Dealing with delicate issues in waveform calculations

Luis Lehner[1,*] and Osvaldo M. Moreschi[2,†]

[1]*Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803, USA*
[2]*FaMAF, Universidad Nacional de Córdoba, Ciudad Universitaria, (5000) Córdoba, Argentina*

We revisit the calculation of gravitational radiation through the use of Weyl scalars. We point out several possible problems arising from gauge and tetrad ambiguities and ways to address them. Our analysis indicates how relatively simple corrections can be introduced to remove these ambiguities.

I. INTRODUCTION

The definition of gravitational radiation involves a fair amount of work so as to distinguish variations in the metric that are purely coordinate effects from true signals propagating on a background which itself must be well established. At finite (but sufficiently far) distances from an isolated source several approaches exist to do so provided a suitable background can be identified [1–5]. For sufficiently generic cases the unambiguous identification of the gravitational radiation can only be done at future null infinity ($\mathcal{I}^+$). This is possible since when studying an isolated system, future null infinity is the asymptotic region of an asymptotically flat spacetime. There the metric is exactly flat and disturbances around it can be associated with outgoing gravitational waves subject to singling out a preferred frame. This issue arises as there is no unique flat asymptotic metric. In fact there are as many of them as there are supertranslations in the Bondi-Metzner-Sachs group: the asymptotic symmetry group. Nevertheless there is a well-defined notion of gravitational radiation which exploits the algebraic notion of asymptotic flatness [6] which in turn implies the existence of the gravitational radiation fields Ψ_4^0 and Ψ_3^0. The existence of these fields can also be deduced from stronger conditions, like peeling [7,8]. Then, after a careful choice of coordinates, radiation can be unambiguously defined.

Unfortunately, numerical applications dealing with black hole spacetimes cannot yet reach $\mathcal{I}^+$ unless Cauchy-characteristic extraction is employed [9,10] for the extraction procedure or Cauchy-characteristic matching [11,12] (see also [13]) or the conformal formulations [14] are fully realized to model the whole spacetime. Without adopting any of these options a commonly employed approach for obtaining gravitational waves relies on applying the infrastructure developed for $\mathcal{I}^+$ at finite distances. Here, several compromises are made as natural quantities defined at $\mathcal{I}^+$ need to be translated somehow to finite distances where they need not be well defined.

The purpose of this article is to point out difficulties that likely arise in this approach and a route to address them.

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*lehner@phys.lsu.edu
†moreschi@fis.uncor.edu

II. REVIEW OF THE APPROACH

The calculation of gravitational radiation at $\mathcal{I}^+$ is based on the foundational works of Bondi [15] and Sachs [16] as an approximation expansion at infinite distances and by the work of Penrose [17] as a geometrical construct at the boundary of the physical spacetime. Under reasonable assumptions one can show that the spacetime is asymptotically flat and time-dependent perturbations close to the boundary are either due to gravitational waves or coordinate effects. In order to disentangle coordinate with physical effects, care must be taken in dealing with the asymptotic symmetry group and fixing a convenient frame with respect to which inertial observers can define radiation.

Our goal here is not to revisit this approach (of which descriptions have been presented in, e.g., [8,18–22]), but simply to state the crucial ingredients which are missing when the strategy is employed at finite distances. In the next sections we will carry out an analysis of what corrections are required when key assumptions described below are missing.

These ingredients are,

(i) Peeling is assumed, this means that suitable Weyl curvature components behave asymptotically in a well-defined manner.

(ii) Outgoing null hypersurfaces, parametrized by u, intersect $\mathcal{I}^+$ defining a sequence of S^2 surfaces.

(iii) Each of these surfaces is conformal to a unit sphere metric; off this surface (into the spacetime) the departure from it is of lower order. Namely the angular metric in a neighborhood of $\mathcal{I}^+$ can be expressed as $g_{AB} = r^2 h_{AB} = r^2(q_{AB}/V^2 + c_{AB}/r + O(r^{-2}))$, with V a conformal factor, q_{AB} a unit sphere metric, and r a suitably defined radial distance.

(iv) A null tetrad $\{\ell^a, n^a, m^a, \bar{m}^a\}$ satisfying $\ell^a n_a = -m^a \bar{m}_a = 1$ (with all other products being 0). The spacetime metric can be expressed as $g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}$, and suitable projections of the Weyl tensor can be defined.

(v) Using standard [23,24] conventions for the Riemann tensor and spinor dyad, particularly useful scalars obtained from them are

$$\Psi_4 = C_{abcd}n^a\bar{m}^b n^c \bar{m}^d \tag{1}$$

$$\Psi_3 = C_{abcd}\ell^a n^b \bar{m}^c \bar{n}^d \tag{2}$$

$$\Psi_2 = C_{abcd}\ell^a m^b \bar{m}^c n^d \tag{3}$$

$$\sigma = m^a m^b \nabla_a l_b. \tag{4}$$

(vi) A suitable (Bondi type) expansion in terms of $1/r$ coupled to coordinates chosen such that $V = 1$, $g^0_{ur} = 1$ and $g^0_{uA} = 0$ (x^A labeling angular coordinates at $u = $ const, $r \rightarrow \infty$) gives rise to several important relations [6,8]. In particular,

$$\Psi^0_4 = -\ddot{\bar{\sigma}}^0 \tag{5}$$

$$\Psi^0_3 = -\eth\dot{\bar{\sigma}}^0 \tag{6}$$

$$\dot{\Psi}^0_2 = \eth\Psi^0_3 + \sigma^0\Psi^0_4, \tag{7}$$

where the supra-index "0" indicates leading order in an expansion in the radial coordinate and $\eth$ is the edth operator [23] of the unit sphere.

With this structure at hand, the following results are obtained. Given any section S at future null infinity, the Bondi momentum is given by

$$P^a = -\frac{1}{4\pi}\int_S \hat{l}^a(\Psi^0_2 + \sigma^0\dot{\bar{\sigma}}^0)dS^2, \tag{8}$$

where $\hat{l}^a = (1, \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ when expressed in standard angular coordinates (θ, ϕ). The so-called Bondi mass M is the timelike component of this vector; namely

$$M = -\frac{1}{4\pi}\int_S (\Psi^0_2 + \sigma^0\dot{\bar{\sigma}}^0)dS^2. \tag{9}$$

It is interesting to note that the Bondi momentum can also be expressed in the terms of the Psi supermomentum [25] $\Psi \equiv \Psi^0_2 + \sigma^0\dot{\bar{\sigma}}^0 + \eth^2\bar{\sigma}^0$ by

$$P^a = -\frac{1}{4\pi}\int_S \hat{l}^a\Psi dS^2, \tag{10}$$

since it has a couple of useful properties; namely, it is real, $\bar{\Psi} = \Psi$, and its time derivative is simply

$$\dot{\Psi} = \dot{\sigma}^0\dot{\bar{\sigma}}^0. \tag{11}$$

Therefore, the time variation of the Bondi momentum, in terms of time Bondi coordinate, is just given by

$$\dot{P}^a = -\frac{1}{4\pi}\int_S \hat{l}^a\dot{\sigma}^0\dot{\bar{\sigma}}^0dS^2. \tag{12}$$

Notice that due to relation (5) one might choose to replace any appearance of $\dot{\sigma}^0$ by the time integral of Ψ^0_4. This is commonly done in numerical simulations [26–35];

however, one should notice that (5) is only valid in terms of a Bondi tetrad and coordinate system.

