Invariant conserved currents in gravity theories: Diffeomorphisms and local gauge symmetries

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(Received 31 August 2007; published 27 December 2007)

Previously, we developed a general method to construct invariant conserved currents and charges in gravitational theories with Lagrangians that are invariant under spacetime diffeomorphisms and local Lorentz transformations. This approach is now generalized to the case when the local Lorentz group is replaced by an arbitrary local gauge group. The particular examples include the Maxwell and Yang-Mills fields coupled to gravity with Abelian and non-Abelian local internal symmetries and the metric-affine gravity in which the local Lorentz spacetime group is extended to the local general linear group.

DOI: 10.1103/PhysRevD.76.124030 PACS numbers: 04.20.Cv, 04.20.Fy, 04.50.+h

I. INTRODUCTION

As is well known, the Noether theorem establishes a relation between the symmetries and conservation laws of a physical model. It tells that conserved currents arise from the invariance of the classical action when the fields are transformed under the action of the symmetry groups. There are two types of symmetries: the internal ones [like the Abelian $U(1)$ phase transformations, or the non-Abelian isotopic $SU(2)$ transformations, as well as their generalizations] that act in spaces of internal degrees of freedom, and the external symmetries that act on the spacetime manifold itself and on its related geometrical structures. In particular, gravity theories are normally based on the covariance principle, which in technical terms means that the action is invariant under spacetime diffeomorphisms. Since diffeomorphisms are generated by vector fields, one can expect that every vector field should give rise to conserved quantities.

In the previous paper [1], we have proposed a general definition of invariant conserved quantities for gravity theories with general coordinate and local Lorentz symmetries. More exactly, we have demonstrated that, indeed, every vector field ξ on spacetime generates, in any dimension n, for any Lagrangian of gravitational plus matter fields and for any (minimal or nonminimal) type of interaction, a current $\mathcal{J}[\xi]$ with the following properties: (i) the current $(n-1)$-form $\mathcal{J}[\xi]$ is constructed from the Lagrangian and the generalized field momenta, (ii) it is conserved, $d\mathcal{J}[\xi]=0$, when the field equations are satisfied, (iii) $\mathcal{J}[\xi]=d\Pi[\xi]$ "on shell," (iv) the current $\mathcal{J}[\xi]$, the superpotential $\Pi[\xi]$, and the conserved charge $\mathcal{Q}[\xi]=\int\mathcal{J}[\xi]$ are invariant under diffeomorphisms and the local Lorentz group. This construction generalizes the results known for models that are invariant under the diffeomorphism group only [2] and improves and clarifies the earlier facts about the conserved quantities associated with a vector field [3–13] that were discovered for specific Lagrangians (usually, for the Hilbert-Einstein one) and for specific types of vector fields (usually, for Killing or generalized Killing ones). It is also possible to extend the analysis to the case of quasi-invariant models that are described by Lagrangians which change by a total derivative under the action of the symmetry groups [14,15]. In this case, however, there seems to be no clear way to define invariant conserved currents.

Several remarks are in order. We use the physical terminology throughout this paper. In particular, we refer to the equation $dJ=0$ as a conservation law. Being an $(n-1)$-form, in the local coordinates $\{x^i\}$ the current can be expanded as $J=\mathcal{J}^i\epsilon_i$ with respect to the natural basis $\epsilon_i=\frac{1}{(n-1)!}\epsilon_{ij_1\dots j_{n-1}}dx^{j_1}\wedge\dots\wedge dx^{j_{n-1}}$ of the $(n-1)$-forms (where $\epsilon_{i_1\dots i_n}$ is the totally antisymmetric Levi-Civita symbol on an n-manifold). Then $dJ=0$ is equivalent to the divergence equation $\partial_i\mathcal{J}^i=0$ for the components of the vector density $\mathcal{J}^i$. In physics, $\mathcal{J}^i$ is called a conserved current when it satisfies $\partial_i\mathcal{J}^i=0$. In the mathematical language, a form with the property $dJ=0$ is called closed, but we prefer to use the standard physical terminology.

It is well known that in diffeomorphism-invariant models one can associate a conserved current to a vector field ξ on the spacetime manifold. This can be done in various ways. One example is to consider a symmetric energy-momentum tensor $T_j{}^i$ and a *Killing* vector field $\xi=\xi^i\partial_i$ that generates an isometry of the spacetime. Since the energy-momentum tensor is *covariantly* conserved in diffeomorphism-invariant theories, one then straightforwardly verifies that $\mathcal{J}^i=\xi^jT_j{}^i\sqrt{-g}$ is a conserved current, i.e., $\partial_i\mathcal{J}^i=0$. A further example is provided by a general scheme [2] in which a conserved current $(n-1)$-form is derived for any solution of a diffeomorphism-invariant model even when ξ is not a Killing field. All such

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$(n-1)$-form currents are scalars under general coordinate transformations, as well as the corresponding charges derived from them. They proved to be useful for the computation of the total energy and angular momentum of various gravitational field configurations, and for the discussion of the thermodynamic laws of the black holes in gravity models.

The interpretation of the vector field ξ is an important geometrical and physical issue. For example, when ξ is timelike, the corresponding charge has the meaning of the energy of the gravitating system with respect to an observer moving along the integral lines of ξ, with 4-velocity $u = \xi/|\xi|$, cf. [16]. In this way, the dependence of the charges on ξ describes the usual dependence of the energy of a system on the choice, and on the dynamics, of a physical observer. In particular, the invariant charge $\mathcal{Q}[\xi] = \int_S \mathcal{J}^i \epsilon_i$ is then the integral of the *projection* of the energy-momentum density along the vector ξ. This charge reduces to the usual expression $\int_S T_0{}^0\sqrt{-g}dx^1 \wedge \ldots \wedge dx^{n-1}$ in coordinates adapted to ξ such that $\xi = \partial_0$, and the hypersurface S is defined by $x^0 =$ constant. Such an approach to the definition of the energy of a gravitating system as a scalar (that is, *invariant* depending on some vector field) closely follows the well-known construction for point particles (see Sec. 2.8, and, in particular, Eq. (2.29) of [17]).

However, the situation becomes more complicated when, besides the diffeomorphism symmetry, the gravitational model is also invariant under some additional gauge group. For example, there is a large class of theories which are invariant under the local Lorentz group $SO(1, n-1)$, including the gauge gravity models [6], the supergravity, and the so-called first order formulation of standard general relativity. The problem of defining the corresponding conserved quantities associated with a vector field was analyzed previously [6–12] for specific Lagrangians (usually, for the Hilbert-Einstein one) and for specific types of vector fields (usually, for Killing or generalized Killing ones). Moreover, the resulting conserved quantities were often discovered to be *not invariant* under the local Lorentz group (e.g., in [12]).

We have shown in [1] that for models with arbitrary Lorentz-invariant Lagrangians it is possible to define *invariant* conserved currents for every vector field ξ. These conserved currents do not depend on the coordinate system or the tetrad frame used to compute them. They depend only on the field configuration and on the choice of the vector field ξ.

In the present paper, we further develop our approach and replace the local Lorentz group with an arbitrary gauge Lie group G. This includes two subcases: (i) the group G acts in a space of internal degrees of freedom, and (ii) the group G acts on the spacetime manifold and the local Lorentz group is its subgroup $SO(1, n-1) \in G$. The particular choice of the general linear group $G = GL(n, R)$ is of special importance since this group underlies the gauge formulation of the so-called metric-affine gravity (MAG) theory. As a result, we give a general construction of the invariant conserved quantities for gravity theories with general coordinate (diffeomorphism) and local gauge G symmetries. We again show that for every vector field ξ on spacetime, and any Lagrangian of gravitational plus matter fields there exists a conserved current $(n-1)$-form that is a true scalar under both diffeomorphisms and G.

There is an important difference between the cases when G is an internal symmetry group and when it is the general linear group. In the latter case, i.e., in the framework of MAG, there exist a well-defined way to construct the conserved current by making use of the Yano choice of the generalized Lie derivative that appears in the Lagrange-Noether derivation of the conservation law. Such a Yano derivative (see [18], and cf. the previous derivations in [1]) is always defined in MAG in terms of the coframe, which is a dynamical field (''translational potential'') in this approach. In contrast to this situation, in the models where G is an internal symmetry group, one cannot come up with a suitable counterpart of the Yano derivative, in general. Sometimes, a similar construction is possible, and we explicitly demonstrate this for the case of $G = U(1)$, for the models that include a Higgs-type complex scalar field. However, at the moment it is unclear whether an appropriate non-Abelian generalization can be found with the help of a suitable Higgs multiplet.

Previously [1], we demonstrated that our approach works nicely for the calculation of the total mass (energy) and the total angular momentum for the solutions of the gravitational field equations without and with torsion. The corresponding resulting values of the mass and angular momentum were shown to be consistent with the calculations obtained by alternative methods, see for example [19,20] and the references therein. In order to test the generalized formalism, we apply it to the analogous computation of the mass and angular momentum of the exact solutions in MAG with nontrivial torsion and nonmetricity.

