pi}_{\mu\alpha}(\varphi\bar{\pi}_{\alpha\nu}\varphi),\end{aligned} \qquad (6.4)$$

where $\hat{\pi}_{\mu\nu} = \hat{\partial}_\mu\hat{\partial}_\nu - \hat{\delta}_{\mu\nu}\hat{\partial}^2$ and $\bar{\pi}_{\mu\nu} = \bar{\partial}_\mu\bar{\partial}_\nu - \bar{\delta}_{\mu\nu}\bar{\partial}^2$. The first three terms of (6.4) correspond to the Noether tensor (6.3), while the rest collects the improvement terms, identically conserved. Define the weighted trace

$$\Theta \equiv \tilde{T}_{\hat{\mu}\hat{\mu}} + \frac{1}{n}\tilde{T}_{\bar{\mu}\bar{\mu}}.$$

Using the field equations, it is easy to show that $\tilde{T}_{\mu\nu}$ is conserved and that its weighted trace Θ vanishes in the physical spacetime dimension $d = \hat{d} + \bar{d}$. Moreover, $\tilde{T}_{\mu\nu}$ is conserved also in the continued spacetime dimension. The coefficients of the improvement terms are chosen so that in the free-field limit Θ vanishes also in the continued dimension $D = \hat{D} + \bar{D}$. Finally, it is straightforward to check that the weighted trace Θ coincides with the divergence (6.2) of the current J'^μ.

Anomaly. We need to write Θ in terms of renormalized operators. When we differentiate a renormalized correlation function with respect to λ or Λ_L we obtain a renormalized correlation function containing additional insertions of $-\partial S/\partial\lambda$ or $-\partial S/\partial\Lambda_L$, respectively. Thus, $-\partial S/\partial\lambda$ and $-\partial S/\partial\Lambda_L$ are renormalized operators. Following a standard procedure [19] we can find which operators $\mathcal{O}$ they are the renormalized versions of. In the minimal subtraction scheme, it is sufficient to express the renormalized operators as bare operators $\mathcal{O}_B$ plus poles. Schematically,

$$\text{finite} = \mathcal{O}_B + \text{poles} \Rightarrow \text{finite} = [\mathcal{O}],$$

where $[\mathcal{O}]$ denotes the renormalized version of the operator $\mathcal{O}$. We find

$$\begin{aligned}\frac{\partial S}{\partial\lambda} &= \text{finite} = \frac{1}{\hat{\beta}_\lambda}\left(\gamma_\varphi[E_\varphi] - \eta_L\Lambda_L\frac{\partial S}{\partial\Lambda_L} - \varepsilon\frac{\lambda_B}{4!\Lambda_{BL}^2}\int\varphi_B^4\right) = \frac{\mu^\varepsilon}{4!\Lambda_L^2}\int[\varphi^4], \\ -\frac{1}{2}\Lambda_L\frac{\partial S}{\partial\Lambda_L} &= \text{finite} = \frac{1}{2\Lambda_{BL}^2}\int(\bar{\triangle}\varphi_B)^2 + \frac{\lambda_B}{4!\Lambda_{BL}^2}\int\varphi_B^4 \\ &= \frac{1}{2\Lambda_L^2}\int[(\bar{\triangle}\varphi)^2] + \frac{\lambda\mu^\varepsilon}{4!\Lambda_L^2}\int[\varphi^4],\end{aligned}$$

where $[E_\varphi] = \int\varphi(\delta S/\delta\varphi)$ is the operator that counts the number of φ-insertions. Thus,

$$\begin{aligned}\int\Theta &= -\int\varepsilon\frac{\lambda_B\varphi_B^4}{4!\Lambda_{BL}^2} \\ &= \frac{(\hat{\beta}_\lambda - 2\lambda\eta_L)\mu^\varepsilon}{4!\Lambda_L^2}\int[\varphi^4] - \frac{\eta_L}{\Lambda_L^2}\int[(\bar{\triangle}\varphi)^2] \\ &\quad - \gamma_\varphi[E_\varphi].\end{aligned}$$

