BPS Z_N string tensions, sine law and Casimir scaling, and integrable field theories

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We consider a Yang-Mills-Higgs theory with spontaneous symmetry breaking of the gauge group $G \to U(1)^r \to C_G$, with C_G being the center of G. We study two vacua solutions of the theory which produce this symmetry breaking. We show that for one of these vacua, the theory in the Coulomb phase has the mass spectrum of particles and monopoles which is exactly the same as the mass spectrum of particles and solitons of two-dimensional affine Toda field theory, for suitable coupling constants. That result holds also for $\mathcal{N} = 4$ super Yang-Mills theories. On the other hand, in the Higgs phase, we show that for each of the two vacua the ratio of the tensions of the BPS Z_N strings satisfy either the Casimir scaling or the sine law scaling for $G = SU(N)$. These results are extended to other gauge groups: for the Casimir scaling, the ratios of the tensions are equal to the ratios of the quadratic Casimir constant of specific representations; for the sine law scaling, the tensions are proportional to the components of the left Perron-Frobenius eigenvector of Cartan matrix K_{ij} and the ratios of tensions are equal to the ratios of the soliton masses of affine Toda field theories.

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I. INTRODUCTION

In $SU(N)$ QCD, it is believed that the confinement of particles in strong coupling regime happens by formation of chromoelectric flux tubes, which we shall call QCD strings, carrying charge on the discrete group Z_N. There are different ways to try to understand this phenomenon (for nice reviews see [1,2]). In particular it is believed that particle confinement in strong coupling could be a phenomenon dual to the monopole confinement in weak coupling. For some time, it was thought that QCD string could be dual to strings solutions appearing in (effective) theories with broken $U(1)$ gauge group. However, it seems it does not give the right spectrum of mesons [1]. For this reason, we have studied many properties of topological Z_N strings solutions and monopole confinement in the Higgs phase of theories with simple gauge groups G (*without* $U(1)$ factors) [3–5]. More recently some works [6] also appeared analyzing semilocal strings [7] with gauge group $SU(N) \times U(1)$ and global flavor symmetry $SU(N)_{\rm flavor}$.

Our motivation for considering chromomagnetic Z_N strings in the Higgs phase produced by breaking of gauge groups G without $U(1)$ factors is that the QCD's chromoelectric strings in confining phase should be formed only by fields with $SU(3)$ color charges and not $U(1)$ gauge fields. We consider general gauge groups G since it allows us to consider more general and direct arguments which may clarify some fundamental results common to different groups. We also hope that these results might be useful for lattice calculation of chromoelectric strings for groups other than $SU(N)$.

Differently from string solutions associated with the breaking of $U(1)$ group which have just one fundamental string and the string tensions proportional to the winding number, for the Z_N strings obtained by breaking simple gauge groups G, we showed that they are associated with weights of representations of the dual group $G^\vee$ and the string tensions seem to be consistent with the tensions of QCD strings as discussed in [5] and in the present work. Therefore, the chomomagnetic Z_N string solutions we consider have features similar to QCD strings. In particular in [5], there were constructed Z_N strings solutions which appear in a theory with a symmetry breaking pattern

$$G \overset{\phi_1^{\rm vac}}{\longrightarrow} U(1)^r \overset{\phi_2^{\rm vac}}{\longrightarrow} C_G, \tag{1}$$

where r is the rank of G and C_G its center. For each weight of G one can construct a Z_N string and we established how these strings are separated into topological sectors for general G. The string flux quantization condition was also obtained and the flux matching between strings and monopoles for *any* group G was shown. The *set* of strings which should be attached to each monopole was shown to belong necessarily to the trivial topological sector which is consistent with the fact that only these configurations can terminate at some point and can break as it happens for the QCD strings. In particular, for $G = SU(N)$, a string solution was constructed for each weight (color) of the N-dimensional fundamental representation. We deter-

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mined the set of strings which should be attached to each non-Abelian monopole which appeared in the Coulomb phase and showed that one could also have a confining system composed by N monopoles besides the monopole-antimonopole system. Since these monopoles have magnetic charges in the adjoint representation of the dual group $G^\vee$ [8], they should be dual to gluons and these confined systems should be dual to the glueballs. Differently from what is sometimes said, the Z_N strings do not necessarily point in a direction in the Cartan subalgebra. However, since the monopoles' magnetic flux is in the direction of the Cartan subalgebra (CSA) [9], we only consider Z_N string solutions with flux in the CSA which are relevant for confinement of these monopoles. This result is analogous to the Abelian dominance observed in QCD.

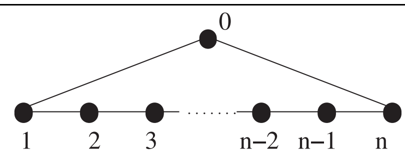
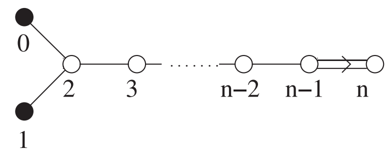
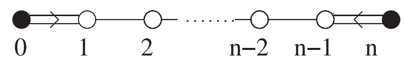
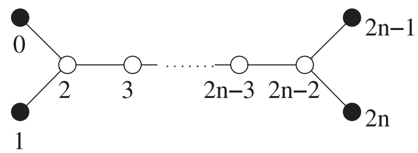
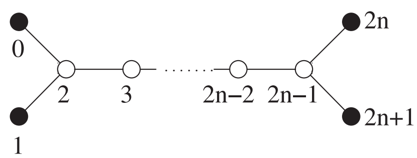
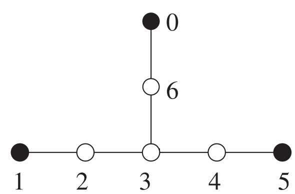
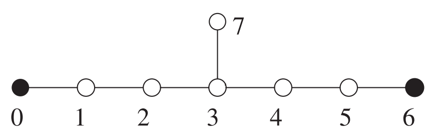
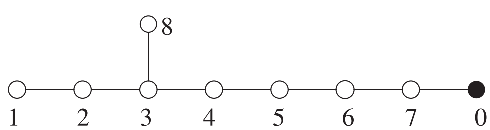
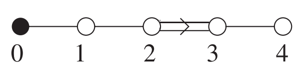
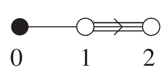
An important quantity in particle confinement in QCD are the string tensions. The spectrum of string tensions has been extensively studied in lattice calculations in recent years [10,11]. The main conjectures for the QCD string tensions are the Casimir scaling [12] and the sine law scaling [13]. In supersymmetric theories it was possible to arrive at these scalings by using analytical calculation. For softly broken $\mathcal{N} = 2$ super Yang-Mills theories with gauge group $G = SU(N)$ and a hypermultiplet in fundamental representation, it was obtained an effective Lagrangian with spontaneously broken $U(1)^{N-1}$ Abelian gauge group and Nielsen-Olesen strings [14] with tensions satisfying the sine law scaling [13]. On the other hand, in [5] a softly broken $\mathcal{N} = 4$ super Yang-Mills theory was considered and the Casimir scaling was obtained for tensions of the BPS Z_N strings when $G = SU(N)$. The sine law scaling was also derived in the M theory description of $\mathcal{N} = 1$ $SU(N)$ super Yang-Mills theory [15] and in the AdS/CFT correspondence [16].

