# Boosted black holes on Kaluza-Klein bubbles

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We construct an exact stationary solution of black-hole–bubble sequence in the five-dimensional Kaluza-Klein theory by using solitonic solution-generating techniques. The solution describes two stationary black holes with topology $S^3$ on a Kaluza-Klein bubble and has a linear momentum component in the compactified direction. We call the solution boosted black holes on Kaluza-Klein bubble because it has the linear momentum. The Arnowitt-Deser-Misner mass and the linear momentum depend on the two boosted velocity parameters of black holes. In the effective four-dimensional theory, the solution has an electric charge which is proportional to the linear momentum. The solution includes the static solution found by Elvang and Horowitz. The small and the big black holes limits are investigated. The relation between the solution and the single boosted black string are considered.

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## I. INTRODUCTION

Kaluza-Klein (KK) theory is the five-dimensional theory of gravity which unifies Einstein's four-dimensional theory of gravity and Maxwell's electromagnetic theory [1,2]. The spacetime is asymptotically the product of the four-dimensional Minkowski spacetime $\mathcal{M}^{3,1}$ and a circle $S^1$. The extra dimension with $S^1$ is compactified too small for us to observe it. This type of compactification of the extra dimensions are also extended to the supergravity theories and the superstrings.

The studies on black holes in the KK theory have attracted much attention since they admit much richer structures than asymptotically flat higher dimensional black holes. For example, such black holes can have the horizons with the topologies of the squashed $S^3$ and the lens space $L(2;1) = S^3/\mathbb{Z}_2$ [3,4]. Another exciting aspects of the KK theory are the existence of KK bubbles. In Ref. [5], a large class of five- and six-dimensional static solutions which describe the sequences of the black holes and KK bubbles were constructed and analyzed. The KK bubble was first found by Witten as the end state of the KK vacuum decay [6]. The first solution of the sequence of black holes and bubbles is the combination of the static black hole and KK bubble [7]. Elvang and Horowitz found and analyzed the two black holes sitting on the KK bubble [8]. The static equilibrium of the spacetime is maintained by the existence of the KK bubbles which balance the attractive force of black holes.

In the previous article, we obtained the new five-dimensional vacuum solution of rotating black holes on the KK bubble [9]. This solution is the extension of the static solution found by Elvang and Horowitz to a stationary solution, which has an Arnowitt-Deser-Misner (ADM) mass and an ADM angular momentum. We used two different types of solution-generating methods to obtain the solution. One is called Bäcklund transformation [10,11], which is basically the technique to generate a new solution of the Ernst equation. The other is the inverse scattering technique, which Belinski and Zakharov [12] developed as an another type of solution-generating technique. In several years, these techniques have been applied to generate and to reproduce five-dimensional black hole solutions with asymptotically flatness [13–22]. The relation between these two methods was examined in the context of the five-dimensional spacetime [23]. It was shown that the two-solitonic solutions generated from an arbitrary diagonal seed by the Bäcklund transformation coincide with those with a single angular momentum generated from the same seed by the inverse scattering method.

In this article we generate another type of stationary solution which describes boosted black holes on the KK-bubble as a vacuum solution in the five-dimensional Einstein equations by using both solitonic methods. It should be noted that this solution cannot be generated by the simple boost transformation of the static solution because the boosted solution always has closed timelike curves around the bubble. Both black holes of the solution are topologically $S^3$ because of the existence of the KK bubble. Also they are considered to be rotating black holes in the context of the rod structure near the horizons. However, the solution has a linear momentum in the compact direction and does not have an ADM angular momentum. This means that the asymptotic observer observes boosted objects in this spacetime. Therefore we call the solution boosted black holes on KK bubble even though the black holes are rotating. In the four-dimensional effective theory, the solution with linear momentum has an electric charge which is proportional to the linear momentum. One of the two black holes has zero Komar angular momentum. This black hole rotates because of the gravitational frame-dragging effect of the other intrinsically rotating black hole. When we take the small limit of the intrinsically rotating black hole, it approaches the maximally rotating black hole. When we increase the size of black holes, we find that the black holes cannot merge if we fix the asymptotic metric. To achieve the limit of single black hole without KK bubble in this solution, we have to redefine the size of KK circle. This limiting solution exactly corresponds to the simply boosted black string [24–27] whose thermodynamical properties are studied recently [28]. It is known that there is an unstable mode of boosted black string [29]. We compare the entropy of the boosted black holes with the boosted black string. The result of this analysis shows that the spontaneous decay of the boosted black string to the boosted black holes on KK bubble is unlikely. The solution with two momentum components will be generated by the inverse scattering method.

This article is organized as follows: In Sec. II, we give a new solution generated by the solitonic methods. We introduce only the construction by the Bäcklund transformation in this section, while the other construction is briefly mentioned in the Appendix. In Sec. III, we investigate the properties of the solution. In Sec. IV, we give the summary and discussion of this article. In the Appendix, we give the solution generated by the inverse scattering method and the relation between these solutions.

## II. SOLUTIONS

At first we briefly present the solution obtained by the Bäcklund transformation which was applied the five-dimensional case [13]. Using this method, we can generate axially symmetric solutions of five-dimensional vacuum Einstein equations. See [13] for the detail of the solution-generating method.

We start from the following form of a seed static metric:

$$ds^2 = e^{-T^{(0)}}[-e^{S^{(0)}}(dt)^2 + e^{-S^{(0)}}\rho^2(d\phi)^2 + e^{2\gamma^{(0)}-S^{(0)}}(d\rho^2+dz^2)] + e^{2T^{(0)}}(d\psi)^2, \tag{1}$$

with seed functions

$$S^{(0)} = U_{\lambda\sigma} - \tilde{U}_{\eta_1\sigma} + 2\tilde{U}_{\eta_2\sigma} = -\tilde{U}_{\lambda\sigma} - \tilde{U}_{\eta_1\sigma} + 2\tilde{U}_{\eta_2\sigma} + \ln\rho \tag{2}$$

$$T^{(0)} = U_{\lambda\sigma} + \tilde{U}_{\eta_1\sigma} = -\tilde{U}_{\lambda\sigma} + \tilde{U}_{\eta_1\sigma} + \ln\rho, \tag{3}$$

where we assume $\eta_1 < \eta_2 < -1 < \lambda < 1$ and $\sigma > 0$. The function $U_d$ is defined as $U_d := \frac{1}{2}\ln[R_d - (z-d)]$ and the function $\tilde{U}_d$ is defined as $\tilde{U}_d := \frac{1}{2}\ln[R_d + (z-d)]$, where $R_d := \sqrt{\rho^2 + (z-d)^2}$. Here we take the coordinate $\phi$ as a Kaluza-Klein compactified direction. As explained later, the solitonic solution has two event horizons at $\eta_1\sigma \le z \le \eta_2\sigma$ and $-\sigma \le z \le \lambda\sigma$ and a Kaluza-Klein bubble at $\eta_2\sigma \le z \le -\sigma$, where the Kaluza-Klein circles shrink to zero. The metric of the solitonic solution can be written in the following form:

$$ds^2 = e^{-T}[-e^S(dt - \omega d\phi)^2 + e^{-S}\rho^2(d\phi)^2 + e^{2\gamma - S}(d\rho^2 + dz^2)] + e^{2T}(d\psi)^2. \tag{4}$$

