

Self-action of a point charge in a wormhole space-timeNail R. Khusnutdinov^{1,2,*} and Ilya V. Bakhmatov^{1,†}¹*Department of Physics, Kazan State University, Kremlevskaya 18, Kazan 420008, Russia*²*Department of Physics, Tatar State University of Humanity and Pedagogic, Tatarstan 2, Kazan 420021, Russia*

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We consider the self-energy and the self-force for an electrically charged particle at rest in the wormhole space-time. We develop a general approach to finding the self-force and apply it to the two specific profiles of the wormhole throat with singular and with smooth curvature. The self-force for these two profiles is found in manifest form; it turns out to be an attractive force. We also find an expression for the self-force in the case of arbitrary symmetric throat profile. Far from the throat the self-force is always attractive.

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I. INTRODUCTION

Wormholes are topological handles that join different regions of the Universe or different universes. The activity in wormhole physics was initiated by the classical paper by Einstein and Rosen [1], and later by Wheeler [2]. The most recent growth of interest in wormholes was connected with the “time machine,” introduced by Morris, Thorne, and Yurtsever in Refs. [3,4]. Their works led to a surge of activity in wormhole physics [5]. The main and yet unsolved problem in wormhole physics is whether wormholes exist or not. The wormhole has to violate energy conditions and thus a source of the wormhole geometry should be exotic matter. One of the examples [6–8] of such exotic matter is vacuum fluctuations of some quantum field, which may violate the energy conditions. Other possible sources are a scalar field with reversed sign of kinetic term [9] and cosmic phantom energy [10,11]. The issue of wormholes’ stability is complicated and requires subtle calculations (see, for example [12]). The problem arises because a wormhole needs some amount of exotic matter which violates energy conditions and has unusual properties. Recently it was found that some observational features of black holes can be closely mimicked by spherically symmetric static wormholes [13] having no event horizon. Some astronomical observations indicate the possible existence of black holes (see, for example, Ref. [14]). It is therefore important to consider possible astronomical evidences of wormholes. There is an interesting observation of quasar Q0957 + 561 which shows the existence of a compact object without an event horizon. Some aspects of the wormholes’ astrophysics were considered in Ref. [15], where it was noted that this massive compact object may correspond to a wormhole of macroscopic size with a strong magnetic field. Matter may go in and come back out through the throat of the wormhole.

In the framework of general relativity there exists a specific type of interaction of particles with gravitating

objects: the gravitationally induced self-interaction force. This force may have considerable effect on wormhole physics. It is well known [16] that in a curved background alongside with the standard Abraham-Lorentz-Dirac self-force there exists a specific force acting on a charged particle. This force is the manifestation of nonlocal essence of the electromagnetic field. It was considered in details in some specific space-times (see Refs. [17,18] for review). For example, in the case of straight cosmic string space-time the self-force appears to be the only form of interaction between the particle and the string. Cosmic string has no Newtonian potential but nevertheless a massless charged particle is repelled by the string [19], whereas a massive uncharged particle is attracted by the string due to the self-force [20]. This conclusion will not change if the string has nontrivial internal structure [21]. In this case a potential barrier appears which prevents the charged particle from penetrating into the string. For grand unified theory (GUT) cosmic strings the potential barrier is $\sim 10^5$ Gev. Thus the self-action effects play an important role in the physics of space-times with topological defects.

The wormhole is an example of the space-time with nontrivial topology. Consideration of the Casimir effect for a sphere that surrounds the wormhole throat demonstrates an unusual behavior of the Casimir force [22]—it may change its sign depending on the radius of the sphere. It is expected that the self-force in the wormhole space-time will show an unusual behavior too. As we will show in subsequent sections the self-force does have an unusual behavior—a charged particle is attracted by the wormhole instead of expected repulsive action. This result is valid for an arbitrary profile of the throat. Therefore charged particles are likely to gather at the wormhole throat.

The organization of this paper is as follows. In Sec. II we briefly discuss the background under consideration and describe all the geometrical characteristics we need. In Sec. III we develop a general approach to calculation of the self-force in this background for an arbitrary profile of the throat and apply it to the two specific kinds of profiles, finding the self-force explicitly. We obtain simple formulae for the self-force in these cases. However, in practice this

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approach is applicable only for simple enough throat profiles, and therefore in Sec. IV we develop an alternative method which is appropriate for an arbitrary symmetric throat profile. We obtain general formulae and find a general expression for the self-force far from the wormhole throat. Section V is devoted to the discussion of the results. Throughout this paper we use units $c = G = 1$.

II. THE BACKGROUND

Let us consider an asymptotically flat wormhole space-time. We choose the line element of this space-time in the following form

$$ds^2 = -dt^2 + d\rho^2 + r^2(\rho)(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

where $t, \rho \in \mathbb{R}$ and $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$. The shape of the wormhole throat is described by the function $r(\rho)$, which we call a throat profile function. This function must have a minimum at $\rho = 0$, and the minimal value at $\rho = 0$ corresponds to the radius, a , of the wormhole throat,

$$r(0) = a, \quad \dot{r}(0) = 0,$$

where a dot denotes a derivative with respect to the radial coordinate ρ . Space-time is naturally divided into two regions in accordance with the sign of ρ . We shall label the space-time region with positive (negative) ρ and functions on this region with the sign “+” (“-”).

The space-time possesses nonzero curvature. The scalar curvature is given by

$$R = -\frac{2(2r\ddot{r} + \dot{r}^2 - 1)}{r^2}.$$

Far from the wormhole throat the space-time becomes Minkowskian,

$$r(\rho)|_{\rho \rightarrow \pm\infty} = \pm\rho. \quad (2)$$

Various kinds of throat profiles have been already considered in another context [7]. The simplest model of a wormhole is that with an infinitely short throat [7],

$$r = a + |\rho|. \quad (3)$$

The space-time is flat everywhere except for the throat, $\rho = 0$, where the curvature has deltalike form,

$$R = -8\frac{\delta(\rho)}{a}.$$

Another wormhole space-time that is characterized by the throat profile

$$r = \sqrt{a^2 + \rho^2} \quad (4)$$

is free of curvature singularities:

$$R = -\frac{2a^2}{(a^2 + \rho^2)^2}.$$

The wormholes with the following profiles of throat

$$r = \rho \coth\frac{\rho}{b} + a - b, \quad r = \rho \tanh\frac{\rho}{b} + a,$$

have a naturally defined length of the throat, which may be described using the parameter b . The point is that for $\rho > b$ the space-time becomes Minkowskian exponentially fast.

III. SELF-ACTION IN THE WORMHOLE SPACE-TIME

Let us consider a charged particle at rest in the point ρ' , θ' , φ' in the space-time with metric (1). The Maxwell equation for zero component of the potential reads

$$\Delta A^0 = -\frac{4\pi e\delta(\rho - \rho')}{r^2(\rho)} \frac{\delta(\theta - \theta')\delta(\varphi - \varphi')}{\sin\theta},$$

where $\Delta = g^{kl}\nabla_k\nabla_l$. Because of static character of the background we set other components of vector potential to be zero. It is obvious that $A^0 = 4\pi eG(\mathbf{x}; \mathbf{x}')$, where the three-dimensional Green's function G obeys the following equation

$$\Delta G(\mathbf{x}; \mathbf{x}') = -\frac{\delta(\rho - \rho')}{r^2(\rho)} \frac{\delta(\theta - \theta')\delta(\varphi - \varphi')}{\sin\theta}.$$

Because of spherical symmetry we may extract the angular dependence (denoting succinctly $\Omega = (\theta, \varphi)$)

$$G(\mathbf{x}; \mathbf{x}') = \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}(\Omega) Y_{lm}^*(\Omega') g_l(\rho, \rho'),$$

and introduce the radial Green's function g_l subject to the equation

$$\ddot{g}_l + \frac{2r'}{r}\dot{g}_l - \frac{l(l+1)}{r^2}g_l = -\frac{\delta(\rho - \rho')}{r^2(\rho)}. \quad (5)$$

