

Fock space, quantum fields, and κ -Poincaré symmetries

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We study the quantization of a linear scalar field, whose symmetries are described by the κ -Poincaré Hopf algebra, via deformed Fock space construction. The one-particle sector of the theory exhibits a natural (Planckian) cutoff for the modes of the field. At the “multiparticle” level the nontrivial coalgebra structure of κ -Poincaré leads to a deformed bosonization in the construction of Fock space states. These physical states carry energy-momentum charges which are divergenceless and obey a deformed dispersion relation.

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I. INTRODUCTION

Local quantum fields on Minkowski space-time provide a successful framework for an effective description of particle physics up to the energies probed by current experiments. The Planck energy ($E_p \sim 10^{28}$ eV) sets the scale at which such a description is believed to break down. Indeed at Planckian regimes the field quanta would probe the background space-time at scales for which quantum-gravitational effects become important and new physics described by a general theory of quantum gravity would take the stage.

Quantum field theories on noncommutative spaces are an example of generalizations of ordinary quantum field theory which find their motivation from certain quantum gravity scenarios and whose new features are expected to emerge in the ultraviolet [1–4]. The view that such theories might enjoy “deformed” relativistic symmetries (rather than breaking standard Poincaré invariance through the emergence of the noncommutativity scale) has become increasingly popular in recent years [5,6]. Such deformed symmetries are generally described by a “quantization” of standard relativistic symmetries in terms of quantum groups (noncocommutative Hopf algebras) [7]. In the present work we focus our attention on the κ -Poincaré Hopf algebra [8,9] as one candidate for quantum group description of relativistic symmetries at the Planck scale [10]. The deep connection between κ -Poincaré and κ -Minkowski space-time, a “Lie algebra”-type of noncommutative space-time, has been known for quite a while [9]. Indeed the κ -Poincaré Hopf algebra can be used to describe the symmetries of such noncommutative space-time [9,11]. Much of the recent interest towards these type of deformed symmetries has been motivated by various arguments which suggest that κ -Poincaré-type of quantum group symmetries might emerge in the low-energy limit of

three (and possibly four) dimensional quantum gravity [12–14] with the dimensionful deformation parameter κ set by the Planck energy. Moreover earlier interest in the study of scenarios with κ -Poincaré symmetries arose from the possibility of detecting experimental signatures of the deformed kinematics associated with such symmetries [15].

Classical field theories with κ -type of deformed symmetries have now been studied extensively [11,16–21], however much work seems to be needed to reach a satisfactory understanding of their quantum counterparts. One of the early proposals for the quantization of a self-interacting scalar field, by one of us and Amelino-Camelia [22], was based on a generalization of the path integral quantization method which in a κ -deformed context seemed more straightforward than the canonical approach (see Sec. IV below for a review of some recent attempts on the quantization of fields with κ -Poincaré symmetries). Recent works on the derivation of Noether charges [16,18] associated with κ -deformed symmetries of classical fields have prompted new attention to the problem of canonical quantization.

In the present paper we discuss a new strategy for the quantization of a free field theory with space-time symmetries described by the κ -Poincaré Hopf algebra. We start from the construction of a one-particle Hilbert space, we define an appropriate Fock space, creation, and annihilation operators associated to a given basis of modes, and we study the properties of energy and momentum charges carried by the quantum states of the theory. The very first step of our approach to quantization will be the introduction of a (positive definite) inner product on the space of solutions of the deformed equation of motion. To this end we borrow the results of [18] in which such an inner product was derived using the tools of symplectic geometry of classical fields. Thus the present analysis can be seen as a natural continuation of the work started in [18].

Below is the plan of the paper. In the following section we review the basic properties of the κ -Poincaré Hopf

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algebra in the so-called bicrossproduct basis [9] and recall some results from our previous work [18] which will turn out to be useful in our construction. In Sec. III we focus on the quantization of free fields. In the first subsection we start with an overview of the usual Hilbert (and Fock) space construction in the presence of standard space-time symmetries. We then proceed, in Sec. III B, with the definition of a one-particle Hilbert space in the context of deformed symmetries described by the κ -Poincaré Hopf algebra and show how a natural cutoff, set by the deformation parameter κ , arises in the construction. In Sec. III C we use the one-particle Hilbert space defined in the previous section and introduce a “deformed bosonization” in order to build the full Fock space of the theory. Then in the following Sec. III D we study the energy and momentum charges carried by the states of our deformed theory. In Sec. IV we review and comment on some recent works related to our results. Section V contains closing remarks and some suggestions for future studies.

II. PRELIMINARIES

A. κ -Poincaré Hopf algebra

In the following we will consider κ -deformations of relativistic symmetries described by a Hopf algebra in which both the coalgebra sector and the product (commutator) of the algebra elements (symmetries generators) are deformed. In particular we will focus on the example of the κ -Poincaré Hopf algebra \mathcal{P}_κ [8] in the *bicrossproduct basis* [9].

It is well known that the standard Poincaré algebra \mathcal{P} can be endowed with the structure of a (trivial) Hopf algebra. If $Y = \{P_\mu, M_{\mu\nu}\}$ are the generators of \mathcal{P} , the (trivial) coproduct, counit, and antipode can be defined, respectively, as

$$\Delta(Y) = Y \otimes 1 + 1 \otimes Y, \quad \epsilon(Y) = 0, \quad S(Y) = -Y. \quad (1)$$

The main feature of the κ -Poincaré Hopf algebra \mathcal{P}_κ is that its coproducts are, in general, noncommutative, in particular, in the bicrossproduct basis they read

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\ \Delta(P_j) &= P_j \otimes 1 + e^{-P_0/\kappa} \otimes P_j, \\ \Delta(M_j) &= M_j \otimes 1 + 1 \otimes M_j, \\ \Delta(N_j) &= N_j \otimes 1 + e^{-P_0/\kappa} \otimes N_j + \frac{\epsilon_{jkl}}{\kappa} P_k \otimes N_l. \end{aligned} \quad (2)$$

One also has deformed commutators for the generators of translations P_0, P_j , rotations M_j , and boosts N_j

$$\begin{aligned} [P_0, P_j] &= 0, \\ [M_j, M_k] &= i\epsilon_{jkl}M_l, \\ [M_j, N_k] &= i\epsilon_{jkl}N_l, \\ [N_j, N_k] &= i\epsilon_{jkl}M_l, \\ [P_0, N_l] &= -iP_l, \\ [P_l, N_j] &= -i\delta_{lj}\left(\frac{\kappa}{2}(1 - e^{-(2P_0/\kappa)}) + \frac{1}{2\kappa}\tilde{P}^2\right) + \frac{i}{\kappa}P_lP_j, \\ [P_0, M_k] &= 0, \\ [P_j, M_k] &= i\epsilon_{jkl}P_l, \end{aligned} \quad (3)$$

where ϵ_{jkl} is the Levi-Civita symbol and j, k, l are spatial indices. The nontrivial antipodes consistent with the coproducts above are

$$\begin{aligned} S(M_l) &= -M_l, \\ S(P_0) &= -P_0, \\ S(P_l) &= -e^{P_0/\kappa}P_l, \\ S(N_l) &= -e^{P_0/\kappa}N_l + \frac{1}{\kappa}\epsilon_{ljk}e^{P_0/\kappa}P_jM_k, \end{aligned} \quad (4)$$

while for the co-units one has

$$\epsilon(P_\mu) = \epsilon(M_j) = \epsilon(N_k) = 0. \quad (5)$$

The (deformed) mass Casimir C_κ of the κ -Poincaré Hopf algebra links the generators of translations and is given by

$$C_\kappa = \left(2\kappa \sinh\left(\frac{P_0}{2\kappa}\right)\right)^2 - \tilde{P}^2 e^{P_0/\kappa}. \quad (6)$$

As mentioned in the introduction the κ -Poincaré Hopf algebra describes the symmetries of κ -Minkowski noncommutative space-time [11]

$$[x_i, t] = \frac{i}{\kappa}x_i, \quad [x_i, x_j] = 0. \quad (7)$$

For further details on κ -Poincaré in the bicrossproduct basis and its relation with κ -Minkowski noncommutative space-time we refer to the literature (see e.g. [11] and references therein).

B. Symplectic geometry of classical κ -field theories

In this section we summarize the key results obtained in [18] where we proposed a description of classical field theories with Hopf algebra of space-time symmetries using the tools of symplectic geometry. The idea behind a symplectic geometrical formulation of classical field theories is to provide a canonical formulation of the phase space which does not spoil the symmetries of the theory. Such a description turns out to be very useful in the deformed field theory case as it gives a clear identification of the Noether charges associated with space-time symmetries and more importantly, for the purposes of the present work, it provides a natural candidate for an inner product

which will lead to the definition of the Hilbert space of our quantum fields.