In the present work we show that for the BPS Z_N strings, one can obtain the Casimir scaling and the sine law scaling by considering two different vacua of the same theory which give rise to the symmetry breaking (1). This result shows that these scalings are not necessarily "universal laws," but they depend on the vacuum which is responsible for the symmetry breaking. From the dual superconductor picture this result may indicate that if the tensions of the QCD strings satisfy one of these scalings, it may be due to a non-Abelian monopole condensate (in the adjoint representation) in one of these two vacua. We also generalize the Casimir and sine law scalings to groups other than $SU(N)$. It is important to note that in [13], the sine law scaling was obtained for the tensions of Z strings which appear due to the spontaneously broken $U(1)^{N-1}$ Abelian gauge group, which gives rise to a different meson spectrum [1] from Z_N strings we consider. Since in this paper we are interested in studying some general properties at the classical level of these Z_N strings which might be useful for QCD and not necessarily confinement in supersymmetric theories, we shall not restrict the potential to be supersymmetric. Similarly to the monopole solutions, for the Z_N string one can construct moduli spaces of solutions. However, for the determination of the properties of the Z_N string solution as string flux and tension, it is not necessary to construct moduli spaces since for all solutions in a moduli space these properties are the same.

In this paper we introduce in Secs. II and III some general results for Bogomol'nyi-Prasad-Sommerfield (BPS) Z_N string and Lie algebra which will be used in the following sections. In Sec. IV we obtain two different vacuum solutions which give rise to the symmetry breaking (1) for any group G. The first stage of the symmetry breaking corresponds to the Coulomb phase which is analyzed in Sec. V. In particular we show that for one of the vacua, the mass spectrum of particles and monopoles of the four-dimensional theories in the Coulomb phase is exactly the same as the mass spectrum of particles and solitons of two-dimensional affine field Toda theories (ATFTs), if the couplings of the two theories satisfy some suitable relations. That result holds also for $\mathcal{N} = 4$ super Yang-Mills theories. Then, in Sec. VI we analyze the Higgs phase. We start reviewing the construction of the Z_N strings solutions, where for each weight of the dual gauge group $G^\vee$ one can construct a solution, and how these solutions are classified in topological sectors for any group G. Next we obtain the BPS string tension for each topological sector and show that, depending on the vacuum, the ratios of the tensions satisfy the sine law scaling or the Casimir scaling when the gauge group is $SU(N)$. These scalings are generalized to other groups, and, in particular, the tensions which appear in the sine law scaling are identified with components $x_i^{(1)}$ of left Perron-Frobenius eigenvector of the Cartan matrix K_{ij}, and the ratios of tensions are equal to the ratios of soliton masses of the corresponding affine Toda field theory, for any gauge group G.

Differently from the $SU(n)$ group which, at fixed k and large n, the Casimir and the sine law scaling coincide, for $G = \mathrm{Spin}(2n)$ [the universal covering group of $SO(2n)$], the Casimir and the sine law scaling give different results in the large n limit. With the generalization of the Casimir and sine law scaling for the Z_N strings to other gauge groups, it could be interesting to analyze, using lattice calculation, the chromoelectric string tensions of QCD for groups other than $SU(n)$, as for example $G = \mathrm{Spin}(2n)$. It is important to note that the Casimir scaling and the sine law scaling we obtained are lower bounds for the non-BPS Z_N string tensions and they hold exactly only for the BPS Z_N strings, which exist only on the boundary between a type I and type II superconductor. Therefore, the small deviation from the Casimir scaling observed in [11] could be due to fact that QCD strings would not be BPS. Recently the interest on topological solutions in (supersymmetric) field theory [17] has increased. We hope that the results may also be useful for the study of other topological solutions.

TABLE I. Extended Dynkin diagrams, nodes symmetrically related to the node 0 and center groups C_G.

G	Extended Dynkin diagram of g	W_0	C_G
$SU(n+1)$		$0, 1, 2, \ldots, n$	$\mathbb{Z}_{n+1}$
Spin$(2n+1)$		0, 1	$\mathbb{Z}_2$
$Sp(2n)$		$0, n$	$\mathbb{Z}_2$
Spin$(4n)$		$0, 1, 2n-1, 2n$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Spin$(4n+2)$		$0, 1, 2n, 2n+1$	$\mathbb{Z}_4$
E_6		0, 1, 5	$\mathbb{Z}_3$
E_7		0, 6	$\mathbb{Z}_2$
E_8		0	1
F_4		0	1
G_2		0	1

II. BPS Z_N STRINGS

Let us consider Yang-Mills-Higgs theories with arbitrary gauge group G which is simple, connected and simply connected. In order to have strings and confined monopoles we shall consider theories with two complex scalars fields[1] ϕ_s, $s = 1, 2$, in the adjoint representation of G. We also consider that the vacuum solutions $\phi_1^{\rm vac}$, $\phi_2^{\rm vac}$ produce the symmetry breaking (1).

In order to exist stable Z_N string solutions for the symmetry breaking (1), C_G must be nontrivial. Therefore, we shall not consider the groups E_8, F_4, and G_2. In Table I we list the centers of simply connected simple Lie groups. In [3,4] we consider an alternative symmetry breaking in which stable strings exist even in theories with gauge groups with a trivial center.

The Lagrangian of the theory we study is

$$\mathcal{L} = -\tfrac{1}{4}G_{a\mu\nu}G_a^{\mu\nu} + \tfrac{1}{2}(D_\mu\phi_s)_a^*(D^\mu\phi_s)_a - V(\phi,\phi^*), \tag{2}$$

where $D_\mu = \partial_\mu + ie[W_\mu, \ ]$. Let $D_\pm = D_1 \pm iD_2$ and $B_{ai} = -\epsilon_{ijk}G_{ajk}/2$ is the non-Abelian magnetic field. As analyzed in [3,5], the BPS string conditions for a theory with gauge group G without $U(1)$ factors are

$$B_{a3} = \mp d_a, \tag{3}$$

$$D_\mp\phi_s = 0, \tag{4}$$

$$V(\phi,\phi^*) - \frac{1}{2}(d_a)^2 = 0, \tag{5}$$

$$E_{ai} = B_{a1} = B_{a2} = D_0\phi_s = D_3\phi_s = 0, \tag{6}$$

with

$$d_a = \frac{e}{2}(\phi_{sb}^* i f_{abc}\phi_{sc}) - X_a,$$

where X_a is a real scalar quantity which transforms in the adjoint representation with dimension of mass. From its transformation properties we can consider

$$X_a = \frac{em}{2}\,\mathrm{Re}(\phi_{1a}), \tag{7}$$

where m is a mass parameter which is considered to be nonnegative. This term was introduced in [3] as a generalization of the Fayet-Iliopolous term in the sense that it is responsible for the symmetry breaking which gives rise to stable string solutions for theories with non-Abelian gauge groups. In this case, the string tension satisfies [3,5]

$$T \geq \frac{me}{2}|\phi_1^{\rm vac}||\Phi_{\rm st}|, \tag{8}$$

where

$$\Phi_{\rm st} = \frac{1}{|\phi_1^{\rm vac}|}\int d^2x[\mathrm{Re}(\phi_1)_a B_{3a}] \tag{9}$$

is the string flux, with the integral being taken in the plane orthogonal to the string. The equality in Eq. (8) happens only for the BPS strings satisfying the conditions (3)–(6). In order to fulfill (5), we shall consider

$$V(\phi,\phi^*) = \tfrac{1}{2}(d_a)^2. \tag{10}$$

Note that condition (5) does not restrict the potential to have this form. In [3–5] it was considered different potentials.

[1] Note that, if one only wants string solutions, it is enough to have only one complex scalar.