The function $T$ is derived from the seed functions

$$T = -\tilde{U}_{\lambda\sigma} + \tilde{U}_{\eta_1\sigma} + \ln\rho. \tag{5}$$

The other metric functions for the five-dimensional metric (4) are obtained by using the formulas shown by [30]

$$e^S = e^{S^{(0)}}\frac{A}{B}, \tag{6}$$

$$\omega = 2\sigma e^{-S^{(0)}}\frac{C}{A} - C_1, \tag{7}$$

$$e^{2\gamma} = C_2(x^2-1)^{-1}Ae^{2\gamma'}, \tag{8}$$

where $C_1$ and $C_2$ are constants and $A$, $B$, and $C$ are given by

$$A = \frac{1}{(2\sigma)^2}\{(e^{2\tilde{U}_{-\sigma}} + e^{2U_\sigma})(e^{2\tilde{U}_\sigma} + e^{2U_{-\sigma}})(1+ab)^2 - (e^{2\tilde{U}_{-\sigma}} - e^{2\tilde{U}_\sigma})(e^{2U_\sigma} - e^{2U_{-\sigma}})(b-a)^2\},$$

$$B = \frac{1}{(2\sigma)^2}\{[(e^{2\tilde{U}_{-\sigma}} + e^{2U_\sigma}) + (e^{2\tilde{U}_\sigma} + e^{2U_{-\sigma}})ab]^2 + [(e^{2\tilde{U}_{-\sigma}} - e^{2\tilde{U}_\sigma})a - (e^{2U_\sigma} - e^{2U_{-\sigma}})b]^2\},$$

$$C = \frac{1}{(2\sigma)^3}\{(e^{2\tilde{U}_{-\sigma}} + e^{2U_\sigma})(e^{2\tilde{U}_\sigma} + e^{2U_{-\sigma}})(1+ab)((e^{2U_\sigma} - e^{2U_{-\sigma}})b - (e^{2\tilde{U}_{-\sigma}} - e^{2\tilde{U}_\sigma})a) + (e^{2\tilde{U}_{-\sigma}} - e^{2\tilde{U}_\sigma})(e^{2U_\sigma} - e^{2U_{-\sigma}})(b-a)((e^{2\tilde{U}_{-\sigma}} + e^{2U_\sigma}) - (e^{2\tilde{U}_\sigma} + e^{2U_{-\sigma}})ab)\}.$$

The functions $a$ and $b$, which are auxiliary potential to obtain the new Ernst potential by the transformation, are given by

$$a = \alpha\sqrt{\frac{(e^{2\tilde{U}_{-\sigma}} + e^{2U_\sigma})(e^{2U_\sigma} - e^{2U_{-\sigma}})}{(e^{2\tilde{U}_\sigma} + e^{2U_{-\sigma}})(e^{2\tilde{U}_{-\sigma}} - e^{2\tilde{U}_\sigma})}}\frac{e^{\tilde{U}_{\lambda\sigma}}}{e^{2U_\sigma} + e^{2\tilde{U}_{\lambda\sigma}}} \times \frac{e^{\tilde{U}_{\eta_1\sigma}}}{e^{2U_\sigma} + e^{2\tilde{U}_{\eta_1\sigma}}}\left(\frac{e^{2U_\sigma} + e^{2\tilde{U}_{\eta_2\sigma}}}{e^{\tilde{U}_{\eta_2\sigma}}}\right)^2, \tag{9}$$

$$b = \beta\sqrt{\frac{(e^{2\tilde{U}_{-\sigma}} + e^{2U_\sigma})(e^{2\tilde{U}_{-\sigma}} - e^{2\tilde{U}_\sigma})}{(e^{2\tilde{U}_\sigma} + e^{2U_{-\sigma}})(e^{2U_\sigma} - e^{2U_{-\sigma}})}}\frac{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\lambda\sigma}}}{e^{\tilde{U}_{\lambda\sigma}}} \times \frac{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\eta_1\sigma}}}{e^{\tilde{U}_{\eta_1\sigma}}}\left(\frac{e^{\tilde{U}_{\eta_2\sigma}}}{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\eta_2\sigma}}}\right)^2. \tag{10}$$

In addition the function $\gamma'$ is obtained as

$$\begin{aligned}\gamma' = {}& \gamma'_{\sigma,\sigma} + \gamma'_{-\sigma,-\sigma} + \gamma'_{\lambda\sigma,\lambda\sigma} + \gamma'_{\eta_1\sigma,\eta_1\sigma} + \gamma'_{\eta_2\sigma,\eta_2\sigma} \\ & - 2\gamma'_{\sigma,-\sigma} - \gamma'_{\sigma,\lambda\sigma} - \gamma'_{\sigma,\eta_1\sigma} + 2\gamma'_{\sigma,\eta_2\sigma} + \gamma'_{-\sigma,\lambda\sigma} \\ & + \gamma'_{-\sigma,\eta_1\sigma} - 2\gamma'_{-\sigma,\eta_2\sigma} - \gamma'_{\lambda\sigma,\eta_1\sigma} - \gamma'_{\lambda\sigma,\eta_2\sigma} \\ & - \gamma'_{\eta_1\sigma,\eta_2\sigma} + \tilde{U}_{\sigma} - \tilde{U}_{-\sigma} - 2\tilde{U}_{\lambda\sigma} + \tilde{U}_{\eta_1\sigma} \\ & + \tilde{U}_{\eta_2\sigma} + \ln\rho, \end{aligned} \tag{11}$$

where

$$\gamma'_{cd} = \tfrac{1}{2}\tilde{U}_c + \tfrac{1}{2}\tilde{U}_d - \tfrac{1}{4}\ln[R_c R_d + (z-c)(z-d) + \rho^2]. \tag{12}$$

The constants $C_1$ and $C_2$ are chosen as follows:

$$C_1 = 0, \qquad C_2 = \frac{1}{(1+\alpha\beta)^2}, \tag{13}$$

to avoid the global boost of the spacetime and to set the period of $\psi$ to $2\pi$, respectively. Also the integration constants $\alpha$ and $\beta$ should be decided as

$$\alpha^2 = \frac{(1-\lambda)(1-\eta_1)}{(1-\eta_2)^2}, \qquad \beta = 0, \tag{14}$$

to remove the singularity at $z = \sigma$ on $z$-axis and closed timelike curves around the bubble, respectively.

## III. PROPERTIES

In this section, we investigate the properties of the solution satisfying the conditions (13) and (14). At first, we study the asymptotic structure, the asymptotic charge of the solution and the geometry of two black-hole horizons and a bubble. Next, to consider the effects of the linear momentum, we investigate several limits of the solution and the entropy of it.

### A. Asymptotic structure

In order to investigate the asymptotic structure of the solution, let us introduce the coordinate $(r, \theta)$ defined as

$$\rho = r\sin\theta, \qquad z = r\cos\theta, \tag{15}$$

where $0 \le \theta < 2\pi$ and $r$ is a four-dimensional radial coordinate in the neighborhood of the spatial infinity. For the large $r \longrightarrow \infty$, each component behaves as

$$g_{tt} \simeq -1 + \frac{(2-\eta_1+\eta_2)\sigma}{r}, \tag{16}$$

$$g_{\rho\rho} = g_{zz} \simeq 1 + \frac{(\lambda-\eta_1)\sigma}{r}, \tag{17}$$

$$g_{t\phi} \simeq -\frac{2\alpha\sigma}{r}, \tag{18}$$

$$g_{\phi\phi} \simeq 1 + \frac{(2+\eta_2-\lambda)\sigma}{r}, \tag{19}$$

$$g_{\psi\psi} \simeq r^2\sin^2\theta\left(1 - \frac{(\eta_1-\lambda)\sigma}{r}\right). \tag{20}$$

Hence, the leading order of the metric takes the form

$$ds^2 \simeq -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\psi^2) + d\phi^2. \tag{21}$$

Therefore, the spacetime has the asymptotic structure of the direct product of the four-dimensional Minkowski spacetime and $S^1$. The $S^1$ at infinity is parametrized by $\phi$ and the size $\Delta\phi$ is given in III C.