We start with a classical field theory with standard (Poincaré) relativistic symmetries. We will consider a real massless scalar field with Lagrangian

$$L = \int_M \frac{1}{2} (\partial_\alpha \Phi \partial^\alpha \Phi), \quad (8)$$

where M is the space-time manifold on which the theory is defined which in the undeformed case is just standard Minkowski space-time. The phase space of such a theory is the infinite dimensional manifold Γ whose elements are the pairs $\{\Phi, \Pi\}$, where $\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}}$ is the momentum (density) canonically conjugate to the field Φ . The definition of Π requires singling out a time coordinate t , a choice that obscures the relativistic invariance of the theory. The crucial observation [23] is that there is a one-to-one correspondence between the points of the classical phase space and the solutions of the Klein-Gordon equation of motion thanks to the uniqueness properties of the initial value problem for such equation in Minkowski space. This allows one to *identify* points in phase space with solutions of the classical equation of motion i.e. the phase space Γ is the space of solutions \mathcal{S} . The next step in the formalism is the introduction of a symplectic structure through the definition of a nondegenerate two-form ω . To do so one needs to specify the differential structure (functions, tangent ($T\mathcal{S}$), and cotangent ($T^*\mathcal{S}$) spaces) on \mathcal{S} . For details we refer to [18] and references therein. The nondegenerate, symplectic two-form ω one needs is given by

$$\omega \equiv \delta \left(\int_\Sigma d\sigma_\alpha J^\alpha \right) \quad (9)$$

with the ‘‘symplectic potential current’’ J^α defined by

$$J^\alpha \equiv \frac{\partial L}{\partial (\Phi_{,\alpha})} \delta \Phi(x), \quad (10)$$

where $\delta \Phi(x) \in T_\Phi^* \mathcal{S}$ and δ is the exterior derivative. It can be easily shown [23] that ω is Poincaré invariant and for a vector field tangent to the orbit of space-time symmetry $V_s \in TS$

$$\omega(V_s) = -\delta(Q_s), \quad (11)$$

where Q_s is the generator of the symmetry. In particular for a choice of Σ as the standard $t = 0$ spacelike hypersurface (9) reduces to the more familiar expression for the symplectic form on the phase space manifold

$$\omega = \frac{1}{2} \int_{\Sigma_t} \delta \Pi \wedge \delta \Phi \quad (12)$$

and

$$\omega(V_s) = \frac{1}{2} \int_{\Sigma_t} (\delta \Pi(V_s) \delta \Phi - \delta \Pi \delta \Phi(V_s)). \quad (13)$$

\mathcal{S} , as a vector space, is isomorphic to the tangent space at

any given point $T_\Phi \mathcal{S}$ and through this correspondence the two-form ω and the one-form $\omega(V_s)$ can be used to define, respectively, a bilinear and a linear functional on \mathcal{S} . The bilinear will induce a symplectic product on \mathcal{S}

$$\omega(\Phi_1, \Phi_2) = \frac{1}{2} \int_{\Sigma_t} (\Pi_1 \Phi_2 - \Phi_1 \Pi_2) \quad (14)$$

and the linear functional $\omega(V_s)$ evaluated on a solution Φ will give the (conserved) value of the Noether charge associated with the symmetry

$$\omega(V_s)(\Phi) = \frac{1}{2} \int_{\Sigma_t} ((\mathcal{L}_{(V_s)} \Pi) \Phi - \Pi (\mathcal{L}_{(V_s)} \Phi)) = -Q_s, \quad (15)$$

where the contraction of a one-form with the Killing vector field V_s is replaced by the Lie derivative¹ of $\Phi \in \mathcal{S}$ and its canonically conjugate momentum Π . When the Killing vector fields correspond to (external) space-time symmetries, $Y \in \mathcal{P}$, we will use the following notation for the action of the Lie-algebra element associated with the symmetry on functions

$$\mathcal{L}_Y \Phi = Y \triangleright \Phi \quad (16)$$

to stress the fact that \mathcal{S} is an ∞ -dimensional representation of the Poincaré algebra.

On the space of complex solutions of the (massless) Klein-Gordon equation, $\mathcal{S}^\mathbb{C}$, our symplectic structure will define a Hermitian inner product

$$(\Phi_1, \Phi_2) = -2i\omega(\Phi_1^*, \Phi_2), \quad (17)$$

where Φ_i^* denotes the complex conjugate of $\Phi \in \mathcal{S}^\mathbb{C}$. It will be useful for our purposes to write such a product in an explicit covariant form as

$$(\Phi_1, \Phi_2) = \int \frac{d^4 p}{(2\pi)^3} \delta(C(p)) \frac{p_0}{|p_0|} \tilde{\Phi}_1^*(-p) \tilde{\Phi}_2(p), \quad (18)$$

where $\tilde{\Phi}_i(p)$, $\tilde{\Phi}_i^*(p)$ are the Fourier transforms of Φ_i , Φ_i^* and $C(p)$ is the relativistic ‘‘mass’’ Casimir which sets the Fourier parameters p_μ on shell according to the equation of motion. Noether charges associated with space-time translations can be expressed in terms of the action of the symmetry generators P_μ acting on the fields Φ using the conserved symplectic product

$$Q_\mu = \frac{1}{2} (\Phi, P_\mu \triangleright \Phi). \quad (19)$$

We now turn our attention to the deformed case. In the standard case plane waves on shell are solutions to the Klein-Gordon equation and provide a basis for the standard space of solutions. In the κ -Poincaré Hopf algebra symmetry case plane waves will still be solutions of the de-

¹From the expression of the Lie derivative of a general p -form $\mathcal{L}_{V_s} F = (\delta F)(V_s) + \delta(F(V_s))$ which for a function (0-form) implies $\mathcal{L}_{V_s} f = (\delta f)(V_s)$.

formed Klein-Gordon equation [24]

$$\mathcal{C}_\kappa(P)\Phi = 0. \quad (20)$$

However wave exponentials labeled by a given value of the spatial momentum will combine according to the coproduct structure of the nontrivial coalgebra sector of the Hopf algebra. Seen from a dual space-time point of view [22] this corresponds to introducing a $*$ -multiplication for the algebra of functions on space-time coordinates. Roughly speaking any such $*$ -structure corresponds to a different choice of normal ordering for functions of noncommuting coordinates or to a different choice of basis for the generators of the κ -Poincaré (Hopf) algebra [11]. Unlike the standard Lie algebra symmetry case, in κ -Poincaré in the bicrossproduct basis, for each mode \vec{p} the positive and negative roots of the deformed mass Casimir $\mathcal{C}_\kappa(p)$ will not be equal in modulus. In fact there will be two roots²:

$$\omega^\pm(\vec{p}) = \kappa \log(\Omega(\vec{p})_\pm), \quad (21)$$

where $\Omega_\pm(\vec{p})$ is a function of $|\vec{p}|$ and in our specific case reads

$$\Omega(\vec{p})_\pm = \frac{1}{1 \mp \frac{|\vec{p}|}{\kappa}}. \quad (22)$$

Notice how the positive root becomes complex for trans-Planckian ($|\vec{p}| > \kappa$) modes. In the limit $\kappa \rightarrow \infty$ each solution approaches the value of the standard roots $\pm\omega = \pm|\vec{p}|$.

We denote with $\mathcal{S}_\kappa^\mathbb{C}$ the spaces of (complex) solutions of the κ -Klein-Gordon equation (20). The set of plane waves $\{\phi_{\vec{p}^\pm}^\kappa\}$, with the subscript \vec{p}^\pm specifying plane waves whose momentum labels are on the positive or negative deformed mass-shell, provides a basis for the associated vector space $\mathcal{S}_\kappa^\mathbb{C}$. This can be easily seen [18] introducing a linear map

$$\mathfrak{m}: \mathcal{S}_\kappa^\mathbb{C} \rightarrow \mathcal{S}^\mathbb{C} \quad (23)$$

which associates every element of $\mathcal{S}_\kappa^\mathbb{C}$ to its ‘‘classical’’ counterpart in $\mathcal{S}^\mathbb{C}$. Positive and negative energy solutions are related by *deformed* complex conjugation which on plane waves acts as follows:

$$\bar{\phi}_{\vec{p}^+}^\kappa = \phi_{-\vec{p}^+}^\kappa = \phi_{\vec{k}^-}^\kappa \quad (24)$$

with \vec{k} given by $\vec{k} = -\vec{p}e^{(\omega^+/\kappa)}$ and the label $\dot{\vec{p}}_\pm$ is a short-hand notation meaning that we are taking the antipode (4) of a four-momentum on shell in the plane wave.

² $\mathcal{C}_\kappa(p) = 0$ has a countable infinite set of complex roots. As will become clear from our considerations below, each pair of such complex roots will correspond to a copy of the same deformed space of solutions. For our purposes the restriction to a single pair of roots, which in the limit $\kappa \rightarrow \infty$ reproduces the standard positive and negative roots, will be enough to construct a model of field theory with *deformed* relativistic symmetries.

