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Regularized braneworlds of arbitrary codimension

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We consider a thick p-brane embedded in an n-dimensional spacetime possessing radial symmetry in the directions orthogonal to the brane. We first consider a static brane, and find a general fine-tuning relationship between the brane and bulk parameters required for the brane to be flat. We then consider the cosmology of a time-dependent brane in a static bulk, and find the Friedmann equation for the brane scale factor $a(t)$. The singularities that would ordinarily arise when considering arbitrary codimensions are avoided by regularizing the brane, giving it a finite profile in the transverse dimensions. However, since we consider the brane to be a strictly local defect, we find that the transverse dimensions must have infinite volume, and hence gravity cannot be localized on the brane without resorting to some infrared cutoff.

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I. INTRODUCTION

The idea that our Universe might be a four-dimensional subspace embedded in a higher dimensional manifold has been exhaustively studied over the past decade; see for example [1–15] and references therein (and previously; see [16–18]). The most prominent model to have arisen since the inception of so-called "braneworld" theories is the Randall Sundrum (RS) model. In Refs. [4,5], a thin, static, pure tension brane is embedded in five-dimensional anti-de Sitter (AdS) space. It was found that a flat hypersurface could be embedded into the background space, but only if the brane tension was fine-tuned to the bulk cosmological constant. Moreover, it was found that gravity could be localized in the vicinity of the four-dimensional static brane. This important result meant that observers on a brane would feel standard four-dimensional gravity at sufficiently low energies, where any massive bulk modes would be highly suppressed. In addition, this model offered a possible solution to the hierarchy problem.

Simple codimension one braneworld models such as the Randall Sundrum setup typically involve a static brane in a static higher dimensional space. However, the cosmological generalization to a time-dependent brane has been considered by many authors since (see Ref. [10] and references therein, for example). For codimension one objects, modifications to the standard four-dimensional Friedmann equation have been found; specifically, one obtains a new radiationlike term that can be considered as a contribution to the brane evolution due to the bulk. In addition, one finds another modification to standard four-dimensional cosmology which is quadratic in the brane energy density. This implies that for large energy densities on the brane, one would experience a completely different brane evolution than would be the case in conventional four-dimensional gravity.

Progress in determining the cosmological evolution of codimension two branes has also been made [19–25]. Initially seen as promising candidates for solving the cosmological constant problem, such models run into difficulties when one attempts to introduce matter on an otherwise pure tension brane. Specifically one obtains a singularity at the position of the brane, when considering more general equations of state for matter localized on the source. However, by treating the introduction of matter as a perturbation around a known static six-dimensional braneworld model, it has been found in Ref. [21] that if a thick brane is considered, then we can obtain standard late-time Friedmann-Robertson-Walker (FRW) behavior. Thickening the brane is one method of regularizing codimension two objects, but there are others (introducing Gauss-Bonnet terms for example [22,26]).

Branes of arbitrary codimension have not been studied in general (although see, for example, Refs. [27–29]), for a number of reasons. In particular, one cannot simply embed a thin $(3 + 1)$-brane in an n-dimensional space, since one generically finds naked singularities in curvature invariants at the position of the brane. This problem can be resolved in a number of ways; for example, one could thicken the brane, effectively smearing any singularities over a small region of the transverse space. This is the approach that we will take, that is we will replace the singular brane energy momentum tensor with some smoothed distribution. We then define effective four-dimensional quantities by integrating the full field equations over the codimensions, averaged by the brane profile. Our approach follows Refs. [9,30,31], where a similar regularization was considered.

This paper will be concerned with the cosmological evolution of a thick $(3 + 1)$-brane of arbitrary codimension m. We will show that four-dimensional brane cosmology can be obtained for a large class of metrics where a thick $(3 + 1)$-brane is embedded in an n-dimensional background space. Our work is an extension of Refs. [19,32], where the cosmological equations for the brane scale factor $a(t)$ were obtained for codimension one and two branes. In Ref. [19] it was found that conventional late-time cosmology can be obtained on a four-dimensional brane in a six-dimensional space. Our calculation closely follows these works, and we find a corresponding set of equations for a brane of arbitrary codimension. The method used in Ref. [19] to derive the cosmological equations will generalize for higher codimension, and we will show that we can recover standard late-time cosmology regardless of the number of extra dimensions, subject to the assumptions we impose. We will concentrate on the case of a highly symmetric bulk spacetime, in which the brane is static. Some authors [33–35] have considered dropping exact spherical symmetry in the bulk and studying the resulting brane field equations. This scenario is considerably more complicated than the models considered here, since the brane will generically experience a force when the bulk is not symmetric, and we will not consider such behavior.

One important aspect of braneworld models is the requirement that conventional four-dimensional gravity must be obtainable on the brane. For this to occur, we must have some mechanism that localizes gravity in the vicinity of the brane. One way in which this can be achieved is to have a transverse space with finite volume, as in the RS model. However, in this paper we will be considering local defects of arbitrary codimension, and it has been found in Ref. [36] that for such models the transverse dimensions must have infinite volume. This means that gravity cannot be localized on the brane without introducing some large distance cutoff when we integrate over the transverse dimensions to obtain four-dimensional gravity. This cutoff could arise by introducing a second brane, for example. We will consider this issue in Sec. VI.

The paper will proceed as follows. In the following section we review some important definitions and general braneworld results that we will need for the rest of the paper. In Sec. III, we find the field equations for a particular class of static, spherically symmetric metrics. We then introduce a time-dependent metric in Sec. V and obtain an evolution equation for the brane scale factor, and show that our result correctly reduces to the codimension one and two examples as found in Refs. [19,32]. We write our result as a modified four-dimensional Friedman equation, and doing so we find a nonstandard cosmological equation with a dark radiation term present. The origin of this dark radiation term is explored in the appendix.

II. FORMALISM

We begin with some important results that will be required in subsequent sections. In this paper, we will be considering codimension m, p-branes (p being the number of spacetime dimensions of the brane) embedded in n-dimensional background spacetimes, hence $m = n - p$. Specifically, we will consider two different metrics. The first is a static, n-dimensional metric with spherical symmetry in the extra dimensions. It takes the form

$$ds^2 = f(r)g_{AB}(x)dx^A dx^B - dr^2 - \alpha^2(r)\gamma_{ab}(y)dy^a dy^b, \quad (1)$$

where $f(r)$ and $\alpha(r)$ are functions of the radial coordinate r only, and γ_{ab} is the metric of the unit $(m-1)$ sphere, so the extra dimensions are radially symmetric. The metric $g_{AB}(x)$ is the $(3+1)$-dimensional brane metric, and hence capital Latin indices $(A, B, \ldots)$ run over the standard $(3+1)$ coordinates. The metric γ_{ab} is that of the $(m-1)$ sphere (that is all of the brane-orthogonal coordinates except the radial coordinate r), and hence lower case Latin indices $(a, b, \ldots)$ run over $(m-1)$ of the codimensions. Finally, we will use the notation that Greek indices $(\mu, \nu, \ldots)$ run over all n-dimensional coordinates. The full n-dimensional metric is $G_{\mu\nu}$, and in subsequent sections we will frequently need the measure $\sqrt{G}$, which is given by

$$\sqrt{G} = [f(r)]^2 \alpha^{m-1}\sqrt{g\gamma}, \quad (2)$$

where g and γ are the determinants of g_{AB} and γ_{ab}, respectively.