Our general notations are the same as in [21]. In particular, we use the Latin indices $i, j, \ldots$ for local holonomic spacetime coordinates and the Greek indices $\alpha, \beta, \ldots$ label (co)frame components. Particular frame components are denoted by hats, $\hat{0}$, $\hat{1}$, etc. As usual, the exterior product is denoted by $\wedge$, while the interior product of a vector ξ and a p-form Ψ is denoted by $\xi\rfloor\Psi$. The vector basis dual to the frame 1-forms ϑ^α is denoted by e_α and they satisfy $e_\alpha\rfloor\vartheta^\beta = \delta^\beta_\alpha$. Using local coordinates x^i, we have $\vartheta^\alpha = h^\alpha_i dx^i$ and $e_\alpha = h^i_\alpha \partial_i$. We define the volume n-form by $\eta := \vartheta^{\hat{0}} \wedge \cdots \wedge \vartheta^{\hat{n}}$. Furthermore, with the help of the interior product we define $\eta_\alpha := e_\alpha\rfloor\eta$, $\eta_{\alpha\beta} := e_\beta\rfloor\eta_\alpha$, $\eta_{\alpha\beta\gamma} := e_\gamma\rfloor\eta_{\alpha\beta}$, etc., which are bases for $(n-1)$-, $(n-2)$- and $(n-3)$-forms, etc., respectively. Finally, $\eta_{\alpha_1\cdots\alpha_n} = e_{\alpha_n}\rfloor\eta_{\alpha_1\cdots\alpha_{n-1}}$ is the Levi-Civita tensor density.

The η-forms satisfy the identities

$$\vartheta^\beta \wedge \eta_\alpha = \delta_\alpha^\beta \eta, \tag{1.1}$$

$$\vartheta^\beta \wedge \eta_{\mu\nu} = \delta_\nu^\beta \eta_\mu - \delta_\mu^\beta \eta_\nu, \tag{1.2}$$

$$\vartheta^\beta \wedge \eta_{\alpha\mu\nu} = \delta_\alpha^\beta \eta_{\mu\nu} + \delta_\mu^\beta \eta_{\nu\alpha} + \delta_\nu^\beta \eta_{\alpha\mu}, \tag{1.3}$$

$$\vartheta^\beta \wedge \eta_{\alpha\gamma\mu\nu} = \delta_\nu^\beta \eta_{\alpha\gamma\mu} - \delta_\mu^\beta \eta_{\alpha\gamma\nu} + \delta_\gamma^\beta \eta_{\alpha\mu\nu} - \delta_\alpha^\beta \eta_{\gamma\mu\nu}, \tag{1.4}$$

etc. The line element $ds^2 = g_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$ is defined by the spacetime metric $g_{\alpha\beta}$ of signature $(+, -, \cdots, -)$.

II. GENERAL FORMALISM IN CONDENSED NOTATION

Let us consider the most general case of a gravitational theory with the diffeomorphism and an additional arbitrary gauge symmetry. As a matter of fact, all gravitational theories are special cases of metric-affine gravity with the gravitational field described by the three basic variables: the metric $g_{\alpha\beta}$, the coframe ϑ^α, and the linear connection $\Gamma_\alpha{}^\beta$. In addition, the nongravitational sector of the theory contains the usual matter fields ψ (scalars and spinors of any rank that describe massive particles) and the gauge fields A of a certain internal symmetry group (these are one-forms with values in the corresponding Lie algebra, or p-forms, in general).

Taking this into account, we denote the total symmetry group by G. It is defined as a direct product of the *internal symmetry* group and the *spacetime symmetry* group. The former can be any Abelian or non-Abelian Lie group, acting via a suitable representation on the multiplet of matter fields. The spacetime symmetry group can be a Lorentz, linear, conformal, de Sitter group or other, acting on the geometric objects on the spacetime manifold. Accordingly, we will denote the gauge fields of internal symmetries A together with the gravitational connection Γ by a collective gauge field $\mathcal{A}^a$ which is a one-form taking values in the Lie algebra $\mathcal{G}$ of the group G. The index a runs through both the gravitational and nongravitational sectors and labels the corresponding infinitesimal parameters ε^a of $\mathcal{G}$. In a similar way, we will denote all the covariant fields of the model by a collective field Ψ. The latter includes the gravitational sector (for example, $g_{\alpha\beta}$, ϑ^α) and the matter fields ψ^A. Finally, we collect *all* the fields of the model into a single object $\Phi^I = \{\Psi, \mathcal{A}\}$, where the index I runs over all the components of the fields in all sectors.

After these preliminaries, we can describe the generalization of the construction [1] of conserved currents for theories invariant under diffeomorphisms and an arbitrary gauge group G. As before, we start with the total Lagrangian n-form $V^{\rm tot}(\Phi^I, d\Phi^I)$ and define the generalized field momenta and the energy terms by

$$H_I := -\frac{\partial V^{\rm tot}}{\partial d\Phi^I}, \qquad E_I := \frac{\partial V^{\rm tot}}{\partial \Phi^I}. \tag{2.1}$$

The total variation of the Lagrangian then reads

$$\delta V^{\rm tot} = \delta\Phi^I \wedge \mathcal{F}_I - d(\delta\Phi^I \wedge H_I), \tag{2.2}$$

where the variational derivative is defined by

$$\mathcal{F}_I := \frac{\delta V^{\rm tot}}{\delta \Phi^I} = (-1)^{p(I)} DH_I + E_I. \tag{2.3}$$

Here $p(I)$ denotes the rank (in the exterior sense) of the corresponding sector of the collective field.

The Eq. (2.2) describes how the Lagrangian changes under the change of the fields. This includes two cases: (i) when the variations of the fields are arbitrary, and (ii) when the fields are transformed under the action of the symmetry group. In the first case, when the action is stationary (i.e., $\delta V^{\rm tot} = 0$) under the *arbitrary* variations $\delta\Phi^I$, one finds from (2.2) the field equations

$$\mathcal{F}_I = 0. \tag{2.4}$$

However, in the second case the variation $\delta\Phi^I$ is not arbitrary, and the invariance of the action gives rise to the conservation laws. The total variation of the field variables under diffeomorphisms and the gauge symmetry reads

$$\delta\Phi^I = \ell_{(\varsigma\xi)}\Phi^I + \delta_{(\varsigma\varepsilon)}\Phi^I =: \varsigma\mathcal{L}_{\xi,\varepsilon}\Phi^I. \tag{2.5}$$

Here ς is an arbitrary infinitesimal constant parameter, $\ell_\xi = \xi\rfloor d + d\xi\rfloor$ is the ordinary Lie derivative, and the second term describes the gauge transformation

$$\delta_{\varsigma\varepsilon}\Phi^I = \varsigma[\varepsilon^a(\rho_a)^I{}_J\Phi^J - (\sigma_a)^I d\varepsilon^a]. \tag{2.6}$$

This reduces to the well-known law for the case of the Lorentz symmetry. The generators ρ_a satisfy the commutation relation $[\rho_a, \rho_b]^I{}_J = f^c{}_{ab}(\rho_c)^I{}_J$ with the structure constants $f^c{}_{ab}$ of the Lie algebra $\mathcal{G}$. It is worthwhile to note that $(\sigma_a)^I = \delta_a^I$ is a unit matrix in the gauge sector and is trivial in the covariant sector.

We will call $\mathcal{L}_{\xi,\varepsilon}\Phi^I$ defined by (2.5) a generalized Lie derivative of the multifield Φ^I.

After these preliminary steps, we can derive the conservation law and introduce the conserved current. Namely, for the general models under consideration, we define the generalized current

$$J[\xi, \varepsilon] := \xi\rfloor V^{\rm tot} + (\mathcal{L}_{\xi,\varepsilon}\Phi^I) \wedge H_I. \tag{2.7}$$

As for the case of the local Lorentz symmetry (cf. [1]), it satisfies

$$dJ[\xi, \varepsilon] \equiv (\mathcal{L}_{\xi,\varepsilon}\Phi^I) \wedge \mathcal{F}_I. \tag{2.8}$$

This is just the total variation formula (2.2) in a different form. By using the Noether identities (which we do not write down here explicitly, see the subsequent discussion of the internal symmetry in Sec. III and of the general linear group in Sec. V), we can recast this current as

$$J[\xi, \varepsilon] \equiv d(\Xi^I[\xi, \varepsilon] \wedge H_I) + \Xi^I[\xi, \varepsilon] \wedge \mathcal{F}_I. \tag{2.9}$$

Here we denoted

$$\Xi^I[\xi, \varepsilon] := \xi\rfloor\Phi^I - \varepsilon^a \delta^I_a. \tag{2.10}$$

When the field Eqs. (2.4) are satisfied, the generalized current (2.7) is conserved, $dJ[\xi, \varepsilon] = 0$, and hence one can define the corresponding charge by the integrals

$$Q[\xi, \varepsilon] = \int_S J[\xi, \varepsilon] = \int_{\partial S} \Xi^I[\xi, \varepsilon] \wedge H_I \tag{2.11}$$

over a spacelike $(n-1)$-hypersurface S with an $(n-2)$-dimensional boundary ∂S. The Eq. (2.11) arises, as usual, when we integrate the conservation law $dJ[\xi, \varepsilon] = 0$ over the n-volume domain with the boundary $S_1 + S_2 + T$, where S_1 and S_2 are $(n-1)$-dimensional spacelike hypersurfaces (which correspond to the arbitrary time values t_1 and t_2, respectively) and T is a timelike surface that connects them. Then assuming that the fields satisfy the boundary conditions such that $\int_T J = 0$, we find that $Q[\xi, \varepsilon] = \int_{S(t)} J[\xi, \varepsilon]$ is constant. Using subsequently (2.9), one finds (2.11) with the help of the Stokes theorem. We will always assume the appropriate asymptotic behavior of the geometric and matter fields that makes the conserved charges well-defined objects. Since we deal with a general scheme without specifying a Lagrangian, the necessary boundary conditions appropriate for all theories cannot be given explicitly. They depend on the particular model and should be chosen after a case-by-case inspection of a theory.