Both in the standard and in the κ -deformed case the spaces of positive and negative energy solutions can be shown to be isomorphic to the spaces of functions on the positive and negative mass-shell, respectively, through (noncommutative) Fourier transforms [11]. As a vector space, $\mathcal{S}_\kappa^\mathbb{C}$ can be endowed with a symplectic structure if we require \mathfrak{m} to be a *Poisson map* (see [18]). Such symplectic structure defines a bilinear functional $\omega_\kappa(\Phi_1^\kappa, \Phi_2^\kappa)$ from which one can define the inner product³ $(\Phi_1, \Phi_2)_\kappa = -2i\omega_\kappa(\Phi_1^*, \Phi_2)$ which written explicitly reads

$$\begin{aligned} (\Phi_1, \Phi_2)_\kappa &= \int \frac{d^4p}{(2\pi)^3} \delta(\mathcal{C}_\kappa(p)) \text{sign}(e^{p_0/\kappa} - 1) \\ &\quad \times e^{(3p_0/\kappa)} \tilde{\Phi}_1^*(\dot{-}p) \tilde{\Phi}_2(p) \end{aligned} \quad (25)$$

in which $\mathcal{C}_\kappa(p)$ is the deformed mass Casimir of the κ -Poincaré Hopf algebra. If in Eq. (25) above we restrict to modes in the range $|\vec{p}| \in [0, \kappa)$ the factor $\text{sign}(e^{p_0/\kappa} - 1)$ can be written in the more familiar form $p_0/|p_0|$ and thus we have

$$\begin{aligned} (\Phi_1, \Phi_2)_\kappa &= \int \frac{d^4p}{(2\pi)^3} \delta(\mathcal{C}_\kappa(p)) \\ &\quad \times \frac{p_0}{|p_0|} e^{(3p_0/\kappa)} \tilde{\Phi}_1^*(\dot{-}p) \tilde{\Phi}_2(p). \end{aligned} \quad (26)$$

Energy and momentum charges associated with κ -deformed translation symmetries for each $\Phi \in \mathcal{S}_\kappa^\mathbb{C}$ will be given by⁴

$$Q_\mu^\kappa = \frac{1}{2}(\Phi, P_\mu^\kappa \triangleright \Phi)_\kappa. \quad (27)$$

III. QUANTUM FIELDS

Now we move on to the quantization of the linear (non-interacting) field theory described in the previous sections. We will first recall some basic facts about canonical quantization in the usual special relativistic framework keeping in mind the covariant phase space formalism used above.

A. Standard Hilbert space construction

The ‘‘one-particle’’ Hilbert space of a free quantum (scalar) field theory can be constructed starting from the complexified space of classical solutions of the equations of motion $\mathcal{S}^\mathbb{C}$ [25]. In order to turn $\mathcal{S}^\mathbb{C}$ into a Hilbert space one needs to define an inner product. The natural choice would be the conserved product (18), that preserves the Poincaré symmetry structure, but it turns out that such a

³Here we drop the superscript κ for the fields, being now obvious that in the following we refer to elements of $\mathcal{S}_\kappa^\mathbb{C}$.

⁴The definition of the action of translational symmetries on fields $P_\mu^\kappa \triangleright \Phi$ involves a choice of deformed differential calculus. Such a choice is not unique (see [17,20] for examples of Noether analyses with a 5-d differential calculus). There appears to be no obstruction in adapting the results presented here to any choice of differential calculus.

product fails to be positive definite on all $\mathcal{S}^{\mathbb{C}}$, thus one has to restrict to a subspace of $\mathcal{S}^{\mathbb{C}}$ on which (18) is positive definite. One possible choice⁵ is to restrict to the subspace spanned by *positive frequency* solutions which we denote with $\mathcal{S}^{\mathbb{C}^+}$. The Hilbert space completion of $\mathcal{S}^{\mathbb{C}^+}$ under (18) is the one-particle Hilbert space of the theory \mathcal{H} . Using Fourier transforms it can be shown that \mathcal{H} (and thus $\mathcal{S}^{\mathbb{C}^+}$ as a vector space) is isomorphic to the Hilbert space of square integrable functions in momentum space on the positive mass-shell. One also has that $\mathcal{S}^{\mathbb{C}}$ is spanned by \mathcal{H} and its complex conjugate $\bar{\mathcal{H}}$ and that for $\xi^+ \in \mathcal{H}$, $\xi^- \in \bar{\mathcal{H}}$ one has $(\xi^+, \xi^-) = 0$.

From \mathcal{H} one can construct the (symmetric) Fock space $\mathcal{F}_s(\mathcal{H})$ as follows: denote $\mathcal{H}^n = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \dots \otimes \mathcal{H}}_{n\text{-times}}$ (with $\mathcal{H}^0 = \mathbb{C}$); if $\sigma \in P_n$ is a permutation in the permutation group of n -elements and $\xi_1^+ \otimes \dots \otimes \xi_n^+$ is a basis of \mathcal{H}^n

$$\sigma(\xi_1^+ \otimes \dots \otimes \xi_n^+) = \xi_{\sigma(1)}^+ \otimes \dots \otimes \xi_{\sigma(n)}^+ \quad (28)$$

one defines

$$S_n = \frac{1}{n!} \sum_{\sigma \in P_n} \sigma \quad (29)$$

then the bosonic (symmetric) Fock space is given by

$$\mathcal{F}_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty} S_n \mathcal{H}^n. \quad (30)$$

We denote a generic element $\Psi \in \mathcal{F}_s(\mathcal{H})$ as $\Psi = (a, a_1, a_2, \dots)$ with $a \in \mathbb{C}$, $a_1 \in \mathcal{H}$, $a_2 \in \mathcal{H} \otimes_s \mathcal{H}$, etc. Observables are self-adjoint operators on $\mathcal{F}_s(\mathcal{H})$. For each ‘‘positive frequency’’ solution or ‘‘one-particle’’ state $\xi \in \mathcal{H}$ one defines the *annihilation operator* $a(\bar{\xi}): \mathcal{F}_s(\mathcal{H}) \rightarrow \mathcal{F}_s(\mathcal{H})$

$$a(\bar{\xi})\Psi = (\bar{\xi} \cdot a_1, \sqrt{2}\bar{\xi} \cdot a_2, \sqrt{3}\bar{\xi} \cdot a_3, \dots), \quad (31)$$

where $\bar{\xi} \in \bar{\mathcal{H}}$ is the complex conjugate vector associated to ξ and $\bar{\xi} \cdot a_n$ is the ‘‘contraction,’’ via inner product, of the first element of each tensor product of the symmetrized sum appearing in a_n with $\bar{\xi}$. The vacuum state $|0\rangle \in \mathcal{F}_s$ is defined by

$$a(\bar{\xi})|0\rangle = 0 \quad \forall \xi \in \mathcal{H}. \quad (32)$$

The creation operator, adjoint to $a(\bar{\xi})$ is defined by

$$a^+(\xi)\Psi = (0, \xi a, \sqrt{2}\xi \otimes_s a_1, \sqrt{3}\xi \otimes_s a_2, \dots). \quad (33)$$

⁵The choice of a subspace of $\mathcal{S}^{\mathbb{C}}$ to be used as a Hilbert space is actually not unique. Any subspace satisfying appropriate conditions (see e.g. [25]) would work. Most interesting properties of quantum fields on curved space-time (and for noninertial observers in flat space-time) can be traced back to this observation.

If $\{\xi_i\}$ is an orthonormal basis of our (separable) Hilbert space \mathcal{H} then the quantum field operator can be written as

$$\hat{\Phi} = \sum_i (\xi_i a(\bar{\xi}_i) + \bar{\xi}_i a^+(\xi_i)); \quad (34)$$

when such basis elements are (positive energy) plane waves $\phi_{\vec{p}}^+$ and we are restricting to a box of spatial size L one obtains the familiar expression⁶

$$\hat{\Phi} = \sum_{\vec{p}} (\phi_{\vec{p}}^+ a_{\vec{p}} + \bar{\phi}_{\vec{p}}^+ a_{\vec{p}}^\dagger), \quad (35)$$

with $a_{\vec{p}}^\dagger = a_{-\vec{p}}$, as we are dealing with a real scalar field.

In the next section we will see how the above construction must be modified in the presence of deformations of the algebra of relativistic symmetries. To this end it is useful to remind the reader how standard symmetries act on the algebra of scalar (quantum) fields equipped with the standard multiplication ‘‘ \cdot ’’. As a symmetry of our theory $Y \in \mathcal{P}$ will act on an element $\Psi \in \mathcal{F}_s(\mathcal{H})$ as $Y \triangleright \Psi$ and, formally, on the operators defined on the Fock space through $Y \blacktriangleright a$. We would like to establish a relation between these two actions. First let us consider the action of a Poincaré group transformation on $\Psi \in \mathcal{F}_s(\mathcal{H})$:

$$U(\Lambda, \varepsilon)(a\Psi) = a_{(\Lambda, \varepsilon)} \Psi_{(\Lambda, \varepsilon)}. \quad (36)$$

For an infinitesimal transformation $U(\Lambda, \varepsilon)(a\Psi) \approx 1 - i\delta^c Y_c$ (with δ^c infinitesimal parameters and $Y_c \in \mathcal{P}$) one has from (36) that

$$(Y \blacktriangleright a)\Psi = (Y \triangleright a\Psi) - a(Y \triangleright \Psi) \quad (37)$$

which can be rewritten formally as $Y \blacktriangleright a = [Y, a]$. One can verify, using the above relation, that the following property must hold

$$Y \blacktriangleright (a \circ b) = \circ[\Delta Y \blacktriangleright (a \otimes b)] = \sum (Y_{(1)} \blacktriangleright a) \circ (Y_{(2)} \blacktriangleright b), \quad (38)$$

where ΔY is the trivial coproduct of the standard Poincaré (Hopf) algebra (see (1)), \circ is the composition law for operators acting on $\mathcal{F}_s(\mathcal{H})$, and in the last term we used Sweedler’s notation for the coproduct. The equation above is telling us that there is a nontrivial relation between the coproduct and the composition law for operators. In particular if the coproduct is deformed or ‘‘twisted’’ then one has to modify \circ to a deformed ‘‘star product’’ \star . The need for a deformed composition law for operators is well known for the case of θ -twisted Poincaré algebra in which only the coproduct structure is deformed [26–29]. As we will see in the next sections something similar emerges in the κ -Poincaré case in which the nontrivial coalgebra

⁶From a rigorous mathematical point of view the ‘‘field operator’’ is an operator valued distribution. In the case of linear scalar field (i.e. a noninteracting theory) our lack of rigor will be harmless.

structure exhibits deformed (noncocommutative) coproduct as well as a deformed action of symmetry generators.