III. MATHEMATICAL RESULTS

Let us start by giving some conventions and useful mathematical results which will be used later. Let $\mathfrak{g}$ be the Lie algebra associated with the group G. Let us adopt the Cartan-Weyl basis in which

$$\mathrm{Tr}(H_iH_j) = \delta_{ij}, \qquad \mathrm{Tr}(E_\alpha E_\beta) = \frac{2}{\alpha^2}\delta_{\alpha+\beta},$$

where the trace is taken in the adjoint representation. The generators H_i, $i = 1, 2, \ldots, r$ form a basis for the CSA $\mathfrak{h}$. In this basis, the commutation relations read

$$[H_i, E_\alpha] = (\alpha)^i E_\alpha, \qquad [E_\alpha, E_{-\alpha}] = \frac{2\alpha}{\alpha^2}\cdot H, \tag{11}$$

where α are roots and the upper index in $(\alpha)^i$ means the component i of α. Then, α_i and λ_i, $i = 1, 2, \ldots, r$, are, respectively, the simple roots and fundamental weights of $\mathfrak{g}$, and

$$\alpha_i^\vee = \frac{2\alpha_i}{\alpha_i^2}, \qquad \lambda_i^\vee = \frac{2\lambda_i}{\alpha_i^2} \tag{12}$$

are the simple co-roots and fundamental co-weights, which satisfy the relations

$$\alpha_i\cdot\lambda_j^\vee = \alpha_i^\vee\cdot\lambda_j = \delta_{ij}. \tag{13}$$

$\alpha_i^\vee$ and $\lambda_i^\vee$ are simple roots and fundamental weights of the dual algebra $\mathfrak{g}^\vee$. Moreover,

$$\alpha_i = K_{ij}\lambda_j \tag{14}$$

where

$$K_{ij} = \frac{2\alpha_i\cdot\alpha_j}{\alpha_j^2}$$

is the Cartan matrix associated with $\mathfrak{g}$.

The fundamental weights form a basis for the weight lattice of G,

$$\Lambda_w(G) = \left\{\omega = \sum_{i=1}^r n_i\lambda_i, \quad n_i \in \mathbb{Z}\right\}. \tag{15}$$

This lattice includes as a subset, the root lattice of G,

TABLE II. Some (nonnormalized) left Perron-Frobenius eigenvectors $x_i^{(1)}$ and Coxeter numbers h.

$SU(n+1)$ $(h = n+1)$	$\mathrm{Spin}(2n)$ $(h = 2n-2)$	E_6 $(h = 12)$
$x_1^{(1)} = \sin(\pi/h)$	$x_1^{(1)} = \sin(\pi/h)$	$x_1^{(1)} = \sin(\pi/h)$
$x_2^{(1)} = \sin(2\pi/h)$	$x_2^{(1)} = \sin(2\pi/h)$	$x_2^{(1)} = \sin(2\pi/h)$
$x_3^{(1)} = \sin(3\pi/h)$	$\vdots$	$x_3^{(1)} = \sin(3\pi/h)$
$\vdots$	$x_{n-2}^{(1)} = \sin[(n-2)\pi/h]$	$x_4^{(1)} = \sin(2\pi/h)$
$x_{n-1}^{(1)} = \sin[(n-1)\pi/h]$	$x_{n-1}^{(1)} = 1/2$	$x_5^{(1)} = \sin(\pi/h)$
$x_n^{(1)} = \sin(n\pi/h)$	$x_n^{(1)} = 1/2$	$x_6^{(1)} = \sin(8\pi/h) - \sin(2\pi/h)$

$$\Lambda_r(G) = \left\{\beta = \sum_{i=1}^{r} n_i\alpha_i, \quad n_i \in \mathbb{Z}\right\}, \tag{16}$$

which has the simple roots α_i as basis. Similarly, the fundamental co-weights $\lambda_i^\vee$ are basis of the weight lattice of the dual group[2] $G^\vee$

$$\Lambda_w(G^\vee) = \left\{\omega = \sum_{i=1}^{r} n_i\lambda_i^\vee, \quad n_i \in \mathbb{Z}\right\}, \tag{17}$$

which is also called the co-weight lattice of G and which has the root lattice of the dual group $G^\vee$ (or co-root lattice of G)

$$\Lambda_r(G^\vee) = \left\{\beta = \sum_{i=1}^{r} n_i\alpha_i^\vee, \quad n_i \in \mathbb{Z}\right\} \tag{18}$$

as subset.

Let

$$I_{ij} = 2\delta_{ij} - K_{ij},$$

which is called the incident matrix. One can show that the eigenvalues of I_{ij} are

$$\lambda(\nu) = 2\cos\frac{\pi\nu}{h},$$

where h is the Coxeter number of $\mathfrak{g}$ and ν are the exponents of $\mathfrak{g}$. For $\mathfrak{g} = su(n)$, the exponents are $\nu = 1, 2, \ldots, n-1$ and $h = n$.

Let $x_i^{(\nu)}$ be the left eigenvector of I_{ij} associated with the eigenvalue $\lambda(\nu)$. Then

$$y_i^{(\nu)} = \alpha_i^2 x_i^{(\nu)}/2$$

is the right eigenvector of I_{ij} with same eigenvalue $\lambda(\nu)$. We shall adopt the normalization

$$x_i^{(\mu)} y_i^{(\nu)} = \delta_{\mu\nu}.$$

[2]We shall consider the dual group $G^\vee$ as the covering group associated with the dual algebra $\mathfrak{g}^\vee$.

Clearly $x^{(\mu)}$ and $y^{(\mu)}$ are also eigenvectors of the Cartan matrix with

$$K_{ij}y_j^{(\nu)} = 2\left(1 - \cos\frac{\pi\nu}{h}\right)y_i^{(\nu)} = 4\sin^2\frac{\pi\nu}{2h}y_i^{(\nu)}. \tag{19}$$

The incident matrix I_{ij} has strictly positive entries and it is irreducible since we are considering $\mathfrak{g}$ simple. Therefore, we can apply the Perron-Frobenius theorem which says that for a given nonnegative irreducible matrix M, there exists a eigenvalue μ such that $\mu \geq |\lambda|$ for all eigenvalues λ of M and this eigenvalue μ can be associated with strictly positive left and right eigenvectors. Since for any algebra $\mathfrak{g}$, $\nu = 1$ is always the smallest exponent and $\nu = h - 1$ is the largest one, then

$$\lambda(1) = 2\cos\frac{\pi}{h}$$

is the largest eigenvalue of I_{ij}, and we conclude that the corresponding eigenvector components $x_i^{(1)}$ and $y_i^{(1)}$ never vanish and can be taken positive. The other eigenvectors necessarily have negative components. Therefore, we call $x_i^{(1)}$ ($y_i^{(1)}$) the components of the left (right) Perron-Frobenius eigenvector of I_{ij} and K_{ij}. Some of these (nonnormalized) vectors are listed in Table II, using the Dynkin diagram numbering convention of Table I.

IV. VACUUM SOLUTIONS

Let us now analyze some vacuum solutions of our theory. A vacuum solution ϕ^{vac} breaks the gauge group G to a subgroup G_ϕ which consist of the group elements which commutes with ϕ^{vac}. Considering that ϕ^{vac} can be embedded in a Cartan subalgebra, we can write

$$\phi^{\mathrm{vac}} = v \cdot H, \tag{20}$$

where v is an r component real vector which can be expanded in the basis of the fundamental co-weight vectors

$$v = v_i\lambda_i^\vee. \tag{21}$$

If all coefficients v_i do not vanish, then G is broken to the maximal torus $U(1)^r$ [18] associated with the CSA $\mathfrak{h}$ which corresponds to the first symmetry breaking in (1). The compact $U(1)$ factors are associated with the group elements[3]

$$\exp\{2\pi i a^{(i)} \lambda_i^\vee \cdot H\}, \qquad i = 1, 2, \ldots, r, \tag{22}$$

where $a^{(i)}$ are real parameters.