### B. Mass and momentum

Next, we compute the total mass and the linear momentum of the spacetime. It should be noted that since the asymptotic structure is $\mathcal{M}^{3,1} \times S^1$, the ADM mass and momentum are given by the surface integral over the spatial infinity with the topology of $S^2 \times S^1$. In order to compute these quantities, we introduce asymptotic Cartesian coordinates $(x, y, z, \phi)$, where $x = \rho\cos\psi$ and $y = \rho\sin\psi$. Then, the ADM mass and momentum in the $\phi$ direction are given by

$$M_{\rm ADM} = \frac{1}{16\pi}\int_{S^2\times S^1}(\partial_j h_{ij} - \partial_i h_{jj})dS_i, \tag{22}$$

$$P = -\frac{1}{16\pi}\int_{S^2\times S^1}\partial_i h_{0\phi}dS_i \tag{23}$$

respectively. (See, for example, [31].) Here $h_{\mu\nu}$ is deviation from the five-dimensional flat metric $\eta_{\mu\nu}$ near infinity,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{24}$$

The Latin indices $i$, $j$ run $x$, $y$, $z$ and $\phi$ and the Greek indices $\mu$, $\nu$, $\alpha$ and $\beta$ label $t, x, y, z$ and $\phi$. Then, the ADM mass of the solution is computed as

$$M_{\rm ADM} = \frac{(\lambda - 2\eta_1 + \eta_2 + 2)\sigma}{4}\Delta\phi. \tag{25}$$

It should be noted that the ADM mass is non-negative. The linear momentum becomes

$$P = -\frac{\alpha\sigma}{2}\Delta\phi. \tag{26}$$

The boosted black string has an electric charge which is proportional to the linear momentum in the four-dimensional effective theory. Because the asymptotic behavior of the solution is the same as the boosted black string, we can define the electric charge of the solution as

$$Q_4 = \alpha\sigma. \tag{27}$$

### C. Black holes and bubble

Here, for the solution, we consider the rod structure developed by Harmark [32] and Emparan and Reall [7]. The rod structure at $\rho = 0$ is illustrated in Fig. 1. (i) The finite timelike rod $[\eta_1\sigma, \eta_2\sigma]$ and $[-\sigma, \lambda\sigma]$ denote the locations of black-hole horizons. These timelike rods have directions $v_1 = (1, \Omega_1, 0)$ and $v_2 = (1, \Omega_2, 0)$. We call $\Omega_1$ and $\Omega_2$ boost velocity parameters. These are given by

$$\Omega_1 = \frac{2\alpha}{1-\eta_1}, \tag{28}$$

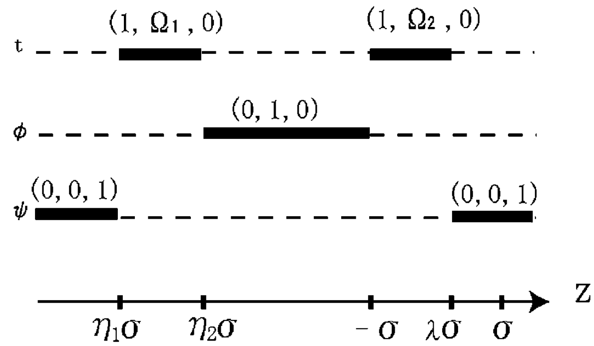


FIG. 1. Rod structure of boosted black holes on a Kaluza-Klein bubble. The finite timelike rods $[\eta_1\sigma, \eta_2\sigma]$ and $[-\sigma, \lambda\sigma]$ correspond to black holes with boost velocities $\Omega_1$ and $\Omega_2$, respectively. The finite spacelike rod $[\eta_2\sigma, -\sigma]$ denotes a Kaluza-Klein bubble, where Kaluza-Klein circles shrink to zero.

for $\eta_1\sigma < z < \eta_2\sigma$ and

$$\Omega_2 = \frac{\alpha(1-\eta_2)^2}{2(1-\eta_1)}, \tag{29}$$

for $-\sigma < z < \lambda\sigma$. Here, it should be noted that $\Omega_1$ and $\Omega_2$ have the same signature and $|\Omega_1| < |\Omega_2|$ for $\eta_2 < -1$. Therefore, two black holes are boosted along the same direction. (ii) The finite spacelike rod $[\eta_2\sigma, -\sigma]$ which corresponds to a Kaluza-Klein bubble has the direction $v = (0, 1, 0)$. In order to avoid conical singularity for $z \in [\eta_2\sigma, -\sigma]$ and $\rho = 0$, $\phi$ has the periodicity of

$$\frac{\Delta\phi}{2\pi} = \lim_{\rho\to 0}\sqrt{\frac{\rho^2 g_{\rho\rho}}{g_{\phi\phi}}} = 2\sigma\frac{\eta_2+1}{\eta_2-1}\sqrt{\frac{(\lambda-\eta_1)(\lambda-\eta_2)(\eta_1-1)}{\eta_1+1}}. \tag{30}$$

(iii) The semi-infinite spacelike rods $[-\infty, \eta_1\sigma]$ and $[\lambda\sigma, \infty]$ have the direction $v = (0, 0, 1)$. In order to avoid conical singularity, $\psi$ has the periodicity of

$$\Delta\psi = 2\pi. \tag{31}$$

The local structures around the finite timelike rods are similar to that of the $S^3$ black hole rotating around the $\phi$-plane. Therefore the solution has two rotating $S^3$ event horizons at either end of the KK bubble. As we have confirmed above, however, the solution has the linear momentum in the direction of the KK circle and not the angular momentum. In this meaning we call the solution boosted black holes on KK bubble.

Here, we write the induced metrics of the event horizons and the bubble. For $\eta_1\sigma < z < \eta_2\sigma$, the induced metric becomes

$$g_{tt} = \frac{\Omega_1^2(z-\sigma)(z+\sigma)(z-\eta_2\sigma)}{(z-\lambda\sigma)(z+\sigma)^2 - \Omega_1^2(z-\eta_1\sigma)(z-\eta_2\sigma)^2}, \tag{32}$$

$$g_{t\phi} = -\frac{\Omega_1(z-\sigma)(z+\sigma)(z-\eta_2\sigma)}{(z-\lambda\sigma)(z+\sigma)^2 - \Omega_1^2(z-\eta_1\sigma)(z-\eta_2\sigma)^2}, \tag{33}$$

$$g_{zz} = \frac{\sigma^2(1-\eta_1)(\eta_2-\eta_1)(\lambda-\eta_1)((z-\lambda\sigma)(z+\sigma)^2 - \Omega_1^2(z-\eta_1\sigma)(z-\eta_2\sigma)^2)}{(1+\eta_1)(z-\sigma)(z+\sigma)(z-\eta_1\sigma)(z-\eta_2\sigma)(z-\lambda\sigma)}, \tag{34}$$

$$g_{\phi\phi} = \frac{(z-\sigma)(z+\sigma)(z-\eta_2\sigma)}{(z-\lambda\sigma)(z+\sigma)^2 - \Omega_1^2(z-\eta_1\sigma)(z-\eta_2\sigma)^2}, \tag{35}$$

$$g_{\psi\psi} = -4(z-\eta_1\sigma)(z-\lambda\sigma). \tag{36}$$

Since the $\phi$ circles shrink to zero at $z = \eta_2\sigma$ and $\psi$ circles shrink to zero at $z = \eta_1\sigma$, the spatial cross section of this black-hole horizon is topologically $S^3$. The area of the event horizon is

$$A_1 = 4\pi\sigma^2\sqrt{\frac{(1-\eta_1)(\lambda-\eta_1)(\eta_2-\eta_1)^3}{-1-\eta_1}}\Delta\phi = 16\pi^2\sigma^3\frac{(1-\eta_1)(-1-\eta_2)(\eta_2-\eta_1)^{3/2}(\lambda-\eta_1)(\lambda-\eta_2)^{1/2}}{(-1-\eta_1)(1-\eta_2)}. \tag{37}$$