In the next section, we will need the following decompositions of the n-dimensional Ricci tensor $R_{\mu\nu}$

$$R_{AB} = R^{(g)}_{AB} - \frac{1}{2}g_{AB}(\nabla_a\nabla^a + \nabla_r\nabla^r)f(r) - \frac{p-2}{4}g_{AB}f(r)^{-1}\nabla_r f(r)\nabla^r f(r), \quad (3)$$

$$R_{ab} = R^{(\gamma)}_{ab} - \frac{p}{2}f^{-1}\nabla_a\nabla_b f + \frac{p}{4}f^{-2}(\nabla_a f)(\nabla_b f), \quad (4)$$

$$R_{rr} = R^{(\gamma)}_{rr} - \frac{p}{2}f^{-1}\nabla_r\nabla_r f + \frac{p}{4}f^{-2}(\nabla_r f)(\nabla_r f), \quad (5)$$

where we will take $p = 4$ in this paper, but is in general $p = g_{AB}g^{AB}$, and $R^{(g)}_{AB}$ is the Ricci tensor constructed from $g_{AB}(x)$. $R^{(\gamma)}_{ab}$ and $R^{(\gamma)}_{rr}$ are the components of the Ricci tensor constructed from the metric

$$ds^2_{[\mathrm{m}]} = -dr^2 - \alpha(r)^2\gamma_{ab}(y)dy^a dy^b. \quad (6)$$

In the next section, we will find that the Einstein equations for our metric (1) can be written in a very simple form in terms of the extrinsic curvatures of the subspaces with metrics g_{AB} and γ_{ab}. For objects of codimension greater than one the extrinsic curvature is defined as [37]

$$K_{\mu\nu}{}^{\rho} \equiv \eta_\nu{}^\sigma \eta_\mu{}^\beta \nabla_\beta \eta_\sigma{}^\rho, \quad (7)$$

where ∇_β preserves the full n-dimensional background metric, and $\eta_\nu{}^\sigma$ projects tensors tangentially to the brane. For our metric, $\eta_\nu{}^\sigma$ is given by

$$\eta_\mu{}^\nu = \begin{pmatrix} g_A{}^B & 0 \\ 0 & 0 \end{pmatrix}. \quad (8)$$

The first two indices of the extrinsic curvature tensor are tangential to the brane, and the last is orthogonal. The metrics that we are considering are highly symmetric, and as a result the extrinsic curvature tensor simplifies considerably. Using the metric (1), the only nonzero components of (7) are for $\rho = r$. We will use the extrinsic curvatures for our metric, $K_A{}^{\rm Br}$ and $K_a{}^{\rm br}$, which are given by

$$K_A{}^{\rm Br} \equiv K_A{}^B = \frac{f'}{f} g_A{}^B, \qquad K_a{}^{\rm br} \equiv L_a{}^b = 2\frac{\alpha'}{\alpha}\gamma_a{}^b, \quad (9)$$

where primes denote derivatives with respect to r. We have dropped the third index on the extrinsic curvature tensor, since the only nonzero components of (7) are for $\rho = r$.

The second metric ansatz that we will consider is

$$ds^2 = N^2(r,t)dt^2 - A^2(r,t)g_{ij}(x)dx^i dx^j - dr^2 - \alpha^2(r,t)\gamma_{ab}(y)dy^a dy^b, \quad (10)$$

which is a more general version of (1), and will be used to model a time-dependent brane. As before, we can split the line element (10) into brane tangential and brane-orthogonal components. The "brane" line element, $ds^2_{[b]}$, is given by

$$ds^2_{[b]} = N^2(r,t)dt^2 - A^2(r,t)g_{ij}(x)dx^i dx^j, \quad (11)$$

and for surfaces of constant r is of the form of a FRW metric. We have introduced another set of indices (i, j) in (10), which run over the standard three spatial dimensions. For the metric (10), the measure $\sqrt{G}$ is given by

$$\sqrt{G} = NA^3\alpha^{m-1}\sqrt{\gamma}. \quad (12)$$

The extrinsic curvatures $K_A{}^B$ and $L_a{}^b$ for this metric are

$$K_i{}^j = 2\delta_i{}^j\frac{A'}{A}, \qquad K_t{}^t = 2\frac{N'}{N}, \qquad L_a{}^b = 2\delta_a{}^b\frac{\alpha'}{\alpha}, \quad (13)$$

and taking the trace gives

$$K = 2\frac{N'}{N} + 6\frac{A'}{A}, \qquad L = 2(m-1)\frac{\alpha'}{\alpha}. \quad (14)$$

A. Regularization of the brane

Finally in this section, we explain our method of regularizing higher codimension branes, which follows Refs. [9,30,31,33,38]. For a thin brane, the total energy momentum tensor, $T_{\mu\nu}$, can be decomposed into distinct brane and bulk components. In the thin case, the standard definition of the p-brane energy momentum tensor $\hat{T}_{\mu\nu}$ is

$$\hat{T}_{\mu\nu} = \int \sqrt{g} d^p\sigma \bar{T}_{\mu\nu}\delta^{(n)}[x_\alpha - X_\alpha(\sigma^A)], \quad (15)$$

where $\bar{T}_{\mu\nu}$ is the brane supported energy momentum tensor, σ^A are the brane coordinates, and x_α the n-dimensional background coordinates. The brane is situated at $x_\alpha = X_\alpha(\sigma^A)$.

In this paper we are considering branes of finite thickness. This means that we no longer treat the brane energy momentum tensor $\hat{T}_\mu{}^\nu$ as a singular source as in (15), but rather as some smoothed distribution. Explicitly, we consider the energy momentum tensor

$$\hat{T}_{\mu\nu} = \int \sqrt{g} d^p\sigma \bar{T}_{\mu\nu} D_\epsilon^{(n)}(x - X(\sigma)), \quad (16)$$

where we have replaced the n-dimensional δ-function in (15) with the finite brane profile function $D_\epsilon^{(n)}(x - X(\sigma))$, which is peaked at $x = X(\sigma)$ and falls away sharply from the brane.

In this paper, we will consider the particular simple brane profile

$$D_\epsilon^{(n)} = 1 \quad r < \epsilon \quad (17)$$

$$0 \quad r > \epsilon, \quad (18)$$

where ϵ can be considered as the brane thickness parameter. This profile has been considered previously, see, for example, Refs. [10,32] (and a different brane profile was considered in [33]). Although we use this particular profile, we expect that our results will be approximately valid for a large class of profile functions. Using (17), the brane supported energy momentum tensor $\tilde{T}_{\mu\nu}$ is given by

$$\tilde{T}_\mu{}^\nu = \frac{1}{\sqrt{g}|_{r=\epsilon}} \int d^{m-1}y \int_0^\epsilon \sqrt{G} dr T_\mu{}^\nu. \quad (19)$$

We must also define the bulk energy momentum tensor, $T^{\rm bulk}_{\mu\nu}$. For simplicity, we will consider a cosmological constant only in the bulk, with no additional fields, so $T_{\mu\nu}|^{\rm bulk} = -\Lambda g_{\mu\nu}$, where Λ is the n-dimensional cosmological constant.

Finally, we must discuss how to obtain four-dimensional equations for a thick brane. For thin branes, effective four-dimensional equations are found by taking the full n-dimensional field equations and evaluating them on a surface of constant y_a (the codimensions), at the position of the brane. This is equivalent to taking the full n-dimensional equations, and integrating them over the m codimensions, weighted by the brane profile, which in this case is a delta function. For thick branes, we follow the same procedure; to obtain four-dimensional equations, we take the full n-dimensional equations, integrate them over the codimensions, weighted by the brane profile $D_\epsilon^{(n)}$, which is no longer singular. For the profile (17) used in this paper, our approach corresponds to integrating the field equations over the range $r = (0, \epsilon)$. In the next section, we will perform our method of regularization for a codimension m static brane.

III. STATIC BRANEWORLD MODEL

Let us first consider a braneworld model with a metric of the form (1). This is the simplest generalization of codimension one and two braneworlds that exist in the literature. Our approach follows the work of Refs. [19,32]; we will find that all of the field equations for the metric ansatz

(1), except the (r, r) equation, can be written approximately as total derivatives with respect to r. These equations can be integrated over the brane thickness to obtain a set of junction conditions. We then evaluate the remaining (r, r) field equation at the brane-bulk boundary, that is at $r = \epsilon$, and use our junction conditions to write an equation relating the brane Ricci scalar $R^{(g)}$ to the brane energy momentum tensor $\tilde{T}_{\mu\nu}$. Before continuing we note that static metrics of the form (1) have been considered previously, see for example [36,39].