In [5] we considered a vacuum solution of the form

$$\begin{aligned} \phi_1^{\rm vac} &\propto \delta \cdot H, \qquad \delta \equiv \sum_{j=1}^r \lambda_j^\vee = \frac{1}{2}\sum_{\alpha>0} \alpha^\vee, \\ \phi_2^{\rm vac} &\propto \sum_{i=1}^r \sqrt{c_i}\, E_{-\alpha_i}, \qquad c_i = \sum_{j=1}^r (K_{ij}^{-1}) = \lambda_i \cdot \delta, \end{aligned} \tag{23}$$

where δ is the dual Weyl vector. There was also a third complex field in order for the theory to be supersymmetric which did not produce any extra symmetry breaking. This vacuum configuration produces the symmetry breaking (1). This result follows from the fact that since $\phi_1^{\rm vac}$ belongs to the Cartan subalgebra with all the coefficients of $\lambda_i^\vee$ not vanishing, it produces the first symmetry breaking in (1). Then, as

$$\begin{aligned} &\exp\{2\pi i a^{(i)} \lambda_i^\vee \cdot H\} E_{-\alpha_j} \exp\{-2\pi i a^{(i)} \lambda_i^\vee \cdot H\} \\ &\quad = \exp\{-2\pi i a^{(i)} \delta_{ij}\} E_{-\alpha_j}, \end{aligned}$$

the $U(1)$ group elements (22) will only commute with $\phi_2^{\rm vac}$ if the constants $a^{(i)}$ are integers. Remembering that the center C_G of a group G is formed by group elements

$$\exp 2\pi i \omega \cdot H, \tag{24}$$

where ω is a vector of the co-weight lattice $\Lambda_w(G^\vee)$, we can conclude that $\phi_2^{\rm vac}$ only commutes with the center elements (24) and produces the second symmetry breaking in (1). We showed [5] that for this vacuum configuration, the tensions of the BPS strings satisfy the Casimir scaling when $G = SU(N)$. Let us now analyze some vacuum solutions which produce the same symmetry breaking (1).

From the above example we can conclude that in order to produce the symmetry breaking (1) we can consider a general vacuum solution

$$\begin{aligned} \phi_1^{\rm vac} &= v \cdot H, \qquad v = v_i \lambda_i^\vee, \\ \phi_2^{\rm vac} &= \sum_{l=1}^r b_l E_{-\alpha_l}, \end{aligned} \tag{25}$$

where v_i are nonvanishing real constants and b_l must be nonvanishing complex constants in order for G to be broken to C_G.

[3] No summation is assumed for the index i.

The vacua of our theory are solutions of

$$G_{\mu\nu} = D_\mu \phi_s = V(\phi, \phi^*) = 0.$$

The condition $V(\phi, \phi^*) = 0$ for the potential (10) implies that

$$[\phi_1^\dagger, \phi_1] + [\phi_2^\dagger, \phi_2] = m \mathrm{Re}(\phi_1).$$

Using the configuration (25) in this condition gives the result

$$m (K^{-1})_{ij} v_j = |b_i|^2. \tag{26}$$

From this equation, we conclude that when $m = 0$, then $b_i = 0$ and $\phi_2^{\rm vac} = 0$, and it happens the first symmetry breaking in (1), which corresponds to the Coulomb phase. In order for the second symmetry breaking to happen, all components v_i and b_i must be nonvanishing, and therefore we must have $m \neq 0$, which means that the term X_a, given by Eq. (7), must not vanish.

In principle, Eq. (26) has various solutions depending on G. However, there are two which hold for any G. One solution is

$$v_i = a, \qquad b_i = \sqrt{am \sum_{j=1}^r (K^{-1})_{ij}}, \tag{27}$$

where a is a positive real constant. This solution gives rise to the vacuum (23).

The other solution is to consider that v_j are the components of a right eigenvector of the Cartan matrix K_{ij}. However, the components v_j cannot vanish and from the relation (26) we also see that they cannot be negative since m and the eigenvalues of K_{ij} are positive. Therefore, from the discussion in the previous section we can conclude that v_j can only be proportional to the Perron-Frobenius right eigenvector of K_{ij}. Hence,

$$v_i = a y_i^{(1)}, \qquad b_i = \frac{1}{2\sin\frac{\pi}{2h}} \sqrt{a m y_i^{(1)}} \tag{28}$$

is a solution where a is a positive constant and the corresponding vacuum solution

$$\begin{aligned} \phi_1^{\rm vac} &= a \sum_{i=1}^r y_i^{(1)} \lambda_i^\vee \cdot H, \\ \phi_2^{\rm vac} &= \frac{\sqrt{am}}{2\sin\frac{\pi}{2h}} \sum_{i=1}^r \sqrt{y_i^{(1)}}\, E_{-\alpha_i}, \end{aligned} \tag{29}$$

also produces the symmetry breaking (1). This vacuum is very interesting since it gives rise to the sine law scaling for the ratios of BPS string tensions and a possible connection with affine Toda field theories as we shall see in the next sections.

V. THE COULOMB PHASE

Let us analyze the Coulomb phase which happens when $m = 0$. In this phase there exist free monopoles. For a symmetry breaking produced by an arbitrary vacuum configuration (25), with $v_i \neq 0$ and $b_i = 0$, one can construct monopole solutions for each root α such that $\alpha \cdot v \neq 0$, which has magnetic charge [19]

$$g_\alpha = \frac{1}{|\phi_1^{\text{vac}}|} \oint d^2 S_i [\text{Re}(\phi_1)_a B_a^i] = \frac{2\pi}{e} \frac{v \cdot \alpha^\vee}{|v|}. \tag{30}$$

Since in our case the scalar product $\alpha \cdot v$ with any root α never vanishes, we can then construct monopoles for any root α. The vacuum solution $\phi_1^{\text{vac}} = v \cdot H$ singles out a particular $U(1)$ direction which we call $U(1)_v$. This magnetic charge is equal to the monopole magnetic flux in this $U(1)_v$ direction.

The mass of the BPS monopole associated with a root α is [19]

$$M_\alpha^{\text{mon}} = \frac{4\pi}{e} |v \cdot \alpha^\vee|.$$

However, the stable or fundamental BPS monopoles are the ones associated with the simple roots α_i [20]. In particular, for the vacuum (28), the vector v (25) can be written as

$$v = a y_i^{(1)} \lambda_i^\vee = a x_i^{(1)} \lambda_i = \frac{a}{4\sin^2(\pi/2h)} x_i^{(1)} \alpha_i, \tag{31}$$

where in the last equality was used Eqs. (14) and (19). Therefore, for this vacuum, the masses for the stable BPS monopoles are

$$M_{\alpha_i}^{\text{mon}} = \frac{4\pi}{e} |v \cdot \alpha_i^\vee| = \frac{4\pi a}{e} x_i^{(1)}, \qquad i = 1, 2, \ldots, r. \tag{32}$$

Likewise, for each root α there is a massive gauge particle associated with the step operator E_α, but the stable massive particles are the ones associated with the simple roots α_i with masses [19]

$$M_{\alpha_i}^{\text{W}} = e|v \cdot \alpha_i| = a e y_i^{(1)}, \qquad i = 1, 2, \ldots, r. \tag{33}$$

Let us now see that this spectrum of masses of stable massive particles and monopoles coincide with the spectrum of masses of particles and solitons of affine Toda field theories.

Affine Toda field theories are two-dimensional integrable theories. For each affine Lie algebra $\hat{\mathfrak{g}}$ we can associate an ATFT. For simplicity only the untwisted case will be considered. It has r scalar fields ϕ_i where r is the rank of the Lie algebra $\mathfrak{g}$, from which $\hat{\mathfrak{g}}$ is constructed. The Lagrangian can be written as

$$L = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \mu^2 \sum_{i=0}^{r} n_i e^{\beta \alpha_i \cdot \phi},$$

where μ is a mass parameter and β is an adimensional constant. The scalar product is defined in the r-dimensional space of fields ϕ_i and the integers n_i are defined from the expansion of the highest root ψ in the basis of simple roots:

$$\psi = \sum_{i=1}^{r} n_i \alpha_i. \tag{34}$$

Moreover, we are considering that $\alpha_0 = -\psi$ and $n_0 = 1$. In particular, for $\mathfrak{g} = su(n)$, $n_i = 1$, for $i = 1, 2, \ldots, n-1$.