We can write down the explicit form of the action of the generators of translations and space-time rotations on the creation and annihilation operators, $a_{\vec{k}}, a_{\vec{k}}^\dagger$, from their action on one-particle states. The state $a_{\vec{k}}^\dagger|0\rangle$ is, by construction, a positive energy eigenvector of the momentum operators with

$$P_0 \triangleright (a_{\vec{k}}^\dagger|0\rangle) = \omega(\vec{k})(a_{\vec{k}}^\dagger|0\rangle), \quad P_i \triangleright (a_{\vec{k}}^\dagger|0\rangle) = k_i(a_{\vec{k}}^\dagger|0\rangle). \quad (39)$$

From

$$P_\mu \triangleright (a_{\vec{k}}^\dagger|0\rangle) = (P_\mu \blacktriangleright a_{\vec{k}}^\dagger)|0\rangle + a_{\vec{k}}^\dagger(P_\mu \triangleright |0\rangle) \quad (40)$$

and the invariance of the vacuum under translations we derive

$$P_0 \blacktriangleright a_{\vec{k}}^\dagger = \omega(\vec{k})a_{\vec{k}}^\dagger, \quad P_i \blacktriangleright a_{\vec{k}}^\dagger = k_i a_{\vec{k}}^\dagger. \quad (41)$$

For boosts and rotations one has

$$M_{0j} \blacktriangleright a_{\vec{k}}^\dagger = i\omega(\vec{k})\partial_{k_j} a_{\vec{k}}^\dagger, \quad (42)$$

$$M_{jl} \blacktriangleright a_{\vec{k}}^\dagger = i(k_j \partial_{k_l} - k_l \partial_{k_j})a_{\vec{k}}^\dagger, \quad (43)$$

and analogous expressions, with appropriate sign changes, hold for annihilation operators $a_{\vec{k}}$.

B. κ -one-particle Hilbert space

In the following sections we propose an extension of the standard Fock space construction described above to the case when standard relativistic symmetries described in terms of the Poincaré (Lie) algebra are generalized by the κ -Poincaré deformed (Hopf) algebra of symmetries. We assume that the physical Hilbert space of our quantum field theory consists of a deformation of the standard bosonic Fock space $\mathcal{F}_\kappa(\mathcal{H})$ whose structure is, at this stage, unknown.

Our strategy will be to start from the space of solutions of the κ -Klein-Gordon equation from which, using the inner product (25) (and implicitly the map (23)) described in Sec. II, we will define a κ -one-particle Hilbert space. At this level the nontrivial Hopf algebra structure of the symmetries will impose a natural truncation to sub-Planckian modes in the Hilbert space. In the next section we will also show how the deformed coalgebra structure of κ -Poincaré will require a new concept of symmetrization when constructing “ n -particle” states leading to a non-trivial structure for the Fock space of the theory $\mathcal{F}_\kappa(\mathcal{H})$. After defining a field operator we will apply our results to the calculation of the energy and momentum charges carried by one and two-particle states of our theory and study their properties.

We start with the definition of our “one-particle” Hilbert space. To construct such Hilbert space we consider the inner product (25) and look for a subspace of the space of complex solutions⁷ \mathcal{S}^κ on which such a product is positive definite. In order to do so we give a closer look at the properties of our inner product

$$(\Phi_1, \Phi_2)_\kappa = \int \frac{d^4 p}{(2\pi)^3} \delta(C_\kappa(p)) \text{sign}(e^{p_0/\kappa} - 1) \\ \times e^{(3p_0/\kappa)} \tilde{\Phi}_1^*(\dot{-}p) \tilde{\Phi}_2(p).$$

One can easily verify that $\text{sign}(e^{p_0/\kappa} - 1)$, when p_0 is put on shell, behaves as

$$\text{sign}(e^{p_0/\kappa} - 1) = \frac{p_0}{|p_0|} \theta(\kappa - |\vec{p}|) - \frac{p_0}{|p_0|} \theta(-\kappa + |\vec{p}|) \quad (44)$$

in which θ is the Heaviside function. Now consider an orthonormal plane wave basis $\phi_{\vec{k}} = A e^{ikx}$ in which A is a normalization factor.⁸ Their κ -Fourier transform and its complex conjugate are given by

$$\tilde{\phi}_{\vec{k}}(p) = A(2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}); \quad (45) \\ (\tilde{\phi}_{\vec{k}}(p))^* = A(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{k}),$$

and $\tilde{\phi}_{\vec{k}}^*(\dot{-}p) = A(2\pi)^3 e^{-3\omega^+(\vec{k})/\kappa} \delta^{(3)}(\vec{p} - \vec{k})$. Plugging these expressions in the formula for the inner product

$$(\phi_{\vec{k}_1}, \phi_{\vec{k}_2})_\kappa = (2\pi)^3 A^2 |\vec{k}_1|^{-1} \text{sign}(e^{(k_1^0/\kappa)} - 1) \delta^{(3)}(\vec{k}_1 - \vec{k}_2), \quad (46)$$

where k_1^0 is now on shell and from which we fix the normalization factor to $A = |\vec{k}|^{1/2} (2\pi)^{-3/2}$. As customary, we now put our fields “in the box” so that the modes have finite spacing determined by the spatial extent of the box.

From (44) one can easily see that for modes on the “positive energy” mass-shell ($\omega^+(\vec{k})$) with $|\vec{k}| > \kappa$ the inner product is *no longer positive definite*. Indeed the energies associated with such modes become complex valued. Thus, in order to construct our one-particle Hilbert space, we need to restrict to the subspace $\mathcal{S}_{|\vec{k}| \leq \kappa}^{\kappa+} \subset \mathcal{S}^\kappa$, the space of “positive energy” solutions whose mode components are truncated at the Planck scale κ , on which the inner product is positive definite. Our κ -one-particle Hilbert space \mathcal{H}_κ will be defined by the completion of $\mathcal{S}_{|\vec{k}| \leq \kappa}^{\kappa+} \subset \mathcal{S}^\kappa$ in the inner product (25).

Through the κ -complex conjugation map described in Sec. II B, whose definition combines standard complex conjugation and the antipode transformation, $\mathcal{S}_{|\vec{k}| \leq \kappa}^{\kappa+}$ is mapped into $\mathcal{S}^{\kappa-}$, the space of negative solutions, i.e.

⁷In the following we will use \mathcal{S}^κ instead of $\mathcal{S}_\kappa^\kappa$ to leave room for other indices.

⁸Without loss of generality we take A to be real.

$\bar{\mathcal{S}}_{|\vec{k}|\leq\kappa}^{\kappa+} = \mathcal{S}^{\kappa-}$. Indeed, it can be easily seen that the anti-pode is a one-to-one and onto map only restricting to positive energy modes with $|\vec{k}| \leq \kappa$ or, equivalently, restricting to *purely real* energy modes.

At this point, even though a general description of the Fock space of the theory $\mathcal{F}_\kappa(\mathcal{H})$ is still lacking, we can characterize the creation and annihilation operators relative to a specific basis in terms of their action on the vacuum state they single out in $\mathcal{F}_\kappa(\mathcal{H})$. In particular consider the (normalized) plane wave basis $\{\phi_{\vec{p}}^+\}$ of $\mathcal{S}_{|\vec{k}|\leq\kappa}^{\kappa+} \subset \mathcal{S}^\kappa$ described above. We define the annihilation operators $b_{\vec{p}}$ relative to $\{\phi_{\vec{p}}^+\}$ and a vacuum state $|0\rangle \in \mathcal{F}_\kappa(\mathcal{H})$ such that

$$b_{\vec{p}}|0\rangle \equiv (0, 0, \dots) \quad \forall \vec{p}. \quad (47)$$

The action of the creation operator $b_{\vec{p}}^\dagger$ on the vacuum will be defined by

$$b_{\vec{p}}^\dagger|0\rangle = (0, \phi_{\vec{p}}^+, 0, \dots). \quad (48)$$

Our state $b_{\vec{p}}^\dagger|0\rangle$ will be, according to the well-defined action of momenta on plane waves (see [11]), a positive energy eigenvector of the momentum operators with

$$P_0 \triangleright (b_{\vec{p}}^\dagger|0\rangle) = \omega^+(b_{\vec{p}}^\dagger|0\rangle); \quad P_i \triangleright (b_{\vec{p}}^\dagger|0\rangle) = p_i (b_{\vec{p}}^\dagger|0\rangle) \quad (49)$$

in analogy with the standard case, we have

$$P_0 \blacktriangleright b_{\vec{p}}^\dagger = \omega^+ b_{\vec{p}}^\dagger; \quad P_i \blacktriangleright b_{\vec{p}}^\dagger = p_i b_{\vec{p}}^\dagger, \quad (50)$$

which again we write in a more compact form as

$$P_\mu \blacktriangleright b_{\vec{p}}^\dagger = p_\mu^+ b_{\vec{p}}^\dagger. \quad (51)$$

For the adjoint operators we have, according to the deformed complex conjugation defined in Sec. II B

$$P_\mu \blacktriangleright b_{\vec{p}} = (\div p_\mu^+) b_{\vec{p}}. \quad (52)$$

Now that we have a candidate κ -one-particle Hilbert space we can start building multiparticle states and their associated Fock space.