III. CONNECTION WITH FINITE DISTANCES

The above expressions provide a well-defined infrastructure with which gravitational radiation can be defined. However, to obtain them one has introduced key ingredients, otherwise the expressions would be significantly more involved. In particular without choosing the conformal frame and/or radial distance such that the leading part of the angular metric is indeed that of the unit sphere, several changes arise even when all other properties are satisfied. For instance, if the angular metric were given by $g_{AB} = r^2(q_{AB}/V(u, \theta, \phi)^2 + O(r^{-1}))$ then, instead of (5) one would have

$$\Psi^0_4 = -\ddot{\bar{\sigma}}^0 - \frac{1}{V}\bar{\eth}^2\dot{V} + \frac{2}{V^2}\bar{\eth}\dot{V}\bar{\eth}V - \frac{2}{V^3}\dot{V}(\bar{\eth}V)^2$$
$$+ \frac{1}{V^2}\dot{V}\bar{\eth}^2 V - \frac{3}{V^2}\dot{V}^2\bar{\sigma}^0 + \frac{1}{V}\ddot{V}\bar{\sigma}^0 + \frac{3}{V}\dot{V}\dot{\bar{\sigma}}^0$$
$$= -\ddot{\bar{\sigma}}^0 - \bar{\eth}^2\frac{\dot{V}}{V} - \frac{3}{V^2}\dot{V}^2\bar{\sigma}^0 + \frac{1}{V}\ddot{V}\bar{\sigma}^0 + \frac{3}{V}\dot{V}\dot{\bar{\sigma}}^0. \tag{13}$$

Therefore, if we call $\tilde{\Psi}^0_4$ the inertial (Bondi) radiation field one would have the relation

$$\tilde{\Psi}^0_4 = \frac{1}{V^3}\left(-\ddot{\bar{\sigma}}^0 - \bar{\eth}^2\frac{\dot{V}}{V} - \frac{3}{V^2}\dot{V}^2\bar{\sigma}^0 + \frac{1}{V}\ddot{V}\bar{\sigma}^0 + \frac{3}{V}\dot{V}\dot{\bar{\sigma}}^0\right), \tag{14}$$

where the factor V^{-3} results from one of the corrections which we discuss in this work. Notice that this is not a perverse circumstance, rather it can be generically expected as surfaces at $u = $ const, $r = $ const define a 2-sphere whose metric is always conformally related to the unit sphere.

Certainly, in principle coordinate conditions can be adopted so that at the extraction sphere the simplifying conditions hold; these conditions need not be those that are convenient for the numerical implementations. Indeed, coordinate conditions are exploited in a nontrivial manner to aid in the numerical simulation [36–38]. Therefore, one is then left with having to consider how to correctly identify suitable coordinates and/or a frame to extract the desired quantity. Here, one of two options can be adopted. One approach is motivated from a perturbative point of view and relies in extracting a suitable Kinnersley tetrad [39] from the numerically generated spacetime [40–44]. The so-called Kinnersley tetrads are null tetrads that were constructed for the study of type D vacuum spacetimes and later similarly defined tetrds were applied to the extraction of gravitational radiation. A delicate point in the construction of this tetrad is the suitable fixing of the spin/boost freedom, and ongoing work is directed towards this goal [45,46]

A different approach is to consider the asymptotic structure of asymptotically flat spacetimes and define the analog quantities at finite distances together with the conditions adopted at future null infinity to single out the appropriate frame. This approach, which we adopt here, does not rely on obtaining background quantities correctly; rather one needs only to identify a suitable inertial Bondi frame which will essentially be unambiguous as long as the extraction worldtube is *sufficiently far.* We will not concern ourselves with how far the extraction worldtube must be; rather we assume one is able to place it sufficiently far for the peeling property, in the extracted quantities, of the Weyl scalars to be observed. Under this assumption we analyze what consequences, and most important, modifications one must take into account when calculating physical observables with the coordinates naturally induced on the extraction worldtube in a numerical simulation.

To this end, it will be important to calculate the induced metric on the extraction worldtube and evaluate different quantities which play a key role. We begin by following precisely the same procedure commonly employed in numerical simulations (see for instance [26,30,47]). First a Cartesian extraction worldtube is defined by $x^2 + y^2 + z^2 = R^2$. Observers at this worldtube are defined by their trajectories on the worldtube given by (t, x = const, y = const, z = const). Then, Ψ_4 is calculated on the hypersurface Σ_t (defined by t = const) by suitably adopting a tetrad frame. This tetrad is chosen in a straightforward manner by adopting three orthonormal spatial vectors $\{r^a, \theta^a, \phi^a\}$ on Σ_t (which are the analogs to the radial and tangents vectors to the sphere) and combine them with the unit normal vector N^a to the hypersurface. Once Ψ_4 is obtained, an interpolation on to the extraction sphere is carried out. At this point it is useful to recall that the extraction must be done along null rays, thus any comparison (or refinements of the extracted quantities [48]) at different worldtubes should contemplate suitable time offsets. This offset reflects the arrival time of a null signal from a given internal worldtube to the next.

As mentioned, this procedure need not give rise to a structure fully compatible with those mentioned in Sec. I. In particular the induced angular sphere metric need not be sufficiently close to that of the unit sphere; inertial observers' coordinates might ''shift'' in time around the worldtube and their associated times might tick at different rates. At finite distances these issues are even more relevant since the waves themselves will influence the geometry of the extraction worldtube as they propagate across it. In the next section we discuss the corrections required and will illustrate their application in Sec. V. As we will see, even in linearized problems these issues can play a nontrivial role.

A short summary of what these corrections entail is (in addition to the standard first-step to calculate Ψ_4):

(i) Assuming that the extraction worldtube has been chosen far enough [49], calculate the leading order

behavior Ψ_4^0 from $\Psi_4 = \Psi_4^0 r^{-1} + O(r^{-2})$; where r is an appropriate asymptotic radial coordinate, such that $r = R$ at the extraction worldtube.

(ii) In a similar way, obtain the leading order behavior of the induced metric g_Γ on the sphere at the extraction worldtube defined by t = const, $r = R$, from $g_\Gamma = -qR^2V^{-2} + O(r^1)$, where q is the unit sphere metric.

(iii) Obtain the leading order behavior g_{ur}^0 from the expansion $g_{ur} = g_{ur}^0 + O(r^{-1})$, which measures the time observers of Ψ_4 have.

(iv) Obtain the leading order behavior g_{uA}^0 from the expansion $g_{uA} = g_{uA}^0 + O(r^{-1})$, which measures the coordinate shift in time observers of Ψ_4 have around the worldtube.

(v) Bondi's radiation $\tilde{\Psi}_4^0$ is then obtained by relatively simple correction factors from the knowledge of $\{g_{ur}^0, g_{uA}^0, V\}$. In the particular case where $g_{uA}^0 \simeq 0$ the expression is

$$\tilde{\Psi}_4^{\,0} = \frac{1}{(g_{ur}^0)^2 V^3} \Psi_4^0. \qquad (15)$$

IV. ASYMPTOTIC STRUCTURE OF ASYMPTOTICALLY FLAT SPACETIMES

In this section we examine which corrections must be taken into account to remove the ambiguities described previously. To this end, we will examine how the relevant quantities transform among different coordinate systems compatible with the asymptotic structure. Then, by adopting one of these systems to be a Bondi one, we will be able to extract the correcting terms. For the sake of simplicity in the derivation we adopt the spinor-calculus approach, though the results obtained are completely independent of this technique.