The functions ε^a parametrize a *family* of conserved currents (2.7) and charges (2.11) associated with a vector field ξ. In order to select *invariant* conserved quantities, we will have to specialize to a particular choice of ε. The trivial choice $\varepsilon^a = 0$ yields a *noninvariant* current and charge. Indeed, then $\mathcal{L}_{\{\xi,\varepsilon\}}\Psi = \ell_\xi\Psi$, and the last term in (2.7) is not gauge invariant, since the ordinary Lie derivative ℓ_ξ of a covariant object is not covariant under gauge transformations. In [1] it was shown that one can define covariant conserved currents (and charges) with the help of an appropriate choice of $\varepsilon(\xi)$. Unfortunately, there seems to be no general recipe how to choose $\varepsilon(\xi)$ for an arbitrary internal and external symmetry group. Accordingly, the situation should be studied on a case-by-case basis. In the next sections we analyze separately the models with internal gauge symmetries (Sec. III) and the metric-affine gravity (Sec. V).

III. CONSERVED CURRENTS IN GAUGE THEORIES

First we consider the class of theories that are invariant under the diffeomorphism group and an arbitrary local (gauge) Lie group G. We denote the corresponding Lie algebra $\mathcal{G}$. Its generators ρ_a, $a = 1, \ldots, \dim(\mathcal{G})$, satisfy the commutator relations

$$[\rho_a, \rho_b] = f^c{}_{ab}\rho_c, \tag{3.1}$$

with the structure constants $f^c{}_{ab}$. We also consider matter p-form fields Ψ^A transforming covariantly under the action of G, such that

$$\delta_\varepsilon \Psi^A = -\varepsilon^a (\rho_a)^A{}_B \Psi^B, \tag{3.2}$$

where $\varepsilon^a(x)$ are the parameters of the transformation, and $(\rho_a)^A{}_B$ denote the matrix representation of the generators ρ_a acting in the vector space of the matter fields Ψ^A. The covariant derivative is then defined by

$$D\Psi^A := d\Psi^A - A^a (\rho_a)^A{}_B \wedge \Psi^B, \tag{3.3}$$

where A^a denotes the gauge field potential one-form. The corresponding gauge field strength reads

$$F^a := dA^a - \tfrac{1}{2} f^a{}_{bc} A^b \wedge A^c. \tag{3.4}$$

The latter, as usual, may be derived from the commutator of the covariant derivatives, $DD\Psi^A = -F^a(\rho_a)^A{}_B\Psi^B$. For completeness, let us recall the transformation laws of the potential and the field strength:

$$\delta_\varepsilon A^a = -d\varepsilon^a + f^a{}_{bc} A^b \varepsilon^c, \tag{3.5}$$

$$\delta_\varepsilon F^a = -\varepsilon^b f^a{}_{bc} F^c. \tag{3.6}$$

Now, let $V(\Psi^A, A^a, D\Psi^A, F^a)$ be a general Lagrangian n-form. We define

$$H_A := -\frac{\partial V}{\partial D\Psi^A}, \qquad E_A := \frac{\partial V}{\partial \Psi^A}, \tag{3.7}$$

$$H_a := -\frac{\partial V}{\partial F^a}, \qquad E_a := \frac{\partial V}{\partial A^a}. \tag{3.8}$$

Then a total variation of the Lagrangian is

$$\delta V = \delta\Psi^A \wedge \mathcal{F}_A + \delta A^a \wedge \mathcal{F}_a - d(\delta\Psi^A \wedge H_A + \delta\Psi^a \wedge H_a), \tag{3.9}$$

where we introduced the variational derivatives

$$\mathcal{F}_A := \frac{\delta V}{\delta \Psi^A} = (-1)^p DH_A + E_A, \qquad \mathcal{F}_a := \frac{\delta V}{\delta A^a} = -DH_a + E_a. \tag{3.10}$$

Here p denotes the rank (in the exterior sense) of the matter field Ψ^A. Assuming that the action is stationary for the arbitrary variations of the fields, from the above we derive the system of field equations

$$\mathcal{F}_A = 0, \qquad \mathcal{F}_a = 0. \tag{3.11}$$

We assume that the action of the theory is invariant under diffeomorphism and gauge transformations. The total *infinitesimal symmetry variation* of the dynamical fields then consists of two terms:

$$\delta\Psi^A = \varsigma\mathcal{L}_{\{\xi,\varepsilon\}}\Psi^A := \ell_{(\varsigma\xi)}\Psi^A + \delta_{(\varsigma\varepsilon)}\Psi^A, \tag{3.12}$$

$$\delta A^a = \varsigma\mathcal{L}_{\{\xi,\varepsilon\}}A^a := \ell_{(\varsigma\xi)}A^a + \delta_{(\varsigma\varepsilon)}A^a. \tag{3.13}$$

The first terms on the right-hand sides come from a diffeomorphism generated by a vector field ξ, and ℓ_ξ is the Lie derivative along that field.

Putting $\xi = 0$ and assuming ε^a completely arbitrary, we straightforwardly derive from (3.9) the Noether identities for the gauge symmetry:

$$E_a \equiv (\rho_a)^A{}_B\Psi^B \wedge H_B - f^b{}_{ca}A^c \wedge H_b, \tag{3.14}$$

$$dE_a - (\rho_a)^A{}_B\Psi^B \wedge \mathcal{F}_A + f^b{}_{ca}A^c \wedge \mathcal{F}_b \equiv 0. \tag{3.15}$$

Analogously, putting $\varepsilon^a = 0$ and assuming an arbitrary vector field ξ, we find from (3.9) the Noether identities for the diffeomorphism symmetry:

$$e_\alpha\rfloor V + e_\alpha\rfloor d\Psi^A \wedge H_A + e_\alpha\rfloor dA^a \wedge H_a \equiv e_\alpha\rfloor\Psi^A \wedge E_A + e_\alpha\rfloor A^a \wedge E_a, \tag{3.16}$$

$$e_\alpha\rfloor d\Psi^A \wedge \mathcal{F}_A + (-1)^p e_\alpha\rfloor\Psi^A \wedge d\mathcal{F}_A + e_\alpha\rfloor dA^a \wedge \mathcal{F}_a - e_\alpha\rfloor A^a \wedge d\mathcal{F}_a \equiv 0. \tag{3.17}$$

The last identity is equivalent to

$$\ell_{e_\alpha}\Psi^A \wedge \mathcal{F}_A + \ell_{e_\alpha}A^a \wedge \mathcal{F}_a \equiv d(e_\alpha\rfloor\Psi^A \wedge \mathcal{F}_A + e_\alpha\rfloor A^a \wedge \mathcal{F}_a). \tag{3.18}$$

After these preliminaries, we are now in a position to derive the generalized conserved current associated with any vector field ξ. The condition of the invariance of the theory under a general variation (3.12) follows directly from (3.9) and reads

$$d(\xi\rfloor V) = (\mathcal{L}_{\{\xi,\varepsilon\}}\Psi^A) \wedge \mathcal{F}_A + (\mathcal{L}_{\{\xi,\varepsilon\}}A^a) \wedge \mathcal{F}_a - d[(\mathcal{L}_{\{\xi,\varepsilon\}}\Psi^A) \wedge H_A + (\mathcal{L}_{\{\xi,\varepsilon\}}A^a) \wedge H_a]. \tag{3.19}$$

Introducing the current $(n-1)$-form

$$J[\xi, \varepsilon] := \xi\rfloor V + (\mathcal{L}_{\{\xi,\varepsilon\}}\Psi^A) \wedge H_A + (\mathcal{L}_{\{\xi,\varepsilon\}}A^a) \wedge H_a, \tag{3.20}$$

we see from (3.19) that

$$dJ[\xi, \varepsilon] = (\mathcal{L}_{\{\xi,\varepsilon\}}\Psi^A) \wedge \mathcal{F}_A + (\mathcal{L}_{\{\xi,\varepsilon\}}A^a) \wedge \mathcal{F}_a. \tag{3.21}$$

Hence, this current is conserved, $dJ[\xi, \varepsilon] = 0$, for any ξ and ε^a, when the field Eqs. (3.11) are satisfied.

Using (3.12) and (3.20), and the Noether identities of the diffeomorphism symmetry (3.16), (3.17), and (3.18), we rewrite the current (3.20) as

$$J[\xi, \varepsilon] = d\Pi[\xi, \varepsilon] + \xi\rfloor\Psi^A \wedge \mathcal{F}_A + \Xi^a[\xi, \varepsilon] \wedge \mathcal{F}_a. \tag{3.22}$$

Here we introduced the superpotential $(n-2)$-form

$$\Pi[\xi, \varepsilon] := \xi\rfloor\Psi^A \wedge H_A + \Xi^a[\xi, \varepsilon] \wedge H_a \tag{3.23}$$

and defined [cf. the general definition (2.10)]

$$\Xi^a[\xi, \varepsilon] := \xi\rfloor A^a - \varepsilon^a. \tag{3.24}$$

Accordingly, on the solutions of the field Eqs. (3.11), the conserved charge is computed as an integral over an $(n-2)$-boundary:

$$Q[\xi, \varepsilon] = \int_S J[\xi, \varepsilon] = \int_{\partial S}\Pi[\xi, \varepsilon]. \tag{3.25}$$

As before, the functions ε^a parametrize a family of conserved currents and charges associated with a vector field ξ. These conserved quantities are not scalars, in general. They are invariant under the diffeomorphisms, but they become invariant under local gauge transformations only for certain special choices of the parameters $\varepsilon^a(\xi)$. One choice that is always possible is to take a *nondynamical* (or background) gauge field $\bar{A}^a$ and define $\varepsilon^a = \xi\rfloor\bar{A}^a$. Then $\Xi^a = \xi\rfloor(A^a - \bar{A}^a)$ is obviously a gauge-covariant quantity, and consequently the conserved current and charge are true scalars.