We represent the solution of this equation in the following form

$$g_l = \theta(\rho - \rho')\Psi_1(\rho)\Psi_2(\rho') + \theta(\rho' - \rho)\Psi_1(\rho')\Psi_2(\rho), \quad (6)$$

where functions Ψ are the solutions of the corresponding homogeneous equation

$$\ddot{\Psi} + \frac{2r'}{r}\dot{\Psi} - \frac{l(l+1)}{r^2}\Psi = 0, \quad (7)$$

satisfying the boundary conditions

$$\lim_{\rho \rightarrow +\infty} \Psi_1 = 0, \quad \lim_{\rho \rightarrow +\infty} \Psi_2 = \infty. \quad (8)$$

If one substitutes (6) to (5) the condition for the Wronskian emerges:

$$W(\Psi_1, \Psi_2) = \Psi_1\dot{\Psi}_2 - \dot{\Psi}_1\Psi_2 = \frac{1}{r^2(\rho)}. \quad (9)$$

We consider the radial equation in domains $\rho > 0$ and $\rho < 0$ and obtain a pair of independent solutions ϕ^1, ϕ^2 for each of these domains. We do not need to consider two domains if it is possible to construct solutions that are C^1 -smooth over all space. However, in many cases this is impossible. For two kinds of the throat profile considered below we may easily construct solutions for the two domains separately (but not for all space). After that a procedure developed here allows to construct a C^1 -smooth solution over all space. Because of condition (2) we obtain asymptotically

$$\phi^1|_{\rho \rightarrow +\infty} = \rho^l, \quad \phi^2|_{\rho \rightarrow +\infty} = \rho^{-l-1}.$$

The Wronskian of these solutions has the following form

$$W(\phi_{\pm}^1, \phi_{\pm}^2) = \frac{A_{\pm}}{r^2(\rho)} \quad (10)$$

with some constants A_{\pm} . Let us consider two different solutions:

$$\Psi_1 = \begin{cases} \alpha_+^1 \phi_+^1 + \beta_+^1 \phi_+^2, & \rho > 0 \\ \alpha_-^1 \phi_-^1 + \beta_-^1 \phi_-^2, & \rho < 0 \end{cases}$$

$$\Psi_2 = \begin{cases} \alpha_+^2 \phi_+^1 + \beta_+^2 \phi_+^2, & \rho > 0 \\ \alpha_-^2 \phi_-^1 + \beta_-^2 \phi_-^2, & \rho < 0 \end{cases}$$

The Wronskian condition (10) implies the constraint on the coefficients:

$$\alpha_{\pm}^1 \beta_{\pm}^2 - \beta_{\pm}^1 \alpha_{\pm}^2 = \frac{1}{A_{\pm}}.$$

The general solution of the matching conditions, $\Psi_+(0) = \Psi_-(0), \dot{\Psi}_+(0) = \dot{\Psi}_-(0)$, has the following form

$$\alpha_+ = \alpha_- \frac{W(\phi_-^1, \phi_+^2)}{W(\phi_+^1, \phi_+^2)} \Big|_0 - \beta_- \frac{W(\phi_+^2, \phi_-^2)}{W(\phi_+^1, \phi_+^2)} \Big|_0,$$

$$\beta_+ = \alpha_- \frac{W(\phi_+^1, \phi_-^1)}{W(\phi_+^1, \phi_+^2)} \Big|_0 + \beta_- \frac{W(\phi_+^1, \phi_-^2)}{W(\phi_+^1, \phi_+^2)} \Big|_0,$$

or vice versa

$$\alpha_- = +\alpha_+ \frac{W(\phi_+^1, \phi_-^2)}{W(\phi_-^1, \phi_-^2)} \Big|_0 + \beta_+ \frac{W(\phi_+^2, \phi_-^2)}{W(\phi_-^1, \phi_-^2)} \Big|_0,$$

$$\beta_- = -\alpha_+ \frac{W(\phi_+^1, \phi_-^1)}{W(\phi_-^1, \phi_-^2)} \Big|_0 + \beta_+ \frac{W(\phi_+^1, \phi_+^2)}{W(\phi_-^1, \phi_-^2)} \Big|_0.$$

To satisfy the conditions (8) we consider two specific solutions that emerge when we set $\alpha_+^1 = 0$ and $\alpha_-^2 = 0$. These solutions have the following form:

$$\Psi_1 = \begin{cases} \beta_+^1 \phi_+^2, & \rho > 0 \\ \alpha_-^1 \phi_-^1 + \beta_-^1 \phi_-^2, & \rho < 0 \end{cases}$$

$$\Psi_2 = \begin{cases} \alpha_+^2 \phi_+^1 + \beta_+^2 \phi_+^2, & \rho > 0 \\ \beta_-^2 \phi_-^2, & \rho < 0 \end{cases}$$

where

$$\alpha_-^1 = \beta_+^1 \frac{W(\phi_+^2, \phi_-^2)}{W(\phi_-^1, \phi_-^2)} \Big|_0, \quad \beta_-^1 = \beta_+^1 \frac{W(\phi_+^1, \phi_+^2)}{W(\phi_-^1, \phi_-^2)} \Big|_0,$$

and

$$\alpha_+^2 = -\beta_-^2 \frac{W(\phi_+^2, \phi_-^2)}{W(\phi_+^1, \phi_+^2)} \Big|_0,$$

$$\beta_+^2 = +\beta_-^2 \frac{W(\phi_+^1, \phi_+^2)}{W(\phi_+^1, \phi_+^2)} \Big|_0.$$

The Wronskian condition reads

$$-\beta_+^1 \alpha_+^2 = \frac{1}{A_+}, \quad \alpha_-^1 \beta_-^2 = \frac{1}{A_-}.$$

Taking into account the above relations we obtain the radial Green's function in the following form

(1) $\rho > \rho' > 0$

$$g_l^{(1)}(\rho, \rho') = -\frac{1}{A_+} \phi_+^2(\rho') \phi_+^1(\rho) + \frac{1}{A_+} \frac{W(\phi_+^1, \phi_-^2)}{W(\phi_+^2, \phi_-^2)} \Big|_0 \phi_+^2(\rho') \phi_+^2(\rho) \quad (11a)$$

(2) $0 < \rho < \rho'$

$$g_l^{(2)}(\rho, \rho') = g_l^{(1)}(\rho', \rho) \quad (11b)$$

(3) $\rho < \rho'$ and $\rho' > 0, \rho < 0$

$$g_l^{(3)}(\rho, \rho') = \frac{1}{A_+} \frac{W(\phi_+^1, \phi_+^2)}{W(\phi_+^2, \phi_-^2)} \Big|_0 \phi_+^2(\rho') \phi_-^2(\rho) \quad (11c)$$

(4) $\rho > \rho'$ and $\rho' < 0, \rho > 0$

$$g_l^{(4)}(\rho, \rho') = g_l^{(3)}(\rho', \rho) \quad (11d)$$

(5) $\rho' < \rho < 0$

$$g_l^{(5)}(\rho, \rho') = \frac{1}{A_-} \phi_-^2(\rho') \phi_-^1(\rho) + \frac{1}{A_-} \frac{W(\phi_-^1, \phi_+^2)}{W(\phi_+^2, \phi_-^2)} \Big|_0 \phi_-^2(\rho') \phi_-^2(\rho) \quad (11e)$$

(6) $\rho < \rho' < 0$

$$g_l^{(6)}(\rho, \rho') = g_l^{(5)}(\rho', \rho), \quad (11f)$$

where the constants A_{\pm} may be found from the relation (10):

$$A_{\pm} = W(\phi_{\pm}^1, \phi_{\pm}^2)r^2(\rho) \quad (12)$$

at an arbitrary point ρ .