To begin, we write the decompositions of the background Ricci tensor $R_\mu{}^\nu$, (3)–(5), in terms of the extrinsic curvature tensors $K_A{}^B$ and $L_a{}^b$,

$$\sqrt{G}R_A{}^B = \frac{\sqrt{G}}{f}R^{(g)}{}_A{}^B + \frac{1}{2}(\sqrt{G}K_A{}^B)' = \frac{\sqrt{G}}{M^{n-2}}\left(T_A{}^B - \delta_A{}^B\frac{T}{n-2}\right), \tag{20}$$

$$\sqrt{G}R_a{}^b = \frac{\sqrt{G}}{\alpha^2}(m-2)\delta_a{}^b + \frac{1}{2}(\sqrt{G}L_a{}^b)' = \frac{\sqrt{G}}{M^{n-2}}\left(T_a{}^b - \delta_a{}^b\frac{T}{n-2}\right), \tag{21}$$

$$\sqrt{G}R_r{}^r = \sqrt{G}\left(\frac{1}{2}(L' + K') + \frac{1}{4}(K_C{}^D K_D{}^C + L_a{}^b L_a{}^b)\right) = \frac{\sqrt{G}}{M^{n-2}}\left(T_r{}^r - \frac{T}{n-2}\right), \tag{22}$$

where primes denote derivatives with respect to the radial coordinate r. M is the n-dimensional fundamental mass scale. Note that Eq. (21) does not exist for codimension one branes; we will return to this specific case shortly.

We now assume that derivatives tangential to the brane (that is derivatives with respect to the x_A coordinates) will be negligible compared to derivatives in the brane-orthogonal directions. This implies that we can neglect the term $R^{(g)}{}_A{}^B/f$ in (20), which then becomes

$$\sqrt{G}R_A{}^B \approx \frac{1}{2}(\sqrt{G}K_A{}^B)' = \frac{\sqrt{G}}{M^{n-2}}\left(T_A{}^B - \delta_A{}^B\frac{T}{n-2}\right) \tag{23}$$

and we see that the (A, B) components of the Ricci tensor can be approximately written as a total derivative with respect to r. The next step is to integrate this expression over the brane thickness $0 < r < \epsilon$. The integral of $(\sqrt{G}K_A{}^B)'$ over the brane thickness will give us two terms, one at $r = 0$ and one at $r = \epsilon$. We must choose our boundary conditions at $r = 0$ carefully so that the metric is regular there, since a poor choice will generally create naked singularities in curvature invariants at the origin. For this reason we choose $\alpha(0) = 0$ and $\partial_r\alpha|_{r=0} = 1$, since this corresponds to the m extra dimensions taking the form

$$ds^2_{[m]} = -dr^2 - \alpha^2(r)\gamma_{ab}dy^a dy^b = -dr^2 - r^2 d\Omega^2_{[m-1]}, \tag{24}$$

at $r = 0$. In other words, the extra dimensions are simply Minkowski space at the origin, and hence the metric is regular here. Integrating (23) and using these boundary conditions, we obtain

$$K_A{}^B|_\epsilon = \frac{2M_b^{m-1}}{\Omega^{[m-1]}M^{n-2}}\left(\tilde{T}_A{}^B - \delta_A{}^B\frac{\tilde{T}}{n-2}\right), \tag{25}$$

where we have defined $M_b = \alpha(\epsilon)^{-1}$. This result has been derived previously in Refs. [19,32] for codimension one and two branes, and an explanation as to the relationship between these total derivatives and the underlying symmetries of the background space is given in Ref. [40]. Note that if we consider the case $m = 1$, and take the limit $\epsilon \rightarrow 0$ we recover the five-dimensional junction conditions.

We now consider (21). Unlike (20), there is no total derivative in (21), due to the presence of the $\sqrt{G}R^{(\gamma)}{}_a{}^b/\alpha^2$ term. To proceed, we integrate (21) over the region of the transverse space occupied by the brane,

$$\frac{1}{2}\Omega^{[m-1]}\int_0^\epsilon(\sqrt{G}L_a{}^b)'dr = \frac{\Omega^{[m-1]}}{2M_b^{m-1}}\sqrt{g}|_\epsilon L_a{}^b|_\epsilon = \frac{1}{M^{n-2}}\int d^{m-1}ydr\sqrt{G} \times\left(T_a{}^b - \delta_a{}^b\frac{T}{n-2}\right) - (m-2) \times\delta_a^b\int d^{m-1}ydr\sqrt{G}\frac{1}{\alpha^2}, \tag{26}$$

which is valid for $m > 2$. The problematic term is the last one on the right-hand side of (26). Since we are only integrating over the small transverse region occupied by the brane (that is in the region $r < \epsilon$), we can find an approximate solution to the above equation. To do so, we note that we have chosen our boundary conditions at $r = 0$ such that $\alpha(r) = r$ for small r. In addition, we also impose the boundary condition $\frac{df}{dr}|_{r=0} = 0$, which is also required for a regular solution. We will assume that over the small brane region $(0, \epsilon)$, that $\alpha \approx r$ and $f(r) \approx$ const. In making this approximation we find

$$L_a{}^b|_{r=\epsilon} \approx \frac{2M_b^{m-1}}{\Omega^{[m-1]}M^{n-2}}\left(\tilde{T}_a{}^b - \delta_a{}^b\frac{\tilde{T}}{n-2}\right) - \frac{2}{\epsilon}\delta_a{}^b, \tag{27}$$

which is valid for $m > 2$. For the cases $m = 1, 2$, the last integral in (26) is not defined; it diverges at $r = 0$. However, this term is absent when we consider codimension one and two branes. We will consider the specific examples of $m = 1, 2$ shortly.

We now have approximate expressions for $K_A{}^B$ and $L_a{}^b$ at $r = \epsilon$, (25) and (27). The next step is to consider the (r, r) Einstein equation,

$$G_r{}^r = R_r{}^r - \frac{1}{2}R = \frac{1}{8}(K_C{}^D K_D{}^C + L_a{}^b L_a{}^b) - \frac{1}{8}(L^2 + K^2) - \frac{1}{4}KL - \frac{1}{2}\left(\frac{R^{(g)}}{f} + \frac{R^{(\gamma)}}{\alpha^2}\right) = \frac{T_r{}^r|_{\rm bulk}}{M^{n-2}}. \quad (28)$$

To obtain an effective four-dimensional field equation from (28), we should perform our regularization procedure and integrate it over the range $r = (0, \epsilon)$. However, for simplicity we will instead evaluate (28) at the surface $r = \epsilon$. We stress that this is only an approximation, and we should apply our regularization procedure to (28). However, given our particular profile function, we expect that evaluating (28) at $r = \epsilon$ is sufficient to obtain an approximate brane equation. The reason why we use this approximation is that by virtue of the "junction conditions" (25) and (27), we know $K_A{}^B$ and $L_a{}^b$ in terms of the brane energy momentum tensor $\tilde{T}_\mu{}^\nu$ at $r = \epsilon$. We consider the cases $m = 1$, $m = 2$, and $m > 2$ separately.