We shall consider that β is imaginary which implies that the theory has a degenerated vacuum and solitons interpolating these vacua. In this case, the theory has r species of particles and r species of solitons [21,22], one for each node of the Dynkin diagram of $\mathfrak{g}$. The particle masses are [23–25]

$$M_i^{\text{part}} = \mu|\beta|\sqrt{2h} y_i^{(1)}, \qquad i = 1, 2, \ldots, r. \tag{35}$$

Here h is the Coxeter number of $\mathfrak{g}$. The soliton masses are [21,22]

$$M_i^{\text{sol}} = \frac{2h}{|\beta|^2} \frac{2}{\alpha_i^2} M_i^{\text{part}} = \frac{\mu(2h)^{3/2}}{|\beta|} x_i^{(1)}, \qquad i = 1, 2, \ldots, r. \tag{36}$$

Each soliton species may have many topological charges, with same masses.

One can easily check that the spectrum of masses of particles and solitons of an ATFT associated with the affine algebra $\hat{\mathfrak{g}}$ coincide, respectively, with the spectrum of masses of stable massive gauge particles and BPS monopoles of our theory in the Coulomb phase with the gauge group associated with the algebra $\mathfrak{g}$, if the couplings of the two theories satisfy the relations

$$\frac{e^2}{4\pi} = \frac{|\beta|^2}{2h}, \tag{37}$$

$$a = \frac{\mu h}{\sqrt{\pi}}. \tag{38}$$

Note that this result holds also for Yang-Mills theories with gauge groups E_8, F_4, and G_2 which, although do not have stable Z_N strings in the Higgs phase, they have monopoles in the Coloumb phase. This result indicates a possible relation or "duality" between ATFTs in two dimensions and Yang-Mills-Higgs in four dimensions with the vacuum

$$\phi_1^{\text{vac}} = a \sum_{i=1}^{r} y_i^{(1)} \lambda_i^\vee \cdot H, \qquad \phi_2^{\text{vac}} = 0. \tag{39}$$

One must observe that all these mass spectra we mentioned are at the classical level. Therefore, these possible relations probably only hold exactly (i.e., at the quantum level) when these theories are embedded in supersymmetric theories, as usual. Since ϕ_1^{vac} is in Cartan subalgebra, it is direct to see that the field configuration (39), together with an extra field $\phi_3^{\rm vac} = 0$, is a vacuum solution of the bosonic part of the $\mathcal{N} = 4$ potential

$$V = \frac{1}{2}\,{\rm Tr}\bigg[\frac{e}{2}\sum_{s=1}^{3}([\phi_s^*, \phi_s])\bigg]^2$$

and therefore gives rise to the same mass spectrum (32) and (33) for the gauge particles and BPS monopoles in $\mathcal{N} = 4$ superYang-Mills theoreis.

It is interesting to note that [26] also observed a relation between BPS mass spectra for some two- and four-dimensional theories. On the other hand, a relation between non-Abelian monopoles and conformal invariant Toda theory was shown in [27,28]. In those works it was shown that for a particular spherically symmetric BPS monopole associated with the maximal $SU(2)$ subalgebra, $T_3 = \delta \cdot H$, $T_\pm = \sum_{i=1}^{r}\sqrt{\delta\cdot\lambda_i}\,E_{\pm\alpha_i}$ [like the vacuum configuration (23)], the monopole's radial function satisfies the equation of motion of conformal Toda field theory.

Note that our theory in the Coulomb phase when embedded in an $\mathcal{N} = 4$ super Yang-Mills theory should satisfy the Montonen-Olive duality [8], with the monopoles and particles of the theory with gauge group G and coupling e being mapped, respectively, to the particles and monopoles of the theory with gauge group $G^\vee$ and coupling $4\pi/e$. Therefore, combining the above duality with Montonen-Olive duality should imply a duality between ATFT associated with $\hat{\mathfrak{g}}$ with coupling constants (μ, β) and ATFT associated with $\widehat{\mathfrak{g}^\vee}$ with coupling constants $(\mu, 2h/\beta)$. This is consistent with the classical spectrum of the masses of these theories. One can see that fact remembering that if K_{ij} is the Cartan matrix associated with the algebra $\mathfrak{g}$ then the transposed $(K^T)_{ij}$ is the Cartan matrix of the dual algebra $\mathfrak{g}^\vee$ and the right (left) vectors of K_{ij} are left (right) vectors of $(K^T)_{ij}$. Therefore, the mass spectrum for particles (solitons) of the ATFT associated with $\hat{\mathfrak{g}}$ with coupling constants (μ, β) is the same as the mass spectrum for solitons (particles) of the ATFT associated with $\widehat{\mathfrak{g}^\vee}$ with coupling constants $(\mu, 2h/\beta)$. However, one must keep in mind that each soliton species has many different topological charges. Therefore, similar to the Yang-Mills theory in four dimensions, this duality probably should hold only when ATFT is embed in a supersymmetric theory where the number of particles is increased.

VI. THE HIGGS PHASE: THE SINE LAW AND THE CASIMIR SCALING

When $m \neq 0$, G is broken to its center C_G, which corresponds to the Higgs phase and there exist Z_N string solutions. In [5] we analyzed many properties of these solutions for the vacuum given by Eq. (23). Let us extend these results for a general vacuum configuration given by Eq. (25) which breaks G to its center C_G. In order to have finite string tension, asymptotically these solutions have the form

$$\begin{aligned}\phi_s(\varphi, \rho\to\infty) &= g(\varphi)\phi_s^{\rm vac}g(\varphi)^{-1}, \qquad s = 1, 2,\\ W_I(\varphi, \rho\to\infty) &= -\frac{1}{ie}(\partial_I g(\varphi))g(\varphi)^{-1},\end{aligned} \tag{40}$$

where $\phi_s^{\rm vac}$ are the vacuum solutions (25), ρ and φ are the radial and angular coordinates, and the capital Latin letters I, J denote the coordinates 1 and 2 orthogonal to the string. In order for the configuration to be single valued, $g(\varphi + 2\pi)g(\varphi)^{-1} \in C_G$. Considering

$$g(\varphi) = \exp i\varphi M,$$

where M is a generator of $\mathfrak{g}$, it results that $\exp 2\pi i M \in C_G$. From this condition we can consider

$$M = \omega\cdot H$$

with $\omega \in \Lambda_w(G^\vee)$. Then, the asymptotic form of the Z_N string solution can be written as

$$\begin{aligned}\phi_1(\varphi, \rho\to\infty) &= v\cdot H,\\ \phi_2(\varphi, \rho\to\infty) &= \sum_{i=1}^{r} b_i\{\exp(-i\varphi\omega\cdot\alpha_i)\}E_{-\alpha_i},\\ W_I(\varphi, \rho\to\infty) &= \frac{\epsilon_{IJ}x^J}{e\rho^2}\,\omega\cdot H, \qquad I = 1, 2.\end{aligned} \tag{41}$$