Other choices are also possible, in general, with a covariant Ξ^a constructed from the dynamical fields available in a particular model. The simplest is a "natural" choice with $\varepsilon^a = \xi\rfloor A^a$ but then $\Xi^a = 0$, and the corresponding contribution to the current is trivial. As another example, recall [1] that we have demonstrated how one can use the coframe field in order to define the so-called Yano derivative which leads to the invariant conserved currents. A similar construction is outlined in the next section for the Abelian gauge field model.

IV. ABELIAN MODEL: DEFINING A COVARIANT Ξ

Let us consider an Abelian model with the local gauge symmetry group $G = U(1)$. Then, besides the choice $\varepsilon = \xi\rfloor\bar{A}$, a nontrivial field $\varepsilon(\xi)$ can be defined, provided a $U(1)$-covariant scalar field ϕ is available. We assume that under a $U(1)$ transformation with parameter λ (with $\lambda \in [0, 2\pi]$), the field ϕ transforms as $\phi' = e^{i\lambda}\phi$. Using ϕ, we can construct $\varepsilon(\xi)$ from the assumption that

$$\mathcal{L}_{\xi,\varepsilon}\phi = 0. \tag{4.1}$$

A 0-form ϕ (a complex scalar field) usually plays the role of a Higgs field in such models. The condition (4.1) reads explicitly

$$\ell_\xi\phi + i\varepsilon\phi = 0 \tag{4.2}$$

By multiplying this by $\phi^\dagger$, we solve (4.2) for ε. This yields

$$\varepsilon = i\frac{\phi^\dagger \ell_\xi \phi}{\phi^\dagger \phi}. \tag{4.3}$$

The solution ε is real, for every ξ, when $\phi^\dagger\phi = \text{const}$. Then we can always choose the normalization such that $\phi^\dagger\phi = 1$, and recast (4.3) as

$$\begin{aligned}\varepsilon &= \frac{i}{2}[\phi^\dagger(\ell_\xi\phi) - (\ell_\xi\phi^\dagger)\phi]\\ &= \frac{i}{2}\xi\rfloor[\phi^\dagger(d\phi) - (d\phi^\dagger)\phi].\end{aligned} \tag{4.4}$$

It is interesting to notice that the right-hand side of (4.4) is proportional to the $U(1)$ current of a free complex scalar field ϕ.

A. Complex scalar field on a background spacetime

Consider now the model with a complex scalar field ψ coupled to the electromagnetic field A and the gravitational field. The latter can be treated as a curved background, since at the moment we are concerned primarily with the aspects of the local $U(1)$ invariance. The covariant derivative is given, as usual, by $D\psi = d\psi + iA\psi$. Let us take the total Lagrangian $V^{\rm tot} = V^{\rm tot}(\psi, D\psi, dA)$, where we have written explicitly only the dynamical fields ψ, A, whereas a fixed metric g is assumed. Since ψ is a 0-form, the superpotential (3.23) reduces to

$$\Pi[\xi, \varepsilon] = (\xi\rfloor A - \varepsilon)H, \qquad H = -\frac{\partial V}{\partial F}. \tag{4.5}$$

Clearly, in this case the natural choice $\varepsilon = \xi\rfloor A$ leads to trivial conserved quantities.

However, we can use the dynamical (Higgs-type) field ψ to construct a nontrivial conserved quantity using (4.4). If we take

$$\phi = \frac{\psi}{\sqrt{\psi^\dagger\psi}}, \tag{4.6}$$

then $\phi^\dagger\phi = 1$ and we obtain

$$\varepsilon = \frac{i}{2}\xi\rfloor\left[\frac{\psi^\dagger(d\psi) - (d\psi^\dagger)\psi}{\psi^\dagger\psi}\right] = \frac{m}{e}\frac{1}{\psi^\dagger\psi}\xi\rfloor j_{\rm free}, \tag{4.7}$$

where $j_{\rm free} = \frac{ie}{2m}[\psi^\dagger(d\psi) - (d\psi^\dagger)\psi]$ is the electric current density one-form of a free complex scalar field ($\hbar = c = 1$). Using (4.7), we find

$$\Xi = \xi\rfloor A - \varepsilon = -\frac{m}{e}\frac{1}{\psi^\dagger\psi}\xi\rfloor j, \tag{4.8}$$

where now $j = \frac{ie}{2m}[\psi^\dagger(D\psi) - (D\psi^\dagger)\psi]$ is the invariant current of the field ψ interacting with the electromagnetic field. Thus, we obtain finally

$$\Pi[\xi] = -\frac{m}{e}\frac{1}{\psi^\dagger\psi}(\xi\rfloor j)H. \tag{4.9}$$

The corresponding conserved quantity $Q[\xi] = \int_{\partial S}\Pi[\xi]$ is gauge invariant and a scalar under general coordinate transformations.

V. INVARIANT CONSERVED CURRENTS FOR METRIC-AFFINE GRAVITY

The geometry of MAG is described by the *curvature* two-form $R_\alpha{}^\beta$, the *nonmetricity* one-form $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, and the *torsion* two-form $T^\alpha := D\vartheta^\alpha$ which are the gravitational field strengths for the linear connection $\Gamma_\alpha{}^\beta$, metric $g_{\alpha\beta}$, and coframe ϑ^α, respectively. The corresponding physical sources are the three-forms of canonical energy-momentum Σ_α and hypermomentum $\Delta_\beta{}^\alpha$. The latter includes the dilation, shear, and spin currents associated to matter. The field equations and the formalism are comprehensively described in [21,22].

The MAG theory is invariant under diffeomorphisms and the local general linear group $G = GL(n, R)$ that acts on the geometric objects (that describe the gravitational field) as

$$\begin{aligned}\delta\vartheta^\alpha = \varepsilon^\alpha{}_\beta\vartheta^\beta, \qquad \delta\Gamma_\beta{}^\alpha = -D\varepsilon^\alpha{}_\beta,\\ \delta g_{\alpha\beta} = -\varepsilon^\gamma{}_\alpha g_{\gamma\beta} - \varepsilon^\gamma{}_\beta g_{\alpha\gamma},\end{aligned} \tag{5.1}$$

and on the matter fields ψ^A as

$$\delta\psi^A = \varepsilon^\alpha{}_\beta(\rho^\beta{}_\alpha)^A_B\psi^B. \tag{5.2}$$

Here the elements of the matrix $\varepsilon^\alpha{}_\beta(x)$ are the n^2 arbitrary local parameters, and $\rho^\beta{}_\alpha$ are the generators of the general linear group in a corresponding representation.

A. Lagrangian formalism

We assume that the total Lagrangian n-form $V^{\rm tot}(g_{\alpha\beta}, dg_{\alpha\beta}, \vartheta^\alpha, d\vartheta^\alpha, \Gamma_\alpha{}^\beta, d\Gamma_\alpha{}^\beta, \psi^A, d\psi^A)$ is *invariant* under local transformations (5.1). Then one can verify [21] that it always has the form

$$V^{\rm tot} = V^{\rm tot}(g_{\alpha\beta}, Q_{\alpha\beta}, \vartheta^\alpha, T^\alpha, R_\alpha{}^\beta, \psi, D\psi), \tag{5.3}$$

where $D\psi^A$ denotes the covariant exterior derivative of the matter field ψ^A. In accordance with the general scheme, we denote

$$\begin{aligned}\mathcal{H}_\alpha := -\frac{\partial V^{\rm tot}}{\partial T^\alpha}, \qquad \mathcal{H}^\alpha{}_\beta := -\frac{\partial V^{\rm tot}}{\partial R_\alpha{}^\beta},\\ \mathcal{M}^{\alpha\beta} := -2\frac{\partial V^{\rm tot}}{\partial Q_{\alpha\beta}}.\end{aligned} \tag{5.4}$$

Furthermore, we also introduce

$$\mathcal{E}_\alpha := \frac{\partial V^{\rm tot}}{\partial\vartheta^\alpha}, \qquad \mathcal{E}^\alpha{}_\beta := \frac{\partial V^{\rm tot}}{\partial\Gamma_\alpha{}^\beta}, \qquad \mu^{\alpha\beta} := 2\frac{\partial V^{\rm tot}}{\partial g_{\alpha\beta}}. \tag{5.5}$$

Then a general variation of the total Lagrangian reads

$$\delta V^{\rm tot} = \delta\vartheta^\alpha \wedge \mathcal{F}_\alpha + \delta\Gamma_\alpha{}^\beta \wedge \mathcal{F}^\alpha{}_\beta + \frac{1}{2}\delta g_{\alpha\beta} f^{\alpha\beta} + \delta\psi^A \wedge \mathcal{F}_A + d\Big(-\delta\vartheta^\alpha \wedge \mathcal{H}_\alpha - \delta\Gamma_\alpha{}^\beta \wedge \mathcal{H}^\alpha{}_\beta + \frac{1}{2}\delta g_{\alpha\beta}\mathcal{M}^{\alpha\beta} + \delta\psi^A \wedge \frac{\partial V^{\rm tot}}{\partial D\psi^A}\Big), \tag{5.6}$$

where we have defined the variational derivatives with respect to the gravitational potentials:

$$\mathcal{F}_\alpha := \frac{\delta V^{\rm tot}}{\delta\vartheta^\alpha} = -D\mathcal{H}_\alpha + \mathcal{E}_\alpha, \tag{5.7}$$

$$\mathcal{F}^\alpha{}_\beta := \frac{\delta V^{\rm tot}}{\delta\Gamma_\alpha{}^\beta} = -D\mathcal{H}^\alpha{}_\beta + \mathcal{E}^\alpha{}_\beta, \tag{5.8}$$

$$f^{\alpha\beta} := 2\frac{\delta V^{\rm tot}}{\delta g_{\alpha\beta}} = -D\mathcal{M}^{\alpha\beta} + \mu^{\alpha\beta}, \tag{5.9}$$

$$\mathcal{F}_A := \frac{\delta V^{\rm tot}}{\delta\psi^A} = \frac{\partial V^{\rm tot}}{\partial\psi^A} - (-1)^p\frac{\partial V^{\rm tot}}{\partial D\psi^A}. \tag{5.10}$$