We begin by determining key relations between different frames, to do so it is convenient to have at hand some basic equations of the asymptotic structure.

A. Basic variables

The asymptotic geometry can be expressed in terms of a complex null tetrad ($\ell^a, m^a, \bar{m}^a, n^a$) with the properties:

$$g_{ab}\ell^a n^b = -g_{ab}m^a \bar{m}^b = 1 \qquad (16)$$

and all other possible scalar products being zero, the metric can be expressed by

$$g_{ab} = \ell_a n_b + n_a \ell_b - m_a \bar{m}_b - \bar{m}_a m_b. \qquad (17)$$

Such a null tetrad is easily related to a dyad of spinors (o^A, ι^A). The relation of the null tetrad with a spinor dyad is given by $\ell^a \Leftrightarrow o^A o^{A'}$, $m^a \Leftrightarrow o^A \iota^{A'}$, $\bar{m}^a \Leftrightarrow \iota^A o^{A'}$ and $n^a \Leftrightarrow \iota^A \iota^{A'}$. Thus, determining how the tetrad transforms one infers the transformation of (o^A, ι^A). In our discussion, to

make a direct contact with the standard literature on the subject we also adopt a null polar coordinate system $(x^0, x^1, x^2, x^3) = (u, r, (\zeta + \bar{\zeta}), \frac{1}{i}(\zeta - \bar{\zeta}))$. However, as opposed to the standard discussions we will not make use of the available coordinate freedom to simplify the treatment at the onset. Rather, we will adopt coordinates consistent with those employed in the extraction procedure at finite distances and deal with the consequences of this choice.

In most numerical applications, one normally works with finite size regions and would like to estimate the asymptotic fields in terms of null tetrads based on a choice of coordinate system. At the extraction worldtube, one assumes that a coordinate system $(t, r, \zeta, \bar{\zeta})$ can be constructed. Let Γ be the timelike surface defined by $(r = R = \text{const})$. On Γ, a null tetrad $\ell, n, m, \bar{m}$ can be defined in the following way. First, define the null function u such that on Γ one has $u = t$, and $\ell = du$ everywhere. The function u is chosen such that the future-directed vector ℓ points outwards with respect to the 2-surfaces $(t = \text{const}, r = R)$, which are, topologically, two-dimensional spheres.

The complex vectors m and $\bar{m}$ are defined to be tangent to the spheres $(t = \text{const}, r = \text{const})$. Furthermore, one can choose m to be proportional to $\frac{\partial}{\partial \zeta}$ and $\bar{m}$ to be proportional to $\frac{\partial}{\partial \bar{\zeta}}$ in the asymptotic region for large r.

Then, by requiring the standard normalization conditions given in (16) one settles the remaining null vector n^a.

The coordinate system $(u, r, \zeta, \bar{\zeta})$ is thus straightforwardly related to the one commonly employed in numerical efforts (t, R, θ, ϕ), though we will continue our discussion with the former as it is the one employed in the standard literature on the subject. The conclusion, however, will be independent of this choice.

Keeping $(\zeta = \text{const})$ and $(\bar{\zeta} = \text{const})$ on the null hypersurface $u = \text{const}$, and increasing r, one moves along a null direction. Since ℓ is contained on the hypersurface $u = \text{const}$, one deduces that ℓ is proportional to $\frac{\partial}{\partial r}$. Then, one can write

$$(\ell^a) = \left(\frac{1}{g_{ur}}\frac{\partial}{\partial r}\right)^a. \tag{18}$$

The appearance of g_{ur} in the denominator is the first correction to be accounted for due to the structure induced at Γ not necessarily conforming to a Bondi frame. The intersection of the null hypersurface $u = \text{const}$ with future null infinity defines a two-dimensional surface denoted by S. A natural null tetrad is then completed with,

$$\ell_a = (du)_a \tag{19}$$

$$m^a = \xi^i \left(\frac{\partial}{\partial x^i}\right)^a \tag{20}$$

$$\bar{m}^a = \bar{\xi}^i \left(\frac{\partial}{\partial x^i}\right)^a \tag{21}$$

$$n^a = \left(\frac{\partial}{\partial u}\right)^a + U\left(\frac{\partial}{\partial r}\right)^a + X^i\left(\frac{\partial}{\partial x^i}\right)^a \tag{22}$$

with $i = 2, 3$ and components ξ^i, U and X^i are

$$\xi^2 = \frac{\xi_0^2}{r} + O\left(\frac{1}{r^2}\right), \qquad \xi^3 = \frac{\xi_0^3}{r} + O\left(\frac{1}{r^2}\right), \tag{23}$$

with

$$\xi_0^2 = \sqrt{2}P_0 V, \qquad \xi_0^3 = -i\xi_0^2, \tag{24}$$

where $V = V(u, \zeta, \bar{\zeta})$ and the square of $P_0 = \frac{(1+\zeta\bar{\zeta})}{2}$ is the conformal factor of the unit sphere;

$$U = rU_{00} + U_0 + \frac{U_1}{r} + O\left(\frac{1}{r^2}\right), \tag{25}$$

where (for the case $g_{ur} = 1$) one has

$$U_{00} = \frac{\dot{V}}{V}, \qquad U_0 = -\frac{1}{2}K_V, \qquad U_1 = -\frac{\Psi_2^0 + \bar{\Psi}_2^0}{2}, \tag{26}$$

where K_V is the Gaussian curvature, given by

$$K_V = 2V\bar{\eth}\eth V - 2\eth V\bar{\eth}V + V^2 \stackrel{=}{=} V\nabla^2 V - \nabla^i V\nabla_i V + V^2$$
$$= V^2\nabla^2 \ln V + V^2, \tag{27}$$

of the 2-metric

$$dS^2 = \frac{1}{V^2 P_0^2} d\zeta d\bar{\zeta}; \tag{28}$$

where the regular conformal metric restricted to S is precisely $\tilde{g}\mid_S = -dS^2$; $\nabla^2 = \nabla^i\nabla_i$ is the Laplacian operator of the unit sphere metric and ∇_i its covariant derivative. Let us emphasize that $V = 1$ makes dS^2 in (28) the unit sphere metric. Finally, the other components of the vector n^a have the asymptotic form

$$X^2 = X_2^0 + O\left(\frac{1}{r^2}\right), \qquad X^3 = X_3^0 + O\left(\frac{1}{r^2}\right). \tag{29}$$

B. Asymptotic gauge freedom, restricted case

At this point, we find it convenient to restrict to a simplified case before describing the general one. We will assume here that $X_2^0 = X_3^0 = 0$, which in an alternative view implies that angular coordinates at Γ do not shift in time as $g_{uA}^0 = 0$. Our discussion will center around the allowed transformations between the induced coordinates at the worldtube $(u, r, \zeta, \bar{\zeta})$ and the Bondi system $(\tilde{u}, \tilde{r}, \tilde{\zeta}, \bar{\tilde{\zeta}})$.

1. Coordinate and tetrad transformations I

Let us consider the main gauge freedom admitted in our calculation which is of the form

$$\tilde{u} = \alpha(u, \zeta, \bar{\zeta}) + \frac{\tilde{u}_1(u, \zeta, \bar{\zeta})}{r} + O\left(\frac{1}{r^2}\right), \qquad (30)$$

$$\tilde{r} = \frac{r}{w(u, \zeta, \bar{\zeta})} + O(r^0), \qquad (31)$$

$$\tilde{\zeta} = \zeta + O\left(\frac{1}{r}\right), \qquad (32)$$

with $\dot{\alpha} > 0$. The possible further transformation of the coordinates of the sphere $(\zeta, \bar{\zeta})$ into itself is not needed at this point.