A. Codimension one

For $m = 1$ there are no $L_a{}^b$ components of the extrinsic curvature, and in addition there is no $R^{(\gamma)}/\alpha^2$ term in (28). Therefore using (25) and evaluating (28) at $r = \epsilon$, we find

$$-\frac{1}{2}\frac{R^{(g)}}{f(\epsilon)} + \frac{A_1^2}{8(n-2)^2}[((n-2)^2\tilde{T}_A{}^B\tilde{T}_B{}^A + B_1(\tilde{T}_A{}^A)^2) + (2B_1\tilde{T}_r{}^r\tilde{T}_A{}^A + C_1(\tilde{T}_r{}^r)^2)] = -\frac{\Lambda}{M^{n-2}}, \quad (29)$$

where the constants A_1, B_1, and C_1 are given by

$$A_1 = \frac{1}{M^{n-2}}, \qquad B_1 = 1 - p, \qquad C_1 = p - p^2. \quad (30)$$

We note that the brane energy momentum tensor now has a nonzero component $\tilde{T}_r^r$ in the transverse direction. This component is zero for a thin brane.

B. Codimension two

The case $m = 2$ is unique; to see why we return to Eq. (21), and integrate over the region $r < \epsilon$. Evaluating this integral yields two terms, one at $r = 0$ and the other at $r = \epsilon$,

$$\frac{\Omega^{[m-1]}}{2}\int_0^\epsilon (\sqrt{G}L_a{}^b)' dr = \frac{\Omega^{[m-1]}}{2}[(\sqrt{G}L_a{}^b)|_{r=\epsilon} - (\sqrt{G}L_a{}^b)|_{r=0}]. \quad (31)$$

When $m > 2$ the term $(\sqrt{G}L_a{}^b)|_{r=0}$ in (31) is zero by virtue of the boundary condition $\alpha(0) = 0$. However, for $m = 2$, $(\sqrt{G}L_a{}^b)|_{r=0} \neq 0$ and we have to include a boundary term at the origin. Hence, accounting for this extra term, Eq. (28), evaluated at $r = \epsilon$, is

$$-\frac{1}{2}\frac{R^{(g)}}{f(\epsilon)} + \frac{A_2^2}{8(n-2)^2}[((n-2)^2\tilde{T}_A{}^B\tilde{T}_B{}^A + B_2(\tilde{T}_A{}^A)^2) + ((n-2)^2\tilde{T}_a{}^b\tilde{T}_b{}^a + B_2(\tilde{T}_a{}^a)^2)] + \frac{A_2^2}{8(n-2)^2}[2B_2(\tilde{T}_a{}^a + \tilde{T}_r{}^r)\tilde{T}_A{}^A + 2B_2\tilde{T}_a{}^a\tilde{T}_r{}^r + C_2(\tilde{T}_r{}^r)^2] + \frac{1}{2}A_2 M_{\rm b}^{m-1}\frac{\sqrt{g}|_{r=0}}{\sqrt{g}|_{r=\epsilon}}\tilde{T}_a{}^a + \frac{1}{2}A_2 M_b^{m-1}\frac{\sqrt{g}|_{r=0}}{\sqrt{g}|_{r=\epsilon}}\tilde{T}_r{}^r = -\frac{\Lambda}{M^{n-2}}, \quad (32)$$

where the constants A_2, B_2, and C_2 are given by

$$A_2 = \frac{M_{\rm b}}{\pi M^{n-2}}, \qquad B_2 = -p, \qquad C_2 = p - p^2 - 2p. \quad (33)$$

We have confirmed that Eqs. (29) and (32) agree with the results of Refs. [19,32].

C. Higher codimension

Now, the term $R^{(\gamma)}/\alpha^2$ is not zero, and we find the relation

$$-\frac{R^{(g)}}{2f(\epsilon)} + \frac{A_m^2}{8(n-2)^2}[((n-2)^2\tilde{T}_A{}^B\tilde{T}_B{}^A + B_m(\tilde{T}_A{}^A)^2) + ((n-2)^2\tilde{T}_a{}^b\tilde{T}_b{}^a + B_m(\tilde{T}_a{}^a)^2)] + \frac{A_m^2}{8(n-2)^2}[2B_m(\tilde{T}_a{}^a + \tilde{T}_r{}^r)\tilde{T}_A{}^A + 2B_m\tilde{T}_a{}^a\tilde{T}_r{}^r + C_m(\tilde{T}_r{}^r)^2] - \frac{A_m}{2\epsilon}\tilde{T}_a{}^a - \frac{A_m(m-1)}{2\epsilon}\tilde{T}_r{}^r - \frac{(m-1)(m-2)}{\epsilon^2} = -\frac{\Lambda}{M^{n-2}}, \quad (34)$$

where A_m, B_m, and C_m are given by

$$A_m = \frac{2M_{\rm b}^{m-1}}{\Omega^{[m-1]}M^{n-2}}, \qquad B_m = 2 - m - p, \qquad C_m = p - p^2 + (m-1) - (m-1)^2 - 2p(m-1), \quad (35)$$

for $m > 2$. The Eq. (34) gives the brane Ricci scalar $R^{(g)}$ in terms of the bulk cosmological constant Λ and the "four-dimensional" energy momentum tensor $\tilde{T}_{\mu\nu}$. We will return to Eqs. (29), (32), and (34) shortly.

Next, we consider the (A, a), (A, r) components of $R_{\mu\nu}$ evaluated at $r = \epsilon$, which are simply given by $R_{Aa} = 0$, and $R_{Ar} = 0$. For our particular metric, $R_{Ar} = 0$ tells us that $G_{Ar} = 0$, which implies $T_{Ar}|_{\rm bulk} = 0$. As this component of the energy momentum tensor describes the flow of energy from the brane to the bulk, it appears that for braneworld models with a metric given by (1) we will obtain no loss of energy into the bulk. However this conclusion is not necessarily true if the brane thickness is variable, as was discussed in Ref. [19].

IV. SPECIAL CASES OF STATIC BRANEWORLDS

The Eqs. (29), (32), and (34) that were derived in the previous section can act as generalized fine-tuning relations for branes of codimension $m = 1$, $m = 2$, and $m > 2$, respectively. To see this, we set $R^{(g)} = 0$. Taking the $m = 1$ case as an example, we therefore see that the fine-tuning relation

$$[((n-2)^2 \tilde{T}_A{}^B \tilde{T}_B{}^A + B_1(\tilde{T}_A{}^A)^2) + (2B_1 \tilde{T}_r{}^r \tilde{T}_A{}^A + C_1(\tilde{T}_r{}^r)^2)] = -\frac{8(n-2)^2}{A_1^2 M^{n-2}} \Lambda \tag{36}$$

must hold to ensure the brane is flat. If we take the background space to be five dimensional, and consider a pure tension brane, then when we take the $\epsilon \to 0$ limit, we find $\tilde{T}_r{}^r \to 0$ and $\tilde{T}_A{}^B \to -\frac{1}{2}\delta_A{}^B \lambda$, where λ is the brane tension. Using these results in (36), we find the fine-tuning relation

$$\Lambda = \frac{\lambda^2}{24M^3}, \tag{37}$$

which is equivalent to the standard Randall Sundrum fine-tuning condition. There is a sign discrepancy between this fine-tuning and the RS model, but this is because we are considering the metric,

$$ds_{[5]}^2 = f(r) g_{\mu\nu} dx^\mu dx^\nu - dr^2, \tag{38}$$

which is five-dimensional de Sitter space, not AdS as in the RS model. If we consider an AdS space, we set $\Lambda \to -\Lambda$ and the correct fine-tuning is recovered.

Whilst (37) is valid for a thin brane, for a thick brane we have the fine-tuning relation (36), which we expect will depend on the brane profile. For a pure tension brane, we can write $\hat{T}_A^B = -\frac{1}{2}\lambda \delta_A^B D_\epsilon(y)$, where $D_\epsilon(y)$ is the brane profile. If we normalize the brane profile appropriately such that

$$\int_{-\epsilon}^{\epsilon} \sqrt{G} D_\epsilon(y) dy = 1, \tag{39}$$

then we can write the effective four-dimensional energy momentum tensor as $\tilde{T}_A^B = -\frac{1}{2}\lambda \delta_A^B$, as before. It follows that the fine-tuning relation (36) has the form

$$\Lambda = \frac{\lambda^2}{24M^3} - \frac{\lambda \tilde{T}_r{}^r}{6M^3} + \frac{(\tilde{T}_r{}^r)^2}{6M^3}. \tag{40}$$

We see for a thick brane the fine-tuning relation is modified by the $\tilde{T}_r{}^r$ components of the brane energy momentum tensor.