For $-\sigma < z < \lambda\sigma$, the induced metric takes the following form:

$$g_{tt} = \frac{\Omega_2^2(z-\sigma)(z+\sigma)(z-\eta_2\sigma)}{(z-\lambda\sigma)(z-\eta_2\sigma)^2 - \Omega_2^2(z+\sigma)^2(z-\eta_1\sigma)}, \tag{38}$$

$$g_{t\phi} = -\frac{\Omega_2(z-\sigma)(z+\sigma)(z-\eta_2\sigma)}{(z-\lambda\sigma)(z-\eta_2\sigma)^2 - \Omega_2^2(z+\sigma)^2(z-\eta_1\sigma)}, \tag{39}$$

$$g_{zz} = -\frac{4\sigma^2(1-\eta_1)(\eta_1-\lambda)(\eta_2-\lambda)((z-\lambda\sigma)(z-\eta_2\sigma)^2 - \Omega_2^2(z+\sigma)^2(z-\eta_1\sigma))}{(1-\eta_2)^2(1+\lambda)(z-\sigma)(z+\sigma)(z-\eta_1\sigma)(z-\eta_2\sigma)(z-\lambda\sigma)}, \tag{40}$$

$$g_{\phi\phi} = \frac{(z-\sigma)(z+\sigma)(z-\eta_2\sigma)}{(z-\lambda\sigma)(z-\eta_2\sigma)^2 - \Omega_2^2(z+\sigma)^2(z-\eta_1\sigma)}, \tag{41}$$

$$g_{\psi\psi} = -4(z-\eta_1\sigma)(z-\lambda\sigma). \tag{42}$$

Since the $\phi$ circles shrink to zero at $z = -\sigma$ and $\psi$ circles shrink to zero at $z = \lambda\sigma$, the spatial cross section of this black-hole horizon is also topologically $S^3$. The area of this event horizon is

$$\begin{aligned} A_2 &= 8\pi\sigma^2 \frac{\sqrt{(1-\eta_1)(\lambda-\eta_1)(\lambda-\eta_2)(\lambda+1)}}{1-\eta_2}\Delta\phi \\ &= 32\pi^2\sigma^3 \frac{(1-\eta_1)(-1-\eta_2)(\lambda-\eta_1)(\lambda-\eta_2)(\lambda+1)^{1/2}}{(1-\eta_2)^2(-1-\eta_1)^{1/2}}. \end{aligned} \tag{43}$$

For $\eta_2\sigma < z < -\sigma$, the induced metric on the bubble can be written in the form

$$g_{tt} = -\frac{(z+\sigma)(z-\eta_2\sigma)}{(z-\sigma)(z-\eta_1\sigma)}, \tag{44}$$

$$g_{t\phi} = -\frac{\sigma^2\Omega_2\rho^2}{(z+\sigma)(z-\eta_2\sigma)(z-\lambda\sigma)}, \tag{45}$$

$$g_{zz} = \sigma^2 \frac{(1-\eta_1)(1+\eta_2)^2(\lambda-\eta_1)(\lambda-\eta_2)(z-\sigma)}{(1+\eta_1)(1-\eta_2)^2(z+\sigma)(z-\eta_2\sigma)(z-\lambda\sigma)}, \tag{46}$$

$$g_{\phi\phi} = -\frac{(z-\sigma)\rho^2}{4(z-\eta_2\sigma)(z-\lambda\sigma)(z+\sigma)}, \tag{47}$$

$$g_{\psi\psi} = -4(z-\eta_1\sigma)(z-\lambda\sigma). \tag{48}$$

The $\phi$ circle vanishes for $z \in [\eta_2\sigma, -\sigma]$ and $\rho = 0$, which means that there exists a Kaluza-Klein bubble in this region. Since the $\psi$ circle does not vanish at $z = \eta_2\sigma$ and $z = \lambda\sigma$, this bubble on the time slice is topologically a cylinder $S^1 \times R$. Therefore, there exists a Kaluza-Klein bubble between two boosted black holes with topology of $S^3$. The proper distance between the two black holes is

$$s = \sigma\frac{(\eta_2+1)}{(\eta_2-1)}\sqrt{\frac{(\eta_1-1)(\lambda-\eta_1)(\lambda-\eta_2)}{(\eta_1+1)}} \times \int_{\eta_2\sigma}^{-\sigma} dz\sqrt{-\frac{z-\sigma}{(z+\sigma)(z-\eta_2\sigma)(z-\lambda\sigma)}}. \tag{49}$$

The Kaluza-Klein bubble is significant to keep the balance of two black holes and achieve the solution without any strut structures and singularities. This property resembles that of the solution given by Elvang and Horowitz [8] and the extension of it with rotation [9]. In the next subsection, we will show that the static limit of the solution coincides with the solution given by Elvang and Horowitz.

Because the horizons rotate in the local point of view, there appear ergo regions around the horizons. In Fig. 2 we numerically plot the ergo regions for $\eta_1 = -3$, $\eta_2 = -2$ and $\lambda = 0.9, 0, -0.9$. The right black hole has a larger ergo region than the left black hole. These two ergo regions join for the case of a sufficiently small and fast right black hole. These ergo regions cannot adhere to the bubble because $g_{tt}$ of Eq. (44) is negative for $\eta_2\sigma < z < -\sigma$. This can be explained by the fact that the bubble plays a role of rotational plane of black holes.

In Fig. 3 we examine the relation between the ergo region of the left black hole and the separation of the black holes. The ergo region of the left black hole gradually shrinks as the black holes are away from each other. This fact suggests that the left black hole rotates by the influence of the intrinsic rotation of the right black hole. This is consistent with the fact that the ratio of the velocities of the horizons only depends on the parameter $\eta_2$,

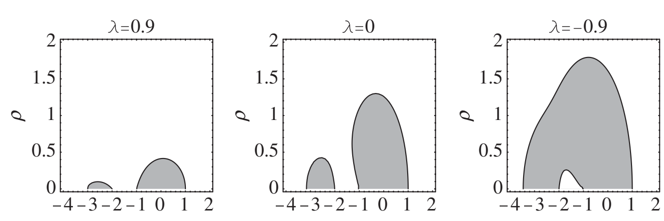


FIG. 2. Ergo regions for the cases of $\lambda = 0.9$, 0, $-0.9$ with $\eta_1 = -3$, $\eta_2 = -2$, and $\sigma = 1$. In the shaded region the values of $g_{tt}$ are positive.

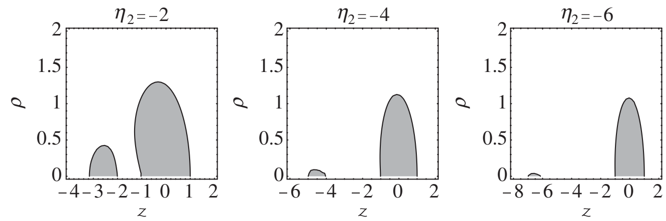


FIG. 3. Ergo regions for the cases of $\eta_2 = -2, -4, -6$ with $\lambda = 0$, $\eta_1 = \eta_2 - 1$, and $\sigma = 1$. In the shaded region the values of $g_{tt}$ are positive.

$$\frac{\Omega_1}{\Omega_2} = \frac{4}{(1-\eta_2)^2}, \tag{50}$$

and decreases according to $\eta_2 \to -\infty$. To show that the rotation of the left black hole is a consequence of gravitational frame dragging, we calculate the Komar angular momenta of black holes [22]:

$$J_{\mathrm{Komar},i} = \frac{1}{16\pi}\int_{H_i} dz d\psi d\phi \frac{1}{\sqrt{-\det g}} g_{zz} g_{\psi\psi} \times [-g_{\phi\phi}\partial_\rho g_{t\phi} + g_{t\phi}\partial_\rho g_{\phi\phi}]. \tag{51}$$

We obtain the Komar angular momenta of left and right black holes as

$$J_{\mathrm{Komar},1} = 0, \qquad J_{\mathrm{Komar},2} = -\frac{\alpha\sigma}{2}\Delta\phi, \tag{52}$$

respectively. The left black hole has no intrinsic rotation. The Komar angular momentum of the right black hole is exactly the same as the linear momentum of the spacetime.