The condition $g^{\tilde{u}\tilde{r}} = 1$ in a Bondi system imposes the relation

$$w = \frac{\dot{\alpha}}{g^0_{ur}}. \qquad (33)$$

This asymptotic coordinate transformation is associated to a corresponding null tetrad transformation, which to leading orders is given by

$$\tilde{\ell} = d\tilde{u} = \dot{\alpha}du + \alpha_\zeta d\zeta + \alpha_{\bar{\zeta}}d\bar{\zeta} + O\left(\frac{1}{r}\right)$$

$$= \dot{\alpha}\ell - \frac{\eth_V\alpha}{r}\bar{m} - \frac{\bar{\eth}_V\alpha}{r}m + O\left(\frac{1}{r}\right), \qquad (34)$$

$$\tilde{n} = \frac{\partial}{\partial\tilde{u}} + O\left(\frac{1}{r}\right) = \frac{1}{\dot{\alpha}}\frac{\partial}{\partial u} + O\left(\frac{1}{r}\right) = \frac{1}{\dot{\alpha}}n + O\left(\frac{1}{r}\right), \qquad (35)$$

$$\tilde{m} = \frac{\sqrt{2}\tilde{P}}{\tilde{r}}\frac{\partial}{\partial\tilde{\zeta}} + O\left(\frac{1}{r^2}\right)$$

$$= \frac{\sqrt{2}P_0\tilde{V}w}{r}\left(-\frac{\alpha_\zeta}{\dot{\alpha}}\frac{\partial}{\partial u} + \frac{\partial}{\partial\zeta}\right) + O\left(\frac{1}{r^2}\right)$$

$$= -\frac{\sqrt{2}P_0\tilde{V}w}{r}\frac{\alpha_\zeta}{\dot{\alpha}}n + \frac{\tilde{V}w}{V}m + O\left(\frac{1}{r^2}\right); \qquad (36)$$

since the metric expressed in terms of the new null tetrad must coincide with the metric expressed in terms of the original null tetrad, it is deduced that

$$\tilde{V} = \frac{V}{w} = \frac{Vg^0_{ur}}{\dot{\alpha}}; \qquad (37)$$

therefore

$$\tilde{m} = m - \frac{\eth_V\alpha}{r\dot{\alpha}}n + O\left(\frac{1}{r^2}\right). \qquad (38)$$

The null tetrad transformation equations can be used to write the leading order transformation relations for the spinor dyad associated to the null tetrad [22,23]; namely

$$\tilde{o}^A = \sqrt{\dot{\alpha}}\left(o^A - \frac{\eth_V\alpha}{r\dot{\alpha}}\iota^A\right) \qquad (39)$$

and

$$\tilde{\iota}^A = \frac{1}{\sqrt{\dot{\alpha}}}\iota^A, \qquad (40)$$

where $\eth_V$ is the edth operator of the metric (28). Taking into account higher order transformations would include an equation of the form

$$\tilde{\iota}^A = \frac{1}{\sqrt{\dot{\alpha}}}(\iota^A + ho^A), \qquad (41)$$

where in principle h could be of order $O(r^0)$.

The regular dyad at future null infinity in terms of the spacetime one, can be given by

$$\hat{o}^A = \Omega^{-1}o^A, \qquad (42)$$

$$\hat{\iota}^A = \iota^A. \qquad (43)$$

Then, the transformed regular dyad at future null infinity is given by

$$\hat{\tilde{o}}^A = \tilde{\Omega}^{-1}\tilde{o}^A = \frac{r}{w}\sqrt{\dot{\alpha}}\left(o^A - \frac{\eth_V\alpha}{r\dot{\alpha}}\iota^A\right)$$

$$= \frac{1}{\sqrt{\dot{\alpha}}}\left(\hat{o}^A - \frac{\eth_V\alpha}{\dot{\alpha}}\hat{\iota}^A\right), \qquad (44)$$

$$\hat{\tilde{\iota}}^A = \tilde{\iota}^A = \frac{1}{\sqrt{\dot{\alpha}}}\hat{\iota}^A, \qquad (45)$$

where we are using $\Omega = \frac{1}{r}$.

As a side comment, while the use of the dyad frame might seem to be an unnecessary complication at this stage, we note that the calculation of Ψ_4 only involves one of the dyad spinors, namely ι^A (for completeness we also include o^A). As we will see in the following section, by obtaining the asymptotic behavior of ι^A, it will straightforwardly provide the transformation properties we seek. Certainly one could directly consider the transformation of the frame itself, but the power of the spinor formalism allows for carrying the required calculations in a simpler manner. The answers, of course, are exactly the same.

2. Transformation of Ψ_4^0

We can now easily calculate the component Ψ_4 of the Weyl tensor, in leading orders, with respect to the new null tetrad, obtaining

$$\tilde{\Psi}_4^0 = \tilde{\Omega}^{-1}\Psi_{ABCD}\hat{\tilde{\iota}}^A\hat{\tilde{\iota}}^B\hat{\tilde{\iota}}^C\hat{\tilde{\iota}}^D = \frac{1}{w\dot{\alpha}^2}\Psi_4^0 = \frac{g^0_{ur}}{\dot{\alpha}^3}\Psi_4^0. \qquad (46)$$

The appearance of the factor $\tilde{\Omega}^{-1}$ in this equation is related to the asymptotic properties of the Weyl tensor in a general asymptotically flat spacetime [6], when expressed in terms of a regular tetrad or a regular dyad. More specifically, a regular tetrad (with hat) at future null infinity is related to the standard tetrad in the asymptotic region by

$$\hat{\ell}_a = \ell_a, \qquad (47)$$

$$\hat{m}_a = \Omega m_a, \tag{48}$$

$$\hat{n}_a = \Omega^2 n_a, \tag{49}$$

and

$$\hat{\ell}^{\,a} = \Omega^{-2}\ell^a, \tag{50}$$

$$\hat{m}^{\,a} = \Omega^{-1}m^a, \tag{51}$$

$$\hat{n}^{\,a} = n^a, \tag{52}$$

where Ω is the conformal factor that defines the regular metric $\hat{g}_{ab} = \Omega^2 g_{ab}$ at future null infinity. Normally one takes $\Omega = \frac{1}{r}$ where r is the affine parameter of a congruence of null geodesics that reach the asymptotic region.

3. Transformations to a Bondi system

Having analyzed how to transform among different frames we can now consider our main task, to relate Ψ_4 calculated in an arbitrary frame to that which would be obtained in a Bondi frame. This frame satisfies $g^{\tilde{u}\tilde{r}} = 1$ and $\tilde{V} = 1$, which implies

$$\dot{\alpha} = V g_{ur}^0; \qquad \dot{\alpha} = w g_{ur}^0, \tag{53}$$

which fixes both w and $\dot{\alpha}$ and indicates how different quantities are to transform. In particular,

$$\tilde{\Psi}_4^0 = \frac{1}{V^3(g_{ur}^0)^2}\Psi_4^0. \tag{54}$$

Thus, knowledge of both g_{ur}^0 and V allows one to obtain $\tilde{\Psi}_4$ in terms of the more directly calculated Ψ_4. We defer to Sec. VI and the discussion of how to obtain g_{ur}^0 and V.