C. κ -Fock space and n -particle states

In the case of standard scalar quantum fields, the Fock space $\mathcal{F}(\mathcal{H})$ built from the direct sum of tensor products of one-particle Hilbert spaces \mathcal{H} is “too large” and one has to resort to a symmetrization, related to the specific type of particle statistics, which leads to the *bosonic* Fock space described in Sec. III A. The redundancy in using $\mathcal{F}(\mathcal{H})$ as the Hilbert space of the theory comes from the fact that if we decide to label the states according to the number of particles and the eigenvalues of the four-momentum operator P_μ , for an n -particle state there will be $n!$ different states in $\mathcal{F}(\mathcal{H})$ which have *the same*

momentum eigenvalue. One thus proceeds to symmetrize such states obtaining a single element belonging to the “physical” Hilbert space $\mathcal{F}_s(\mathcal{H})$. In the κ -deformed case one could try to proceed in an analogous way but such a naive guess turns out to be inconsistent as the new symmetrized states, like e.g. the two-particle state⁹ $(0, 0, 1/\sqrt{2}(\phi_{\vec{p}} \otimes \phi_{\vec{q}} + \phi_{\vec{q}} \otimes \phi_{\vec{p}}), 0, \dots)$, will not be eigenstates of the operator P_μ , in other words the standard “flip” operator $\tau(\phi_{\vec{p}} \otimes \phi_{\vec{q}}) = \phi_{\vec{q}} \otimes \phi_{\vec{p}}$ is no longer “superselected” i.e. does not commute with all the observables. To overcome this difficulty, starting from the two-particle case, we will look for states which have the same momentum eigenvalues, extract the information regarding the type of nontrivial reshuffling of the momentum labels needed to go from one state to the other, and then construct a κ -bosonic Fock space.

The two-mode tensor product state $\phi_{\vec{p}} \otimes \phi_{\vec{q}}$ has an associated four-momentum eigenvalue $p \dot{+} q$ due to the nontrivial coproduct of the P_μ 's. We now look for momentum labels $\sigma_{12}^\kappa(\vec{p})$ and $\sigma_{12}^\kappa(\vec{q})$ such that $\phi_{\sigma_{12}^\kappa(\vec{q})} \otimes \phi_{\sigma_{12}^\kappa(\vec{p})}$ will carry the same four-momentum eigenvalue $p \dot{+} q$. Here σ_{12} indicates that the new momenta are associated to the only nontrivial element of the permutation group S^2 (σ_0^κ being the identity) but *do not* correspond to the standard flip of momentum labels. One can check that

$$\begin{aligned} \sigma_{12}^\kappa(\vec{q}) &= \tilde{q} = (\tilde{q}_0; \tilde{q} e^{-p_0/\kappa}), \\ \sigma_{12}^\kappa(\vec{p}) &= \tilde{p} = (\tilde{p}_0; \tilde{p} e^{\tilde{q}_0/\kappa}) \end{aligned} \quad (53)$$

with¹⁰ $\tilde{q}_0 = \omega^+(\tilde{q} e^{-\omega^+(\vec{p})/\kappa})$ and $\tilde{p}_0 = \omega^+(\tilde{p} e^{\tilde{q}_0/\kappa})$ are such that $\tilde{q} \dot{+} \tilde{p} = p \dot{+} q$. One also easily verifies that such a choice is unique.

We can write down the action of two creation operators on the vacuum obtaining a κ -symmetrized two-particle state¹¹

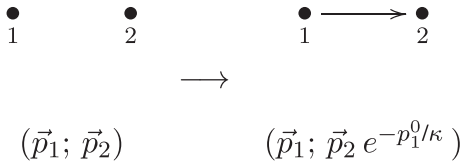
$$\begin{aligned} b_{\vec{p}}^\dagger \star b_{\vec{q}}^\dagger |0\rangle &= (0, 0, 1/\sqrt{2}(\phi_{\vec{p}} \otimes \phi_{\vec{q}} + \phi_{\vec{q}} \otimes \phi_{\vec{p}}), 0, \dots) \\ &= b_{\vec{q}}^\dagger \star b_{\vec{p}}^\dagger |0\rangle. \end{aligned} \quad (54)$$

Notice how we introduced the *non-Abelian* \star composition of creation and annihilation operators, which reflects the noncocommutativity of the coproduct, to stress that $b_{\vec{p}}^\dagger \star b_{\vec{q}}^\dagger$ and $b_{\vec{q}}^\dagger \star b_{\vec{p}}^\dagger$, acting on the vacuum, will create *two different* states in $\mathcal{F}_s^\kappa(\mathcal{H})$. The action of an annihilation operator on the state above will be

⁹In the following we omit the superscript $+$ from $\phi_{\vec{p}}^+$ being at this point obvious that we are considering only “positive energy” plane wave solutions.

¹⁰Remember that, according to (21), the time component of the four-momentum on the positive deformed mass-shell is given $\omega^+(\vec{p}) = -\kappa \log(1 - |\vec{p}|/\kappa)$.

¹¹From now on, in an effort to keep the notation clear, the different modes will be labeled in terms of the four-momentum p instead of the spatial vector \vec{p} and it will be understood that the four-momenta are on shell.


 FIG. 1. Two-particle state with momentum eigenvalue $p_1 + p_2$.

$$b_k \star b_p^\dagger \star b_q^\dagger |0\rangle = (0, (\delta_{\vec{k}\vec{p}} \phi_q + \delta_{\vec{k}\vec{q}} \phi_p), 0, \dots). \quad (55)$$

A pictorial representation of how one gets from $p + q$ to $\vec{q} + \vec{p}$ will be helpful in generalizing the construction to n -mode states. We consider two modes of a single particle state as an ordered couple $(\vec{p}_1; \vec{p}_2)$ and associate to each element of the couple a node of a graph. Drawing an arrow going from node 1 to node 2 corresponding to the ordered couple $(\vec{p}_1; \vec{p}_2)$ will give the two-particle state labeled by the couple $(\vec{p}_1; \vec{p}_2 e^{-p_1^0/\kappa})$ with momentum eigenvalue $p_1 + p_2$ (Fig. 1). Analogously drawing an arrow going from 2 to 1 gives the two-particle state $(\vec{p}_1 e^{-p_2^0/\kappa}; \vec{p}_2)$ with momentum eigenvalue $p_2 + p_1$ (Fig. 2). Notice how summing the elements of the resulting ordered couple in the first case reproduces the spatial component of the coproduct $p_1 + p_2$ and in the second case of $p_2 + p_1$.

“Retracting” an arrow going from node 1 to node 2 corresponding to the couple $(\vec{p}_1; \vec{p}_2)$ is the inverse move of the one described above and will give $(\vec{p}_1; \vec{p}_2 e^{p_1^0/\kappa})$. Retracting an arrow extending from 2 to 1 one goes from $(\vec{p}_1; \vec{p}_2)$ to $(\vec{p}_1 e^{p_2^0/\kappa}; \vec{p}_2)$ (Fig. 3).

In general an n -mode state graph will consist of n nodes. Each node will have l arrows ending on it with $0 \leq l \leq n - 1$ and each value of l will occur only once in the graph. We call l the *weight* of the node. Each one of the n -mode graphs has an associated coproduct that we represent as a set of spatial momenta multiplied by the exponents determined by the coproduct.

Let us see how it is possible to obtain the correct symmetrized tensor product states using the pictorial representation just introduced. Consider the simple example of a two-mode state with momentum eigenvalue $p_1 + p_2 = (\omega_1^+(\vec{p}_1) + \omega_2^+(\vec{p}_2); \vec{p}_1 + \vec{p}_2 e^{-p_1^0/\kappa})$. We can obtain such a coproduct summing the two entries of $(\vec{p}_1; \vec{p}_2 e^{-p_1^0/\kappa})$ or $(\vec{p}_2 e^{-p_1^0/\kappa}; \vec{p}_1)$ (attach the first entry to node 1 and the second entry to node 2). “Undoing” the state in both cases gives the two components of the symmetrized two-particle state given in (53) (see Fig. 4).

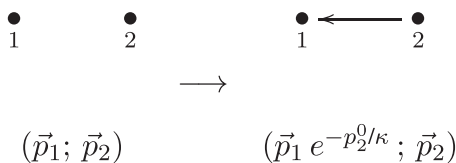
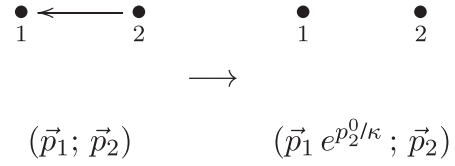
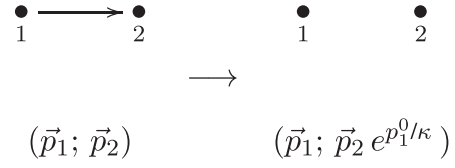

 FIG. 2. Two-particle state with momentum eigenvalue $p_2 + p_1$.


FIG. 3. Undoing two-particle states.