At the end of this subsection we rewrite the ADM mass (25) and the electric charge (27) by using boost velocities $\Omega_1$ and $\Omega_2$. The ADM mass is

$$\frac{M_{\mathrm{ADM}}}{\Delta\phi} = \frac{(\lambda-\eta_1)\sigma}{2}\left(1 + \frac{1-\lambda}{2(\lambda-\eta_1)}\right) + \frac{(1+\eta_2)\sigma}{4} = m\left(1 + \frac{\Omega_1\Omega_2}{2(1-\Omega_1\Omega_2)}\right) + m_0\left(1 - \sqrt{\frac{\Omega_1}{\Omega_2}}\right), \tag{53}$$

where $m = \frac{(\lambda-\eta_1)\sigma}{2}$ and $m_0 = \frac{\sigma}{2}$. The electric charge is

$$Q_4 = \alpha\sigma = m\frac{\Omega_1}{1-\Omega_1\Omega_2}. \tag{54}$$

### D. Static case

In this subsection, we consider the static case, which can be obtained by the choice of the parameter $\lambda = 1$. Then, from Eq. (14) we see that $\alpha$ vanishes. Let us define the parameters $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ as

$$\tilde{a} = \frac{2\lambda + 1 - \eta_2}{2}\sigma, \qquad \tilde{b} = \frac{-1-\eta_2}{2}\sigma, \qquad \tilde{c} = \frac{-1+\eta_2-2\eta_1}{2}\sigma. \tag{55}$$

It should be noted that $\lambda = 1$ is equal to the condition $\sigma = (\tilde{a} - \tilde{b})/2$. Furthermore, let us shift an origin of the $z$-coordinate such that $z \to \tilde{z} := z - (\eta_2 - \lambda)\sigma/2$. Then, we obtain the metric

$$ds^2 = -\frac{(R_{\tilde{b}} - (\tilde{z}-\tilde{b}))(R_{-\tilde{c}} - (\tilde{z}+\tilde{c}))}{(R_{\tilde{a}} - (\tilde{z}-\tilde{a}))(R_{-\tilde{b}} - (\tilde{z}+\tilde{b}))}dt^2 + (R_{\tilde{a}} - (\tilde{z}-\tilde{a}))(R_{-\tilde{c}} - (\tilde{z}+\tilde{c}))d\psi^2 + \frac{R_{-\tilde{b}} - (\tilde{z}+\tilde{b})}{R_{\tilde{b}} - (\tilde{z}-\tilde{b})}d\phi^2 + \frac{Y_{\tilde{a},-\tilde{c}}Y_{\tilde{b},-\tilde{b}}}{4R_{\tilde{a}}R_{\tilde{b}}R_{-\tilde{b}}R_{-\tilde{c}}} \times \sqrt{\frac{Y_{\tilde{a},\tilde{b}}Y_{-\tilde{b},-\tilde{c}}}{Y_{\tilde{a},-\tilde{b}}Y_{\tilde{b},-\tilde{c}}}}\frac{R_{\tilde{a}} - (\tilde{z}-\tilde{a})}{R_{-\tilde{c}} - (\tilde{z}+\tilde{c})}(d\rho^2 + d\tilde{z}^2), \tag{56}$$

where the coordinate $z$ in the definition of $R_d$ is replaced with $\tilde{z}$. This coincides with the solution obtained by Elvang and Horowitz [8], which describes static black holes on the Kaluza-Klein bubble.

### E. Small black holes

In this subsection we consider the small black holes limit of the solution. The small limit of the left black hole is achieved by $\eta_1 = \eta_2 - \epsilon$, where $0 < \epsilon \ll 1$. Introducing the new coordinate $\theta$ through $z = \eta_2\sigma - \epsilon\sigma\sin^2\theta$, the horizon geometry of the left black hole becomes

$$ds^2_{\mathrm{bh1}} = 4\epsilon(\lambda - \eta_2)\sigma^2(d\theta^2 + \sin^2\theta d\bar{\phi}^2 + \cos^2\theta d\psi^2), \tag{57}$$

where we rescale the $\phi$ coordinate as

$$\frac{\phi}{\bar{\phi}} = 2\sigma\frac{\eta_2+1}{\eta_2-1}\sqrt{\frac{(\lambda-\eta_1)(\lambda-\eta_2)(\eta_1-1)}{\eta_1+1}}. \tag{58}$$

This shows that the left small black hole is a round three sphere as similar as the static case. To make the right black hole small, we take $\lambda = -1 + \epsilon$. Because the $\alpha^2$ is a decreasing function of $\lambda$, it is expected that the effect of the linear momentum is the most significant. By defining $z = -\sigma + \epsilon\sigma\sin^2\theta$, the horizon geometry of the right black hole becomes

$$ds^2_{\mathrm{bh2}} = \frac{8(1-\eta_1)(-1-\eta_2)^2\sigma^2}{(1-\eta_2)^2}\left(\cos^2\theta d\theta^2 + \frac{\sin^2\theta}{\cos^2\theta}d\bar{\phi}^2\right) + 4\epsilon(-1-\eta_1)\sigma^2\cos^2\theta d\psi^2. \tag{59}$$

This is similar to the horizon geometry of the extremal Myers-Perry black hole with single rotation.

The fact that the black holes have a different limit also argues that the left black hole is rotating indirectly as a result of the rotation of right black hole.

### F. Big black holes

Next we consider the big black holes limit. At first we show the impossibility of merging two black holes, keeping the radius of KK circle at the infinity. To let the black holes come close to each other, we take the limit $\eta_2 \to -1$. In this limit the ratio of the separation between the two black holes to the size of the circle at the infinity becomes

$$\frac{s}{\Delta\phi} \approx \frac{1}{4}\sqrt{\frac{2}{1+\lambda}}. \tag{60}$$

Now we examine the two big black holes limit and show that the geometries of the horizons and the bubble become similar forms of the static solution. The bubble metric including the time direction becomes

$$ds^2_{\rm bubble} = 4(-1-\eta_1)(1+\lambda)\sigma^2 d\psi^2 + \frac{\epsilon^2(1-\eta_1)(\lambda-\eta_1)\sigma^2}{2(-1-\eta_1)} \times \left(d\eta^2 - \frac{\sin^2\eta}{4(1-\eta_1)(\lambda-\eta_1)\sigma^2}dt^2\right), \tag{61}$$

where $z = -\sigma - \frac{\epsilon\sigma}{2}(1+\cos\eta)$ with $0<\eta<\pi$. The geometry of the bubble is a flat cylinder with large radius and small height the same as the static case. By defining $z = \frac{\lambda+\eta_1}{2}\sigma + \frac{\lambda-\eta_1}{2}\sigma\cos\theta$, the horizon metric becomes

$$ds^2_{\rm bh1,2} = \frac{1-\eta_1}{\lambda-\eta_1}d\phi^2 + (\lambda-\eta_1)^2\sigma^2(d\theta^2+\sin^2\theta d\psi^2), \tag{62}$$

where $\eta_1\sigma < z \ll \eta_2\sigma$ for the left black hole and $-\sigma \ll z < \lambda$ for the right. This is a product of a circle and round two sphere similar to the static solution. Both horizons have the same velocity parameters,

$$\Omega_1 = \Omega_2 = \sqrt{\frac{1-\lambda}{1-\eta_1}}, \tag{63}$$

in this limit. Starting from a boosted black sting we can configure the two large $S^3$ black hole with rotation separated by the KK bubble by the similar way of deformation of Elvang and Horowitz [5]. In the next subsection we compare the entropy of the boosted black string to the entropy of the boosted black holes on the KK bubble.