C. Coordinate and tetrad transformations

After examining the simpler case, we concentrate now on the general case; namely, we will include the possibility of $X_0^A \neq 0$. As before, the task at hand is to transform to an asymptotic Bondi coordinate $(\tilde{u}, \tilde{r}, \tilde{\zeta}, \bar{\tilde{\zeta}})$ and tetrad frame $(\tilde{\ell}, \tilde{n}, \tilde{m}, \bar{\tilde{m}})$. This transformation is of the form

$$\tilde{u} = \alpha(u, \zeta, \bar{\zeta}) + \frac{\tilde{u}_1(u, \zeta, \bar{\zeta})}{r} + O\left(\frac{1}{r^2}\right), \tag{55}$$

$$\tilde{r} = \frac{r}{w(u, \zeta, \bar{\zeta})} + O(r^0), \tag{56}$$

$$\tilde{\zeta} = \tilde{\zeta}_0(u, \zeta, \bar{\zeta}) + O\left(\frac{1}{r}\right), \tag{57}$$

with $\dot{\alpha} > 0$.

If one assumes that ζ is an stereographic coordinate of the conformal unit sphere ($t = $ const, $r = $ const) it is only necessary to consider an angular transformation of the form

$$\tilde{\zeta} = \tilde{\zeta}_0(u, \zeta) + O\left(\frac{1}{r}\right). \tag{58}$$

The stronger statement is as follows.

The stereographic coordinates of the sphere can be thought of as a map of the extended complex plane. In the asymptotic sphere defined by this setting, the angle transformation, at $u = $ const, must be conformal. But, a map of the extended complex plane onto itself is conformal if and only if it is a Möbius transformation; that is, a transformation of the form

$$\tilde{\zeta} = \frac{a\zeta + b}{c\zeta + d}, \tag{59}$$

with $ab - bc \neq 0$.

We proceed in two stages.

First, note that the contravariant metric components for the standard Bondi-like coordinate system are given by equations (3.13–18) of [6], whose inverse is given by equations (3.19–24) of the same reference. The only difference is that for the general tetrad here considered

$$g^{ur} = \frac{1}{g_{ur}}, \tag{60}$$

with g_{ur} not necessarily being equal to 1. In order to deduce the required relations we start from the null tetrad defined by (19),

$$(\ell^a) = \left(A\frac{\partial}{\partial r}\right)^a, \tag{61}$$

(20) and (22). Then the contravariant components of the metric (the inverse metric), are given by

$$g^{uu} = 0, \tag{62}$$

$$g^{ur} = A, \tag{63}$$

$$g^{ui} = 0, \tag{64}$$

$$g^{rr} = 2AU, \tag{65}$$

$$g^{ri} = AX^i, \tag{66}$$

$$g^{ij} = -(\xi^i\bar{\xi}^j + \bar{\xi}^i\xi^j). \tag{67}$$

Then the metric is given by

$$g_{ur} = \frac{1}{A}, \tag{68}$$

$$g_{rr} = 0, \tag{69}$$

$$g_{ri} = 0, \tag{70}$$

$$g_{uu} = -2\frac{U}{A} + X^iX^jg_{ij}, \tag{71}$$

$$g_{ui} = -g_{ij}X^j, \tag{72}$$

$$g_{ij} = (g^{ij})^{-1} = -d\epsilon_{ik}\epsilon_{jl}(\xi^i\bar{\xi}^j + \bar{\xi}^i\xi^j), \tag{73}$$

with $i, j, k, l = 2, 3$, $d = \det(g_{ij})$, $\epsilon_{ij} = -\epsilon_{ji}$ and $\epsilon_{23} = 1$. In particular, defining the quantity

$$\lambda = \epsilon_{ij}\xi^i\bar{\xi}^j, \tag{74}$$

one has that

$$d = \frac{1}{|\lambda|^2}. \tag{75}$$

We will be interested in calculating the transformation of the radiation field component Ψ_4^0; which depends on the transformation properties of the dyad spinor ι^A. This can be read off from the transformation properties of the tetrad vector n. A straightforward calculation reveals that the one-form n_a is

$$
\begin{aligned}
(n_a) &= (g_{ab}n^b) \\
&= g_{ur}dr + g_{uu}du + g_{ui}dx^i + Ug_{ru}du \\
&\quad + X^i g_{iu}du + X^i g_{ij}dx^j \\
&= \frac{1}{A}dr + \left(-2\frac{U}{A} + X^iX^jg_{ij} + U\frac{1}{A} \right. \\
&\quad \left. + X^i(-g_{ij}X^j)\right)du + (-g_{ij}X^j + X^jg_{ij})dx^i \\
&= \frac{1}{A}(dr - Udu). \tag{76}
\end{aligned}
$$

In the Bondi frame the following relations are satisfied:

$$(\tilde{\ell}_a) = d\tilde{u}, \tag{77}$$

$$(\tilde{m}_a) = \frac{1}{\lambda}\epsilon_{ij}\bar{\xi}^i\tilde{X}^j d\tilde{u} - \frac{1}{\lambda}\epsilon_{ij}\bar{\xi}^i d\tilde{x}^j, \tag{78}$$

$$(\tilde{n}_a) = d\tilde{r} - \tilde{U}d\tilde{u}. \tag{79}$$

Additionally, in terms of the original coordinate system, the one-forms associated to the Bondi coordinates, to leading order, are given by

$$d\tilde{u} = \dot{\alpha}du + \alpha_\zeta d\zeta + \alpha_{\bar{\zeta}}d\bar{\zeta} + O\left(\frac{1}{r}\right), \tag{80}$$

$$d\tilde{r} = \frac{1}{w}dr - \frac{r}{w^2}(\dot{w}du + w_\zeta d\zeta + w_{\bar{\zeta}}d\bar{\zeta}) + O(r^0), \tag{81}$$

and

$$d\tilde{\zeta} = \dot{\tilde{\zeta}}du + \tilde{\zeta}_\zeta d\zeta + \tilde{\zeta}_{\bar{\zeta}}d\bar{\zeta} + O\left(\frac{1}{r}\right). \tag{82}$$

Therefore from the contribution to the metric of $\tilde{\ell}_a\tilde{n}_b$, one deduces that

$$\dot{\alpha}\frac{1}{w} = g_{ur}^0; \tag{83}$$

that is, w is determined by the choice of α and the value of g_{ur}^0.