We can continue in this manner and consider the codimension two case. We first consider a thin, pure tension codimension two brane for which we have $\tilde{T}_a{}^b = \tilde{T}_r{}^r = 0$. For pure tension branes, we can use the property $\tilde{T}_A{}^B = \delta_A{}^B \tilde{T}_C{}^C/4$. If we use this relation, and set $m = 2$, $p = 4$, and $n = 6$ in (32), we find that if we take $R^{(g)} = 0$, then all of the terms on the left-hand side are zero. We are left simply with $\Lambda = 0$ at the position of the brane. If we consider instead a more general bulk energy momentum tensor $T_r{}^r|^{\mathrm{bulk}}$, then this result becomes $T_r{}^r|^{\mathrm{bulk}} = 0$. In [41], a six-dimensional model was considered, and it was found that the bulk energy momentum tensor had to be tuned in such a way that $T_r{}^r|^{\mathrm{bulk}} = 0$ in order to have a flat brane. Hence our result is in agreement with [41], and reflects that fact that to obtain a flat pure tension codimension two brane, the brane tension does not have to be fine-tuned to the bulk energy momentum tensor.

Next, we consider a thick brane, so we cannot assume $\tilde{T}_a{}^b = \tilde{T}_r{}^r = 0$. We find that (32) now reads

$$\frac{A_2^2}{8(n-2)^2}[((n-2)^2 \tilde{T}_a{}^b \tilde{T}_b{}^a + B_2(\tilde{T}_a{}^a)^2) + 2B_2(\tilde{T}_a{}^a + \tilde{T}_r{}^r)\tilde{T}_A{}^A + 2B_2 \tilde{T}_a{}^a \tilde{T}_r{}^r + C_2(\tilde{T}_r{}^r)^2] + \frac{1}{2} A_2 M_{\mathrm{b}}^{m-1} \frac{\sqrt{g}|_{r=0}}{\sqrt{g}|_{r=\epsilon}} \tilde{T}_a{}^a + \frac{(m-1)}{2} A_2 M_{\mathrm{b}}^{m-1} \frac{\sqrt{g}|_{r=0}}{\sqrt{g}|_{r=\epsilon}} \tilde{T}_r{}^r = -\frac{\Lambda}{M^{n-2}}. \tag{41}$$

Now if we assume that the $\tilde{T}_a{}^b$ and $\tilde{T}_r{}^r$ components of the brane energy momentum tensor are small, then we find the following approximate fine-tuning relation that must be satisfied in order for a thick codimension two brane to be flat,

$$-\frac{M_{\mathrm{b}}^2}{16\pi^2 M^8}(\tilde{T}_a{}^a + \tilde{T}_r{}^r)\tilde{T}_A{}^A + \frac{M_{\mathrm{b}}^2}{2\pi M^4} \frac{\sqrt{g}|_{r=0}}{\sqrt{g}|_{r=\epsilon}} (\tilde{T}_a^a + \tilde{T}_r^r) = -\frac{\Lambda}{M^4}, \tag{42}$$

where we have taken $p = 4$. We see that to obtain a flat brane in this case, we must tune $\tilde{T}_A{}^A$ to the bulk energy momentum tensor (unless we have a model in which $\tilde{T}_a{}^a = -\tilde{T}_r{}^r$, in which case we have $\Lambda \approx 0$, or more generally $T_r{}^r|^{\mathrm{bulk}} = 0$, as in the thin brane case.)

V. COSMOLOGICAL SOLUTION

Having considered metrics of the form (1) in the previous section, we now perform an analogous calculation for the time-dependent metric (10). We initially incorporate a time dependence in the extra dimensions by writing $\alpha(r, t)$ as a function of time. However, to perform the same calculation as above, we find that we must remove this time dependence. This amounts to assuming that the extra dimensions are static.

Calculating the Ricci tensor $R_\mu{}^\nu$ for the metric (10) as we did in the previous section gives

$$R_i{}^j = \frac{1}{2}\frac{(NA^{p-1}\alpha^{m-1}K_i^j)'}{NA^{p-1}\alpha^{m-1}} + \frac{R^{(g)}{}_i{}^j}{A^2} - \frac{(m-1)}{\alpha}\nabla^j\nabla_i\alpha = \frac{1}{M^{n-2}}\left(T_i{}^j - \delta_i{}^j\frac{T}{n-2}\right), \quad (43)$$

$$R_t{}^t = \frac{1}{2}\frac{(NA^{p-1}\alpha^{m-1}K_t^t)'}{NA^{p-1}\alpha^{m-1}} + \frac{R^{(g)}{}_t{}^t}{A^2} - \frac{(m-1)}{\alpha}\nabla^t\nabla_t\alpha = \frac{1}{M^{n-2}}\left(T_t{}^t - \delta_t{}^t\frac{T}{n-2}\right), \quad (44)$$

$$R_a{}^b = \frac{1}{2}\frac{(NA^{p-1}\alpha^{m-1}L_a{}^b)'}{NA^{p-1}\alpha^{m-1}} - \frac{(m-2)}{\alpha^2}\delta_a{}^b\nabla_\mu\alpha\nabla^\mu\alpha - \frac{\delta_a{}^b}{\alpha}\nabla^\mu\nabla_\mu\alpha + \frac{(m-2)}{\alpha^2}\delta_a{}^b = \frac{1}{M^{n-2}}\left(T_a{}^b - \delta_a{}^b\frac{T}{n-2}\right), \quad (45)$$

$$R_r{}^r = \frac{1}{2}(K' + L') + \frac{1}{4}(K_A{}^BK_B{}^A + L_a{}^bL_b{}^a), \quad (46)$$

where the extrinsic curvatures are defined in (13). Primes denote derivatives with respect to the radial coordinate r, and dots derivatives with respect to time. We have defined the Ricci tensors of the metrics g_{ij} and γ_{ab} as $R^{(g)}{}_i{}^j$ and $R^{(\gamma)}{}_a{}^b$ respectively, and the covariant derivatives ∇_μ preserve the metric $g_{\mu\nu}$.

The final equation that we will need is

$$G_{Ar} = R_{Ar} = -\frac{(m-1)\partial_A\alpha'}{\alpha} + \frac{(m-1)\partial_B\alpha}{2\alpha}K^B{}_A + \frac{1}{2}\nabla^B(K_{AB} - g_{AB}K) = \frac{T^{\text{bulk}}_{Ar}}{M^{n-2}}. \quad (47)$$

which will give us a conservation equation for the brane energy momentum tensor $\tilde{T}_{\mu\nu}$.

We will now make a number of assumptions in order to obtain an equation for $A(t, r)$ at the surface of the brane. We begin as before, by assuming that brane tangential derivatives are small, in particular, we will neglect the terms $\nabla^j\nabla_i\alpha$ and $\nabla_t\nabla^t\alpha$ in Eqs. (43)–(45). With this assumption we have removed the time dependence of the extra dimensions, and so our solution will be valid only when the transverse dimensions are static or weakly time dependent.