When the action is demanded to be stationary with respect to arbitrary variations of the variables, we find from (5.6) the system of *field equations*:

$$\mathcal{F}_\alpha = 0, \qquad \mathcal{F}^\alpha{}_\beta = 0, \qquad f^{\alpha\beta} = 0, \qquad \mathcal{F}_A = 0. \tag{5.11}$$

B. Noether identities for the local $GL(n, R)$ and diffeomorphisms

Substituting (5.1) and (5.2) into (5.6), one derives the *Noether identities for the local linear symmetry*:

$$\vartheta^\alpha \wedge \mathcal{F}_\beta + D\mathcal{F}^\alpha{}_\beta - f^\alpha{}_\beta + (\rho^\alpha{}_\beta)^A_B\psi^B \wedge \mathcal{F}_A \equiv 0, \tag{5.12}$$

$$\mathcal{E}^\alpha{}_\beta + \vartheta^\alpha \wedge \mathcal{H}_\beta + \mathcal{M}^\alpha{}_\beta - (\rho^\alpha{}_\beta)^A_B\psi^B \wedge \frac{\partial V^{\rm tot}}{\partial D\psi^A} \equiv 0. \tag{5.13}$$

The second identity can be used as a tool for the practical calculation of $\mathcal{E}^\alpha{}_\beta$ (in order to circumvent a difficult direct computation). On the other hand, the identity (5.12) shows that the so-called 0th field equation of MAG, Eq. (5.9), is a consequence of the 1st and the 2nd field equations, (5.7) and (5.8) and of the equation of motion of matter (5.10).

For a *diffeomorphism* generated by an arbitrary vector field ξ, we find:

$$\ell_\xi V^{\rm tot} = \frac{1}{2}(\ell_\xi g_{\alpha\beta})\mu^{\alpha\beta} - \frac{1}{2}(\ell_\xi Q_{\alpha\beta}) \wedge \mathcal{M}^{\alpha\beta} + (\ell_\xi\vartheta^\alpha) \wedge \mathcal{E}_\alpha - (\ell_\xi T^\alpha) \wedge \mathcal{H}_\alpha - (\ell_\xi R_\alpha{}^\beta) \wedge \mathcal{H}^\alpha{}_\beta + (\ell_\xi\psi^A) \wedge \frac{\partial V^{\rm tot}}{\partial\psi^A} + (\ell_\xi D\psi^A) \wedge \frac{\partial V^{\rm tot}}{\partial D\psi^A}. \tag{5.14}$$

This equation does not look invariant under the action of the local $GL(n, R)$. However, if we make an actual linear transformation of all the fields in (5.14) by substituting (5.1) and (5.2), we find that the right-hand side is changed by

$$(\xi\rfloor d\varepsilon^\beta{}_\alpha)\Big[\vartheta^\alpha \wedge \mathcal{F}_\beta + D\mathcal{F}^\alpha{}_\beta - f^\alpha{}_\beta + (\rho^\alpha{}_\beta)^A_B\psi^B \wedge \mathcal{F}_A - D\Big(\mathcal{E}^\alpha{}_\beta + \vartheta^\alpha \wedge \mathcal{H}_\beta + \mathcal{M}^\alpha{}_\beta - (\rho^\alpha{}_\beta)^A_B\psi^B \wedge \frac{\partial V^{\rm tot}}{\partial D\psi^A}\Big)\Big]. \tag{5.15}$$

This is zero in view of the Noether identities (5.12) and (5.13).

C. Generalized Lie derivatives and covariant Noether identities

As a result, we can proceed like in the previous paper [1], and take an arbitrary $gl(n, R)$-valued 0-form $B_\alpha{}^\beta(\xi)$ and add to (5.14) a *zero term*,

$$B_\alpha{}^\beta\Big[\vartheta^\alpha \wedge \mathcal{F}_\beta + D\mathcal{F}^\alpha{}_\beta - f^\alpha{}_\beta + (\rho^\alpha{}_\beta)^A_B\psi^B \wedge \mathcal{F}_A - D\Big(\mathcal{E}^\alpha{}_\beta + \vartheta^\alpha \wedge \mathcal{H}_\beta + \mathcal{M}^\alpha{}_\beta - (\rho^\alpha{}_\beta)^A_B\psi^B \wedge \frac{\partial V^{\rm tot}}{\partial D\psi^A}\Big)\Big]. \tag{5.16}$$

As is easily verified, this addition is *equivalent* to the replacement of the usual Lie derivative ℓ_ξ with a *generalized Lie derivative* $L_\xi := \ell_\xi + B_\beta{}^\alpha\rho^\beta{}_\alpha$ when applied to all geometric and matter fields. This generalized derivative will be *covariant* provided $B_\alpha{}^\beta$ transforms according to

$$B'_\alpha = (L^{-1})^\rho{}_\alpha B_\rho{}^\gamma L^\beta{}_\gamma - (L^{-1})^\gamma{}_\alpha(\xi\rfloor dL^\beta{}_\gamma), \tag{5.17}$$

when the coframe is changed as $\vartheta^\alpha \to \vartheta'^\alpha = L^\alpha{}_\beta\vartheta^\beta$, with $L^\alpha{}_\beta \in GL(n, R)$.

There are three convenient choices: (i) $B_\beta{}^\alpha = \xi\rfloor\Gamma_\beta{}^\alpha$ with the dynamical linear connection $\Gamma_\beta{}^\alpha$, (ii) $B_\beta{}^\alpha = \xi\rfloor\overset{\{\}}{\Gamma}{}_\beta{}^\alpha$ with the Riemannian connection $\overset{\{\}}{\Gamma}{}_\beta{}^\alpha$, and (iii) the Yano choice $B_\beta{}^\alpha = -e_\beta\rfloor\ell_\xi\vartheta^\alpha$. Each of these options give rise to the covariant Lie derivatives:

$$\text{Ł}_\xi := \ell_\xi + \xi\rfloor\Gamma_\beta{}^\alpha\rho^\beta{}_\alpha = \xi\rfloor D + D\xi\rfloor, \tag{5.18}$$

$$\overset{\{\}}{Ł}_{\xi} := \ell_{\xi} + \xi\rfloor\overset{\{\}}{\Gamma}_{\beta}{}^{\alpha}\rho^{\beta}{}_{\alpha} = \xi\rfloor\overset{\{\}}{D} + \overset{\{\}}{D}\xi\rfloor, \tag{5.19}$$

$$\mathcal{L}_{\xi} := \ell_{\xi} - \Theta_{\beta}{}^{\alpha}\rho^{\beta}{}_{\alpha}, \quad \text{with} \quad \Theta_{\alpha}{}^{\beta} := e_{\alpha}\rfloor\ell_{\xi}\vartheta^{\beta}. \tag{5.20}$$

We are free to use in (5.14) any generalized (covariant) Lie derivative instead of the usual (noncovariant) ℓ_{ξ}. One can straightforwardly demonstrate that the choice (5.18) yields the following Noether identities:

$$D\mathcal{F}_{\alpha} \equiv e_{\alpha}\rfloor T^{\beta}\wedge\mathcal{F}_{\beta} + e_{\alpha}\rfloor R_{\gamma}{}^{\beta}\wedge\mathcal{F}^{\gamma}{}_{\beta} - \tfrac{1}{2}(e_{\alpha}\rfloor Q_{\beta\gamma})f^{\beta\gamma} + (e_{\alpha}\rfloor D\psi^{A})\wedge\mathcal{F}_{A} + (-1)^{p}(e_{\alpha}\rfloor\psi^{A})\wedge D\mathcal{F}_{A}, \tag{5.21}$$

$$\mathcal{E}_{\alpha} \equiv e_{\alpha}\rfloor V^{\rm tot} + e_{\alpha}\rfloor T^{\beta}\wedge\mathcal{H}_{\beta} + e_{\alpha}\rfloor R_{\gamma}{}^{\beta}\wedge\mathcal{H}^{\gamma}{}_{\beta} + \frac{1}{2}(e_{\alpha}\rfloor Q_{\beta\gamma})\mathcal{M}^{\beta\gamma} - (e_{\alpha}\rfloor D\psi^{A})\wedge\frac{\partial V^{\rm tot}}{\partial D\psi^{A}} - (e_{\alpha}\rfloor\psi^{A})\wedge\frac{\partial V^{\rm tot}}{\partial\psi^{A}}. \tag{5.22}$$

For the second choice (5.19), in accordance with the above general analysis, see (5.16), a zero term is added that is proportional to

$$N_{\alpha}{}^{\beta} := \overset{\{\}}{\Gamma}_{\alpha}{}^{\beta} - \Gamma_{\alpha}{}^{\beta}. \tag{5.23}$$

This quantity is known as *distortion* one-form. In particular, the torsion is recovered from it as $T^{\alpha} = -N_{\beta}{}^{\alpha}\wedge\vartheta^{\beta}$, whereas the nonmetricity arises as $Q_{\alpha\beta} = -2N_{(\alpha\beta)}$. It is straightforward to verify the explicit formula

$$N_{\alpha\beta} = -\tfrac{1}{2}Q_{\alpha\beta} + e_{[\alpha}\rfloor n_{\beta]}, \tag{5.24}$$

$$n_{\beta} := 2T_{\beta} - Q_{\beta\gamma}\wedge\vartheta^{\gamma} - \tfrac{1}{2}e_{\beta}\rfloor(T_{\gamma}\wedge\vartheta^{\gamma}). \tag{5.25}$$