Let us consider the simple case of symmetric throat profile in detail: $r(-\rho) = r(\rho)$. In this case we may set $\phi_{+}^{1,2}(\rho) = \phi_{+}^{1,2}(-\rho)$ and hence $\phi_{+}^{1,2}(0) = \phi_{+}^{1,2}(0)$, $\phi_{+}^{1,2}(0)' = -\phi_{+}^{1,2}(0)'$, and $A_{+} = -A_{-}$. Taking into account these formulae we obtain

$$(1) \quad \rho > \rho' > 0$$

$$g_l^{(1)}(\rho, \rho') = -\frac{1}{A_{+}} \phi_{+}^2(\rho') \phi_{+}^1(\rho) + \frac{1}{A_{+}} \frac{W_{+}(\phi_{+}^1, \phi_{+}^2)}{W_{+}(\phi_{+}^2, \phi_{+}^2)} \Big|_0 \phi_{+}^2(\rho') \phi_{+}^2(\rho) \quad (13a)$$

$$(2) \quad 0 < \rho < \rho'$$

$$g_l^{(2)}(\rho, \rho') = g_l^{(1)}(\rho', \rho) \quad (13b)$$

$$(3) \quad \rho < \rho' \text{ and } \rho' > 0 \quad \rho < 0$$

$$g_l^{(3)}(\rho, \rho') = -\frac{1}{A_{+}} \frac{W(\phi_{+}^1, \phi_{+}^2)}{W_{+}(\phi_{+}^2, \phi_{+}^2)} \Big|_0 \phi_{+}^2(\rho') \phi_{+}^2(-\rho) \quad (13c)$$

$$(4) \quad \rho > \rho' \text{ and } \rho' < 0 \quad \rho > 0$$

$$g_l^{(4)}(\rho, \rho') = g_l^{(3)}(\rho', \rho') \quad (13d)$$

$$(5) \quad \rho' < \rho < 0$$

$$g_l^{(5)}(\rho, \rho') = g_l^{(1)}(-\rho, -\rho') \quad (13e)$$

$$(6) \quad \rho < \rho' < 0$$

$$g_l^{(6)}(\rho, \rho') = g_l^{(5)}(\rho', \rho). \quad (13f)$$

Here $W_{+}(y_1, y_2) = y_1 \dot{y}_2 + \dot{y}_1 y_2$. Thus we have to write out only $g_l^{(1)}$ and $g_l^{(3)}$ in manifest form.

The Green's function will not yield a finite expression for the self-force. The origin of this divergence is the electromagnetic self-energy of the point particle. Several methods to obtain a finite result are known. The most simple way is to consider total mass as a sum of observed finite mass and an infinite electromagnetic contribution. Usually this procedure is called ‘‘classical renormalization’’ because there is no Planck constant in the divergent term. Dirac [23] suggested to consider the radiative Green's function to calculate the self-force. Since the radiative Green's function is the difference between retarded and advanced Green's functions, the singular con-

tribution cancels out and we obtain a finite result. There is also an axiomatic approach suggested by Quinn and Wald [24]. In the framework of this approach one obtains a finite expression by using a ‘‘comparison’’ axiom. This approach was used in Refs. [25,26] for a specific space-time.

We will use a general approach to renormalization in curved space-time [27], which means the subtraction of the first terms from the DeWitt-Schwinger asymptotic expansion of the Green's function. In general there are two kinds of divergences in this expansion, namely, pole and logarithmic ones [28]. In the three-dimensional case which we are interested in there is only pole divergence, while the logarithmic term is absent. The singular part of the Green's function, which must be subtracted, has the following form (in 3D case)

$$G^{\text{sing}} = \frac{1}{4\pi} \frac{\Delta^{1/2}}{\sqrt{2\sigma}},$$

where σ is half of the square of geodesic distance and Δ is the DeWitt-Morette determinant. If we take the coincidence limit for angular variables then these quantities are easily calculated using the metric (1): $\sigma = (\rho - \rho')^2/2$ and $\Delta = 1$. Thus to carry out renormalization we should subtract the singular part of the Green's function, which has the following form:

$$G^{\text{sing}} = \frac{1}{4\pi} \frac{1}{|\rho - \rho'|}.$$

This is, in fact, the Green's function in Minkowski space-time. This approach was used many times in different curved backgrounds (e.g. see [18]). Now we are in a position to consider several specific cases in detail and after that to proceed to the general profile of the throat.

A. Profile $r = a + |\rho|$

Two linearly independent solutions, ϕ^1, ϕ^2 , are given by

$$\phi_{\pm}^1 = r(\pm\rho)^l = (a \pm \rho)^l, \quad (14a)$$

$$\phi_{\pm}^2 = a^{2l+1} r(\pm\rho)^{-l-1} = a^{2l+1} (a \pm \rho)^{-l-1}, \quad (14b)$$

with the Wronskian

$$W(\phi_{\pm}^1, \phi_{\pm}^2) = \mp(2l+1) \frac{a^{2l+1}}{r^2(\rho)}. \quad (15)$$

Therefore the solutions we need have the following form

$$\Psi_1 = \beta_+^1 \begin{cases} \phi_{+}^2, & \rho > 0 \\ \frac{2(l+1)}{2l+1} \phi_{+}^1 - \frac{1}{2l+1} \phi_{-}^2, & \rho < 0 \end{cases}$$

$$\Psi_2 = \beta_-^2 \begin{cases} \frac{2(l+1)}{2l+1} \phi_{+}^1 - \frac{1}{2l+1} \phi_{+}^2, & \rho > 0 \\ \phi_{-}^2, & \rho < 0 \end{cases}$$

From Eqs. (9) and (15) we obtain an additional constraint

$$\beta_+^1 \beta_-^2 = \frac{1}{a^{2l+1} 2(l+1)}.$$

Thus we have found the radial part of the Green's function in manifest form

$$(2l+1)g_l^{(1)}(\rho, \rho') = \frac{r^l(\rho')}{r^{l+1}(\rho)} - \frac{1}{2(l+1)} \frac{a^{2l+1}}{r^{l+1}(\rho)r^{l+1}(\rho')}, \quad (16a)$$

$$(2l+1)g_l^{(3)}(\rho, \rho') = \frac{2l+1}{2(l+1)} \frac{a^{2l+1}}{r^{l+1}(\rho)r^{l+1}(\rho')}. \quad (16b)$$

To calculate the full Green's function and the potential we turn to the relation

$$\sum_{m=-l}^l Y_{lm}(\Omega) Y_{lm}^*(\Omega') = \frac{2l+1}{4\pi} P_l(\cos\gamma),$$

by using which we obtain

$$G(x; x') = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\gamma) g_l(\rho, \rho').$$

Here $\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varphi - \varphi')$ is a cosine of an angle between two points on a sphere. Now we use two series expansions:

$$\sum_{k=0}^{\infty} t^k P_k(x) = \frac{1}{\sqrt{1-2tx+t^2}},$$

$$\sum_{k=0}^{\infty} \frac{t^{k+1}}{k+1} P_k(x) = \ln \left| 1 + \frac{2t}{1-t+\sqrt{t^2-2tx+1}} \right|,$$

which allow us to obtain the Green's function in manifest form

$$4\pi G^{(1)}(x; x') = \frac{1}{\sqrt{r(\rho)^2 - 2r(\rho)r(\rho')\cos\gamma + r(\rho')^2}} - \frac{1}{2a} \ln \left| 1 + \frac{2t}{1-t+\sqrt{t^2-2t\cos\gamma+1}} \right|,$$

$$4\pi G^{(3)}(x; x') = \frac{t}{a\sqrt{t^2-2t\cos\gamma+1}} - \frac{1}{2a} \ln \left| 1 + \frac{2t}{1-t+\sqrt{t^2-2t\cos\gamma+1}} \right|,$$

where $t = \frac{a^2}{r(\rho)r(\rho')}$.