We proceed as in the previous section. We neglect derivatives tangential to the brane, and integrate (43)–(45) over the brane thickness $0 < r < \epsilon$, using the approximations $N(t, r = 0) \approx N(t, r = \epsilon) = 1$ and $\alpha(t, r < \epsilon) \approx r$. We find

$$K_i{}^j|_{r=\epsilon} = \frac{2M_{\rm b}^{m-1}}{\Omega^{[m-1]}M^{n-2}}\left(\tilde{T}_i{}^j - \delta_i{}^j\frac{\tilde{T}}{n-2}\right), \quad (48)$$

$$K_t{}^t|_{r=\epsilon} = \frac{2M_{\rm b}^{m-1}}{\Omega^{[m-1]}M^{n-2}}\left(\tilde{T}_t{}^t - \delta_t{}^t\frac{\tilde{T}}{n-2}\right), \quad (49)$$

$$L_a{}^b|_{r=\epsilon} = \frac{2M_{\rm b}^{m-1}}{\Omega^{[m-1]}M^{n-2}}\left(\tilde{T}_a{}^b - \delta_a{}^b\frac{\tilde{T}}{n-2}\right) - \frac{2}{\epsilon}\delta_a{}^b, \quad (50)$$

which are valid for $m \geq 3$. As in the previous section, for codimension two objects we obtain an additional term in (50), a boundary term at $r = 0$ of the form $(\sqrt{G}L_a{}^b)|_{r=0}$. We will check throughout though that our results can give the results of [19] for a codimension two brane.

The next step is to evaluate the (r, r) Einstein equation at $r = \epsilon$,

$$G_r{}^r|_{r=\epsilon} = -\frac{1}{2}R^{(g)} - \frac{1}{2}\frac{(m-1)(m-2)}{\alpha^2} + \frac{1}{8}(K_C{}^DK_D{}^C - K^2) + \frac{1}{8}(L_a{}^bL_b{}^a - L^2) - \frac{1}{4}LK = -\frac{\Lambda}{M^{n-2}}. \quad (51)$$

We assume that $A(r = 0) \approx A(r = \epsilon) = a(t)$ is the brane scale factor. We also set the energy momentum tensor on the brane $\tilde{T}_\mu{}^\nu$ as

$$\tilde{T}_\mu{}^\nu = \text{diag}(\rho, -p_{\rm br}, -p_{\rm br}, -p_{\rm br}, -p_{\rm r}, -p_{\rm bk}, \ldots, -p_{\rm bk}), \quad (52)$$

where ρ is the energy density, $p_{\rm br}$ the brane pressure components, and $p_{\rm r}$ and $p_{\rm bk}$ the extra-dimensional pressure components. There are $(m-1)$ $p_{\rm bk}$ terms in (52). Using (52) in (48)–(50), the matching conditions become

$$K_i{}^j|_{r=\epsilon} = -\frac{2M_{\rm b}^{m-1}}{\Omega^{[m-1]}M^{n-2}(n-2)}\delta_i{}^j[(n-1-p)p_{\rm br} + \rho - p_{\rm r} - (m-1)p_{\rm bk}], \quad (53)$$

$$K_t{}^t|_{r=\epsilon} = \frac{2M_{\rm b}^{m-1}}{\Omega^{[m-1]}M^{n-2}(n-2)}[(n-3)\rho + (p-1)p_{\rm br} + p_{\rm r} + (m-1)p_{\rm bk}], \quad (54)$$

$$L_a{}^b|_{r=\epsilon} = -\frac{2M_{\rm b}^{m-1}}{\Omega^{[m-1]}M^{n-2}(n-2)}\delta_a{}^b[(n-1-m)p_{\rm bk} + \rho - (p-1)p_{\rm br} - p_{\rm r}] - \frac{2}{\epsilon}\delta_a{}^b, \quad (55)$$

and the G_r^r Einstein equation is

$$G_r{}^r|_{r=\epsilon} = 3\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) + \frac{A_m^2}{8(n-2)^2}((n-2)^2\tilde{T}_A{}^B\tilde{T}_B{}^A + B_m(\tilde{T}_A{}^A)^2 + (n-2)^2\tilde{T}_a{}^b\tilde{T}_b{}^a + B_m(\tilde{T}_a{}^a)^2) + \frac{A_m^2}{8(n-2)^2} \times (2B_m(\tilde{T}_a{}^a + \tilde{T}_r{}^r)\tilde{T}_A{}^A + 2B_m\tilde{T}_a{}^a\tilde{T}_r{}^r + C_m(\tilde{T}_r{}^r)^2) - \frac{A_m}{2\epsilon}\tilde{T}_a{}^a - \frac{(m-1)A_m}{2\epsilon}\tilde{T}^r{}_r - \frac{(m-1)(m-2)}{\epsilon^2} = -\frac{\Lambda}{M^{n-2}}. \quad (56)$$

Now, using the brane energy momentum tensor (52), we obtain the following equation for the brane scale factor

$$3\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) = -\frac{\Lambda}{M^{n-2}} - \frac{A_m^2}{8(m+2)}[(1+m)(\rho + p_{\rm br})^2 + (2m-4)p_{\rm br}(p_{\rm br} - \rho) + 3(m-1)p_{\rm bk}^2] + -\frac{A_m^2}{8(m+2)}[(2(m-1)p_{\rm bk} + 2p_{\rm r})(\rho - 3p_{\rm br}) - 2(m-1)p_{\rm bk}p_{\rm r} - (m+3)p_{\rm r}^2] + \frac{(m-1)(m-2)}{\epsilon^2} - \frac{(m-1)A_m}{2\epsilon}(p_{\rm r} + p_{\rm bk}). \quad (57)$$

This equation describes the evolution of the scale factor $a(t)$ at the surface $r = \epsilon$ of a four-dimensional, codimension m brane, subject to the approximations and assumptions that we have made thus far. Equation (57) is valid for $m > 2$; as we have discussed in the previous section the codimension one and two cases are unique and should be considered separately. Before continuing with the general case, we briefly discuss the previously studied $m = 1, 2$ examples.

For the codimension one case, there is no $L_a{}^b$ field equation, and (57) reads

$$3\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) = -\frac{\Lambda}{M^{n-2}} - \frac{1}{12M^6}[(\rho + p_{\rm br})^2 - p_{\rm br}(p_{\rm br} - \rho) + p_{\rm r}(\rho - 3p_{\rm br}) - 2p_{\rm r}^2], \quad (58)$$

which agrees with the results of Ref. [32], where a thick codimension one brane was considered. Next, we consider the codimension two case. Once again, a slight complication arises from a boundary term at $r = 0$. We write the modified (a, b) field equation as

$$L_a{}^b|_{r=\epsilon} = -\frac{M_{\rm b}}{4\pi M^4}\delta_a{}^b[3p_{\rm bk} + \rho - 3p_{\rm br} - p_{\rm r}] + \frac{\sqrt{g}|_{r=0}}{\sqrt{g}|_{r=\epsilon}}M_{\rm b}, \quad (59)$$

and hence for $m = 2$ the Eq. (51) becomes

$$3\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) = -\frac{\Lambda}{M^4} - \frac{M_{\rm b}^2}{32\pi^2M^8}[3(\rho + p_{\rm br})^2 + 3p_{\rm bk}^2 + 2(p_{\rm bk} + p_{\rm r})(\rho - 3p_{\rm br}) - 2p_{\rm bk}p_{\rm r} - 5p_{\rm r}^2] + \frac{M_{\rm b}^2}{2\pi M^4}(p_{\rm r} + p_{\rm bk})\frac{\sqrt{g}|_{r=0}}{\sqrt{g}|_{r=\epsilon}}. \quad (60)$$

Once again, this result agrees with the results in Ref. [19].