To achieve the single boosted black string solution starting from the boosted black holes, we have to redefine the size of the KK circle $\Delta\phi$ in addition to taking the limit $\eta_2 \to -1$. This solution corresponds to the electric charged black hole in the effective four-dimensional theory [25–27]. It can be easily confirmed that the ADM mass (53) and the electric charge (54) become well-known forms because $\Omega_1 = \Omega_2$ when $\eta_2 = -1$. The $3\times3$ metric functions become the following form:

$$g_{tt} = -\frac{R_{\lambda\sigma}+R_{\eta_1\sigma}-(2-\lambda-\eta_1)\sigma}{R_{\lambda\sigma}+R_{\eta_1\sigma}+(\lambda-\eta_1)\sigma}, \tag{64}$$

$$g_{t\phi} = -\frac{2\sqrt{(1-\lambda)(1-\eta_1)}\sigma}{R_{\lambda\sigma}+R_{\eta_1\sigma}+(\lambda-\eta_1)\sigma}, \tag{65}$$

$$g_{\phi\phi} = \frac{R_{\lambda\sigma}+R_{\eta_1\sigma}+(2-\lambda-\eta_1)\sigma}{R_{\lambda\sigma}+R_{\eta_1\sigma}+(\lambda-\eta_1)\sigma}, \tag{66}$$

$$g_{\psi\psi} = \rho^2\frac{R_{\lambda\sigma}+R_{\eta_1\sigma}+(\lambda-\eta_1)\sigma}{R_{\lambda\sigma}+R_{\eta_1\sigma}-(\lambda-\eta_1)\sigma}. \tag{67}$$

Introducing the Schwarzschild radial coordinate as

$$r_s = \frac{R_{\lambda\sigma}+R_{\eta_1\sigma}}{2} + m, \tag{68}$$

we can derive a familiar expression of the boosted black string from the solitonic solution.

### G. Entropy

It is known that the boosted black strings have an unstable mode [29]. In this subsection we compare the areas of the boosted black holes on the KK bubble and the boosted black string in the context of the final state of the instability of black string.

At first we write the area of the boosted black string in its mass and charge. This can be done by calculating the area of the horizon directly from the boosted black string metric or by taking the limit $\eta_2 \to -1$ for the areas of horizons normalized by $\Delta\phi$. The result is

$$\frac{A_{\rm BS}}{\Delta\phi} = 4\pi\sigma^2 V^{1/2}W^{3/2}, \tag{69}$$

where

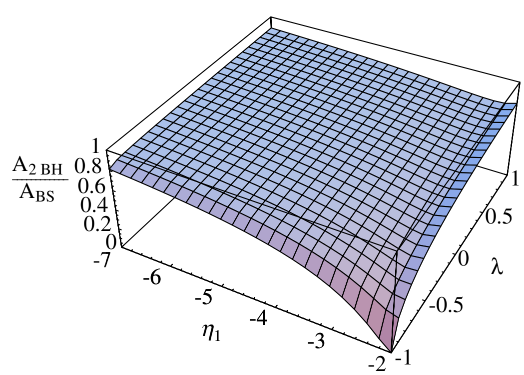


FIG. 4 (color online). Plot of $A_{2\rm BH}/A_{\rm BS}$ for $\eta_2 = -2$. This ratio is less than unity.

$$V = \frac{M_{\rm ADM}}{\sigma\Delta\phi} + \sqrt{\left(\frac{M_{\rm ADM}}{\sigma\Delta\phi}\right)^2 + \frac{2Q_4^2}{\sigma^2}},$$
$$W = 3\frac{M_{\rm ADM}}{\sigma\Delta\phi} - \sqrt{\left(\frac{M_{\rm ADM}}{\sigma\Delta\phi}\right)^2 + \frac{2Q_4^2}{\sigma^2}}. \tag{70}$$

In Fig. 4 we plot the ratio of $A_{2\rm BH} = A_1 + A_2$ to $A_{\rm BS}$ for the case of $\eta_2 = -2$. We find that the ratio is less than unity entirely and vanishes for the small black holes limit $\eta_1 = -2$ and $\lambda = -1$. This result does not depend on the value of $\eta_2$. Therefore we cannot expect the boosted black string to spontaneously transform into the boosted black holes on the KK bubble.

Next we compare the ratio $A_{2\rm BH}/A_{\rm BS}$ with the static case. We fix the ratios of the length of finite rods. Here we consider the case of equal length timelike rods $\tilde{a} = x\tilde{b}$ and $\tilde{c} = x\tilde{b}$. In this case the parameter $\eta_2$ is written by $x$ and $\alpha$ as

$$\eta_2 = \frac{x^2 - 4\alpha^2 + 3 + 4\sqrt{x^2 + \alpha^2(1 - x^2)}}{1 - x^2 - 4\alpha^2}. \tag{71}$$

The other parameters $\eta_1$ and $\lambda$ are obtained from

$$\eta_1 = \tfrac{1}{2}(-1 + \eta_2 - x(-1 - \eta_2)) \tag{72}$$

and

$$\lambda = \tfrac{1}{2}(-1 + \eta_2 + x(-1 - \eta_2)). \tag{73}$$

In Fig. 5, we plot the ratio $A_{2\rm BH}/A_{\rm BS}$ for the several values of $\alpha$. The static case corresponds to $\alpha = 0$. It can be seen that the ratio becomes a slightly larger value by the effect of the linear momentum.

## IV. SUMMARY AND DISCUSSION

Using the solitonic solution-generating methods, we generated a new exact solution which describes a pair of boosted black holes in the compact direction on a Kaluza-Klein bubble as a vacuum solution in the five-dimensional Kaluza-Klein theory. This solution cannot be obtained by the simple boost transformation of the static black holes on the Kaluza-Klein bubble. We also investigated the properties of this solution, particularly, its asymptotic structure, the geometry of the black-hole horizons and the Kaluza-Klein bubble, and several limits of the solution. The asymptotic structure is the $S^1$ bundle over the four-dimensional Minkowski spacetime. Two black holes have the topological structure of $S^3$ and the bubble is topologically $S^1 \times R$. The solution describes the physical situation such that two black holes have the boost velocity of the same direction and the bubble plays a role in holding two black holes. The ADM mass and the linear momentum of the solution can be written by the two boosted velocity parameters. For the local observer near the horizon, the black holes are considered as the rotating $S^3$ black holes. One of the two black holes has intrinsic rotation. The other rotates by the effect of the gravitational frame dragging. In the static case, it coincides with the solution found by Elvang and Horowitz. In the small black holes limit, one black hole approaches an extremal black hole and the other approaches the round $S^3$ sphere. No matter how large the size of the horizons increase, the black holes cannot merge each other if the size of the KK circle is fixed. To construct the single boosted black string from the solution, we have to redefine the size of the KK circle after the limit of $\eta_2 \to -1$. It cannot be expected that the boosted black hole spontaneously breaks down to the boosted black holes on KK bubble from the comparison of areas between the black holes on KK bubble and the black string for the same asymptotic charges.