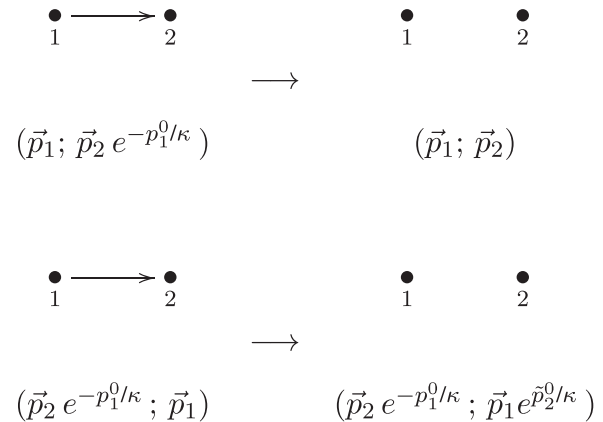
This procedure can be easily generalized to any n -mode state: the $n!$ degeneracy of states will be given by attaching the n addenda of the coproduct to each of the n nodes of the graph (total of $n!$ possible ways) and then retracting the arrows in the following sequence:

1. retract the arrow going from the node with weight 0 and the one with weight 1;
2. retract the 2 arrows going from the weight 0 nodes and the weight 2 node;

...

- $n - 1$. retract the $n - 1$ arrows going from the weight 0 nodes to the weight $n - 1$ node.

Let us watch the above algorithm at work in the example of a 3-mode state. We focus on the symmetrized 3-mode state with momentum eigenvalue $p_1 + p_2 + p_3 = (\omega_1^+(\vec{p}_1) + \omega_2^+(\vec{p}_2) + \omega_3^+(\vec{p}_3); \vec{p}_1 + \vec{p}_2 e^{-p_1^0/\kappa} + \vec{p}_3 e^{-p_1^0/\kappa - p_2^0/\kappa})$ represented by the graph in Fig. 5. We are looking for the $3!$ tensor product states belonging to $\mathcal{H}_\kappa \otimes \mathcal{H}_\kappa \otimes \mathcal{H}_\kappa$ with the same momentum eigenvalue $p_1 + p_2 + p_3$. These states will be obtained by attaching each one of the elements of the triple $(\vec{p}_1, \vec{p}_2 e^{-p_1^0/\kappa},$


 FIG. 4. Obtaining the two states which contribute to the two-particle symmetrized state with momentum eigenvalue $p_1 + p_2$.

$\vec{p}_3 e^{-p_1^0/\kappa - p_2^0/\kappa}$ to the nodes of the graph in Fig. 5 in all the $3!$ possible ways, which correspond to the permutations of the elements of the triple, and then undoing the arrows according to the algorithm above. We can label each of the $3!$ addenda of the κ -symmetrized state with elements of the permutation group of three elements. For example σ_{12}^κ will stand for the element obtained starting from the permutation which exchanges the first and the second term in the triple, σ_{123}^κ to the one which exchanges the first and the second terms, and then the second and the third, etc. The triple obtained from the identity element σ_0^κ (i.e. \vec{p}_1 attached to node 1, $\vec{p}_2 e^{-p_1^0/\kappa}$ attached to node 2, etc.) would be $(\vec{p}_1, \vec{p}_2, \vec{p}_3)$ and is the analogous of the first couple appearing on the right-hand side of Fig. 4. Putting $\alpha_i = \frac{|\vec{p}_i|}{\kappa}$ one can write down the whole set of tensor products which summed will give the symmetrized 3-particle state

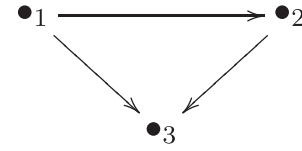


FIG. 5. The three-particle state with momentum eigenvalue $p_1 + p_2 + p_3$.

$$\sigma_0^\kappa = (\vec{p}_1, \vec{p}_2, \vec{p}_3), \quad (56)$$

$$\sigma_{23}^\kappa = (\vec{p}_1, (1 - \alpha_2)\vec{p}_3, (1 - \alpha_3(1 - \alpha_2))^{-1}\vec{p}_2), \quad (57)$$

$$\sigma_{12}^\kappa = ((1 - \alpha_1)\vec{p}_2, (1 - \alpha_2(1 - \alpha_1))^{-1}\vec{p}_1, \vec{p}_3), \quad (58)$$

$$\begin{aligned} \sigma_{123}^\kappa &= ((1 - \alpha_1)\vec{p}_2, (1 - \alpha_2(1 - \alpha_1))^{-1}(1 - \alpha_1)(1 - \alpha_2)\vec{p}_3, (1 - \alpha_2(1 - \alpha_1))^{-1} \\ &\quad \times (1 - \alpha_3(1 - \alpha_2(1 - \alpha_1))^{-1}(1 - \alpha_1)(1 - \alpha_2))^{-1}\vec{p}_1), \end{aligned} \quad (59)$$

$$\begin{aligned} \sigma_{13}^\kappa &= ((1 - \alpha_1)(1 - \alpha_2)\vec{p}_3, (1 - \alpha_3(1 - \alpha_1)(1 - \alpha_2))^{-1}(1 - \alpha_1)\vec{p}_2, (1 - \alpha_3(1 - \alpha_1)(1 - \alpha_2))^{-1} \\ &\quad \times (1 - \alpha_2(1 - \alpha_3((1 - \alpha_1)(1 - \alpha_2))^{-1}(1 - \alpha_1))^{-1}\vec{p}_1), \end{aligned} \quad (60)$$

$$\begin{aligned} \sigma_{132}^\kappa &= ((1 - \alpha_1)(1 - \alpha_2)\vec{p}_3, (1 - \alpha_3(1 - \alpha_1)(1 - \alpha_2))^{-1}\vec{p}_1, (1 - \alpha_3(1 - \alpha_1)(1 - \alpha_2))^{-1} \\ &\quad \times (1 - \alpha_1(1 - \alpha_3(1 - \alpha_1)(1 - \alpha_2))^{-1})^{-1}(1 - \alpha_1)\vec{p}_2). \end{aligned} \quad (61)$$

To clarify the above formulae note that using the same notation the elements of the symmetrized 2-mode state would have been written as

$$\sigma_0^\kappa = (\vec{p}_1, \vec{p}_2), \quad (62)$$

$$\sigma_{12}^\kappa = ((1 - \alpha_1)\vec{p}_2, (1 - \alpha_2(1 - \alpha_1))^{-1}\vec{p}_1). \quad (63)$$

The 3-mode state with momentum eigenvalue $p_1 + p_2 + p_3$ can finally be written as

$$b_{p_1}^\dagger \star b_{p_2}^\dagger \star b_{p_3}^\dagger |0\rangle = (0, 0, 0, 1/\sqrt{6}(\sigma_0^\kappa + \sigma_{23}^\kappa + \sigma_{12}^\kappa + \sigma_{123}^\kappa + \sigma_{13}^\kappa + \sigma_{132}^\kappa), 0, \dots), \quad (64)$$

where each σ^κ corresponds to an ordered triple above which in turn corresponds to a tensor product of plane wave solutions with given momentum labels $(\vec{p}_1, \vec{p}_2, \vec{p}_3) \rightarrow \phi_{p_1} \otimes \phi_{p_2} \otimes \phi_{p_3}$. Applying an annihilation operator we have

$$b_p \star b_{p_1}^\dagger \star b_{p_2}^\dagger \star b_{p_3}^\dagger |0\rangle = (0, 0, 1/\sqrt{2}[\delta_{\vec{p}, \vec{p}_1}(\bar{\sigma}_0^\kappa + \bar{\sigma}_{23}^\kappa) + \delta_{\vec{p}, (1 - \alpha_1)\vec{p}_2}(\bar{\sigma}_{12}^\kappa + \bar{\sigma}_{123}^\kappa) + \delta_{\vec{p}, (1 - \alpha_1)(1 - \alpha_2)\vec{p}_3}(\bar{\sigma}_{13}^\kappa + \bar{\sigma}_{132}^\kappa)], \dots), \quad (65)$$

where $\bar{\sigma}^\kappa$ stands for one of the addenda in (64) with the first factor in the tensor product removed. We have written down these states explicitly as we will need such expressions in order to calculate the Noether charges carried by multiparticle states in the following section.

Having a general prescription for the construction of n -particle, κ -symmetrized, states from the one-particle Hilbert space \mathcal{H}_κ , we can define the deformed Fock space of the theory to be

$$\mathcal{F}_\kappa(\mathcal{H}) = \bigoplus_{n=0}^{\infty} S_n^\kappa \mathcal{H}_\kappa^n, \quad (66)$$

where

$$S_n^\kappa = \frac{1}{n!} \sum_{\sigma \in P_n} \sigma^\kappa \quad (67)$$

and to each permutation $\sigma \in P_n$ is associated with a non-

trivial reshuffling of modes σ^κ

$$\sigma^\kappa(\phi_{k_1} \otimes \cdots \otimes \phi_{k_n}) = \phi_{\sigma^\kappa(k_1)} \otimes \cdots \otimes \phi_{\sigma^\kappa(k_n)} \quad (68)$$

according to the rules described above. Now that we have the full Hilbert space of our theory we can write down the deformed field operator acting on it

$$\hat{\Phi} = \sum_{\vec{p}, |\vec{p}| \leq \kappa} (\phi_p b_{\vec{p}} + \bar{\phi}_p b_{\vec{p}}^\dagger) \quad (69)$$

and the κ -deformed counterpart of the reality condition for creation and annihilation operators reads

$$b_{\vec{p}}^\dagger = b_{-\vec{p}} \quad (70)$$

according to the notation for the antipode of modes on shell introduced in Sec. II B.