Let us consider the case when the observation point, ρ , is far from both the wormhole throat and from the position of a particle, ρ' ; that is we set $\rho \rightarrow \infty$. We consider $\rho' > 0$. In this limit we have

$$\rho > 0: 4\pi G(x; x') = \frac{1}{\rho} - \frac{a}{2\rho(a+\rho')} + O(\rho^{-2}), \quad (17a)$$

$$\rho < 0: 4\pi G(x; x') = -\frac{a}{2\rho(a+\rho')} + O(\rho^{-2}). \quad (17b)$$

From the first of these expressions we observe that the potential of the particle contains an additional term with the same ρ dependence in addition to the standard Coulomb part. The same problem arises for a particle in the Schwarzschild space-time [29]. To solve this problem it was suggested to add a solution of the corresponding homogeneous equation to the solution with a ‘‘bad’’ asymptotical behavior [30]. However, in our case there is no need for such a modification: the potential (17) correctly describes the real potential of the particle at rest. Indeed, let us consider the Gauss theorem in the space-time of the wormhole. We place the particle at the point ρ' and enclose this charge by two spheres, namely, $S_1: \rho = R$ and $S_2: \rho = -R$. The Gauss theorem reads

$$4\pi e = \iint_{S_1} \mathbf{E} n dS + \iint_{S_2} \mathbf{E} n dS,$$

where $\mathbf{E} = -\nabla A_0$ and $n = (\pm 1, 0, 0)$ for S_1 and S_2 , correspondingly. Therefore we have

$$4\pi e = - \iint_{S_1} \partial_\rho A_0|_{\rho=R} dS + \iint_{S_2} \partial_\rho A_0|_{\rho=-R} dS. \quad (18)$$

Taking into account formulae (17) it is easy to show that the Gauss theorem is satisfied and we do not need to add a solution of the homogeneous equation to correct the potential. Thus the potential has bad behavior as the distance between the observation point and the particle increases

$$A_0 = 4\pi e G(\rho; \rho') \approx \frac{e}{\rho} \left[1 - \frac{a}{2(\rho' + a)} \right]$$

instead of the expected e/ρ dependence. In fact, the observation of a charge e from a large distance would yield the result $e(1 - \frac{a}{2(\rho'+a)})$. The explanation is very easy—some electric field lines have gone through the throat to another, invisible domain of space-time and we must take these field lines into account in order to formulate the Gauss theorem correctly. Hence the second sphere S_2 emerges. The Gauss theorem relates the charge of the particle to the electric field flux density. In Fig. 1 the surfaces of the constant potential A_0 of the particle are plotted.

Now let us restrict ourselves to the + part of space-time and consider the situation when $\Omega = \Omega'$. We have the following expression for the Green's function

$$G(\rho; \rho') = \frac{1}{4\pi} \frac{1}{|\rho - \rho'|} + \frac{1}{8\pi a} \ln \left[1 - \frac{a^2}{r(\rho)r(\rho')} \right].$$

Therefore, after renormalization we obtain

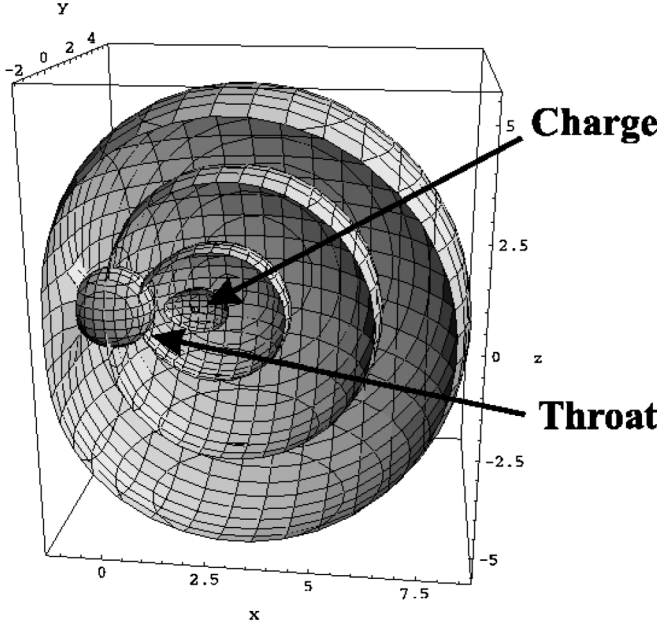


FIG. 1. Several surfaces of constant potential are shown. The charge $e = 1$ is at the point $x = 2, y = z = 0$. The small sphere is the throat of the wormhole, $\rho = 0$. We observe that some surfaces go under the throat to another universe.

$$G^{\text{ren}}(\rho; \rho') = \frac{1}{8\pi a} \ln \left[1 - \frac{a^2}{r(\rho)r(\rho')} \right].$$

The potential at the position of the particle reads

$$\Phi = \frac{e}{2a} \ln \left[1 - \frac{a^2}{(a + \rho)^2} \right],$$

and the self-potential is

$$U = \frac{e^2}{4a} \ln \left[1 - \frac{a^2}{(a + \rho)^2} \right]. \quad (19)$$

Some limiting cases are

$$\begin{aligned} U|_{\rho \rightarrow 0} &= \frac{e^2}{4a} \ln \frac{2\rho}{a}, & U|_{\rho \rightarrow \infty} &= -\frac{e^2 a}{4\rho^2} + \frac{a^2}{2\rho^3}, \\ U|_{a \rightarrow 0} &= -\frac{ae^2}{4(a + \rho)^2}. \end{aligned} \quad (20)$$

The self-force

$$\mathbf{F} = -\nabla U$$

has the radial component only

$$F^\rho = -\partial_\rho U = -\frac{ae^2}{2r(\rho)^3} \frac{1}{1 - \frac{a^2}{r(\rho)^2}}.$$

The self-force is always attractive; it turns into infinity at the throat and goes down monotonically to zero as $\rho \rightarrow \infty$. We may compare this expression with its analog for Schwarzschild space-time with Schwarzschild radius $r_s = a$:

$$F^r = +\frac{ae^2}{2r^3} \sqrt{1 - \frac{a^2}{r^2}}.$$

Important observations are

- (1) The self-force in the wormhole space-time has an opposite sign—it is attractive.
- (2) Far from the wormhole throat and from the black hole we have the same results but with opposite signs

$$F_{wh}^\rho = -\frac{ae^2}{2\rho^3}, \quad F_{bh}^\rho = +\frac{ae^2}{2\rho^3}.$$

- (3) At the Schwarzschild radius $r_s = a$ the self-force equals zero, whereas at the wormhole throat it tends to infinity. The latter discrepancy originates in the selected throat profile function that leads to the curvature singularity at the throat.

B. Profile $r = \sqrt{a^2 + \rho^2}$

The radial equation (7) has two following linearly independent solutions

$$\phi_+^1 = c_1^+ P_l(z), \quad (21a)$$

$$\phi_+^2 = c_2^+ Q_l(z), \quad (21b)$$

with Wronskian

$$W(\phi_+^1, \phi_+^2) = ic_1^+ c_2^+ \frac{a}{r^2}. \quad (22)$$

Here P_l and Q_l are the Legendre polynomials of the first and second kind, and $z = i\rho/a$. In the special case when $c_1^+ = \frac{(-ia)^l l!}{(2l-1)!!}$, $c_2^+ = \frac{(2l+1)!! a^l}{(-i)^{l+1} l!}$ the solutions are real functions and far enough from the wormhole we have

$$\phi_+^1|_{\rho \rightarrow +\infty} = \rho^l, \quad \phi_+^2|_{\rho \rightarrow +\infty} = a^{2l+1} \rho^{-l-1},$$

as in the above example. We should note that solutions given by Eq. (21) are not C^1 -smooth over all space. The point is that the above functions have the following symmetry properties [31]

$$P_l(-z) = (-1)^l P_l(z), \quad Q_l(-z) = (-1)^{l+1} Q_l(z).$$

These conditions imply that the functions are not C^1 -smooth at the throat $\rho = 0$. However, the procedure developed above yields a C^1 -smooth solution provided that we have solutions in domains $\rho > 0$ and $\rho < 0$. For $\rho < 0$ we choose

$$\phi_-^1 = c_1^- P_l(-z), \quad (23a)$$

$$\phi_-^2 = c_2^- Q_l(-z), \quad (23b)$$

with Wronskian

$$W(\phi_-^1, \phi_-^2) = -ic_1^- c_2^- \frac{a}{r^2}. \quad (24)$$

To simplify these expressions we take $c_1^- = c_1^+$, $c_2^- = c_2^+$. The final result does not depend on the specific choice because the formulae involve only the ratio of the function's values at the points ρ and $\rho = 0$.