As we see, the symmetric part of the distortion is determined by the nonmetricity, while the skew-symmetric part is constructed from the two-form n_{α}. The latter has the property

$$n_{\alpha}\wedge\vartheta^{\alpha} = \tfrac{1}{2}T_{\alpha}\wedge\vartheta^{\alpha}. \tag{5.26}$$

The corresponding (total and Riemannian) curvature two-forms are related via

$$R_{\alpha}{}^{\beta} = \overset{\{\}}{R}_{\alpha}{}^{\beta} - \overset{\{\}}{D}N_{\alpha}{}^{\beta} + N_{\gamma}{}^{\beta}\wedge N_{\alpha}{}^{\gamma}. \tag{5.27}$$

Using these facts, we can verify that the diffeomorphism Noether identity can be recast in the alternative form

$$\overset{\{\}}{D}(\mathcal{F}_{\alpha} - \mathcal{F}^{\gamma}{}_{\beta}e_{\alpha}\rfloor N_{\gamma}{}^{\beta}) \equiv (e_{\alpha}\rfloor\overset{\{\}}{R}_{\gamma}{}^{\beta} - \overset{\{\}}{Ł}_{\alpha}N_{\gamma}{}^{\beta})\wedge\mathcal{F}^{\gamma}{}_{\beta} + (e_{\alpha}\rfloor\overset{\{\}}{D}\psi^{A})\wedge\mathcal{F}_{A} + (-1)^{p}(e_{\alpha}\rfloor\psi^{A})\wedge\overset{\{\}}{D}\mathcal{F}_{A}. \tag{5.28}$$

The identities (5.21) and (5.28) are equivalent, but in contrast to the usual (5.21), the alternative form (5.28) is less known. The Lorentz-covariant version of this identity was derived previously in [23] using a different method, see also [1]. Although actually (5.28) was used in [24] for the analysis of the dynamics of test matter in MAG, the derivation of this identity is published here for the first time.

D. Yano derivative and the invariant conserved current for MAG

Directly from the definition (5.20), we can calculate the Yano derivative for the covariant derivative of the matter field, of the torsion, nonmetricity and curvature:

$$\mathcal{L}_{\xi}D\psi^{A} = D(\mathcal{L}_{\xi}\psi^{A}) + (\mathcal{L}_{\xi}\Gamma_{\beta}{}^{\alpha})\wedge(\rho^{\beta}{}_{\alpha})^{A}_{B}\psi^{B}, \tag{5.29}$$

$$\mathcal{L}_{\xi}T^{\alpha} = D(\mathcal{L}_{\xi}\vartheta^{\alpha}) + (\mathcal{L}_{\xi}\Gamma_{\beta}{}^{\alpha})\wedge\vartheta^{\beta}, \tag{5.30}$$

$$\mathcal{L}_{\xi}Q_{\alpha\beta} = -D(\mathcal{L}_{\xi}g_{\alpha\beta}) + (\mathcal{L}_{\xi}\Gamma_{\alpha}{}^{\gamma})g_{\gamma\beta} + (\mathcal{L}_{\xi}\Gamma_{\beta}{}^{\gamma})g_{\alpha\gamma}, \tag{5.31}$$

$$\mathcal{L}_{\xi}R_{\alpha}{}^{\beta} = D(\mathcal{L}_{\xi}\Gamma_{\alpha}{}^{\beta}). \tag{5.32}$$

Furthermore, it is straightforward to find the explicit expressions for the Yano derivatives of the geometric and matter fields:

$$\mathcal{L}_{\xi}\vartheta^{\alpha} = D\xi^{\alpha} + \xi\rfloor T^{\alpha} - \Xi_{\beta}{}^{\alpha}\vartheta^{\beta} \equiv 0, \tag{5.33}$$

$$\mathcal{L}_{\xi}\Gamma_{\alpha}{}^{\beta} = D\Xi_{\alpha}{}^{\beta} + \xi\rfloor R_{\alpha}{}^{\beta}, \tag{5.34}$$

$$\mathcal{L}_{\xi}g_{\alpha\beta} = -\xi\rfloor Q_{\alpha\beta} + 2\Xi_{(\alpha\beta)}, \tag{5.35}$$

$$\mathcal{L}_{\xi}\psi^{A} = D(\xi\rfloor\psi^{A}) + \xi\rfloor D\psi^{A} - \Xi_{\alpha}{}^{\beta}(\rho^{\alpha}{}_{\beta})^{A}_{B}\psi^{B}. \tag{5.36}$$

Here we denote, as usual, $\xi^{\alpha} := \xi\rfloor\vartheta^{\alpha}$, and

$$\Xi_{\alpha}{}^{\beta} := \xi\rfloor\Gamma_{\alpha}{}^{\beta} + \Theta_{\alpha}{}^{\beta}. \tag{5.37}$$

This is the MAG version of the general definition (2.10).

Now, we replace in (5.14) the ordinary Lie derivatives ℓ_{ξ} with the Yano derivatives $\mathcal{L}_{\xi}$, make use of (5.29), (5.30), (5.31), and (5.32), and take into account the Noether identity (5.13). Then we can recast the identity (5.14) as

$$\mathcal{L}_{\xi}\Gamma_{\alpha}{}^{\beta}\wedge\mathcal{F}^{\alpha}{}_{\beta} + \tfrac{1}{2}\mathcal{L}_{\xi}g_{\alpha\beta}\wedge f^{\alpha\beta} + \mathcal{L}_{\xi}\psi^{A}\wedge\mathcal{F}_{A} - d\mathcal{J}[\xi] = 0, \tag{5.38}$$

where we introduced the scalar $(n-1)$-form

$$\mathcal{J}[\xi] := \xi\rfloor V^{\rm tot} + \mathcal{L}_\xi\Gamma_\alpha{}^\beta \wedge \mathcal{H}^\alpha{}_\beta - \frac{1}{2}\mathcal{L}_\xi g_{\alpha\beta} \wedge \mathcal{M}^{\alpha\beta} - \mathcal{L}_\xi\psi^A \wedge \frac{\partial V^{\rm tot}}{\partial D\psi^A}. \tag{5.39}$$

Note that neither (5.38) nor (5.39) contain the terms proportional to the Yano derivative of the coframe (cf. with [1]) because the latter is zero, cf. (5.33). By making use of the Yano derivative (5.33), (5.34), (5.35), and (5.36), and taking into account the Noether identities (5.13) and (5.22), we recast this current into the equivalent form

$$\mathcal{J}[\xi] = d\left(\xi^\alpha\mathcal{H}_\alpha + \Xi_\alpha{}^\beta\mathcal{H}^\alpha{}_\beta - \xi\rfloor\psi^A \wedge \frac{\partial V^{\rm tot}}{\partial D\psi^A}\right) + \xi^\alpha\mathcal{F}_\alpha + \Xi_\alpha{}^\beta\mathcal{F}^\alpha{}_\beta + \xi\rfloor\psi^A \wedge \mathcal{F}_A. \tag{5.40}$$

When the gravitational and matter variables satisfy the field Eqs. (5.11), we find that the current (5.39) is conserved, $d\mathcal{J}[\xi] = 0$, for any vector field ξ. By construction, this conserved current is invariant under both diffeomorphism and local linear transformations. Moreover, when (5.11) are satisfied, the current $(n-1)$-form is expressed in terms of a superpotential $(n-2)$-form, (5.40). The corresponding conserved current then can be calculated via the integral

$$\mathcal{Q}[\xi] = \int_{\partial S}\left(\xi^\alpha\mathcal{H}_\alpha + \Xi_\alpha{}^\beta\mathcal{H}^\alpha{}_\beta - \xi\rfloor\psi^A \wedge \frac{\partial V^{\rm tot}}{\partial D\psi^A}\right) \tag{5.41}$$

over an $(n-2)$-dimensional boundary ∂S of an $(n-1)$-hypersurface S.

VI. EXAMPLES OF MAG SOLUTIONS

Let us now apply the general formalism to the exact solutions that can be obtained with the help of the so-called triplet ansatz in the class of models with quadratic Lagrangians in $n = 4$ dimensions. Such solutions are described, for instance, in [25–30]. The most general scheme for the triplet ansatz technique was developed in [26], and the overview of the exact solutions in MAG models can be found in [31].

A. Gravitational field equations

We consider the Lagrangian [26] that generalizes the models studied in [25,28,30,32,33],

$$\begin{aligned} V_{\rm MAG} = \frac{1}{2\kappa}\Big[& -a_0(R^{\alpha\beta}\wedge\eta_{\alpha\beta} - 2\lambda\eta) + T^\alpha \wedge {}^*\left(\sum_{I=1}^{3} a_I{}^{(I)}T_\alpha\right) + 2\left(\sum_{I=2}^{4} c_I{}^{(I)}Q_{\alpha\beta}\right) \\ & \wedge \vartheta^\alpha \wedge {}^*T^\beta + Q_{\alpha\beta} \wedge {}^*\left(\sum_{I=1}^{4} b_I{}^{(I)}Q^{\alpha\beta}\right) \\ & + b_5({}^{(3)}Q_{\alpha\gamma}\wedge\vartheta^\alpha)\wedge{}^*({}^{(4)}Q^{\beta\gamma}\wedge\vartheta_\beta)\Big] \\ & - \frac{1}{2}z_4 R^{\alpha\beta}\wedge{}^{*(4)}Z_{\alpha\beta}. \end{aligned} \tag{6.1}$$

Here, the coupling constants $a_0, \ldots, a_3, c_2, c_3, c_4, b_1, \ldots, b_5, z_4$ are dimensionless, $\kappa := 8\pi G/c^3$ is the standard Einstein gravitational constant, and λ is the cosmological constant. The segmental curvature is denoted by ${}^{(4)}Z_{\alpha\beta} := \frac{1}{4}g_{\alpha\beta}R_\gamma{}^\gamma$; it is a purely *post*-Riemannian piece.