One can see that for each element ω in the co-weight lattice $\Lambda_w(G^\vee)$ we can associate a string solution. Let us review how the Z_N string solutions are associated with distinct topological sectors of $\Pi_1(G/C_G)$ [5] for a general group G. In order to do that we must remember that since $\Lambda_r(G^\vee)$ is a sublattice (or subgroup) of $\Lambda_w(G^\vee)$, we can define the quotient $\Lambda_w(G^\vee)/\Lambda_r(G^\vee)$ by identifying points $\Lambda_w(G^\vee)$ which differ by an element of the co-root lattice $\Lambda_r(G^\vee)$. In [9], it was shown that

$$C_G \simeq \Lambda_w(G^\vee)/\Lambda_r(G^\vee). \tag{42}$$

On the other hand, as explained in detail in [29], the center group C_G is isomorphic to the symmetry group W_0 of the extended Dynkin diagram formed by the transformations τ where the node 0 is not fixed, but mapped to another node $j = \tau(0)$. The elements of W_0 may be labeled by those nodes symmetrically related to the node 0, as shown in Table I, and are represented by black nodes in the extended Dynkin diagrams. As a consequence, the quotient (42) can be represented by the cosets

$$\begin{gathered}\Lambda_r(G^\vee), \qquad \lambda_{\tau(0)}^\vee + \Lambda_r(G^\vee),\\ \lambda_{\tau^2(0)}^\vee + \Lambda_r(G^\vee), \ldots, \qquad \lambda_{\tau^n(0)}^\vee + \Lambda_r(G^\vee),\end{gathered} \tag{43}$$

where the weights $\lambda_{\tau^q(0)}$ are associated with nodes related to the node 0 by a symmetry transformation. Such weights are called the minimal weights of $\mathfrak{g}$. A fundamental weight λ_k is minimal if $\lambda_k^\vee\cdot\psi = 1$, where ψ is the highest root

(34). One can lift τ to an automorphism of the Lie algebra $\mathfrak{g}$ and show that the center group elements (24), with ω belonging to a given coset in (43) are associated with the same center element of C_G [30]. In other words, the co-weights in a coset are associated with the same center element. The representations of $G^\vee$ with these weights are said to be in the same N-ality. When ω belongs to $\Lambda_r(G^\vee)$, the group element (24) is the identity since when it acts on any weight state $|\lambda\rangle$ of any representation of G,

$$\exp\{2\pi i\omega \cdot H\}|\lambda\rangle = \exp\{2\pi i\omega \cdot \lambda\}|\lambda\rangle = |\lambda\rangle,$$

using the fact that $\omega \in \Lambda_r(G^\vee)$, $\lambda \in \Lambda_w(G)$ and the orthonormality condition (15).

Since the topological sectors of the strings solutions are given by

$$\Pi_1(G/C_G) = C_G,$$

we can conclude that the Z_N string solutions (41) are separated in topological sectors according to the coset (43) in which ω belongs [5]. When ω belongs to $\Lambda_r(G^\vee)$, the group element (24) is the identity, and the corresponding string solution is in the trivial topological sector.

Let us analyze, for example, how the string solutions are split in topological sectors for the groups E_6, Spin$(2n)$, and $SU(n)$ which are the groups in which the center groups have order greater than two, which are the most interesting. All these groups are simply laced, i.e. $\alpha_i^\vee = \alpha_i$, $\lambda_i^\vee = \lambda_i$, $G^\vee = G$ and so the weight and the co-weight lattice are the same. Table I lists the elements of W_0, from which we obtain how the weight lattice split in cosets.

For $G = E_6$, the weight lattice split in the three cosets

$$\Lambda_r(E_6), \qquad \lambda_1 + \Lambda_r(E_6), \qquad \lambda_5 + \Lambda_r(E_6), \tag{44}$$

and the group elements (24) with ω belonging to each of these three cosets are associated with the three elements of Z_3, the center of E_6. λ_1 and λ_5 are the highest weights of the representations 27 and $\overline{27}$.

For $G = \text{Spin}(2n)$, the universal covering group of $SO(2n)$, with $n \geq 4$, the weight lattice split in the four cosets,

$$\begin{gathered}\Lambda_r(\text{Spin}(2n)), \qquad \lambda_1 + \Lambda_r(\text{Spin}(2n)),\\ \lambda_{n-1} + \Lambda_r(\text{Spin}(2n)), \qquad \lambda_n + \Lambda_r(\text{Spin}(2n)),\end{gathered} \tag{45}$$

where λ_1 is the highest weight of the n-dimensional vector representation and λ_{2n-1} and λ_{2n} are the highest weights of the spinor representations of Spin$(2n)$. Then, the group elements (24) with ω belonging to each of these cosets are associated with the four elements of the center group of Spin$(2n)$, $Z_2 \times Z_2$ when n is even or Z_4 when n is odd.

For $G = SU(n)$, the weight lattice split in n cosets,

$$\begin{gathered}\Lambda_r(SU(n)), \qquad \lambda_1 + \Lambda_r(SU(n)),\\ \lambda_2 + \Lambda_r(SU(n)), \ldots, \qquad \lambda_{n-1} + \Lambda_r(SU(n)),\end{gathered} \tag{46}$$

where λ_k is the fundamental weight associated with the representation which is the antisymmetric tensor product of k n-dimensional fundamental representations. The group elements (24) with ω belonging to each of these n cosets are associated with the n elements of Z_n. One can see this result explicitly by acting these group elements on the n weight states

$$|\lambda_1\rangle, \qquad \left|\lambda_1 - \sum_{i=1}^k \alpha_i\right\rangle, \qquad k = 1, 2, \ldots, n-1$$

of the n-dimensional representation of $SU(n)$, which results in

$$\begin{aligned}&\exp\{2\pi i[\lambda_m + \Lambda_r(SU(n))] \cdot H\}\left|\lambda_1 - \sum_{i=1}^k \alpha_i\right\rangle\\ &= \exp\{2\pi i\lambda_m \cdot \lambda_1\}\left|\lambda_1 - \sum_{i=1}^k \alpha_i\right\rangle\\ &= \exp\left\{2\pi i\frac{m}{n}\right\}\left|\lambda_1 - \sum_{i=1}^k \alpha_i\right\rangle.\end{aligned}$$

The same result holds when acting on $|\lambda_1\rangle$.

From the string asymptotic form (41) one can propose an ansatz for all the fields in the whole space. However, for us it is only important to consider the ansatz that $\phi_1(\varphi, \rho)$ is constant in the whole space and equal to its asymptotic value, i.e.,

$$\phi_1(\varphi, \rho) = v \cdot H. \tag{47}$$

In particular for the BPS strings, one can deduce this configuration from the BPS equation $D_\mp\phi_1 = 0$ and asymptotic form (41) [5].

For the string associated with a vector ω given by Eqs. (41) and (47), the string flux (9) is

$$\begin{aligned}\Phi_{\rm st} &= \frac{1}{|\phi_1^{\rm vac}|}\int d^2x[\text{Re}(\phi_1)_a B_{3a}]\\ &= -\frac{1}{|v|}\oint dl_I \,\text{Tr}[v \cdot HW_I] = \frac{2\pi}{e}\frac{v \cdot \omega}{|v|}.\end{aligned} \tag{48}$$

Similarly to the monopole flux (30), this is the string flux in the $U(1)_v$ direction generated by $v \cdot H$.

In the Higgs phase, the monopoles magnetic lines cannot spread radially over space. However, since any co-root $\alpha^\vee$ can be expanded as an integer linear combination of co-weights, we can conclude that any monopole flux (30) can be always as an integer linear combination of string fluxes

(48). Note that this result holds for an arbitrary vacuum (25) with nonvanishing v_i, extending therefore our previous result. Then, the monopole magnetic lines form a set of Z_N strings and monopoles become confined as analyzed in detail in [5]. Note that this flux matching happens not only with respect to $v \cdot H$, which is the generator of $U(1)_v$, but for any other Cartan generator H_i, $i = 1, 2, \ldots, r$. Then a pair of monopole-antimonopole associated with a co-root $\alpha^\vee$ would be confined by a string associated with $\omega = \alpha^\vee$.

From the string flux (48) we can obtain that the lower bound for the tension (8) for a string associated with $\omega = \lambda_k^\vee - \beta^\vee$, with $\beta^\vee \in \Lambda_r(G^\vee)$, is

$$T_\omega \geq \pi m|v \cdot \omega| = \pi m|v \cdot (\lambda_k^\vee - \beta^\vee)|. \tag{49}$$

The bound holds for the BPS strings. We shall only consider here the BPS strings.