To proceed further we use the formulae (we take into account the limit as $\rho \rightarrow 0$ along the imaginary axis from above)

$$\begin{aligned}
 P_l(0) &= \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2} - \frac{l}{2})\Gamma(1 + \frac{l}{2})}, \\
 P'_l(0) &= -\frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2} + \frac{l}{2})\Gamma(-\frac{l}{2})}, \\
 Q_l(0) &= \frac{\sqrt{\pi}}{2} e^{-i(\pi/2)(l+1)} \frac{\Gamma(\frac{1}{2} + \frac{l}{2})}{\Gamma(1 + \frac{l}{2})}, \\
 Q'_l(0) &= \sqrt{\pi} e^{-i(\pi/2)l} \frac{\Gamma(1 + \frac{l}{2})}{\Gamma(\frac{1}{2} + \frac{l}{2})}.
 \end{aligned}$$

With the help of these relations it is easy to show that

$$\left. \frac{W_+(\phi_+, \phi_+^2)}{W_+(\phi_+^2, \phi_+^2)} \right|_0 = \frac{i}{\pi} \frac{c_1^+}{c_2^+}, \tag{25}$$

$$\left. \frac{W(\phi_+, \phi_+^2)}{W_+(\phi_+^2, \phi_+^2)} \right|_0 = (-1)^{l+1} \frac{i}{\pi} \frac{c_1^+}{c_2^-}, \tag{26}$$

$$A_{\pm} = \pm i c_1^+ c_2^{\pm} a. \tag{27}$$

Now we use the formulae (13) and (24) to obtain the radial Green's function

$$\begin{aligned}
 g_l^{(1)}(\rho, \rho') &= \frac{i}{a} P_l(z') Q_l(z) + \frac{1}{\pi a} Q_l(z) Q_l(z'), \\
 g_l^{(3)}(\rho, \rho') &= \frac{1}{\pi a} (-1)^{l+1} Q_l(-z) Q_l(z').
 \end{aligned} \tag{28}$$

Therefore

$$\begin{aligned}
 G^{(1)}(\rho, \rho') &= \frac{1}{4\pi a} \sum_{l=0}^{\infty} (2l+1) \\
 &\quad \times \left\{ i P_l(z') Q_l(z) + \frac{1}{\pi} Q_l(z') Q_l(z) \right\},
 \end{aligned}$$

and

$$G^{(3)}(\rho, \rho') = \frac{1}{4\pi^2 a} \sum_{l=0}^{\infty} (2l+1) (-1)^{l+1} Q_l(z') Q_l(-z),$$

where $z = i\rho/a$ and $z' = i\rho'/a$. Now the Heine formula [31]

$$\sum_{l=0}^{\infty} (2l+1) P_l(z') Q_l(z) = \frac{1}{z - z'}$$

yields

$$\begin{aligned}
 4\pi G^{(1,2)}(\rho, \rho') &= \frac{1}{|\rho - \rho'|} + \frac{1}{\pi a} \sum_{l=0}^{\infty} (2l+1) Q_l(z') Q_l(z), \\
 4\pi G^{(3)}(\rho, \rho') &= \frac{1}{\pi a} \sum_{l=0}^{\infty} (2l+1) (-1)^{l+1} Q_l(z') Q_l(-z).
 \end{aligned}$$

To find these series in closed form we use the integral representation of the Legendre function of the second kind [31]

$$Q_l(z) = \frac{1}{2} \int_{-1}^1 \frac{P_l(t)}{z - t} dt,$$

which yields

$$\begin{aligned}
 \sum_{l=0}^{\infty} (2l+1) Q_l(ix') Q_l(ix) &= -\frac{\arctan x - \arctan x'}{x - x'}, \\
 \sum_{l=0}^{\infty} (2l+1) (-1)^{l+1} Q_l(ix') Q_l(-ix) \\
 &= -\frac{-\arctan x + \arctan x' - \pi}{-x + x'}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 4\pi G^{(1,2)}(\rho, \rho') &= \frac{1}{|\rho - \rho'|} - \frac{1}{\pi} \frac{\arctan \frac{\rho}{a} - \arctan \frac{\rho'}{a}}{\rho - \rho'}, \\
 4\pi G^{(3)}(\rho, \rho') &= -\frac{1}{\pi} \frac{-\arctan \frac{\rho}{a} + \arctan \frac{\rho'}{a} - \pi}{-\rho + \rho'}.
 \end{aligned}$$

Far from the wormhole throat we have

$$\begin{aligned}
 \lim_{\rho \rightarrow +\infty} 4\pi G^{(1,2)}(\rho, \rho') &= \frac{1}{\rho} + \frac{1}{\rho} \left[-\frac{1}{2} + \frac{1}{\pi} \arctan \frac{\rho'}{a} \right], \\
 \lim_{\rho \rightarrow -\infty} 4\pi G^{(3)}(\rho, \rho') &= \frac{1}{\rho} \left[-\frac{1}{2} + \frac{1}{\pi} \arctan \frac{\rho'}{a} \right],
 \end{aligned}$$

hence the potential far from the charge kept at a point ρ' has the following form

$$\begin{aligned}
 A_{(1)}^0 &= \frac{e}{\rho} + \frac{e}{\rho} \left[-\frac{1}{2} + \frac{1}{\pi} \arctan \frac{\rho'}{a} \right], \\
 A_{(3)}^0 &= \frac{e}{\rho} \left[-\frac{1}{2} + \frac{1}{\pi} \arctan \frac{\rho'}{a} \right],
 \end{aligned}$$

and again it obeys the Gauss theorem (18). The renormalized Green's function is given by

$$4\pi G(\rho, \rho)^{\text{ren}} = \frac{1}{\pi a} \sum_{l=0}^{\infty} (2l+1) Q_l(z)^2 = -\frac{1}{\pi} \frac{a}{\rho^2 + a^2}.$$

Therefore we obtain the self-potential

$$U = -\frac{e^2}{2\pi} \frac{a}{\rho^2 + a^2} \tag{29}$$

and the self-force

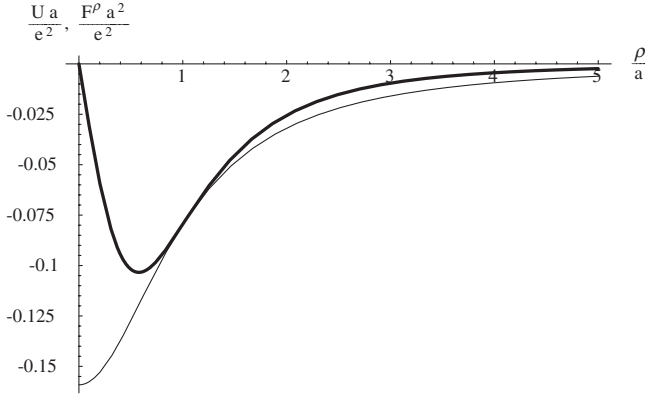


FIG. 2. The thin line is a plot of the potential (29) and the thick line is a plot of the self-force (30). The force has an extremum at the point $\rho^* = a/\sqrt{3}$.

$$F^\rho = \partial_\rho U = -\frac{e^2}{\pi} \frac{a\rho}{(\rho^2 + a^2)^2}. \quad (30)$$

As expected the self-force is everywhere finite and equals zero at the throat. Far from the wormhole we have

$$F^\rho \approx -\frac{e^2}{\pi} \frac{a}{\rho^3}.$$

Thus the self-force is always attractive. It has maximum value at the distance $\rho^* = a/\sqrt{3}$ with magnitude $F_{\max}^\rho = 3\sqrt{3}e^2/16\pi a^2$. Plots of the potential and the self-force are shown in Fig. 2.

IV. GENERAL CASE

In this section we outline our approach to generalization of the results of the previous sections. From now on the profile function $r(\rho)$ will be considered to be an arbitrary symmetric function (the condition of asymptotic flatness given by Eq. (2) is understood).