This Lagrangian is constructed from the irreducible parts of the torsion and nonmetricity. Namely, let us recall that the torsion two-form can be decomposed into three irreducible pieces, $T^\alpha = {}^{(1)}T^\alpha + {}^{(2)}T^\alpha + {}^{(3)}T^\alpha$, where

$${}^{(2)}T^\alpha := \tfrac{1}{3}\vartheta^\alpha \wedge T, \tag{6.2}$$

$${}^{(3)}T^\alpha := -\tfrac{1}{3}{}^*(\vartheta^\alpha \wedge P), \tag{6.3}$$

$${}^{(1)}T^\alpha := T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha. \tag{6.4}$$

The torsion trace (covector) and pseudotrace (axial covector) one-forms are defined, respectively, by

$$T := e_\alpha\rfloor T^\alpha, \qquad P := {}^*(T^\alpha \wedge \vartheta_\alpha). \tag{6.5}$$

Analogously, the nonmetricity one-form can be decomposed into four irreducible pieces, $Q_{\alpha\beta} = {}^{(1)}Q_{\alpha\beta} + {}^{(2)}Q_{\alpha\beta} + {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta}$, with

$${}^{(2)}Q_{\alpha\beta} := \frac{2}{3}{}^*[\vartheta_{(\alpha} \wedge \Omega_{\beta)}], \tag{6.6}$$

$${}^{(3)}Q_{\alpha\beta} := \tfrac{4}{9}[\vartheta_{(\alpha}e_{\beta)}\rfloor\Lambda - \tfrac{1}{4}g_{\alpha\beta}\Lambda], \tag{6.7}$$

$${}^{(4)}Q_{\alpha\beta} := g_{\alpha\beta}Q, \tag{6.8}$$

$${}^{(1)}Q_{\alpha\beta} := Q_{\alpha\beta} - {}^{(2)}Q_{\alpha\beta} - {}^{(3)}Q_{\alpha\beta} - {}^{(4)}Q_{\alpha\beta}. \tag{6.9}$$

Here the shear covector part and the Weyl covector are, respectively,

$$\Lambda := \vartheta^\alpha e^\beta\rfloor \not{Q}_{\alpha\beta}, \qquad Q := \tfrac{1}{4}g^{\alpha\beta}Q_{\alpha\beta}, \tag{6.10}$$

where $\not{Q}_{\alpha\beta} = Q_{\alpha\beta} - Qg_{\alpha\beta}$ is the traceless piece of the nonmetricity. The two-form Ω^α is defined by $\Omega_\alpha := \Theta_\alpha - \frac{1}{3}e_\alpha\rfloor(\vartheta^\beta \wedge \Theta_\beta)$ with $\Theta_\alpha := {}^*(\not{Q}_{\alpha\beta} \wedge \vartheta^\beta)$. We can prove that $e_\alpha\rfloor\Omega^\alpha = 0$ and $\vartheta_\alpha \wedge \Omega^\alpha = 0$.

In order to write down the field Eqs. (5.7), (5.8), and (5.11), we need the field momenta (5.4) and the generalized potentials (5.5). A straightforward computation yields for the Lagrangian (6.1):

$$\mathcal{M}^{\alpha\beta} = -\frac{2}{\kappa}\Big[{}^*\Big(\sum_{I=1}^{4} b_I^{(I)}Q^{\alpha\beta}\Big) + \frac{1}{2}b_5\Big(\vartheta^{(\alpha}\wedge{}^*(Q\wedge\vartheta^{\beta)}) - \frac{1}{4}g^{\alpha\beta *}(3Q+\Lambda)\Big) + c_2\vartheta^{(\alpha}\wedge{}^{*(1)}T^{\beta)} + c_3\vartheta^{(\alpha}\wedge{}^{*(2)}T^{\beta)} + \frac{1}{4}(c_3-c_4)g^{\alpha\beta *}T\Big], \tag{6.11}$$

$$\mathcal{H}_\alpha = -\frac{1}{\kappa}{}^*\Big[\Big(\sum_{I=1}^{3} a_I{}^{(I)}T_\alpha\Big) + \Big(\sum_{I=2}^{4} c_I{}^{(I)}Q_{\alpha\beta}\wedge\vartheta^\beta\Big)\Big], \tag{6.12}$$

$$\mathcal{H}^\alpha{}_\beta = \frac{a_0}{2\kappa}\eta^\alpha{}_\beta + z_4{}^*({}^{(4)}Z^\alpha{}_\beta), \tag{6.13}$$

and, in accordance with the Noether identities (5.13) and (5.22),

$$\mathcal{E}_\alpha = e_\alpha\rfloor V_{\rm MAG} + (e_\alpha\rfloor T^\beta)\wedge\mathcal{H}_\beta + (e_\alpha\rfloor R_\beta{}^\gamma)\wedge\mathcal{H}^\beta{}_\gamma + \tfrac{1}{2}(e_\alpha\rfloor Q_{\beta\gamma})\mathcal{M}^{\beta\gamma}, \tag{6.14}$$

$$\mathcal{E}^\alpha{}_\beta = -\vartheta^\alpha\wedge\mathcal{H}_\beta - \mathcal{M}^\alpha{}_\beta. \tag{6.15}$$

As we already mentioned, the equation which arises from the variation of the Lagrangian with respect to the metric turns out to be redundant.

B. Triplet ansatz and effective system

The triplet ansatz specifies a particular structure for the post-Riemannian sector of the MAG model. Namely, it is assumed that this sector is totally described by the three covectors: the one-form of the torsion trace T, the Weyl one-form Q, and the nonmetricity one-form Λ, defined in (6.5) and (6.10). Moreover, they are all proportional to an auxiliary one-form:

$$Q = k_0 A_{\rm MAG}, \qquad \Lambda = k_1 A_{\rm MAG}, \qquad T = k_2 A_{\rm MAG}, \tag{6.16}$$

with constant coefficients k_0, k_1, k_2; the one-form $A_{\rm MAG}$ is a new variable to be determined from the field equations. Accordingly, ${}^{(1)}T^\alpha = {}^{(3)}T^\alpha = 0$ and ${}^{(1)}Q_{\alpha\beta} = {}^{(2)}Q_{\alpha\beta} = 0$.

Substituting the triplet ansatz into the MAG field equations, we find the explicit form of the above coefficients in terms of the coupling constants of the MAG Lagrangian:

$$k_0 = \Big(\frac{a_2}{2} - a_0\Big)(8b_3 + a_0) - 3(c_3 + a_0)^2, \tag{6.17}$$

$$k_1 = -9\Big[(a_0 - b_5)\Big(\frac{a_2}{2} - a_0\Big) + (c_3 + a_0)(c_4 + a_0)\Big], \tag{6.18}$$

$$k_2 = \tfrac{3}{2}[3(a_0 - b_5)(c_3 + a_0) + (8b_3 + a_0)(c_4 + a_0)], \tag{6.19}$$

and in addition,

$$-4k_0 b_4 + \frac{k_1}{2}b_3 + k_2 c_4 + \frac{a_0}{2}k = 0, \tag{6.20}$$

with $k := 3k_0 - k_1 + 2k_2$.

As a result, the MAG field Eqs. (5.7) and (5.8), for the Lagrangian (6.1) reduce to the effective Einstein-Maxwell system:

$$\tfrac{1}{2}\tilde{R}^{\mu\nu}\wedge\eta_{\alpha\mu\nu} - \lambda\eta_\alpha = \kappa\Sigma_\alpha^{\rm MAG}, \tag{6.21}$$

$$d^*F_{\rm MAG} = 0. \tag{6.22}$$

Here $F_{\rm MAG} = dA_{\rm MAG}$, and the effective energy-momentum form reads

$$\Sigma_\alpha^{\rm MAG} = \frac{Y_{\rm MAG}}{2}[F_{\rm MAG}\wedge(e_\alpha\rfloor{}^*F_{\rm MAG}) - {}^*F_{\rm MAG}\wedge(e_\alpha\rfloor F_{\rm MAG})]. \tag{6.23}$$

The effective "vacuum constant" is defined by $Y_{\rm MAG} := -z_4k_0^2/a_0$, and the tilde as usual denotes the objects constructed from the Riemannian (Christoffel) connection.