The result agrees with the Callan-Symanzik Eq. (5.5), which can be expressed as

$$\left\langle \int \Theta\varphi(x_1)\cdots\varphi(x_k)\right\rangle = \mu\frac{\partial}{\partial\mu}\langle\varphi(x_1)\cdots\varphi(x_k)\rangle.$$

Indeed,

$$\int\Theta = -\mu\frac{\partial S}{\partial\mu} = \hat{\beta}_\lambda\frac{\partial S}{\partial\lambda} + \eta_L\Lambda_L\frac{\partial S}{\partial\Lambda_L} - \gamma_\varphi[E_\varphi].$$

VII. NONRELATIVISTIC THEORIES

Nonrelativistic theories can be studied along the same lines. The action contains only a single time derivative $\hat{\partial}$,

$$\mathcal{L} = \bar{\varphi}\left(\hat{\partial} + \frac{\bar{\triangle}}{2m} + \xi\frac{\bar{\triangle}^2}{m^2} + \cdots\right)\varphi + \zeta\bar{\varphi}^2\bar{\triangle}\varphi^2 + \cdots + \lambda(\bar{\varphi}\varphi)^2 + \cdots$$

so the theory is more divergent. The dimensional-regularization is not easy to use, since there is no simple way to continue the single-derivative term $\bar{\varphi}\,\hat{\partial}\,\varphi$ to complex dimensions. Thus we assume an ordinary cutoff regularization.

The propagator is defined by the term $\bar{\varphi}\,\hat{\partial}\,\varphi$ plus the Lagrangian quadratic term with the highest number of $\bar{\partial}$-derivatives, say $2n$,

$$\mathcal{L}_{\text{free}} = \bar{\varphi}\left(\hat{\partial} + \frac{\bar{\partial}^{2n}}{\Lambda_L^{2n-1}}\right)\varphi.$$

For the purposes of renormalization, the other quadratic terms, if present, can be treated perturbatively, as explained in Sec. II. Thus the nonrelativistic propagator is the inverse of a homogeneous weighted polynomial of degree 1 and weight $1/n$. The integral measure has weighted degree $đ = 1 + (d-1)/(2n)$. A Feynman diagram G with E total external legs, I propagators and $v_N^{(\alpha)}$ N-leg vertices of weighted degrees $\delta_N^{(\alpha)}$ is a weighted function of degree

$$\omega(G) = Lđ - I + \sum_{(N,\alpha)}\delta_N^{(\alpha)}v_N^{(\alpha)}.$$

Formulas (2.3) still hold. We have

$$\omega(G) = đ - \frac{E}{2}(đ-1) + \sum_{(N,\alpha)}\left[\delta_N^{(\alpha)} + \left(\frac{N}{2}-1\right)đ - \frac{N}{2}\right]v_N^{(\alpha)}.$$

Renormalizable theories are those that contain the vertices with

$$\delta_N^{(\alpha)} \leq \frac{N}{2} - \left(\frac{N}{2}-1\right)đ. \qquad (7.1)$$

Strictly renormalizable theories are those that have

$$\delta_N^{(\alpha)} = \frac{N}{2} - \left(\frac{N}{2}-1\right)đ.$$

Polynomiality requires now

$$đ > 1,$$

which ensures also that $\omega(G)$ decreases when the number of external legs increases. The maximal number of legs is

$$N_{\max} = \left[\frac{2đ}{đ-1}\right]. \qquad (7.2)$$

It is straightforward to check that $E = N$ implies

$$\omega(G) \leq đ - \frac{N}{2}(đ-1),$$

so by (7.1) the type of vertex that subtracts the divergence of G is already present in the Lagrangian, which proves renormalizability. No terms with more than one time derivative are turned on by renormalization.