In QCD with gauge group $SU(N)$, it is believed that the chromoelectric flux tubes carries charge in the center Z_N of the gauge group, but their tension in general could depend on the representation of the sources [2]. However, it is believed that for long enough strings it becomes energetically favorable for a pair of gluons to pop out to bind with the quark and antiquark charges. For all representations associated with the same center element [i.e. in the same N-ality of $SU(N)$], the energetically most favorable representation of the quark-gluon bound state will be the lowest-dimensional representation. There are mainly two conjectures for the ratios of these asymptotic tensions: the Casimir scaling and the sine law scaling.

As we have seen, the Z_N strings associated with the same center element are those with ω in the same coset. In general, they do not have the same tensions as can be seen in Eq. (49). For a long enough string associated with ω, it could pop out a pair of monopole-antimonopole confined by a string associated with a co-root $\alpha^\vee$ as described above and the string associated with ω would decay to a string associated with $\omega - \alpha^\vee$, which clearly is in the same coset as ω and therefore associated with the same center element. From the monopole mass (32) and string tension bound (49) we obtain that the threshold length l^{th} for this decay to happen is

$$l^{\text{th}} = \frac{2M_\alpha^{\text{mon}}}{T_{\alpha^\vee}} \leq \frac{8}{em}.$$

The Z_N strings can have different tensions for weights in a same representation. Therefore, in order to compare with results of QCD strings we shall associate with a representation the tension of its hightest weight. In order to determine the smallest tension in the same topological sector it is convenient to write the vector v (21) in the simple root basis:

$$v = u_i\alpha_i, \qquad u_i = \frac{2}{\alpha_i^2}(K^{-1})_{ij}v_j,$$

where all the entries of K^{-1} are positive. A highest co-weight can be written as $\omega = p_i\lambda_i^\vee$ where p_i are integers and $p_i \geq 0$, and the tension (49) of the BPS string associated with ω can be written as

$$T_\omega = \pi m u_i p_i.$$

For the vacua and gauge groups we are considering below, one can check that the highest weight associated with the smallest tension for each topological sector, is the minimal co-weight $\lambda_{\tau^q(0)}^\vee$. We shall call minimal strings the strings associated with minimal co-weights. From Eq. (49) we see that their tensions are

$$T_\omega = \pi m|v \cdot \lambda_{\tau^q(0)}^\vee|. \tag{50}$$

For a theory with vacuum given by Eq. (23), the ratio of BPS minimal string tensions satisfy the Casimir scaling, for the gauge group $G = SU(n)$ [5]. Let us verify that for the vacuum given by Eq. (29), the ratios of tensions give rise to the sine law scaling, when $G = SU(n)$. Let us also analyze how these scalings generalize for other gauge groups. Note that since the groups considered below are simply laced, $\lambda_k^\vee = \lambda_k$.

A. Sine law scaling

For the vacuum with v given by (31), the BPS string tension (50) associated with $\omega = \lambda_k^\vee$ is

$$T_{\lambda_k^\vee} = \frac{\pi m a}{4\sin^2(\pi/2h)} x_k^{(1)}. \tag{51}$$

Therefore, for this vacuum the tensions are proportional to the components of the left Perron-Frobenius eigenvector $x_k^{(1)}$. From Table II we obtain the following BPS minimal string tensions.[4]

a. For $G = SU(n)$

For $SU(n)$, for each fundamental minimal weight λ_k, $k = 1, 2, \ldots, n-1$, we associate a coset and hence a nontrivial string topological sector. The corresponding BPS string tensions are

$$T_{\lambda_k} = \frac{\pi m a}{4\sin^2(\pi/2n)} \sin\frac{k\pi}{n}, \qquad k = 1, 2, \ldots, n-1.$$

Taking the ratios of all tension with the smallest string tension we get the results

$$\frac{T_{\lambda_k}}{T_{\lambda_1}} = \frac{\sin(k\pi/n)}{\sin(\pi/n)}, \qquad k = 1, 2, \ldots, n-1,$$

which is exactly the sine law scaling.

[4]In these results we absorbed a possible normalization constant of $x_i^{(1)}$ redefining the constant a.

b. For $G = \mathbf{Spin}(2n)$, $n \geq 4$

For Spin($2n$), for each fundamental minimal weight λ_1, λ_{2n-1}, λ_{2n}, we associate a coset and hence a nontrivial string topological sector. The corresponding BPS string tensions associated with these weights are

$$T_{\lambda_1} = \frac{a\pi m \sin[\pi/2(n-1)]}{2\sin^2[\pi/4(n-1)]},$$
$$T_{\lambda_{2n-1}} = T_{\lambda_{2n}} = \frac{a\pi m}{4\sin^2[\pi/4(n-1)]}.$$

Note that for $n = 4$, $T_{\lambda_1} = T_{\lambda_{2n-1}} = T_{\lambda_{2n}}$, which is due to the symmetry of the $so(8)$ Dynkin diagram. Taking the ratios of all tensions with the smallest string tension gives the results

$$\frac{T_{\lambda_{2n-1}}}{T_{\lambda_1}} = \frac{T_{\lambda_{2n}}}{T_{\lambda_1}} = \frac{1}{2\sin[\pi/2(n-1)]}. \tag{52}$$

For large n, this ratio gives

$$\frac{T_{\lambda_{2n-1}}}{T_{\lambda_1}} = \frac{T_{\lambda_{2n}}}{T_{\lambda_1}} \to \frac{n}{\pi}. \tag{53}$$

c. For $G = E_6$

For each fundamental minimal weight λ_1 and λ_5 of $G = E_6$ are associated cosets. The BPS string tensions for these weights are

$$T_{\lambda_1} = T_{\lambda_5} = \frac{\pi m a}{4\sin^2(\pi/24)} \sin\frac{\pi}{12}.$$

One can note that for a general gauge group, the string tension (51) associated with $\lambda_k^\vee$ is proportional to $x_k^{(1)}$ and the topological sector is associated with the coset $\lambda_k^\vee + \Lambda_r(G^\vee)$. Similarly, in ATFTs, a soliton species associated with the kth node of the Dynkin diagram has mass proportional to $x_k^{(1)}$, given by Eq. (36), and the topological charge has the form $2\pi i(\lambda_k^\vee + \Lambda_r(G^\vee))/\beta$ [21,22,31]. Taking the ratios of the tensions of the BPS strings associated with the fundamental co-weights (51) (not only the minimal ones) for any gauge group G we obtain

$$\frac{T_{\lambda_i^\vee}}{T_{\lambda_k^\vee}} = \frac{M_i^{\rm sol}}{M_k^{\rm sol}},$$

where $M_i^{\rm sol}$ are the soliton masses (36) of the corresponding affine Toda field theory. Therefore, there may exist some relation between these topological solutions. However, this possible connection between Z_N strings and solitons of ATFTs must yet be clarified. In [32] it was also shown that in $CP(N-1)$ sigma models, the tension between a k-kink and k-antikink also satisfies the sine law scaling for the group $SU(N)$.