The solution obtained here is constructed by only one solitonic transformation from the seed which has one static black hole. It seems that this is the reason why the two black holes have different features, e.g., different small black-hole limits. It is expected that more general solutions which include one static black hole or counterrotating black holes are generated from two solitonic transformations for the seed without the static black hole.

In this article, we concentrated on the black-hole solution with a linear momentum component. The solution with an angular momentum component has been derived in the previous paper [9]. The investigation on the solution with these two components is enormously challenging. In general, the inverse scattering method can generate a solution with two momentum components. We will give such a solution in our future article.

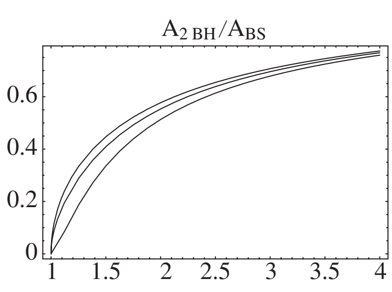


FIG. 5. Plots of $A_{2\rm BH}/A_{\rm BS}$ for the cases of same rod lengths and different velocity parameters. The upper, middle, lower lines correspond to $\alpha^2 = 0.9$, 0.3, 0, respectively.

## ACKNOWLEDGMENTS

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## APPENDIX A: SOLUTIONS GENERATED BY ISM

Following the techniques in Refs. [17,20,23], we construct a new Kaluza-Klein black-hole solution. We consider the five-dimensional stationary and axisymmetric vacuum spacetimes which admit three commuting Killing vectors $\partial/\partial t$, $\partial/\partial\phi$, and $\partial/\partial\psi$, where $\partial/\partial t$ is a Killing vector field associated with time translation, $\partial/\partial\phi$ and $\partial/\partial\psi$ denote spacelike Killing vector fields with closed orbits. In such a spacetime, the metric can be written in the canonical form as

$$ds^2 = g_{ij}dx^i dx^j + f(d\rho^2 + dz^2), \tag{A1}$$

where the metric components $g_{ij}$ and the metric coefficient $f$ are functions which depend on $\rho$ and $z$ only. The metric $g_{ij}$ satisfies the supplementary condition $\det g_{ij} = -\rho^2$. We begin with the following seed:

$$ds^2 = -\frac{R_{\eta_2\sigma} + z - \eta_2\sigma}{R_{\eta_1\sigma} + z - \eta_1\sigma}dt^2 + \frac{R_{\lambda\sigma} + z - \lambda\sigma}{R_{\eta_2\sigma} + z - \eta_2\sigma}d\phi^2 + \frac{R_{\eta_1\sigma} + z - \eta_1\sigma}{R_{\lambda\sigma} + z - \lambda\sigma}\rho^2 d\psi^2 + f(d\rho^2 + dz^2), \tag{A2}$$

where $R_d$ is defined as $R_d := \sqrt{\rho^2 + (z-d)^2}$. The parameters $\eta_1$, $\eta_2$ and $\lambda$ satisfy the inequality $\eta_1 < \eta_2 < \lambda < 1$ and $\sigma > 0$. Instead of solving the L-A pair for the seed metric (A2), it is sufficient to consider the following metric form:

$$ds^2 = -dt^2 + g_2 d\phi^2 + g_3 d\psi^2 + f(d\rho^2 + dz^2), \tag{A3}$$

where $g_2$ and $g_3$ are given by

$$g_2 = \frac{(R_{\eta_1\sigma} + z - \eta_1\sigma)(R_{\lambda\sigma} + z - \lambda\sigma)}{(R_{\eta_2\sigma} + z - \eta_2\sigma)^2},\quad g_3 = \frac{(R_{\eta_2} + z - \eta_2\sigma)^2\rho^2}{(R_{\eta_1\sigma} + z - \eta_1\sigma)(R_{\lambda\sigma} + z - \lambda\sigma)}. \tag{A4}$$

Let us consider the conformal transformation of the two-dimensional metric $g_{AB}(A, B = t, \phi)$ and the rescaling of the $\psi\psi$-component in which the determinant $\det g$ is invariant,

$$g_0 = \mathrm{diag}(-1, g_2, g_3) \rightarrow g_0' = \mathrm{diag}(-\Omega, \Omega g_2, \Omega^{-2}g_3), \tag{A5}$$

where $\Omega$ is the $tt$-component of the seed (A2), i.e.

$$\Omega = \frac{R_{\eta_2\sigma} + z - \eta_2\sigma}{R_{\eta_1\sigma} + z - \eta_1\sigma}. \tag{A6}$$

Then, under this transformation, the three-dimensional metric coincides with the metric (A2). On the other hand, as discussed in [23], under this transformation the physical metric of two-solitonic solution is transformed as

$$g = \begin{pmatrix} g_{AB} & 0 \\ 0 & g_3 \end{pmatrix} \rightarrow g' = \begin{pmatrix} \Omega g_{AB} & 0 \\ 0 & \Omega^{-2}g_3 \end{pmatrix}. \tag{A7}$$

This is why we may perform the transformation (A5) for the two-solitonic solution generated from the seed (A3) in order to obtain the two-solitonic solution from the seed (A2). The generating matrix $\psi_0$ for this seed metric (A3) is computed as follows:

$$\psi_0[\bar\lambda] = \mathrm{diag}(-1, \psi_2[\bar\lambda], \psi_3[\bar\lambda])$$

with

$$\psi_2[\bar\lambda] = \frac{(R_{\eta_1\sigma} + z - \eta_1\sigma + \bar\lambda)(R_{\lambda\sigma} + z - \lambda\sigma + \bar\lambda)}{(R_{\eta_2\sigma} + z - \eta_2\sigma + \bar\lambda)^2},$$

$$\psi_3[\bar\lambda] = \frac{(R_{\eta_2\sigma} + z - \eta_2\sigma + \bar\lambda)^2(\rho^2 - 2z\bar\lambda - \bar\lambda^2)}{(R_{\eta_1\sigma} + z - \eta_1\sigma + \bar\lambda)(R_{\lambda\sigma} + z - \lambda\sigma + \bar\lambda)}.$$

Then, the two-solitonic solution is obtained as

$$\begin{aligned} g_{tt}^{(\mathrm{phys})} &= -\frac{\Omega G_{tt}}{\mu_1\mu_2\Sigma}, \\ g_{t\phi}^{(\mathrm{phys})} &= -g_2\frac{\Omega(\rho^2 + \mu_1\mu_2)G_{t\phi}}{\mu_1\mu_2\Sigma}, \\ g_{\phi\phi}^{(\mathrm{phys})} &= -g_2\frac{\Omega G_{\phi\phi}}{\mu_1\mu_2\Sigma}, \\ g_{\psi\psi}^{(\mathrm{phys})} &= \Omega^{-2}g_3, \\ g_{\phi\psi}^{(\mathrm{phys})} &= g_{t\psi}^{(\mathrm{phys})} = 0, \end{aligned}$$

where the functions $G_{tt}$, $G_{t\phi}$, $G_{\phi\phi}$, and $\Sigma$ are given by

$$\begin{aligned} G_{tt} = &-m_{01}^{(1)2}m_{01}^{(2)2}\psi_2[\mu_1]^2\psi_2[\mu_2]^2(\mu_1 - \mu_2)^2\rho^4 \\ &+ m_{01}^{(1)2}m_{02}^{(2)2}g_2\mu_2^2(\rho^2 + \mu_1\mu_2)^2\psi_2[\mu_1]^2 \\ &+ m_{01}^{(2)2}m_{02}^{(1)2}g_2\mu_1^2(\rho^2 + \mu_1\mu_2)^2\psi_2[\mu_2]^2 \\ &- m_{02}^{(1)2}m_{02}^{(2)2}g_2^2\mu_1^2\mu_2^2(\mu_1 - \mu_2)^2 \\ &- 2m_{01}^{(1)}m_{01}^{(2)}m_{02}^{(1)}m_{02}^{(2)}g_2\psi_2[\mu_1]\psi_2[\mu_2] \\ &\times(\rho^2 + \mu_1^2)(\rho^2 + \mu_2^2)\mu_1\mu_2, \end{aligned} \tag{A8}$$