Let us perform WKB analysis of the defining equation for the radial Green's functions

$$\ddot{\phi} + \frac{2\dot{r}}{r}\dot{\phi} - \frac{l(l+1)}{r^2}\phi = 0. \quad (31)$$

To accomplish this we represent the solution of this equation in the form

$$\phi = e^S, \quad (32)$$

and arrive at the following equation for S :

$$\ddot{S} + \dot{S}^2 + \frac{2\dot{r}}{r}\dot{S} - \frac{\nu^2 - 1/4}{r^2} = 0, \quad (33)$$

where $\nu = l + 1/2$. The next step is to expand S in the following power series in ν [32]:

$$S = \sum_{n=-1}^{\infty} \nu^{-n} S_n. \quad (34)$$

One comment is in order. Equation (33) is valid for an arbitrary $l \geq 0$, whereas the expansion over $1/\nu$ (34) is fulfilled for $l > 0$ only. For $l = 0$ we have $1/\nu = 2$ and the expansion is divergent. Nevertheless, we may easily find g_l for $l = 0$ without turning to the series expansion. Indeed, for $l = 0$ Eq. (31) simplifies considerably (ϕ stands here for the zero mode only):

$$\ddot{\phi} + \frac{2\dot{r}}{r}\dot{\phi} = 0.$$

This equation has two independent solutions:

$$\varphi_+^1 = C_1, \quad \varphi_+^2 = C_2 \int_{\rho_0}^{\rho} \frac{1}{r^2} d\rho + C_3.$$

We take $C_1 = 1$ in order to be in agreement with (14) but this is insufficient. This solution corresponds to the solution $\sim \rho^l$ for $l = 0$. The second solution must tend to zero far from the throat and it corresponds to the solution $\sim \rho^{-l-1}$ for $l = 0$. Therefore we may write

$$\lim_{\rho \rightarrow \infty} \varphi_+^2(\rho) = C_2 \int_{\rho_0}^{\infty} \frac{1}{r^2} d\rho + C_3 = 0,$$

hence

$$C_3 = -C_2 \int_{\rho_0}^{\infty} \frac{1}{r^2} d\rho.$$

Thus we obtain the second solution in the form

$$\varphi_+^2(\rho) = -C_2 \int_{\rho}^{\infty} \frac{1}{r^2} d\rho.$$

We set $C_2 = -a$ to be in agreement with (14). To summarize, we have

$$\varphi_+^1 = 1, \quad \varphi_+^2 = \int_{\rho}^{\infty} \frac{a}{r^2} d\rho, \quad (35)$$

with Wronskian $W(\varphi_+^1, \varphi_+^2) = -a/r^2$. These expressions allow us to obtain zero component of the Green's function

$$g_0^{(1)}(\rho, \rho') = \frac{1}{a} \varphi_+^2(\rho) - \frac{1}{2a} \varphi_+^2(\rho) \frac{\varphi_+^2(\rho')}{\varphi_+^2(0)}.$$

Now that we have separated the $l = 0$ case, it is possible to proceed with $l > 0$ contributions using the series expansion. Substituting (34) to (33) yields the following set of recurring expressions for the functions S_n :

$$\begin{aligned} \dot{S}_{-1} &= \pm \frac{1}{r}, \\ \dot{S}_0 &= -\frac{\ddot{S}_{-1}}{2\dot{S}_{-1}} - \frac{\dot{r}}{r}, \\ \dot{S}_1 &= -\frac{1}{2\dot{S}_{-1}} \left[\ddot{S}_0 + \dot{S}_0^2 + \frac{2\dot{r}}{r}\dot{S}_0 + \frac{1}{4r^2} \right], \\ \dot{S}_{m+1} &= -\frac{1}{2\dot{S}_{-1}} \left[\ddot{S}_m + \sum_{n=0}^m \dot{S}_n \dot{S}_{m-n} + \frac{2\dot{r}}{r}\dot{S}_m \right], \quad m = 1, 2, \dots \end{aligned}$$

The sign “+” in the first equation corresponds to the solution ϕ^1 and sign “-” to the solution ϕ^2 which tends to zero as $\rho \rightarrow \infty$. Following are the manifest expressions for derivatives of the first several functions S_n :

$$\begin{aligned} \dot{S}_{-1} &= \pm \frac{1}{r}, \\ \dot{S}_0 &= -\frac{\dot{r}}{2r} = -\frac{1}{2}(\ln r), \\ \dot{S}_1 &= \pm \frac{1}{4} \left[\ddot{r} + \frac{\dot{r}^2}{2r} - \frac{1}{2r} \right], \\ \dot{S}_2 &= -\frac{1}{8} [r\dot{r}^{(3)} + 2\dot{r}\ddot{r}] = -\frac{1}{8} \left(r\ddot{r} + \frac{1}{2}\dot{r}^2 \right), \\ \dot{S}_3 &= \pm \frac{1}{16} \left[r^2\dot{r}^{(4)} + 4r\dot{r}\ddot{r}^{(3)} + \frac{3}{2}r\dot{r}^2 + \frac{3}{2}\dot{r}^2\ddot{r} + \frac{1}{2}\dot{r} \right. \\ &\quad \left. - \frac{\dot{r}^4}{8r} + \frac{\dot{r}^2}{4r} - \frac{1}{8r} \right], \\ \dot{S}_4 &= -\frac{1}{32} [r^3\dot{r}^{(5)} + 7r^2\dot{r}\ddot{r}^{(4)} + 6r^2\ddot{r}\ddot{r}^{(3)} + 9r\dot{r}^2\ddot{r}^{(3)} + r\dot{r}^{(3)} \\ &\quad + 4r\dot{r}\ddot{r}^2 + 2\dot{r}\ddot{r}] \\ &= -\frac{1}{128} (4r^3\dot{r}^{(4)} + 16r^2\dot{r}\ddot{r}^{(3)} + 4r^2\dot{r}^2 + 4r\dot{r}^2\ddot{r} + 4r\ddot{r} \\ &\quad - \dot{r}^4 + 2\dot{r}^2). \end{aligned}$$

Note that all \dot{S}_{2n} come with only one sign, whereas the signs of \dot{S}_{2n+1} vary according to the choice between ϕ^1 and ϕ^2 . Another important point is that all \dot{S}_{2n} appear to be full derivatives, in contrast to all \dot{S}_{2n+1} . These two points will be essential in the subsequent analysis.

A simple verification of the approach can be done at this point. Taking $r = \sqrt{\rho^2 + a^2}$ we can evaluate the expansion (34) and then compare (32) with the exact solutions (21). Numerical calculation reveals fast convergence of the series to the exact solutions. The series that results when all \dot{S}_{2n+1} are taken with the “-” sign coincides with the solution $i^{n+1}Q_l(z)$ (21b), while the series with the “+” sign corresponds to the following linear combination of (21a) and (21b): $(-i)^n(P_l(z) - \frac{i}{\pi}Q_l(z))$. We would like to note that the result of the previous section would be the same if we had used this linear combination instead of $(-i)^n P_l(z)$.

Let us introduce a pair of indices to help us distinguish between different solutions. From now on S^1 will denote the whole series (34), in which all \dot{S}_{2n+1} are taken with the “+” sign, while S^2 will correspond to the “-” sign. These are essentially the two different solutions that we previously called ϕ^1 and ϕ^2 . We will additionally mark the solutions with “ \pm ” according to the sign of ρ . The Wronskian condition (10) is therefore cast into the form

$$e^{S^{\pm 1}(\rho) + S^{\pm 2}(\rho)} = \frac{A_{\pm}}{r^2(\rho)} \frac{1}{\dot{S}^{\pm 2}(\rho) - \dot{S}^{\pm 1}(\rho)},$$

and evaluating it at $\rho = 0$ we get

$$e^{S^{\pm 1}(0) + S^{\pm 2}(0)} = \frac{A_{\pm}}{a^2} \frac{1}{\dot{S}^{\pm 2}(0) - \dot{S}^{\pm 1}(0)}.$$

Functions \dot{S}_{2n} with even indices have the following structure

$$\dot{S}_{2n}(\rho) = \sum_{k=0}^n s_{2k+1} r^{(2k+1)};$$

each term necessarily contains odd order derivatives of r with respect to ρ . Hence if the profile function $r(\rho)$ is symmetric, $r(-\rho) = r(\rho)$, then $\dot{S}_{2n}(0) = 0$.