Having checked that our Eq. (57) agrees with the well-studied cases $m = 1, 2$, we can now consider the general case $m > 2$. The brane energy momentum tensor contains the usual four-dimensional energy density and pressure terms ρ and $p_{\rm br}$, and in addition nonzero pressure terms $p_{\rm r}$ and $p_{\rm bk}$ in the transverse directions. If we assume that $p_{\rm r}$ and $p_{\rm bk}$ are constant across the brane, we can write (57) as

$$3\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) = \omega_1 + \omega_2(\rho + p_{\rm br})^2 + \omega_3 p_{\rm br}(p_{\rm br} - \rho) + \omega_4(\rho - 3p_{\rm br}), \quad (61)$$

where the constants $\omega_{1,2,3,4}$ are given by

$$\omega_1 = -\frac{\Lambda}{M^{n-2}} - \frac{A_m^2}{8(m+2)}[3(m-1)p_{\rm bk}^2 - 2(m-1)p_{\rm bk}p_{\rm r} - (m+3)p_{\rm r}^2] + \frac{(m-1)(m-2)}{\epsilon^2} - \frac{(m-1)A_m}{2\epsilon}(p_{\rm r} + p_{\rm bk}), \quad (62)$$

$$+ \frac{(m-1)(m-2)}{\epsilon^2} - \frac{(m-1)A_m}{2\epsilon}(p_{\rm r} + p_{\rm bk}), \quad (63)$$

$$\omega_2 = -\frac{A_m^2(1+m)}{8(m+2)}, \quad (64)$$

$$\omega_3 = -\frac{A_m^2(m-2)}{4(m+2)}, \quad (65)$$

$$\omega_4 = -\frac{A_m^2}{4(m+2)}[(m-1)p_{\rm bk} + p_{\rm r}]. \quad (66)$$

From (57), we see that in addition to the standard $(\rho - 3p_{\rm br})$ term, we also have the quadratic terms $(\rho + p_{\rm br})^2$ and $p_{\rm br}(p_{\rm br} - \rho)$. If we take $\rho = T + \rho_m$ and $p_{\rm br} = -T + w\rho_m$, where ρ_m is a small energy component with arbitrary equation of state $p_m = w\rho_m$ and T a constant brane tension, we can expand (61) in powers of ρ_m. Doing so we obtain

$$3\left(\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right) = (\omega_1 + 2T^2\omega_3 + 4\omega_4 T) + (\omega_4 + T\omega_3) \times (1-3w)\rho_m + O(\rho_m^2), \tag{67}$$

which is the standard cosmological equation when considering late-time cosmology (that is when ρ_m is small). The cosmological constant problem in this particular model is why the constant $(\omega_1 + 2T^2\omega_3 + 4\omega_4 T)$ is either zero or very small. There is no reason to expect these terms to cancel one another, suggesting these models are not free from fine-tuning. This is not a surprising result, since we had no reason to expect that self-tuning behavior exists in these models.

Thus we have found that late-time cosmology with quadratic corrections is a generic feature of certain braneworld models, regardless of the codimension, generalizing the arguments made in [19,32] for codimension one and two branes. We note that since we have assumed that $p_{\rm r}$ and $p_{\rm bk}$ are constants, then we must necessarily get an expression of the form

$$3\left(\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right) = \kappa_1(p_{\rm r}, p_{\rm bk}) + \kappa_2(p_{\rm r}, p_{\rm bk})(\rho - 3p_{\rm br}) + \kappa_3(\rho^2, p_{\rm br}^2, \rho p_{\rm br}) + \frac{\Lambda}{M^{n-2}} \tag{68}$$

where κ_1, κ_2 are functions of $p_{\rm r}$ and $p_{\rm bk}$ only, and hence are constants, and κ_3 is a term quadratic in the variables ρ, $p_{\rm br}$. It was remarked in [19] that if we obtained a cosmological equation like

$$3\left(\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right) = F(\rho + p_{\rm br}) + G(p_{\rm r}, p_{\rm bk}, \rho, p_{\rm br}) - \frac{\Lambda}{M^{n-2}}, \tag{69}$$

where F and G are functions of the brane energy density, then if $F(\rho + p_{\rm br})$ was a function linear in $(\rho + p_{\rm br})$ then we would have a potential mechanism for self-tuning. In our setup, the function $F(\rho + p_{\rm br})$ will always be quadratic in $(\rho + p_{\rm br})$.

The above arguments apply when the bulk energy momentum tensor $T_r^r|^{\rm bulk}$ is not a function of ρ, $p_{\rm br}$ and p_r, $p_{\rm bk}$ are constants. If we instead assume that p_r and $p_{\rm bk}$ are related to ρ by the equations of state $p_r = w_r\rho$ and $p_{\rm bk} = w_{\rm bk}\rho$, then we would obtain a cosmological model that differs from (67) [19].

Next, we derive a conservation equation for the brane energy momentum tensor $\tilde{T}^\nu_\mu$. We do so from Eq. (47), evaluated at $r = \epsilon$. Using our matching conditions at the surface of the n-brane, we find that (47) reads

$$\frac{A_m}{2}\nabla^B\tilde{T}_{AB} - \frac{A_m}{2(n-2)}(2-m-p)\nabla_A\tilde{T}^r_r + \frac{(m-1)A_m\partial_B M_b}{2M_b}\left(\tilde{T}^B_A - \delta^B_A\frac{\tilde{T}}{n-2}\right) = \frac{T^{\rm bulk}_{Ar}|_{r=\epsilon}}{M^{n-2}}. \tag{70}$$

We have assumed throughout this paper that the time dependence of the extra dimensions can be neglected to our level of approximation. This means that the $\partial_A M_b$ term in (70) can be neglected. In addition, we assumed that the $\tilde{T}^r_r$ component of the brane energy momentum tensor was approximately constant, which means that we can write (70) as

$$\frac{A_m}{2}\left(\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p_{\rm br})\right) \approx \frac{T^{\rm bulk}_{\rm tr}|_{r=\epsilon}}{M^{n-2}}, \tag{71}$$

and hence, if we have $T^{\rm bulk}_{\rm tr}|_{r=\epsilon} = 0$ then we obtain the standard four-dimensional conservation equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p_{\rm br}) = 0. \tag{72}$$

However, this is only approximate, subject to the assumption that $\partial_t M_b \approx 0$. We also note that if $T^{\rm bulk}_{\rm tr}|_{r=\epsilon} \neq 0$, then the brane energy momentum tensor is not strictly conserved, and we can obtain a flow of energy into the bulk. However, if we assume that $T^{\rm bulk}_{\rm tr}|_{r=\epsilon} = 0$, then from (72) we can write

$$\rho = \rho_c a^{-3(1+w)}, \tag{73}$$

where ρ_c is a constant, and $w = p_{\rm br}/\rho$. If we use (73) in (61), then by multiplying the equation by $\dot{a}a^3$ we can write the left-hand side as a total derivative,

$$\frac{3}{2}\frac{d}{dt}((\dot{a}a)^2) = \omega_1\dot{a}a^3 + (\omega_2(1+w)^2 + \omega_3 w(w-1))\rho_c^2\dot{a}a^{-6(1+w)+3} + \omega_4(1-3w)\rho_c\dot{a}a^{-3(1+w)+3}. \tag{74}$$

This equation can be integrated to give a Friedmann type equation for the brane scale factor,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\omega_1}{6} + \frac{2\omega_4}{3}\rho + \frac{2[\omega_2(1+w)^2 + \omega_3 w(w-1)]}{3[4-6(1+w)]}\rho^2 + \frac{C}{a^4}, \tag{75}$$

where C is an integration constant. The first two terms on the right-hand side of (75) are what we would expect for standard four-dimensional cosmology. The third term is quadratic in the brane energy momentum tensor, and the fourth term is a "dark radiation energy" component [42]. It is curious that we find a dark radiation contribution from the bulk; we might expect that in higher codimension additional terms would be present in the brane field equations. We will return to this point in the appendix.

VI. DISCUSSION

To review our progress; we have written the full n-dimensional Ricci tensor and scalar in terms of $R^{(g)}$, $R^{(\gamma)}$ and the extrinsic curvatures ${K_A}^B$ and ${L_a}^b$ (see (20)–(22)). We then decomposed the total energy momentum tensor into a brane component, strictly localized in the region $r < \epsilon$, and a bulk cosmological constant. Since the brane energy momentum tensor is assumed to be localized in the region $r < \epsilon$, we are considering a local defect.