Let us now see some examples of homogeneous models, beginning from the φ^4-theories. Setting $N_{\max} = 4$ in (7.2) we get

$$\frac{5}{3} < đ \leq 2.$$

For $đ = 2$ we have $d = 2n+1$ and the family of odd-dimensional theories

$$\mathcal{L}_{(1,2n)} = \bar{\varphi}i\hat{\partial}\varphi + \frac{1}{\Lambda_L^{2n-1}}\bar{\varphi}\bar{\partial}^{2n}\varphi + \frac{\lambda}{4\Lambda_L^{2n-1}}(\bar{\varphi}\varphi)^2. \qquad (7.3)$$

Setting $N_{\max} = 6$ we have $7/5 < đ \leq 3/2$. For $đ = 3/2$ we have $d = n+1$. If n is odd we have the family

$$\mathcal{L}_{(1,n)} = \bar{\varphi}i\hat{\partial}\varphi + \frac{1}{\Lambda_L^{2n-1}}\bar{\varphi}\bar{\partial}^{2n}\varphi + \frac{\lambda_6}{36\Lambda_L^{2n-1}}(\bar{\varphi}\varphi)^3.$$

In particular, we see that there exist four-dimensional ($n = 3$) nonrelativistic renormalizable φ^6-theories. If n is even we must include additional vertices,

$$\mathcal{L}_{(1,n)} = \bar{\varphi}i\hat{\partial}\varphi + \frac{1}{\Lambda_L^{2n-1}}\bar{\varphi}\bar{\partial}^{2n}\varphi + \sum_\beta\frac{\lambda_\beta}{4\Lambda_L^{2n-1}}[\bar{\partial}^n\bar{\varphi}^2\varphi^2]_\beta + \frac{\lambda_6}{36\Lambda_L^{2n-1}}(\bar{\varphi}\varphi)^3.$$

VIII. CONCLUSIONS

In this paper we have classified the unitary Lorentz violating renormalizable quantum field theories that can be obtained improving the UV behavior of propagators with the help of higher space derivatives. The removal of divergences is governed by a weighted power-counting criterion. If the Lagrangian has an appropriate form, time derivatives are "protected," in the sense that no higher time derivatives are turned on by renormalization. The so-defined theories are unitarity, but have modified dispersion relations. We have studied their main properties, including the renormalization-group flow and the weighted trace anomaly.

Natural extensions of this work are those that aim to include gauge fields and gravity. Possible applications range from high-energy physics, effective field theory, nuclear physics and the theory of critical phenomena. In the high-energy physics domain, it would be interesting to explore the work-hypothesis that Lorentz invariance is violated at very high energies, to define the ultraviolet limit of quantum gravity, or study new types of Lorentz invariant extensions of the standard model. It would also be interesting to embed the weighted scale invariance into a "weighted conformal group," generalizing the Galilean conformal group that characterizes a class of nonrelativistic theories [20].

APPENDIX A: EXTRA SUBDIVERGENCES

In this appendix we give more details on the extra subdivergences mentioned in Sec. IV. By construction, every row of table (4.7) is free of "ordinary" subdivergences, namely, those originated by the subdiagrams γ. Every column is free of overall divergences. Extra subdivergences are those that occur when a subdiagram γ' is convergent in Q_G, but becomes divergent in one of the $\bar{Q}_\gamma$'s, after replacing γ with its counterterms. Here we prove that the extra subdivergences are automatically subtracted columnwise.