By using the above formulae we may represent the Green’s function (13a) in the following form

$$\begin{aligned} g_l^{(1)}(\rho, \rho') &= -\frac{1}{a^2} \frac{e^{S^{+2}(\rho) - S^{+2}(0) + S^{+1}(\rho') - S^{+1}(0)}}{\dot{S}^{+2}(0) - \dot{S}^{+1}(0)} \\ &\quad + \frac{1}{a^2} \frac{e^{S^{+2}(\rho) - S^{+2}(0) + S^{+2}(\rho') - S^{+2}(0)}}{\dot{S}^{+2}(0) - \dot{S}^{+1}(0)} \\ &\quad \times \frac{\dot{S}^{-2}(0) - \dot{S}^{+1}(0)}{\dot{S}^{-2}(0) - \dot{S}^{+2}(0)}. \end{aligned} \tag{36}$$

Because of symmetry of the profile function, $r(-\rho) = r(\rho)$, we may replace $\dot{S}^{-2}(0) = -\dot{S}^{+2}(0)$. From the above chain we obtain in manifest form:

$$\begin{aligned} S_{-1}^{1,2}(\rho) - S_{-1}^{1,2}(0) &= \pm \int_0^{\rho} \frac{d\rho}{r(\rho)}, \\ S_0^{1,2}(\rho) - S_0^{1,2}(0) &= \frac{1}{2} \ln \frac{a}{r(\rho)}. \end{aligned}$$

Therefore the radial Green’s function reads ($l > 0$)

$$\begin{aligned} g_l(\rho, \rho') &= \frac{1}{2\nu} \frac{e^{-\nu \int_{\rho'}^{\rho} \frac{d\rho}{r}}}{\sqrt{r(\rho)r(\rho')}} \frac{e^{\sum_{n=1}^{\infty} \nu^{-n}(S_n^{+2}(\rho) + S_n^{+1}(\rho'))}}{\sum_{n=0}^{\infty} \nu^{-2n}(a\dot{S}_{2n-1}^{+1}(0))} - \frac{1}{4\nu} \\ &\quad \times \frac{e^{-\nu[\int_0^{\rho} \frac{d\rho}{r} + \int_0^{\rho'} \frac{d\rho'}{r}]} \frac{e^{\sum_{n=1}^{\infty} \nu^{-n}(S_n^{+2}(\rho) + S_n^{+2}(\rho'))}}{\sqrt{r(\rho)r(\rho')}}}{\sum_{n=0}^{\infty} \nu^{-2n}(a\dot{S}_{2n-1}^{+1}(0))} \\ &\quad \times \frac{\sum_{n=0}^{\infty} \nu^{-2n}(-2a\dot{S}_{2n}^{+1}(0))}{\sum_{n=0}^{\infty} \nu^{-n+1}(-a\dot{S}_{n-1}^{+2}(0))}. \end{aligned}$$

The second part here gives zero contribution because \dot{S}_{2n}^{+1} contains only odd order derivatives of r , which are zero at $\rho = 0$. This fact may be verified straightforwardly by performing calculations for the $\sqrt{\rho^2 + a^2}$ profile function. In this case

$$\dot{S}^{+2}(0) + \dot{S}^{+1}(0) = 0.$$

We must take into account that S^{+2} corresponds to $P_l(z)$ and S^{+1} corresponds to $P_l(z) - \frac{i}{\pi}Q_l(z)$. This issue was already addressed above.

Let us now perform summation over l in order to obtain full Green's function. We have

$$\begin{aligned} \sum_{l=0}^{\infty} \frac{2\nu}{4\pi} g_l(\rho, \rho') &= \frac{1}{4\pi} \sum_{l=1}^{\infty} \frac{e^{-\nu \int_{\rho'}^{\rho} \frac{d\rho}{r}} e^{\sum_{n=1}^{\infty} \nu^{-n} (S_n^{+2}(\rho) + S_n^{+1}(\rho'))}}{\sqrt{r(\rho)r(\rho')}} \frac{e^{\sum_{n=0}^{\infty} \nu^{-2n} (a \dot{S}_{2n-1}^{+1}(0))}}{\sum_{n=0}^{\infty} \nu^{-2n} (a \dot{S}_{2n-1}^{+1}(0))}} \\ &+ \frac{1}{4\pi} g_0(\rho, \rho') \\ &= \frac{1}{4\pi} \frac{1}{\sqrt{r(\rho)r(\rho')}} \sum_{k=0}^{\infty} f_k(b) j_k(\rho, \rho') \\ &+ \frac{1}{4\pi} g_0(\rho, \rho'), \end{aligned}$$

where

$$\begin{aligned} f_k(b) &= \sum_{l=1}^{\infty} \frac{1}{\nu^k} e^{-\nu b} = \sum_{l=0}^{\infty} \frac{1}{\nu^k} e^{-\nu b} - 2^k e^{-(1/2)b}, \\ b &= \int_{\rho'}^{\rho} \frac{d\rho}{r(\rho)}. \end{aligned}$$

To obtain the above formula we have permuted the summations over k and over l . The functions f_k are expressed in terms of the function Φ [31]

$$f_k(b) = e^{-(b/2)} \Phi(e^{-b}, k, \frac{3}{2}).$$

The first two functions in manifest form are

$$\begin{aligned} f_0(b) &= \frac{1}{2} \frac{1}{\sinh(\frac{b}{2})} - e^{-(1/2)b}, \\ f_1(b) &= \ln \frac{1 + e^{-(b/2)}}{1 - e^{-(b/2)}} - 2e^{-(1/2)b}. \end{aligned}$$

To calculate the self-force we need the coincidence limit $\rho' \rightarrow \rho$. In this case only the first two functions f_k are divergent

$$\begin{aligned} \frac{f_0(b)}{\sqrt{r(\rho)r(\rho')}} &= \frac{1}{\rho - \rho'} - \frac{1}{r} + O(\rho - \rho'), \\ \frac{f_1(b)}{\sqrt{r(\rho)r(\rho')}} &= -\frac{1}{r} \ln \frac{\rho - \rho'}{4r} - \frac{2}{r} + O(\rho - \rho'), \\ \frac{f_k(b)}{\sqrt{r(\rho)r(\rho')}} &= \frac{1}{r} \zeta_H\left(k, \frac{3}{2}\right) + O(\rho - \rho'), \end{aligned}$$

where ζ_H is the Hurwitz zeta function [31].

The functions j_k cannot be found in closed form for arbitrary index k , however, each function may be found from the expansion over ν . In manifest form we obtain the following expressions for these functions:

$$\begin{aligned} j_0(\rho, \rho') &= 1, \\ j_1(\rho, \rho') &= - \int_{\rho'}^{\rho} \frac{-1 + \dot{r}^2 + 2r\ddot{r}}{8r} d\rho \\ &= - \frac{-1 + \dot{r}^2 + 2r\ddot{r}}{8r} (\rho - \rho') + O((\rho - \rho')^2), \\ j_2(\rho, \rho) &= - \frac{-1 + \dot{r}^2 + 2r\ddot{r}}{8} = \frac{3}{8} a_1 r^2, \\ j_4(\rho, \rho) &= \frac{1}{128} [3 + 3\dot{r}^4 - 12r\ddot{r} - 4r^2\ddot{r}^2 - 2\dot{r}^2(3 + 2r\ddot{r}) \\ &\quad - 32r^2\dot{r}r^{(3)} - 8r^3r^{(4)}], \\ j_6(\rho, \rho) &= \frac{1}{1024} [5 - 5\dot{r}^6 - 30r\ddot{r} - 20r^2\ddot{r}^2 \\ &\quad + 3\dot{r}^4(5 + 6r\ddot{r}) - 192r^2\dot{r}^3r^{(3)} \\ &\quad - 8r^3(\dot{r}^3 + 5r^{(4)}) - \dot{r}^2(15 + 20r\ddot{r} + 12r^2\ddot{r}^2 \\ &\quad + 456r^3r^{(4)}) - 8r^4(11r^{(3)2} + 20\ddot{r}r^{(4)}) \\ &\quad - 16r^2\dot{r}(10(1 + 3r\ddot{r})r^{(3)} + 11r^2r^{(5)}) \\ &\quad - 16r^5r^{(6)}]. \end{aligned}$$

In the above expressions a_1 stands for the first heat kernel coefficient (see review [33]).

All terms of the form $f_{2k+1}j_{2k+1}$ vanish in the coincidence limit. Therefore we obtain

$$\begin{aligned} \sum_{l=0}^{\infty} \frac{2\nu}{4\pi} g_l(\rho, \rho') &= \frac{1}{4\pi} \left[\frac{1}{\rho - \rho'} - \frac{1}{r} + \frac{1}{r} \right. \\ &\quad \times \sum_{k=1}^{\infty} \zeta_H\left(2k, \frac{3}{2}\right) j_{2k}(\rho, \rho) + \frac{1}{a} \varphi_+^2(\rho) \\ &\quad \left. - \frac{1}{2a} \varphi_+^2(\rho) \frac{\varphi_+^2(\rho)}{\varphi_+^2(0)} \right]. \end{aligned}$$

After regularization we arrive at the formula

$$\begin{aligned} U(\rho) &= \frac{e^2}{2} \left[-\frac{1}{r} + \frac{1}{r} \sum_{k=1}^{\infty} \zeta_H\left(2k, \frac{3}{2}\right) j_{2k}(\rho, \rho) + \frac{1}{a} \varphi_+^2(\rho) \right. \\ &\quad \left. - \frac{1}{2a} \varphi_+^2(\rho) \frac{\varphi_+^2(\rho)}{\varphi_+^2(0)} \right]. \end{aligned} \tag{37}$$

This expression is exact and we may use it for an arbitrary throat profile.