B. Casimir scaling

Let us now consider the vacuum (25) and (27) with

$$v = a\sum_{i=1}^r \lambda_i^\vee = a\delta,$$

where a is a real parameter. Then, from (50) we have that the tension of a BPS string for $\omega = \lambda_k^\vee$ is

$$T_{\lambda_k^\vee} = a\pi m \lambda_k^\vee \cdot \delta. \tag{54}$$

This expression can be written [5] in terms of the value of the quadratic Casimir of a representation with fundamental weight $\lambda_k^\vee$ of the dual Lie algebra $\mathfrak{g}^\vee$

$$C(\lambda_k^\vee) = \lambda_k^\vee \cdot (\lambda_k^\vee + 2\delta) \tag{55}$$

as

$$T_{\lambda_k^\vee} = \frac{a\pi m}{2}(C(\lambda_k^\vee) - \lambda_k^\vee \cdot \lambda_k^\vee).$$

Alternatively, from the definition of the dual Weyl vector δ, we can write

$$\lambda_k^\vee \cdot \delta = \frac{2}{\alpha_k^2}\sum_{i=1}^r (K^{-1})_{ki},$$

where $(K^{-1})_{ki}$ is tabulated in any standard Lie algebra book. From this relation one can check explicitly, case by case, that for any fundamental co-weight $\lambda_k^\vee$ which is minimal

$$\lambda_k^\vee \cdot \delta = \frac{h}{2(h+1)} C(\lambda_k^\vee), \tag{56}$$

where h is the Coxeter number of G (which is also the Coxeter number of $G^\vee$) given in Table II. Therefore, the tension (54) for a BPS string for minimal $\lambda_k^\vee$ can be written as

$$T_{\lambda_k^\vee} = a\pi m \frac{h}{2(h+1)} C(\lambda_k^\vee) \tag{57}$$

and the ratio of BPS string tensions associated with any minimal co-weights $\lambda_k^\vee$ and $\lambda_j^\vee$ can be written as

$$\frac{T_{\lambda_k^\vee}}{T_{\lambda_j^\vee}} = \frac{C(\lambda_k^\vee)}{C(\lambda_j^\vee)}, \tag{58}$$

which is a generalization of the Casimir scaling for any group G. However, it is important to emphasize that (57) and (58) hold only for minimal co-weights $\lambda_k^\vee$, otherwise one must use (54).

The relation (56) can be proved in general in the following way: any minimal weight λ_k can be related to the node 0 of the extended Dynkin diagram by a symmetry transformation τ. For each of these transformations [30],

$$\tau(\delta) - \delta = -h\lambda_{\tau(0)}^\vee.$$

One can check this identity by taking scalar products with simple roots. Moreover,

$$\tau(\delta) \cdot \lambda^{\vee}_{\tau(0)} = -\delta \cdot \lambda^{\vee}_{\tau(0)}.$$

Thus,

$$\delta \cdot \lambda^{\vee}_{\tau(0)} = \frac{h}{2} \lambda^{\vee}_{\tau(0)} \cdot \lambda^{\vee}_{\tau(0)}.$$

Therefore, for representations with highest weight $\lambda^{\vee}_k$ minimal, the quadratic Casimir (55) can be written as

$$C(\lambda^{\vee}_k) = 2\left(\frac{h+1}{h}\right)\lambda^{\vee}_k \cdot \delta.$$

Let us now analyze the string tensions (57) and (58) for some gauge groups.

a. For $G = SU(n)$

For $G = SU(n)$, $h = n$ and

$$C(\lambda_k) = \frac{(n+1)k(n-k)}{n}.$$

Therefore, for the minimal fundamental weights λ_k, $k = 1, 2, \ldots, n-1$, the BPS string tensions are

$$T_{\lambda_k} = \frac{a\pi m}{2} k(n-k), \quad \text{for } k = 1, 2, \ldots, n-1,$$

which results in the Casimir scaling for the Z_N BPS strings

$$\frac{T_{\lambda_k}}{T_{\lambda_1}} = \frac{k(n-k)}{n-1}, \tag{59}$$

obtained in [5].

b. For $G = \mathbf{Spin}(2n)$*,* $n \geq 4$

For Spin$(2n)$, $h = 2n - 2$ and,

$$C(\lambda_1) = 2n - 1, \qquad C(\lambda_{n-1}) = C(\lambda_n) = \frac{n(2n-1)}{4}.$$

Therefore, for the minimal fundamental weights λ_1, λ_{n-1}, and λ_n, the BPS string tensions are

$$T_{\lambda_1} = a\pi m(n-1), \qquad T_{\lambda_{n-1}} = T_{\lambda_n} = a\pi m\frac{n(n-1)}{4},$$

which results in the string ratios

$$\frac{T_{\lambda_k}}{T_{\lambda_1}} = \frac{C(\lambda_k)}{C(\lambda_1)} = \frac{n}{4} \quad \text{for } k = n, n-1. \tag{60}$$

Note that for the $G = SU(n)$, for fixed k and large n, the ratio of the tensions coincide, for the Casimir and sine law scalings with $T_{\lambda_k} = kT_{\lambda_1}$ for $n \to \infty$. In contrast, for $G =$ Spin$(2n)$, the Casimir and sine law scalings give different results in the large n limit, as can be seen from Eqs. (53) and (60).

c. For $G = E_6$

For E_6, $h = 12$ and

$$C(\lambda_1) = C(\lambda_5) = \tfrac{52}{3}.$$

Then, the BPS string tensions associated with λ_1 and λ_5 are

$$T_{\lambda_1} = T_{\lambda_5} = 8a\pi m$$

and

$$\frac{T_{\lambda_5}}{T_{\lambda_1}} = \frac{C(\lambda_5)}{C(\lambda_1)} = 1. \tag{61}$$

VII. CONCLUSIONS

In this work we analyzed the Z_N string solutions in Yang-Mills-Higgs theories with simple gauge groups G spontaneously broken to their center C_G. We studied two different vacuum solutions responsible for the symmetry breaking $G \to U(1)^r \to C_G$, for any G. We showed that for one vacuum, in the Coulomb phase, the particles and monopoles of the theory with group G have the same masses as the particles and solitons of the corresponding affine Toda field theory, if the couplings of the two theories satisfy some suitable relations, which may indicate a relation between these theories. The same result holds for $\mathcal{N} = 4$ super Yang-Mills theories. Then, in the Higgs phase, we reviewed the construction of the asymptotic form of the Z_N string solutions and showed the matching of the fluxes of the Z_N strings and monopoles for any G and arbitrary vacuum which produces the symmetry breaking (1). We then showed that for each of the two vacua the ratios of the tensions of the minimal BPS Z_N strings (associated with the representation with smallest tension for each topological sector) satisfy the Casimir scaling and the sine law scaling for $G = SU(N)$, and we extended these scalings for any simple gauge group G, analyzing in particular $G =$ Spin$(2n)$ and $G = E_6$. For the sine law scaling, the tensions are proportional to the components $x_i^{(1)}$ of the left Perron-Frobenius eigenvector of K_{ij} and the ratios of tensions are equal to the ratios of soliton masses of the corresponding affine Toda field theory, for any group gauge G. For the Casimir scaling, we obtained that the ratios of tensions were equal to the ratios of the second Casimir of the fundamental representations associated with the different topological sectors (58). These results show that for the Z_N strings, these scalings are not "universal laws," but they depend on the vacuum which produces the symmetry breaking. From the dual superconductor picture, this result may indicate that tensions of the QCD strings could be due to a non-Abelian monopole condensate in one of these two vacua. It is important to emphasize that the Casimir scaling (57) and (58) hold only for minimal coweights $\lambda^{\vee}_k$, otherwise one must use (54).

The spectrum of QCD string tensions has been extensively studied in recent years in lattice calculations [10,11].

In particular in [11] it was observed that the QCD string tensions lie between the Casimir and sine law scalings (a little above the Casimir scaling). On the other hand, the Casimir scaling (57) and the sine law scaling (51) are lower bounds for the non-BPS Z_N string tensions and they hold exactly only for the BPS Z_N strings, which exist only on the boundary between a type I and a type II superconductor. Therefore, the deviation from the Casimir scaling observed in [11] could be due to the fact that QCD strings would not be BPS.

The properties analyzed so far for the Z_N string solutions indicate that they could be magnetic analogous to QCD strings. We hope that our results may be useful for lattice calculation for analyzing the QCD strings with $G = SU(n)$ and other gauge groups.

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