In order to study the asymptotic transformation of the spinors $\tilde{\iota}^A$, we examine first the tetrad one-form $\tilde{n}_a$.

$$
\begin{aligned}
d\tilde{r} - \tilde{U}d\tilde{u} &= \frac{1}{w}dr - \frac{r}{w^2}(\dot{w}du + w_\zeta d\zeta + w_{\bar{\zeta}}d\bar{\zeta}) + O(r^0) \\
&\quad - \tilde{U}\left(\dot{\alpha}du + \alpha_\zeta d\zeta + \alpha_{\bar{\zeta}}d\bar{\zeta} + O\left(\frac{1}{r}\right)\right) \\
&= \frac{1}{w}\left(dr - \left(\frac{r}{w}\dot{w} + \tilde{U}\dot{\alpha}\right)du\right) \\
&\quad + \frac{r}{w^2}(w_\zeta d\zeta + w_{\bar{\zeta}}d\bar{\zeta}) + \text{(lower orders)}. \tag{84}
\end{aligned}
$$

It is important to note that in the general case, the vector n^a, in the asymptotic region, will have a nonzero component in the direction of the spacelike direction given by the conformal Bondi vectors $\hat{\tilde{m}}$ and $\hat{\bar{\tilde{m}}}$. Then, since in the asymptotic region

$$(\tilde{m}_a) = \frac{\tilde{r}}{\sqrt{2}\tilde{P}_0\tilde{V}}d\bar{\tilde{\zeta}} + O(\tilde{r}^0), \tag{85}$$

one can deduce

$$\hat{n}^a(\hat{\tilde{m}}_a) = \frac{1}{\sqrt{2}\tilde{P}_0\tilde{V}}(\dot{\bar{\tilde{\zeta}}} + X^{\bar{\zeta}}\bar{\tilde{\zeta}}_{\bar{\zeta}}) + O(\tilde{r}^{-1}). \tag{86}$$

Let us note that extending the hypersurface $u = $ const in the asymptotic region, one defines a section S of future null infinity; in which by construction the vectors $\Omega^{-1}m^a$ and $\Omega^{-1}\bar{m}^a$ are tangent. Also, the regular extension of ℓ, namely $\Omega^{-2}\ell^a$, is orthogonal to S. But, since future null infinity is a null hypersurface, one has that n^a, which is a null vector orthogonal to S, must be tangent to future null infinity. Since that regular extension of the Bondi frame to future null infinity is such that $\Omega^{-1}\tilde{m}^a$ and $\Omega^{-1}\bar{\tilde{m}}^a$ will be tangent to future null infinity, one deduces that the function ζ_0 must be chosen such that

$$\dot{\bar{\tilde{\zeta}}} + X^{\bar{\zeta}}\bar{\tilde{\zeta}}_{\bar{\zeta}} = 0, \tag{87}$$

since the vector $\hat{\tilde{m}}_a$ must be orthogonal to n^a at future null infinity.

By studying the asymptotic coordinate transformation one can obtain the leading order behavior of $\tilde{n}^a$, namely

$$
\begin{aligned}
\tilde{n} &= \frac{\partial}{\partial\tilde{u}} + O\left(\frac{1}{r}\right) = \frac{1}{\dot{\alpha} + X_0^\zeta\alpha_\zeta + X_0^{\bar{\zeta}}\alpha_{\bar{\zeta}}}n + O\left(\frac{1}{r}\right) \\
&= \frac{1}{n^0(\alpha)}n + O\left(\frac{1}{r}\right), \tag{88}
\end{aligned}
$$

where n^0 is the leading order behavior of the vector n.

One then can deduce the transformation

$$\tilde{\iota}^A = \frac{1}{\sqrt{n^0(\alpha)}}\,\iota^A. \tag{89}$$

Consequently the radiation component Ψ_4^0 transforms in the following way:

$$\tilde{\Psi}_4^{\,0} = \tilde{\Omega}^{-1}\Psi_{ABCD}\hat{\tilde{\iota}}^A\hat{\tilde{\iota}}^B\hat{\tilde{\iota}}^C\hat{\tilde{\iota}}^D = \frac{1}{w}\frac{1}{(n^0(\alpha))^2}\,\Psi_4^0$$

$$= \frac{g_{ur}^0}{\dot{\alpha}(n^0(\alpha))^2}\,\Psi_4^0. \tag{90}$$

Let us emphasize that $\tilde{\zeta}_0$ is chosen to make $\tilde{X}_0 = 0$, while α is chosen to make $\tilde{V} = 1$.

This complicated transformation of the radiation field indicates the convenience of adapting the numerical code such that $X^i = 0 + O(r^{-1})$, so that $\dot{\tilde{\zeta}} = 0$. Otherwise the determination of α, needed in (90), would be a delicate issue.

V. EXAMPLES

A. Conventions and formulae. Factors of 2

In Sec. I we presented the standard expressions for the radiated four-momentum. Yet, there are differences in factors with the definitions employed in numerical implementations. The difference arises from the frame adopted for the extraction procedure. The analysis presented in this work can be used to clarify these differences. In a Bondi frame, the expression for the radiated four-momentum P^a is

$$\dot{P}^a = -\frac{1}{4\pi}\int_S \hat{l}^a \dot{\sigma}^0 \dot{\bar{\sigma}}^0 dS^2. \tag{91}$$

(with $\dot{}$ denoting a derivative with respect to $\tilde{u}$). This, can be reexpressed in terms of Ψ_4^0 by employing the identity valid in a Bondi frame $\tilde{\Psi}_4^{\,0} = -\ddot{\bar{\sigma}}^0$. We have used the $\tilde{}$ to denote a quantity obtained in a Bondi frame (we do not do that for σ^0 as here we only use it as obtained in the Bondi frame).

$$\dot{P}^a = -\frac{1}{4\pi}\int_S \hat{l}^a \left|\int_{-\infty}^{\tilde{u}}\tilde{\Psi}_4^0 d\tilde{u}'\right|^2 dS^2. \tag{92}$$

In a numerical implementation, as discussed, the readily available Ψ_4^0 differs from the Bondi one by

$$\tilde{\Psi}_4 = \frac{\Psi_4}{(g_{ur}^0)^2 V^3}. \tag{93}$$

Thus

$$\dot{P}^a = -\frac{1}{4\pi}\int_S \hat{l}^a \left|\int_{-\infty}^{\tilde{u}}\frac{\Psi_4}{(g_{ur}^0)^2 V^3}d\tilde{u}'\right|^2 dS^2. \tag{94}$$

Standard approach, boost ambiguity and resulting factors of 2

We recall that the null tetrads are not uniquely defined as they have the boost freedom $\ell^a \to \ell^a\lambda^{-1}$, $n^a \to n^a\lambda$ and spin freedom $m^a \to m^a e^{i\gamma}$ [23]. This freedom can bring about additional factors in the resulting expression for the Weyl scalars; fortunately, a Bondi frame naturally fixes this ambiguity. In what follows we illustrate how this freedom is fixed by ensuring a Bondi tetrad frame is adopted in a simple example. This, in passing, will make evident the different factors encountered in commonly employed formulae. In the extraction procedure it has become customary to introduce a tetrad frame in the following form [47]. First an extraction worldtube is defined as the Cartesian $x^2 + y^2 + z^2 = R^2$. Then, three vectors (labeled by J) $\tilde{v}_J^i$ at a hypersurface slice are adopted in the following way:

$$\tilde{v}_1^i = (-y, x, 0), \qquad \tilde{v}_2^i = (x, y, z)^i,$$

$$\tilde{v}_3^i = \det(\gamma)^{(1/2)}\gamma^{ij}\epsilon_{jlm}v_1^l v_2^m;$$

where γ_{ij} is the induced metric on the spacelike hypersurface at a given time. In our present case, this reduces to

$$\tilde{v}_1^i = \partial_\phi^i, \qquad \tilde{v}_2^i = \partial_r^i, \qquad \tilde{v}_3^i = \partial_\theta^i. \tag{95}$$