Figure 3 shows the exact form of the self-force for the second kind of throat profile and the result obtained from the above formula. We observe that even the first term of the series gives good approximation. In fact both expressions are very close for small and large distances from the wormhole throat; there is small deviation close to the extremum only. For this reason we may write out an approximate expression for the self-potential in which there is contribution from the first heat kernel coefficient

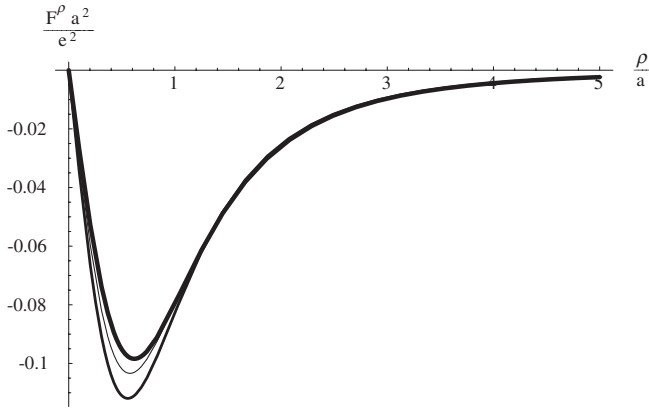


FIG. 3. The thin line is a plot of the exact formula for the self-force given by Eq. (30). The middle thickness line is a plot of the self-force calculated by the above formula up to the first term $k = 1$. The thick line is the same formula up to $k = 2$.

$$U(\rho) \approx \frac{e^2}{2} \left[-\frac{1}{r} + \zeta_H\left(2, \frac{3}{2}\right) \frac{3}{8} a_1 r^2 + \frac{1}{a} \varphi_+^2(\rho) - \frac{1}{2a} \varphi_+^2(\rho) \frac{\varphi_+^2(\rho)}{\varphi_+^2(0)} \right], \quad (38)$$

where $\zeta_H(2, \frac{3}{2}) = \frac{\pi^2}{2} - 4 \approx 0.9348$.

We may obtain a general conclusion from (37). Let us consider large distances from the throat and suppose that the profile function has the following asymptotic series expansion:

$$r = \rho + \sum_{n=0}^{\infty} v_n \rho^{-n}.$$

Each coefficient $j_{2k}(\rho, \rho)$ has the same asymptotic form

$$j_{2k}(\rho, \rho) = -\frac{v_1}{4\rho^2} + O(\rho^{-3}).$$

The relation

$$\sum_{k=1}^{\infty} \zeta_H\left(2k, \frac{3}{2}\right) = \frac{4}{3}$$

yields

$$\frac{1}{r} \sum_{k=1}^{\infty} \zeta_H\left(2k, \frac{3}{2}\right) j_{2k}(\rho, \rho) = -\frac{v_1}{3\rho^3} + O(\rho^{-4}).$$

Putting all expressions together we obtain the following asymptotic expansion for the renormalized Green's function

$$G^{\text{ren}}(\rho; \rho) = -\frac{1}{8\pi\rho^2} \frac{a}{\varphi_+^2(0)} + \frac{1}{4\pi\rho^3} \frac{av_0}{\varphi_+^2(0)} + O(\rho^{-4}),$$

and for the self-potential

$$U = -\frac{e^2}{4\rho^2} \frac{a}{\varphi_+^2(0)} + \frac{e^2}{2\rho^3} \frac{av_0}{\varphi_+^2(0)} + O(\rho^{-4}).$$

Let us compare this result with the cases considered above. For the profile $r = a + |\rho|$ we have $v^0 = a$ and

$$\varphi_+^2(0) = \int_0^{\infty} \frac{ad\rho}{(a+\rho)^2} = 1.$$

Therefore

$$U = -\frac{ae^2}{4\rho^2} + \frac{a^2e^2}{2\rho^3} + O(\rho^{-4}).$$

This expression coincides with the result that follows from the exact formula (20). For the smooth throat profile $\sqrt{\rho^2 + a^2}$ we have $v^0 = 0$ and

$$\varphi_+^2(0) = \int_0^{\infty} \frac{ad\rho}{a^2 + \rho^2} = \frac{\pi}{2}.$$

Therefore

$$U = -\frac{ae^2}{2\pi\rho^2} + O(\rho^{-4})$$

in agreement with Eq. (29).

Thus we may conclude that far from the wormhole throat we have the following expression for the self-potential

$$U = -\frac{e^2}{4\rho^2} \frac{a}{\varphi_+^2(0)} = -\frac{e^2}{4\rho^2} \left[\int_0^{\infty} \frac{d\rho}{r^2(\rho)} \right]^{-1}.$$

Note that it is always negative, hence the self-force is an attractive force. All information about the specific throat profile is encoded in the factor

$$\int_0^{\infty} \frac{d\rho}{r^2(\rho)}.$$

V. CONCLUSION

The aim of the paper was to calculate the self-force that acts on a point charge at rest in the space-time of a wormhole. This calculation has relevance to many problems. First of all the space-time of a wormhole is an example of topologically nontrivial space-time and it is interesting to clear up the role of nontrivial topology of the wormhole space-time. As far as we know this is the first calculation of such kind. The second point is that the self-force may play a serious role in wormhole physics.

We have developed a standard approach to calculation of the self-force. Following it we have obtained the self-force explicitly for two profiles of the wormhole throat. The first

throat shape that was considered possesses a curvature singularity—the space-time is flat everywhere except for the throat. The self-force is everywhere attractive and singular at the throat. The singularity originates in the accepted model of the throat. Far from the wormhole throat the self-force has the same magnitude as in the Schwarzschild space-time with the Schwarzschild radius equal to the throat radius, but the sign of the force is opposite. In the second example we have carried out the calculations for a smooth throat profile. The self-force is zero at the throat, as expected due to symmetry of the space-time with respect to the throat. In both examples we observed that the self-force is attractive, irrespective of the particle position. We suppose that this is inherent to the wormhole with an arbitrary throat profile, and that this is a manifestation of nontrivial topological structure of space-time. The manifest calculations of self-force for the arbitrary throat profile that were performed above show that far from the throat the self-force is always attractive and falls down proportionally to the third power of distance. The magnitude of this force depends on the throat profile.

Let us now speculate about the result obtained. The charged particle in the wormhole space-time is always attracted towards the wormhole's throat. Starting from rest at infinity it will reach the throat with kinetic energy (for the second kind of throat profile considered)

$$\frac{mv^2}{2} = \frac{e^2}{2\pi a} = -U_{\rho=0}.$$

Then it goes through the throat to another universe. The self-force will still attract the particle to the throat and the

velocity of the particle will decrease. The particle will finally stop and the process will repeat. This process is similar to the oscillations of a particle in a potential well. Because of the acceleration the particle will lose its energy for radiation and in the end it will be at rest at the throat. Therefore particles will concentrate at the throat, at the minimum of the self-energy. The intensity of this process is managed by the characteristic value of the self-force, e^2/a^2 , which is smaller for macroscopic wormholes. Therefore the smaller the wormhole's throat, the faster it will gather neighboring charged particles. At the same time particles at the throat with the same sign of charge will repel each other by virtue of the Coulomb force. The magnitude of the force is defined by the same parameter e^2/a^2 . Thus an equilibrium configuration of the particles near the throat is likely to exist. However, in the case of an uncharged particle this picture cannot be used. Usually, the self-force for the massive uncharged particle has opposite sign comparing with electromagnetic charge. The magnitude of the force is much smaller and defined by the parameter Gm^2 instead of e^2 in the electromagnetic case. With reference to our case it means that an uncharged massive particle might be repelled by the wormhole.

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