$$\begin{aligned} G_{\phi\phi} = &\, m_{01}^{(1)2}m_{01}^{(2)2}\mu_1^2\mu_2^2(\mu_1 - \mu_2)^2\psi_2[\mu_1]^2\psi_2[\mu_2]^2 \\ &+ m_{02}^{(1)2}m_{02}^{(2)2}g_2^2(\mu_1 - \mu_2)^2\rho^4 \\ &- m_{01}^{(1)2}m_{02}^{(2)2}g_2\mu_1^2\psi_2[\mu_1]^2(\rho^2 + \mu_1\mu_2)^2 \\ &- m_{01}^{(2)2}m_{02}^{(1)2}g_2\mu_2^2\psi_2[\mu_2]^2(\rho^2 + \mu_1\mu_2)^2 \\ &+ 2m_{01}^{(1)}m_{01}^{(2)}m_{02}^{(1)}m_{02}^{(2)}g_2\mu_1\mu_2\psi_2[\mu_2]\psi_2[\mu_1] \\ &\times(\rho^2 + \mu_1^2)(\rho^2 + \mu_2^2), \end{aligned} \tag{A9}$$

$$\begin{aligned} G_{t\phi} = &\, m_{01}^{(1)}m_{01}^{(2)2}m_{02}^{(1)}\mu_2(\mu_1 - \mu_2)\psi_2[\mu_2]^2\psi_2[\mu_1] \\ &\times(\rho^2 + \mu_1^2) + m_{01}^{(1)}m_{02}^{(1)}m_{02}^{(2)2}g_2\mu_2(\mu_2 - \mu_1) \\ &\times\psi_2[\mu_1](\rho^2 + \mu_1^2) + m_{01}^{(1)2}m_{01}^{(2)}m_{02}^{(2)}\mu_1(\mu_2 - \mu_1) \\ &\times\psi_2[\mu_1]^2\psi_2[\mu_2](\rho^2 + \mu_2^2) \\ &+ m_{01}^{(2)}m_{02}^{(1)2}m_{02}^{(2)}\mu_1 g_2\psi_2[\mu_2](\rho^2 + \mu_2^2)(\mu_1 - \mu_2), \end{aligned} \tag{A10}$$

$$\begin{aligned}\Sigma = {} & m_{01}^{(1)2} m_{01}^{(2)2} \psi_2[\mu_1]^2 \psi_2[\mu_2]^2 (\mu_1 - \mu_2)^2 \rho^2 \\ & + m_{02}^{(1)2} m_{02}^{(2)2} g_2^2 (\mu_1 - \mu_2)^2 \rho^2 \\ & + m_{01}^{(1)2} m_{02}^{(2)2} g_2 \psi_2[\mu_1]^2 (\rho^2 + \mu_1\mu_2)^2 \\ & + m_{02}^{(1)2} m_{01}^{(2)2} g_2 \psi_2[\mu_2]^2 (\rho^2 + \mu_1\mu_2)^2 \\ & - 2 m_{01}^{(1)} m_{01}^{(2)} m_{02}^{(1)} m_{02}^{(2)} g_2 \psi_2[\mu_1] \psi_2[\mu_2] \\ & \times (\rho^2 + \mu_1^2)(\rho^2 + \mu_2^2). \end{aligned} \tag{A11}$$

Here, $\mu_1$ and $\mu_2$ are given by

$$\begin{aligned} \mu_1(\rho, z) &= \sqrt{\rho^2 + (z + \sigma)^2} - (z + \sigma), \\ \mu_2(\rho, z) &= \sqrt{\rho^2 + (z - \sigma)^2} - (z - \sigma). \end{aligned} \tag{A12}$$

We should note that this three-dimensional metric $g_{ij}^{(\mathrm{phys})}$ satisfies the supplementary condition $\det g_{ij} = -\rho^2$. Next, let us consider the coordinate transformation of the physical metric such that

$$t \to t' = t - C_1 \phi, \qquad \phi \to \phi' = \phi, \tag{A13}$$

where $C_1$ is a constant. Under this transformation, the physical metric becomes

$$\begin{aligned} g_{tt}^{(\mathrm{phys})} &\to g_{tt} = g_{tt}^{(\mathrm{phys})}, \\ g_{t\phi}^{(\mathrm{phys})} &\to g_{t\phi} = g_{t\phi}^{(\mathrm{phys})} + C_1 g_{tt}^{(\mathrm{phys})}, \\ g_{\phi\phi}^{(\mathrm{phys})} &\to g_{\phi\phi} = g_{\phi\phi}^{(\mathrm{phys})} + 2C_1 g_{t\phi}^{(\mathrm{phys})} + C_1^2 g_{tt}^{(\mathrm{phys})}. \end{aligned} \tag{A14}$$

Here, we should note that the transformed metric also satisfies the supplementary condition $\det g = -\rho^2$. Though the metric seems to contain the four new parameters $m_{01}^{(1)}$, $m_{01}^{(2)}$, $m_{02}^{(1)}$, and $m_{02}^{(2)}$, it can be written only in term of the ratios

$$\alpha := \frac{m_{02}^{(2)}}{m_{01}^{(2)}}, \qquad \beta := -\frac{m_{01}^{(1)}}{m_{02}^{(1)}}. \tag{A15}$$

Using the parameters $\alpha$ and $\beta$, we can write all components of the metric. The metric function $f(\rho, z)$ takes the following form:

$$\begin{aligned} f = {} & \frac{C_2 Y_{\sigma,-\sigma} Y_{-\sigma,\eta_2\sigma}}{16\sigma^2 Y_{\sigma,\eta_2\sigma}} \\ & \times \sqrt{\frac{Y_{\sigma,\eta_1\sigma} Y_{\sigma,\lambda\sigma} Y_{\eta_1\sigma,\eta_2\sigma} Y_{\lambda\sigma,\eta_1\sigma} Y_{\lambda\sigma,\eta_2\sigma}}{Y_{-\sigma,-\sigma} Y_{\eta_1\sigma,\eta_1\sigma} Y_{\eta_2\sigma,\eta_2\sigma} Y_{\lambda\sigma,\lambda\sigma} Y_{-\sigma,\eta_1\sigma} Y_{-\sigma,\lambda\sigma} Y_{\sigma,\sigma}}} \\ & \times \frac{\Omega Y}{(\rho^2 + \mu_1\mu_2)^4 \mu_1^3 \mu_2 \psi_2[\mu_2]^2}, \end{aligned} \tag{A16}$$

where $C_2$ is an arbitrary constant, $Y_{c,d}$ is defined as $Y_{c,d} := R_c R_d + (z - c)(z - d) + \rho^2$ and the function $Y$ is given by

$$\begin{aligned} Y = {} & \rho^2 [16\beta\sigma^2 \mu_1^2 \mu_2^2 \psi_2[\mu_1] \psi_2[\mu_2] - \alpha g_2 (\mu_1 - \mu_2)^2 \\ & \times (\rho^2 + \mu_1\mu_2)^2]^2 + 16\sigma^2 g_2 \mu_1^2 \mu_2^2 (\rho^2 + \mu_1\mu_2)^4 \\ & \times (\psi_2[\mu_2] + \alpha\beta\psi_2[\mu_1])^2. \end{aligned}$$


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