The next step involves a Gram-Schmidt procedure to construct three orthonormal vectors v_J^i with respect to γ_{ij}. Finally, four spacetime vectors are easily constructed by $r^a = (0, v_2)$, $\theta^a = (0, v_3)$, $\phi^a = (0, v_1)$, which together with N^a (the unit timelike vector normal to the hypersurface) can be employed to construct the tetrad as

$$\ell'^a = \frac{1}{\sqrt{2}}(N^a + r^a) \qquad n'^a = \frac{1}{\sqrt{2}}(N^a - r^a)$$

$$m'^a = \frac{1}{\sqrt{2}}(\theta^a + i\phi^a),$$

with $r^2 = x^2 + y^2 + z^2$. Consider the simplest case of a spacetime whose metric is given by $g_{ab} = \eta_{ab} + h_{ab}$ with h_{ab} a sufficiently fast decaying function at far distances and η_{ab} a flat metric in Cartesian coordinates. In this case, to leading order, the tetrad resulting from the standard procedure is

$$\ell'^a = \frac{1}{\sqrt{2}}(\partial_t^a + \partial_R^a), \tag{96}$$

$$n'^a = \frac{1}{\sqrt{2}}(\partial_t^a - \partial_R^a), \tag{97}$$

$$m'^a = \frac{1}{r\sqrt{2}}\left(\partial_\theta^a + \frac{i}{\sin(\theta)}\partial_\phi^a\right). \tag{98}$$

The main difference with the previous tetrads is that in the former one has used a null coordinate u that in this case would be of the form $u = t - R$, with the first null tetrad vector given by $\ell_a = du_a$. Therefore, one has $\ell'^a = \frac{1}{\sqrt{2}}\ell^a$

in this case and so $\iota'^A = 2^{1/4}\iota^A$. Then, in the calculation of the radiation component of the Weyl tensor one would have $\Psi'_4 = 2\Psi_4$. The expression for the radiated momentum in these coordinates results in

$$\dot{P}^a = -\frac{1}{16\pi} \int_S \hat{l}^a \left| \int_{-\infty}^u \Psi'_4 d\tilde{u} \right|^2 dS^2. \qquad (99)$$

This then explains the factor of 4 difference in the expressions cited in [27,30,31] with that in Eq. (12) (and those in the standard literature [18,26], etc.).

B. Teukolsky waves and wave extraction

As a second example, we adopt linearized waves on flat spacetime as given by the so-called Teukolsky waves [50]. We adopt this example coupled with a possible coordinate transformation to illustrate the differences that may arise when suitable contact with a Bondi frame is missing. We adopt the simplest expression for the line element given by

$$ds^2 = dt^2 - (1 + Af_{rr})dr^2 - 2Bf_{r\theta}drd\theta - (1 + 3Cf^1_{\theta\theta}$$
$$- A)r^2 d\theta^2 - (1 + 3Cf^1_{\phi\phi} - Af^2_{\phi\phi})r^2 \sin(\theta)^2 d\phi^2 \qquad (100)$$

where

$$f_{rr} = 2 - 3\sin(\theta)^2; \qquad f_{r\theta} = -3\sin(\theta)\cos(\theta)$$
$$f^1_{\theta\theta} = -f^1_{\phi\phi} = f^2_{\phi\phi} + 1 = 3\sin(\theta)^2$$

and

$$A = 3\left(\frac{F^{(2)}}{r^3} + \frac{3F^{(1)}}{r^4} + \frac{3F}{r^5}\right) \qquad (101)$$

$$B = -\left(\frac{F^{(3)}}{r^2} + \frac{3F^{(2)}}{r^3} + \frac{6F^{(1)}}{r^4} + \frac{6F}{r^5}\right) \qquad (102)$$

$$C = \frac{1}{4}\left(\frac{F^{(4)}}{r} + \frac{2F^{(3)}}{r^2} + \frac{9F^{(2)}}{r^3} + \frac{21F^{(1)}}{r^4} + \frac{21F}{r^5}\right) \qquad (103)$$

with $F = F(t - r)$, $F^{(n)} = (\frac{d^n F(x)}{dx^n})_{x=t-r}$.

The Riemann tensor for such a line element can be straightforwardly constructed. Since to leading order the line element is just the flat metric, it is straightforward to identify the Bondi frame. A simple calculation gives

$$\tilde{\Psi}^0_4 = \frac{3}{8}\sin(\theta)^2 F^{(6)}(t - r). \qquad (104)$$

On the other hand, we can calculate Ψ_4 with the standard procedure. We consider two cases: (I) calculating Ψ_4 in the coordinates (t, r, θ, ϕ) and (II) considering the transformation $\tilde{r} \rightarrow rg(t)$ which induces a nontrivial $V(t)$. The induced tetrads, to the order that enters in the calculation of the leading order of Ψ_4, are

$$\ell^a = \frac{1}{\sqrt{2}}\left(\partial^a_t + \frac{1}{g}(1 - r\dot{g})\partial^a_R\right),$$

$$n^a = \frac{1}{\sqrt{2}}\left(\partial^a_t - \frac{1}{g}(1 - r\dot{g})\partial^a_R\right),$$

$$m^a = \frac{1}{rg\sqrt{2}}(\partial^a_\theta + i\partial^a_\phi).$$

Taking $g \equiv 1$ gives the tetrad for case (I). A straightforward calculation gives in this case,

$$\Psi^0_4 = \frac{3}{4}F^{(6)}(t - r)\sin(\theta)^2; \qquad (105)$$

while for case (II)

$$\Psi^0_4 = \frac{3}{4}\frac{F^{(6)}(t - rg)\sin(\theta)^2}{g}. \qquad (106)$$

Notice a difference of a factor g^{-1} appearing in case (II) in addition to a factor of 2 when compared to Eq. (104). These factors result from $V = g^{-1}$ and $g^0_{ur} = \sqrt{2}g$. The corrections described in Sec. IV can be used to reconcile these differences and obtain Bondi's expression in a way mostly independent of gauge issues. In the next section we discuss how the different factors involved can be obtained in generic settings.

VI. OBTAINING THE MISSING LINKS

From our previous discussion, it is clear that one must take into account the effects caused by the conformal factor V and the induced metric components g_{ur}, g_{ui}. These later factors can be read off from the expression of the null one-form $\tilde{n}_a$ in the induced null coordinates at Γ. Namely, $l_a = du_a$ defines the function u, and the transformation from the $(t, r, \zeta, \bar{\zeta})$ to the $(u, r, \zeta, \bar{\zeta})$ coordinates can be employed to express n_a in this system. Then, $g^{-1}_{ur} = l^a dr_a$; $X^i = n^a dx^i_a$.

The extraction of the conformal factor V involves more work but can be obtained by considering a conformal transformation of the sphere and evaluating its curvature scalar [51]. Recall that any metric on S^2 is conformally related to that of the unit sphere; thus it must be the case that $\hat{g}_{ij} \equiv g_{ij}R^{-2} = \omega^{-2}q_{ij}$. Thus, the unit sphere metric and the induced metric at the worldtube at a constant time are related by a conformal factor ω. With our conventions then $V = \omega$. A way to solve for this factor is to compute the scalar curvature of the cut S on the worldtube. This gives rise to the following relation

$$\mathcal{R} = 2(V^2 + D^A D_A \ln V), \qquad (107)$$

where $\mathcal{R}$ is the Ricci scalar curvature and D_i is the covariant derivative of the metric $\hat{g}_{ij}$. Let us observe that in two dimensions one has $\mathcal{R} = 2K_V$.