C. Axially symmetric solution

Let us choose standard Boyer-Lindquist coordinates (t, r, θ, φ). Then we straightforwardly can verify that the MAG field equations admit the generalized Kerr-Newman solution with cosmological constant. It is described by the coframe

$$\vartheta^{\hat 0} = \sqrt{\frac{\Delta}{\Sigma}}(cdt - j_0\Omega\sin^2\theta d\varphi), \tag{6.24}$$

$$\vartheta^{\hat 1} = \sqrt{\frac{\Sigma}{\Delta}}dr, \tag{6.25}$$

$$\vartheta^{\hat 2} = \sqrt{\frac{\Sigma}{f}}d\theta, \tag{6.26}$$

$$\vartheta^{\hat 3} = \sqrt{\frac{f}{\Sigma}}\sin\theta[-j_0cdt + \Omega(r^2 + j_0^2)d\varphi], \tag{6.27}$$

and by the one-form that describes post-Riemannian triplet sector

$$A_{\rm MAG} = u\vartheta^{\hat 0}. \tag{6.28}$$

Here $\Delta = \Delta(r)$, $\Sigma = \Sigma(r,\theta)$, $f = f(\theta)$, $u = u(r,\theta)$, and

j_0 and Ω are constant. The field equations yield the following explicit functions:

$$\Delta = (r^2 + j_0^2)\left(1 - \frac{\lambda}{3}r^2\right) - 2mr + \frac{\kappa Y_{\rm MAG}}{2}N^2, \quad (6.29)$$

$$\Sigma = r^2 + j_0^2\cos^2\theta, \quad (6.30)$$

$$f = 1 + \frac{\lambda}{3}j_0^2\cos^2\theta, \quad (6.31)$$

$$u = \frac{Nr}{\sqrt{\Delta\Sigma}}. \quad (6.32)$$

Here $\Omega = (1 + \lambda j_0^2/3)^{-1}$, $m = GM/c^2$, with the arbitrary integration constants M and N.

For completeness, let us write down explicitly the post-Riemannian two-form

$$F_{\rm MAG} = Nd\left(\frac{r}{\sqrt{\Delta\Sigma}}\vartheta^{\hat 0}\right) = \frac{N}{\Sigma^2}[(r^2 - j_0^2\cos^2\theta)\vartheta^{\hat 0}\wedge\vartheta^{\hat 1} - 2j_0 r\cos\theta\vartheta^{\hat 2}\wedge\vartheta^{\hat 3}]. \quad (6.33)$$

D. Invariant conserved charges for the axially symmetric solution

In order to calculate the conserved charge (5.41), we need the explicit translational and linear field momenta. Using the triplet ansatz in (6.12) and (6.13), we find

$$\mathcal{H}_\alpha = -\frac{a_0 k}{3\kappa}{}^*(\vartheta_\alpha\wedge A_{\rm MAG}) = \frac{a_0 k}{3\kappa}e_\alpha\rfloor{}^*A_{\rm MAG}, \quad (6.34)$$

$$\mathcal{H}^\alpha{}_\beta = \frac{a_0}{2\kappa}\eta^\alpha{}_\beta + \frac{z_4}{8}\delta^\alpha_\beta{}^*dQ = \frac{a_0}{2\kappa}\eta^\alpha{}_\beta + \frac{z_4 k_0}{8}\delta^\alpha_\beta{}^*F_{\rm MAG}. \quad (6.35)$$

From (5.33) we have explicitly $\Xi_\beta{}^\alpha = e_\beta\rfloor(D\xi^\alpha + \xi\rfloor T^\alpha)$. However, it is more convenient to start directly from the definition (5.37), and substitute (5.23) in it. We then find

$$\Xi_\beta{}^\alpha = \overset{\{\}}{D}_\beta\xi^\alpha - \xi\rfloor N_\beta{}^\alpha. \quad (6.36)$$

By making use of (5.24), we then can easily compute the contractions:

$$\Xi_\alpha{}^\alpha = \overset{\{\}}{D}_\alpha\xi^\alpha + 2\xi\rfloor Q, \quad (6.37)$$

$$\eta^{\alpha\beta}\Xi_{\alpha\beta} = {}^*[d(\xi_\alpha\vartheta^\alpha) - \xi^\alpha n_\alpha + \tfrac{1}{2}\xi\rfloor(T_\alpha\wedge\vartheta^\alpha)]. \quad (6.38)$$

Furthermore, in the triplet ansatz framework, these formulas reduce to

$$\Xi_\alpha{}^\alpha = \overset{\{\}}{D}_\alpha\xi^\alpha + 2k_0\xi\rfloor A_{\rm MAG}, \quad (6.39)$$

$$\eta^{\alpha\beta}\Xi_{\alpha\beta} = {}^*d(\xi_\alpha\vartheta^\alpha) + \frac{k}{3}\xi\rfloor{}^*A_{\rm MAG}. \quad (6.40)$$

Now we are in a position to compute the conserved invariant charges. Substituting (6.34), (6.39), (6.40), and (5.41), we find the general expression for the invariant charge in the quadratic MAG model with a triplet ansatz:

$$\mathcal{Q}[\xi] = \frac{a_0}{2\kappa}\int_{\partial S}[{}^*d(\xi^\alpha\vartheta_\alpha) + k(\xi]{}^*A_{\rm MAG})] + \frac{z_4 k_0}{8}\int_{\partial S}(\overset{\{\}}{D}_\alpha\xi^\alpha + 2k_0\xi\rfloor A_{\rm MAG})^*F_{\rm MAG}. \quad (6.41)$$

On the axially symmetric solution (6.24), (6.25), (6.26), (6.27), (6.28), (6.29), (6.30), (6.31), and (6.32), the last integral is zero for $\xi = \xi^i\partial_i$ with the constant components ξ^i. The asymptotic behavior of the physical and geometrical quantities is as follows (keeping the leading terms in $1/r$):

$$\xi\rfloor A_{\rm MAG} = \frac{N}{r}(\xi^0 - \xi^3\Omega j_0\sin^2\theta), \quad (6.42)$$

$${}^*A_{\rm MAG} = \frac{Nr}{\Delta}dr\wedge\sin\theta d\theta\wedge(-j_0 c dt + \Omega(r^2 + j_0^2)d\varphi), \quad (6.43)$$

$${}^*F_{\rm MAG} = -N\Omega\sin\theta d\theta\wedge d\varphi. \quad (6.44)$$

Substituting this, we can verify that the second line in (6.41) vanishes, and for $\lambda = 0$ the invariant conserved charges read

$$\mathcal{Q}[\partial_t] = a_0 Mc^2/2, \qquad \mathcal{Q}[\partial_\varphi] = -a_0 Mc j_0. \quad (6.45)$$

When the coupling constant in the MAG Lagrangian (6.1) has its standard value $a_0 = 1$, these conserved charges reduce to the well-known results of Komar [1]. Note that a_0 cannot vanish, otherwise the quadratic MAG model does not have an Einsteinian limit and the triplet ansatz is not applicable. For the solutions with nontrivial cosmological constant $\lambda \neq 0$, the conserved charges are formally infinite. Accordingly, like in the case of the theories with local Lorentz symmetry, a regularization is required. Unfortunately, the usual regularization via relocalization with the help of the topological boundary term is not possible since in MAG there is no analogue [34] of the Euler invariant. We will analyze the regularization problem elsewhere.

VII. DISCUSSION AND CONCLUSION

In this paper, the approach, developed earlier in [1], is generalized to the case when the local Lorentz group is replaced by an arbitrary local gauge group. The scheme includes the Maxwell and Yang-Mills fields coupled to gravity with Abelian and non-Abelian local internal symmetries. However, in the case of internal symmetries there seems to be no natural way to define nontrivial covariant

Lie derivatives. In the gravitational case, on the other hand, the frame field, transforming covariantly under the symmetry group, is the key ingredient to construct covariant generalized (Yano) Lie derivatives and therefore, invariant conserved quantities. The existence of the frame field in gravitational theories can be understood in terms of the so-called soldering procedure. In Sec. III we have studied a particular case in which a scalar field plays a role analogous to the frame field for the external symmetries, allowing to construct $U(1)$-invariant conserved quantities.

Another important case is the metric-affine gravity in which the local Lorentz spacetime group is extended to the local general linear group. We have developed the corresponding general formalism for MAG in Sec. V. This scheme generalizes and refines the partial results available in the earlier literature [35–41]. In order to illustrate how the formalism works, we applied it to the computation of the conserved charges for an exact MAG solution in Sec. VI. The results obtained are consistent with the derivations for the general relativity and for the models with local Lorentz symmetry.

ACKNOWLEDGMENTS

We thank J. G. Pereira for his hospitality at IFT-UNESP where this work was started. This work was partially supported by FAPESP and by DFG, project He 528/21-1 (for Y. N. O.) and by CNPq and by FONDECYT Grant No. 1060939 (for G. F. R.). We also thank F. W. Hehl for his comments on our paper.

APPENDIX A: GENERALIZED LIE DERIVATIVES

Here, we collect some useful identities satisfied by the generalized Lie derivative $\mathcal{L}_{\xi,\varepsilon}$. They generalize the results found in Appendix A of [1].

We defined the generalized Lie derivative by the formula

$$\mathcal{L}_{\xi,\varepsilon}\omega^A := \ell_\xi\omega^A - \varepsilon^a(\rho_a)^A{}_B\omega^B \tag{A1}$$

when acting on any (gauge-)covariant p-form ω^A. We also define the generalized Lie derivative of the gauge field by

$$\mathcal{L}_{\xi,\varepsilon}A^a := \ell_\xi A^a - d\varepsilon^a + f^a{}_{bc}A^b\varepsilon^c. \tag{A2}$$

With these definitions, one can prove that the generalized Lie derivative commutes with the exterior derivative, i.e. $[\mathcal{L}_{\xi,\varepsilon}, d] = 0$, as well as the following identities:

$$\mathcal{L}_{\xi,\varepsilon}A^a \equiv D(\xi\rfloor A^a - \varepsilon^a) + \xi\rfloor F^a, \tag{A3}$$

$$\mathcal{L}_{\xi,\varepsilon}F^a \equiv D(\mathcal{L}_{\xi,\varepsilon}A^a), \tag{A4}$$

$$[\mathcal{L}_{\xi,\varepsilon}, D]\omega^A \equiv (\mathcal{L}_{\xi,\varepsilon}A^a)(\rho_a)^A{}_B \wedge \omega^B. \tag{A5}$$

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