It is useful to have an explicit example in mind, such as the two-loop diagram depicted in Fig. 1, in the four-dimensional φ^4-theory. The diagram is the p-q integral of

$$Q_G = \frac{1}{(p^2+m^2)[(p-k)^2+m^2]} \times \frac{1}{(q^2+m^2)[(q+p+k')^2+m^2]}.$$

The p-integral is convergent, the q-integral is not. The q-subdivergence is subtracted by

$$-\bar{Q}_\gamma = -\frac{1}{(p^2+m^2)[(p-k)^2+m^2]}\frac{1}{(q^2+m^2)^2}. \tag{A1}$$

In this expression, however, the p-integral is divergent. This divergence is what we call an extra subdivergence. The table reads

$$\begin{array}{cc} Q_G & -\bar{Q}_\gamma \\ -\frac{1}{(p^2+m^2)^2}\frac{1}{(q^2+m^2)[(q+p)^2+m^2]} & +\frac{1}{(p^2+m^2)^2}\frac{1}{(q^2+m^2)^2}. \end{array} \tag{A2}$$

In the general case, assume that the subdiagram γ contains l loops and that $\bar{Q}_\gamma$ contains some extra subdivergences. The extra subdivergences can be overall or not. We call them overall if they arise letting all of the remaining $L-l$ loop momenta tend to infinity. They are not overall if they arise letting only a subset of the remaining $L-l$ loop momenta tend to infinity. Proceeding inductively, we can assume that the nonoverall extra subdivergences have already been subtracted away. Thus, we need to consider only the overall extra subdivergences. It is not difficult to see that they are subtracted columnwise in (4.7). Indeed, as in (A1), the integrands that generate extra overall subdivergences factorize (or split into a sum of terms each of which factorizes): one factor is responsible for the extra subdivergence (see the first factor of $-\bar{Q}_\gamma$ in (A1)), while the other factor is the γ-counterterm (see the second factor of $-\bar{Q}_\gamma$ in (A1)). The second factor is the same throughout the column. Thus, the column subtracts away the overall divergence of the first factor, which is precisely the extra subdivergence. Recapitulating, the rows are free of ordinary subdivergences and the columns are free of extra subdivergences and overall divergences. Thus the table (4.7) is convergent. In the example (A2), it is clear that the column of $-\bar{Q}_\gamma$ is p-convergent.

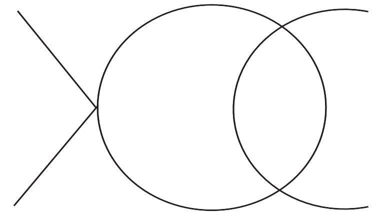

FIG. 1. Simple example of diagram that generates an extra subdivergence.

APPENDIX B: EUCLIDEAN PROPAGATORS

Let us examine some propagators

$$\frac{1}{\hat{p}^2 + \frac{(\bar{p}^2)^n}{\Lambda_L^{2n-2}}}$$

in coordinate space. The Euclidean (2,2)-propagator in four dimensions with $n=2$ reads

$$G_{(2,2)}(\hat{x},\bar{x},\Lambda_L) = \frac{\Lambda_L}{16|\hat{x}|}[I_0(\Lambda_L\bar{x}^2/4|\hat{x}|) - SL_0(\Lambda_L\bar{x}^2/4|\hat{x}|)],$$

where I denotes the modified Bessel function of the first kind, while SL denotes the modified Struve function. For $|\hat{x}| \gg \Lambda_L\bar{x}^2$ and $|\hat{x}| \ll \Lambda_L\bar{x}^2$ we have

$$G_{(2,2)} \sim \frac{\Lambda_L}{16|\hat{x}|} \quad \text{and} \quad G_{(2,2)} \sim \frac{1}{2\pi\bar{x}^2},$$

respectively.

Instead, the Euclidean (1,3)-propagator with $n=2$ reads

$$G_{(1,3)}(\hat{x},\bar{x},\Lambda_L) = \frac{\Lambda_L}{8\pi|\bar{x}|}\mathrm{Erf}\left(\sqrt{\frac{\Lambda_L\bar{x}^2}{4|\hat{x}|}}\right).$$

In the two limits considered above we have the behaviors

$$G_{(1,3)} \sim \frac{\Lambda_L^{3/2}}{8\pi^{3/2}|\hat{x}|^{1/2}} \quad \text{and} \quad G_{(1,3)} \sim \frac{\Lambda_L}{8\pi|\bar{x}|},$$

respectively.

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