This equation can be solved to obtain V. Notice that this equation admits more than one solution due to the rotational group of transformations. One way to solve it and implicitly fix this freedom is motivated in techniques in-

troduced to find apparent horizons in numerical simulation [52,53]. In this approach we express V (or a related quantity) in terms of a spherical-harmonic expansion, and then obtain a recursive relation to solve for the expansion coefficients. We begin by considering [54]

$$W \equiv V^2 = \Sigma_{lm} a_{lm} Y_{lm} \qquad (108)$$

with Y_{lm} spherical harmonics satisfying $\bar{\nabla}^2 Y_{lm} = -l(l+1)Y_{lm}$ with $\bar{\nabla}^2$ the Laplacian operator with respect to the unit sphere metric expressed in the same coordinates as those where $\hat{g}_{ij}$ is known (we will refer to this as $\bar{q}_{ij}$). The solution we seek satisfies $H(W) = 0$ with

$$H = 2W + D^i D_i \ln W - \mathcal{R}. \qquad (109)$$

Next, we consider the equation

$$\rho H(W) + \bar{\nabla}^2 W = \bar{\nabla}^2 W \qquad (110)$$

with ρ a function to be determined in a suitable manner. The solution we seek turns the equation above into an identity. Consider now integrating this equation over the sphere having replaced our anzats for W. This will provide a recursion relation for obtaining the parameters a_{lm} (for $l \neq 0$) as,

$$\int_S \bar{Y}_{lm}(\rho H + \bar{\nabla}^2 W)d\Omega = -l(l+1)a_{lm}. \qquad (111)$$

Thus, starting with trial values for $\{a_{lm}\}$ a new set can be obtained through the equation above. Two ingredients remain to be provided however: one is how a_{00} is to be set and the other a plausibility argument for the convergence of the method. We describe first the latter issue and then discuss the former. This argument relies on concentrating on the principal part of the system $(\rho H + \bar{\nabla}^2 W)$ and to exploit ρ to our advantage. Inspection of these terms allows one to reexpress them as $M^{ij}\bar{D}_i\bar{D}_j W + S$ with

$$M^{ij} = \frac{\rho}{W}\hat{g}^{ij} + \bar{q}^{ij} \qquad (112)$$

$$S = \rho\left(2W - \frac{1}{W}\hat{g}^{ij}C_{ij}^k\bar{D}_k W - \frac{1}{W^2}\hat{g}^{ij}\bar{D}_i W\bar{D}_j W - \mathcal{R}\right) \qquad (113)$$

with $\bar{D}$ the covariant derivative associated with $\bar{q}_{ij}$ and C_{ij}^k the tensor relating the connections $\bar{D}_i$ with D_i. Notice now that Eq. (110) can be formally solved by the iteration scheme

$$W^{(n+1)} = (\bar{\nabla}^2)^{-1}(\rho H + \bar{\nabla}^2)W^{(n)}. \qquad (114)$$

Here the standard argument applies; the Laplacian operator smooths out high frequencies and by properly adjusting M^{ij} such that the highest order derivatives are removed the successive solutions will be smoother which is a requirement for convergence. The factor ρ can be chosen so as to cancel particular terms in the second-order derivatives, or

to try to cancel as much as possible M^{ij} (see the discussion in [52]). For instance, if $\bar{q}^{ij} \simeq q^{ij}$ the trivial choice $\rho = -1$ is ideally suited for this purpose.

Last we comment on how a_{00} is obtained. At the first step of the iteration, a useful choice is induced by assuming $W^{(1)} = a_{00}^1 Y_{00}$. Then, one simply has

$$0 = \int(2W^{(1)} - \mathcal{R})Y_{00}d\Omega = 2a_{00}^1 - \int \mathcal{R}Y_{00}d\Omega. \qquad (115)$$

At subsequent steps, having fixed a_{lm}^{n+1} for $l > 0$, Eq. (111) can be employed to fix a_{00}^{n+1}. However, due to the nonlinearities of the equation this might turn out to be difficult. A simpler, more direct route is to do so from

$$\int_S ((2W^{(n+1)}) + D^i D_i \ln W^{(n)} - \mathcal{R})Y_{00}d\Omega = 0, \qquad (116)$$

which can be used to determine a_{00}^{n+1} after all other coefficients have been obtained.

VII. FINAL COMMENTS

In this work we have reexamined the issue of computing gravitational radiation effects through the use of Weyl scalars. The analysis reveals which correcting factors are to be accounted for if the coordinates adopted do not conform to a Bondi system. To date while these corrections have not been taken into account in numerical efforts, the waveforms obtained both appear quite reasonable and, most important, agreeing across different implementations. What is then the expected contribution of the corrections indicated here or the relevance of the present discussion in light of these observations?

First, at least the formalism presented in this work allows for easily enforcing consistency among different implementations. Namely, while coordinates typically used vary among different efforts, considering the transformation to a Bondi system provides a common frame for the computation. As a result, comparison of obtained signals from different codes would be expedited. Second, the correcting terms will have generically nontrivial angular dependence. Therefore, the decomposition on different multipolar moments would be affected. In the case of nonspinning equal-mass black holes symmetry considerations indicate this might not be a significant issue. On the other hand, for different masses and/or spinning compact objects the contribution should be nontrivial, especially in light of significant kicks observed which indicate a strong direction dependent of the radiated waves.

It is important to stress here that the corrections are case-by-case dependent as they are both sensitive to the gauge and initial data employed. Consequently it is difficult to assess the role the corrections indicated here might have. Nevertheless, consistency with the calculation formalism and a simplified frame of comparison among different implementations are already key reasons for the relevance of the discussion.

Notice that the obtained result for correcting Ψ_4^0 does not contain any contributions from the other Weyl components, like Ψ_2^0 for example. This is related to the way this calculation is carried out, and the assumption of the extraction being done sufficiently far. From a geometrical point of view one can observe that the worldtube defined in a numerical calculation gives rise to a natural null tetrad and coordinate system (that we have discussed here) which in general does not conform with an asymptotic inertial Bondi reference frame. In searching for the relation between these two systems, we explicitly use the asymptotic structure of general asymptotically flat spacetimes [6]. This indicates that a convenient tetrad that has a regular extension to future null infinity is to be chosen so that one of the vectors (say n^a) is asymptotic to the generators of future null infinity. Therefore, asymptotically, both tetrads are related by a null rotation around this null vector. It is probably worth mentioning that from the physical point of view, it is this asymptotic null vector n^a which indicates the appropriate rate of the time coordinate. In other words, it is not sufficient to seek a gauge invariant definition for the radiation field, but one must also take into account what the gravitational radiation detectors will measure, using the physical time of the asymptotic region of the sources. The Bondi frame is defined precisely to satisfy these conditions. However, we stress again the importance of the correct extraction of the leading order behavior Ψ_4^0 of the component Ψ_4 of the Weyl tensor, along null directions. Our treatment assumes this has been performed.

As a side comment we want to stress we have employed a convention based on a (-2) signature following [23,24], which is the standard signature employed in studies of asymptotically flat spacetimes.

Last, short of considering the corrections discussed in this work, as discussed in [55], one could estimate whether these effects may play a role by evaluating the norms $||\mathcal{R} - 2||$, $||g_{ur} - \kappa||$ (with $\kappa = 1$ or $\sqrt{2}$ depending on the boost freedom adopted for ℓ^a) and $||g_{uA}||$. If these norms are at the order of the truncation error in a simulation, then the correcting factors would certainly not be essential. We will examine these issues for different binary systems in a forthcoming work [56].

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