D. Energy and momentum of quantum states

As an application of the formalism introduced we now calculate the energy and momentum charges carried by different states of our theory. The Noether charges described in Sec. II B

$$Q_\mu = \frac{1}{2}(\Phi, P_\mu \triangleright \Phi)_\kappa \quad (71)$$

in the quantum context will become observables, i.e. operators on $\mathcal{F}_\kappa(\mathcal{H})$

$$\hat{Q}_\mu = \frac{1}{2}(\hat{\Phi}, P_\mu \triangleright \hat{\Phi})_\kappa, \quad (72)$$

whose expression depends on the explicit form of the field operator $\hat{\Phi}$ and where $P_\mu \triangleright \hat{\Phi}$ indicates that the generators of translations act on the “function” coefficients of $\hat{\Phi}$. Using the orthogonality properties

$$\begin{aligned} (\phi_p, \bar{\phi}_q) &= 0, & (\phi_p, \phi_q) &= \delta_{\vec{p}\vec{q}}, \\ (\bar{\phi}_p, \bar{\phi}_q) &= -e^{-3\omega^+(\vec{p})/\kappa} \delta_{\vec{p}\vec{q}} \end{aligned} \quad (73)$$

and also

$$\begin{aligned} (\phi_p b_{\vec{p}}, \phi_q b_{\vec{q}}) &= \delta_{\vec{p}\vec{q}} b_{\vec{p}}^\dagger \star b_{\vec{q}}, \\ (\bar{\phi}_p b_{\vec{p}}^\dagger, \bar{\phi}_q b_{\vec{q}}^\dagger) &= -e^{-3\omega^+(\vec{p})/\kappa} \delta_{\vec{p}\vec{q}} b_{\vec{p}} \star b_{\vec{q}}^\dagger, \end{aligned} \quad (74)$$

we can write an explicit expression for the operators \hat{Q}_μ

$$\hat{Q}_\mu = \frac{1}{2} \sum_{\vec{p}, |\vec{p}| \leq \kappa} [p_\mu b_{\vec{p}}^\dagger \star b_{\vec{p}} - e^{-3\omega^+(\vec{p})/\kappa} (-p_\mu) b_{\vec{p}} \star b_{\vec{p}}^\dagger]. \quad (75)$$

We start by calculating the energy-momentum charges carried by the one-particle state

$$|p\rangle = b_{\vec{p}}^\dagger |0\rangle = (0, \phi_p, 0, \dots). \quad (76)$$

Using the explicit form of the operator \hat{Q}_μ given above one has

$$\langle p | \hat{Q}_\mu | p \rangle = \frac{1}{2}(p_\mu^+ - p_\mu^-) - \frac{1}{2} \left(\sum_{\vec{k}, |\vec{k}| \leq \kappa} e^{-3(\omega^+(\vec{k})/\kappa)(\cdot k_\mu^+)} \right), \quad (77)$$

where

$$p_\mu^+ = (\omega^+(\vec{p}), \vec{p}), \quad p_\mu^- = (\omega^-(\vec{p}), -\vec{p}). \quad (78)$$

The last term on the right-hand side of (77) is the analogous of the standard “vacuum energy” and, due to the presence of the exponential factor in the sum and the cutoff κ , turns out to be finite. Once the vacuum contribution has been subtracted the one-particle energy momentum reads

$$\langle p | \hat{Q}_\mu | p \rangle = \frac{1}{2}(p_\mu^+ - p_\mu^-). \quad (79)$$

We can write down an explicit form of the one-particle energy and momentum charges

$$Q_0^{(1-p)} = \frac{1}{2}(p_0^+ - p_0^-), \quad \vec{Q}^{(1-p)} = \vec{p}. \quad (80)$$

It is easily checked that they obey the dispersion relation

$$|\vec{Q}| = \kappa \tanh\left(\frac{Q_0}{\kappa}\right). \quad (81)$$

Notice how, as one would expect in the presence of a cutoff, the modulus of spatial momentum charge is bounded by κ , while the energy charge carried by the state goes to infinity when the modulus of the spatial momentum becomes Planckian.

We now calculate the same charges for a two-particle state. As showed in the previous section, given two modes with momenta labels p and q , we have two 2-particle states labeled by different values of the total four-momentum:

$$b_p^\dagger \star b_q^\dagger |0\rangle = |p \dot{+} q\rangle, \quad b_q^\dagger \star b_p^\dagger |0\rangle = |q \dot{+} p\rangle. \quad (82)$$

Using the expressions for the action of four creation and annihilation operators acting on a vacuum given in the previous subsection one obtains

$$\langle p \dot{+} q | \hat{Q}_\mu | p \dot{+} q \rangle = Q_\mu^{\text{vac}} + [\frac{1}{2}(p_\mu^+ + \tilde{q}_\mu^+) - \frac{1}{2}(p_\mu^- + \tilde{q}_\mu^-)] \quad (83)$$

in which

$$\begin{aligned} \tilde{q}_\mu^+ &= (\omega^+(\vec{q}e^{-p_0^+/\kappa}), \vec{q}e^{-p_0^+/\kappa}), \\ \tilde{q}_\mu^- &= (\omega^-(\vec{q}e^{-p_0^+/\kappa}), -\vec{q}e^{-p_0^+/\kappa}), \end{aligned} \quad (84)$$

and

$$Q_\mu^{\text{vac}} = -\frac{1}{2} \left(\sum_{\vec{k}, |\vec{k}| \leq \kappa} e^{-3(\omega^+(\vec{k})/\kappa)(\cdot k_\mu^+)} \right). \quad (85)$$

The conserved charges for the two-particle state $|p \dot{+} q\rangle$ are then

$$\begin{aligned} Q_0^{(2-p)} &= 1/2(p_0^+ - p_0^-) + 1/2(\tilde{q}_0^+ - \tilde{q}_0^-), \\ Q_i^{(2-p)} &= p_i + e^{-p_0^+/\kappa} q_i. \end{aligned} \quad (86)$$

Note how such charges are different from the corresponding ones carried by the state $|q \dot{+} p\rangle$. In fact this state has an associated energy $Q_0 = 1/2(q_0^+ - q_0^-) + 1/2(\tilde{p}_0^+ - \tilde{p}_0^-)$, with $\tilde{p} = (\omega(\tilde{p}e^{-q_0^+/\kappa}), \vec{p}e^{-q_0^+/\kappa})$, and spatial momentum components $Q_i = q_i + e^{-q_0^+/\kappa} p_i$. This indicates that the two states $|p \dot{+} q\rangle$ and $|q \dot{+} p\rangle$ are truly different physical states.

Using the technology developed so far one could go on and calculate the energy-momentum charges for states with an arbitrary number of particles. For our illustrative purposes here we limit to write explicit formulae up to the two-particle case.

A comment on the possible phenomenological implications of the framework proposed is now in order. Among the original motivations for studying quantum fields with κ -deformed symmetries was the suggestion that the deformed mass Casimir $C_\kappa(p)$ could lead to a Planck-scale modified energy-momentum dispersion relation. Various heuristic arguments showed that such a modified dispersion relation might be responsible for observable effects like e.g. the existence of trans-Greisen-Zatsepin-Kuzmin (GZK) events in the cosmic ray spectrum, time-of-flight delays for gamma ray bursts photons with different energies, or weakened bound on the flux of high energy neutrinos (see e.g. [30]). After the release of the two-year activity data from the Auger cosmic ray observatory [31] the first option has become the most valuable candidate for providing a solid experimental test for theories which predict Planck-scale modifications of kinematical threshold conditions. Our model certainly represents an important step forward in the formulation of a consistent framework which can be falsified by the Auger data. Indeed we provide for the first time an expression for the energy and momentum charges carried by κ -field quanta and show that they obey a dispersion relation (81). Notice, however, that in order to extract quantitative results for kinematical thresholds within our framework one requires the introduction of interactions and a viable proposal for the vertex of the resulting Feynman graphs. A description of canonically quantized interacting fields in a κ -symmetry framework is one of the next challenges on our research agenda.

A remarkable result obtained within the formalism introduced is the expression for the vacuum energy Q_0^{vac} that can be read off (85). As we stressed above, unlike the undeformed case, Q_0^{vac} for κ quantum fields is finite. Moreover it is a well-known fact in standard quantum field theory that, in the presence of boundary conditions, the existence of a vacuum energy leads to macroscopically observable phenomena (e.g. Casimir effect). One then naturally expects that κ -quantum fields would lead to the same effects which differ quantitatively from the standard ones by small ($\frac{1}{\kappa}$ -suppressed) corrective terms. A preliminary study of the Casimir effect in a κ -deformed setting was proposed in [32]. The result found was indeed a

Casimir energy which expanded in terms of the deformation parameter reproduces the standard result plus small corrective terms. The treatment of [32], however, was based on a heuristic quantization of κ -fields and essentially proposed a calculation of the Casimir energy in the presence of a κ -deformed energy-momentum dispersion relation. The derivation of [32] can be adapted to incorporate our rigorous result for the vacuum energy and the result [33] does not differ qualitatively from the ones of [32].

IV. COMPARISON WITH PREVIOUS ANALYSES

Before concluding we review and discuss some recent efforts in constructing quantum fields with κ -Poincaré symmetries.¹²

In [34] the authors attempt a quantization of a real scalar field on κ -Minkowski space-time with symmetries described by the κ -Poincaré algebra in the so-called ‘‘symmetric basis.’’ Such a basis is related to the standard bicrossproduct basis via the mapping $P_i \rightarrow P_i e^{P_0/2\kappa}$. The authors write the following Fourier expansion of the field

$$\phi(\hat{x}) = \frac{1}{(2\pi)^{3/2}} \int d^4 p A(p_0, \vec{p}) \delta(C_2^\kappa - M^2) e^{i p_\mu \hat{x}^\mu},$$

in which M is a mass parameter and C_2^κ represents the mass Casimir of the κ -Poincaré Hopf algebra in the specific basis considered. The Fourier coefficients of the fields are on shell according to the roots of the mass Casimir $p_0^\pm = \pm \omega_\kappa(\vec{p})$ and they are such that $(A(\pm \omega_\kappa(\vec{p}), \vec{p}))^\dagger = A(\mp \omega_\kappa(\vec{p}), -\vec{p})$ holds. Classical fields are promoted to quantum fields in [34] by defining creation and annihilation operators given by

$$\begin{aligned} a_\kappa(\omega_\kappa(\vec{p}), \vec{p}) &= C(\vec{p}) A(\omega_\kappa(\vec{p}), \vec{p}), \\ a_\kappa^\dagger(\omega_\kappa(\vec{p}), \vec{p}) &= C(\vec{p}) A(-\omega_\kappa(\vec{p}), -\vec{p}), \end{aligned}$$

where the factors $C(\vec{p})$ result from the expansion of the delta function which sets the fields on shell.

Moving to the multiparticle sector of the theory the authors adopt a strategy used in various works on field theories with θ -twisted Poincaré symmetries [26–28]. They look for a composition rule \circ for annihilation (and creation) operators which is consistent with the nontrivial coproduct of κ -Poincaré in the symmetric basis

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\ \Delta(P_i) &= P_i \otimes e^{(P_0/2\kappa)} + e^{-(P_0/2\kappa)} \otimes P_i, \end{aligned}$$

i.e. such that the action of the P_μ generators on the annihilation operators is

¹²For the purpose of clarity, in this section, we conform to the notation used, respectively, in each work discussed.

$$\begin{aligned}
P_\mu \triangleright a_\kappa(p_0, \vec{p}) &= p_\mu a_\kappa(p_0, \vec{p}), \\
P_\mu \triangleright a_\kappa^\dagger(p_0, \vec{p}) &= -p_\mu a_\kappa^\dagger(p_0, \vec{p}), \\
P_\mu \triangleright (a_\kappa(p_0, \vec{p}) \circ a_\kappa(q_0, \vec{q})) &= (\Delta^{(1)}(P_\mu) \triangleright a_\kappa(p_0, \vec{p})) \\
&\quad \times (\Delta^{(2)}(P_\mu) \triangleright a_\kappa(q_0, \vec{q})).
\end{aligned}$$

The deformed composition rule derived by the authors is

$$a_\kappa(p) \circ a_\kappa(q) := a_\kappa(p_0, e^{-(q_0/2\kappa)} \vec{p}) a_\kappa(q_0, e^{(p_0/2\kappa)} \vec{q})$$

in which $p_0 = \omega_\kappa(\vec{p})$, $q_0 = \omega_\kappa(\vec{q})$. Using this deformed “multiplication” of operators the appropriate commutators $[\cdot, \cdot]_0$ are then written down and the authors observe that they simply reproduce the standard algebra of creation and annihilation operators. This conclusion is not surprising since the choice of deformed composition of operators above tries to get rid of the nontrivial structure of the coproduct. However a major problem arises if one notices that creation and annihilation operators, acting according to the deformed composition law given above, *do not create or destroy particles which are on shell*,¹³ in fact $M^2 = C_2^\kappa(k_0, \vec{k}) \equiv (2\kappa \sinh(\frac{k_0}{2\kappa}))^2 - \vec{k}^2$ is not satisfied, for example, for the mode $(k_0, \vec{k}) = (p_0, e^{-(q_0/2\kappa)} \vec{p})$ belonging to a 2-particle state. This undesirable feature represents, in our opinion, a major flaw of the quantization of fields proposed in [34].

Two recent works in which quantization of κ -fields is discussed, in the context of an analysis of the blackbody radiation and Unruh effect in κ -Minkowski space-time, are [35,36]. There the authors define a real scalar field as the Fourier decomposition of plane waves which are taken to be the “time to the left” ordered elements of the non-commutative coordinates \hat{x} , i.e. $: e^{-ip \cdot \hat{x}} := e^{-ip^0 \hat{x}^0} e^{i\mathbf{p} \cdot \hat{\mathbf{x}}}$. The basis of the κ -Poincaré Hopf algebra associated with such a normal ordering is related to the standard bicross-product basis by the redefinition $P_i \rightarrow P_i e^{P_0/\kappa}$ and by setting $\kappa \rightarrow -\kappa$. This basis is called by the authors “asymmetric.” In [36] the following field expansion is used

$$\begin{aligned}
\phi(x) &= \int_p e^{-ip \cdot x} f(p) \delta(\mathcal{E}(p)), \\
\mathcal{E}(p) + m^2 &= M_\kappa^2(p) \left(1 + \frac{M_\kappa^2(p)}{4\kappa^2} \right),
\end{aligned}$$

where $M_\kappa^2(p)$ is the mass Casimir relative to the specific basis of the κ -Poincaré algebra considered. The deformed action for a scalar field used in [35,36] is

$$S = \int_{\vec{p}} \varphi^\dagger(p) \Delta_F^{-1}(p) \varphi(p), \quad \Delta_F^{-1}(p) = \mathcal{E}(p) + i\epsilon,$$

¹³On the other hand it easily checked that such a basic requirement is verified in the type of κ -symmetrization we propose in Sec. III C.

where $\Delta_F^{-1}(p)$ is denoted as “the Feynman propagator,” \vec{p} is the antipode, and $p \rightarrow -\vec{p}$ is called the conjugate transformation.

The authors of [35,36] then perform a field quantization promoting the classical function $\Delta_F^{-1}(p)$ to the Feynman propagator of the quantum theory in momentum space. Namely, as the authors write, “it is technically safe to assume” that

$$i\Delta_F(p)|_{\text{on shell}} = \langle \Phi(p) \Phi(p)^\dagger \rangle.$$

The expansion of the quantum field is obtained imposing consistency with the latter relation. The authors then find an algebra of creation and annihilation operators that differs from the standard one by a multiplicative factor $E(\mathbf{q})$, i.e.

$$\begin{aligned}
[a(\mathbf{p}), a^\dagger(\mathbf{q})] &= 2E(\mathbf{p})(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}), \\
[b(\mathbf{p}), b^\dagger(\mathbf{q})] &= 2E(\mathbf{p})(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}).
\end{aligned}$$

The authors repeat their analysis for the same basis of generators of κ -Poincaré considered in [34]. The difference now emerges at the level of the coproduct for the translation generators and at the level of the mass Casimir. The algebra of creation and annihilation operators obtained in this case [36] differs from the one relative to the asymmetric basis by the replacement $E(\mathbf{p}) \rightarrow D(\mathbf{p})$. The possibility of reabsorbing both factors $E(\mathbf{p})$ and $D(\mathbf{p})$ by a suitable renormalization of the creation and annihilation operators is not discussed.

The authors of [35,36] essentially follow the quantization procedure, first proposed in [37], which relies upon the assumption that the function $\Delta_F^{-1}(p)$ can be promoted to the Feynman propagator of the quantum theory. The inconsistencies of such an approach, which are particularly severe in the multiparticle sector, were already pointed out by the same authors of [37]. It is not clear to us how such problems can be resolved in the case of [35,36]. Moreover the analyses reported in [35,36] lack an explicit construction of the Hilbert space of the theory where, as we showed in the present work, the nontrivial coalgebra sector of the κ -Poincaré Hopf algebra plays a crucial role.

V. CONCLUSIONS

We have constructed the Fock space for a free massless scalar field in the presence of deformed symmetries described by the κ -Poincaré Hopf algebra. On such space we have defined the basic observable of the theory, the field operator, and calculated the energy-momentum charges carried by one and two-particle states.

The two main features of the space of quantum states we presented are the existence of Planckian cutoff for the field modes in the one-particle sector and the need of a non-trivial bosonization in the construction of the Fock space of the theory. Moreover the “vacuum energy,” which in usual

quantum field theory is a divergent quantity, due to the presence of the cutoff κ is now finite and, in our particular framework, energy and momentum charges of a single particle state obey a deformed dispersion relation.

To our knowledge this is the first time that a Hilbert space for quantum fields enjoying κ -Poincaré symmetries has been constructed and, remarkably, it provides an example of a quantum field theory in which an ultraviolet cutoff peacefully coexists with its relativistic symmetries. The nontrivial behavior of the theory in the multiparticle sector is not surprising since, for example, it is well known that in the case of θ -Poincaré Hopf algebra symmetries the usual flip operator must be twisted thus leading to a deformed particle statistics [26–28]. However the type of nontrivial bosonization we found, characterized by an “entanglement” of multiparticle modes, radically differs from the more popular framework of twisted statistics and it is a challenge for future studies to investigate the physical consequences of such peculiar behavior. Finally, it is important to notice the emphasis of our approach to quantization on the definition of an inner product, creation and annihilation operators, and field modes. This emphasis

makes the framework developed in the present paper tailored for generalizations to different choices of “time translation” Killing vectors, which define “positive energy” modes for the field, and thus for the study of Unruh-like phenomena in a κ -deformed setting. Such an analysis will be the subject of a forthcoming work.

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