Our work is a particular example of the method outlined in Ref. [37], where brane quantities are defined as the projection of higher dimensional terms onto a four-dimensional subspace. Doing so for the subspace $r = \epsilon$, we have found the approximate cosmological equation (61). In our setup, we see that the four-dimensional Planck mass $M_{\rm pl}^2$ is given by the coefficient of the ρ term in (67), that is

$$M_{\rm pl}^2 = \frac{1}{2(\omega_4 + T\omega_3)}. \tag{76}$$

We note that our definition of the Planck mass differs from the effective action approach considered in (amongst others) Ref. [36], where it is rather defined as

$$M_{\rm pl}^2 = M^{m+2} \int \sqrt{G} d^m y, \tag{77}$$

in other words an integral over the m codimensional space, which for the metric considered in this paper is given by

$$M_{\rm pl}^2 = M^{m+2}\Omega^{[m-1]} \int \alpha^{m-1} dr. \tag{78}$$

The difference between the definitions (76) and (77) reflects the two different philosophies adopted in the literature towards defining four-dimensional quantities. In the effective action approach, four-dimensional quantities are obtained by integrating out the extra dimensions in the full n-dimensional action. However, in this paper we use a tensorial approach, and project n-dimensional tensors tangentially to the brane. For a discussion of the two approaches, see Ref. [43].

As discussed in Refs. [36,44], the codimension $m \geq 3$ case is different to the more commonly studied $m = 1, 2$ cases in the literature. To see the difference, we consider the n-dimensional Ricci scalar for our static metric (1),

$$R = K' + L' + \frac{1}{4}({K_A}^B {K_B}^A + {L_a}^b {L_a}^b) + \frac{1}{4}(K^2 + L^2) + \frac{1}{2}KL + \frac{R^{(g)}}{f} + \frac{(m-1)(m-2)}{\alpha^2}. \tag{79}$$

We see that the last term in (79) is present only when $m > 2$, and represents the curvature of the bulk. We note that this is singular if $\alpha = 0$ anywhere in the transverse space. Thus to avoid singularities in the curvature invariants, we must impose $\alpha \neq 0$ for all $r > 0$. Note that we have imposed $\alpha = 0$ as a boundary condition at $r = 0$, however this is simply a coordinate singularity.

As has been discussed in Ref. [36], the last term in (79) is a problem when we consider localizing gravity on the brane. To see why, we consider the Einstein equations (20)–(22) for our metric (1). In the asymptotic limit $r \to \infty$, we require a solution to the Einstein equations that is anti-de Sitter, that is we require $R \sim$ const. To obtain a solution of this form all terms on the right-hand side of (79) must either asymptote to zero or a constant as $r \to \infty$. We are considering flat branes, so $R^{(g)} = 0$. The ${K_A}^B$ terms asymptote to constants if the warp factor $f(r)$ asymptotes to an exponential, as we expect it to. The problematic term in (79) is the one of the form $(m-1)(m-2)/\alpha^2$, from which we deduce that the function $\alpha(r)$ must either asymptote to infinity or a constant in the limit $r \to \infty$. However, we find that no solution exists to the Einstein equations such that $\alpha \to$ const as $r \to \infty$ and $f(r)$ is a real exponential function. Hence we are left only with the possibility $\alpha \to \infty$ as $r \to \infty$. This removes the problematic α^{-2} term in (21), and we find that a valid solution exists in this case, where both $\alpha(r)$ and $f(r)$ are growing exponentials in r (a result found in Ref. [45]). Since in this case both warp factors are increasing functions as $r \to \infty$, it follows that the transverse space has an infinite volume.

Since we are considering local defects of codimension $m > 2$, the above analysis applies and we will generically obtain transverse spaces with infinite volume. This infinite volume is a problem, since we find that gravity cannot be localized on the brane in such a setup. The reason why we cannot localize gravity is that the zero mode in the graviton spectrum will not be normalizable. To avoid (but not solve) this problem, we could follow Ref. [44] and introduce an infrared cutoff when integrating the zero mode over the extra dimensions. This cutoff could arise as an interbrane separation, for example. In doing so, we would then obtain a finite integral over the transverse dimensions, and it would be possible to recover conventional four-dimensional gravity. Of course, it would be preferable to obtain a model where gravity can be localized on the brane without the need to introduce a cutoff, and it appears that global defects are better suited to achieve this. Alternatively, we could introduce additional fields in the bulk in an attempt to remove this problematic behavior. However, bulk fields have the highly undesirable property of inducing a nontrivial Weyl tensor contribution to the field equations on the brane, and a detailed analysis of the bulk would have to be performed in such a setup.

VII. CONCLUSION

In this paper we have calculated the evolution equation for the scale factor $a(t)$ of a thick, codimension m, 3-brane.

By assuming radial symmetry in the bulk, we have found that many of the Einstein equations approximately admit a first integral, and we have integrated these equations over the brane thickness to obtain a set of approximate junction conditions at the surface of the brane $r = \epsilon$. We then used these junction conditions to write an equation for the evolution of the brane scale factor, $a(t)$, in terms of the brane energy momentum tensor. Since we considered a brane of arbitrary codimension, we were forced to make a large number of assumptions, which we review;

(i) Derivatives tangential to the brane can be neglected. Specifically, we assumed that the bulk metric has only a weak dependence on time, and is static to the level of approximation that we are working at.

(ii) We assumed that in the core of the brane the metric may be approximately written as

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu - dr^2 - r^2 d\Omega^2_{[m-1]}. \quad (80)$$

The boundary conditions that we have used give us this form of the metric at the center of the brane, that is at $r = 0$, and we have assumed that the metric can be approximately written as (80) for $r < \epsilon$.

(iii) We defined four-dimensional quantities as n-dimensional quantities integrated over the brane thickness in the transverse dimensions. For example, the brane energy momentum tensor $\hat{T}_A{}^B$ is given by

$$\hat{T}_A{}^B \equiv \int \sqrt{\gamma} d^{m-1}y \int_0^\epsilon \alpha^{m-1} N A^n T_A{}^B dr. \quad (81)$$

This is not an assumption, but rather a definition. However, there is some ambiguity in defining a four-dimensional quantity when discussing thick braneworlds, and (81) is not unique.

(iv) We have assumed that a solution to the full n-dimensional Einstein equations exists that respects the above assumptions. It is important to stress that we have not found a full solution to the field equations. We expect that a solution exists of the form postulated, subject to the above assumptions.

(v) We have assumed that the brane thickness is time independent, at least to first order. We expect that any matter on the brane will have a backreaction effect on the brane profile, and we have assumed that this effect is negligible. We have also assumed that this thickness is not as small as any fundamental length scale in the model. By this we mean that if the brane is too thin, then we could not use our classical arguments (we would not be able to resolve the thickness of the brane without appealing to quantum mechanics).

Based on these assumptions, we have found the standard cosmological equation plus quadratic terms in the brane energy density for a thick brane of arbitrary codimension. We have also found a general fine-tuning condition required to make the effective four-dimensional cosmological constant small in our model. It depends on the bulk energy momentum tensor, the Ricci scalar of the transverse dimensions, as well as the brane thickness, tension, and transverse energy momentum components $p_{\rm r}$ and $p_{\rm bk}$. We assumed that the bulk energy momentum tensor in our model is simply a constant, and we might expect bulk fields to be present [36]. Introducing new fields into the bulk will affect both the fine-tuning relationship and the four-dimensional Friedmann equation that we have found. Our final result is the conservation equation for the brane energy momentum tensor, (70). We find that the standard four-dimensional conservation equation is obtained, but only if the component $T^{\rm bulk}_{\mu r}|_{r=\epsilon}$ of the bulk energy momentum tensor is zero, a well-known result.

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APPENDIX: COVARIANT BRANEWORLD EQUATIONS

In the appendix, we discuss why we obtain a dark radiation term in the brane Friedmann equation, regardless of the codimension, and thereby produce an alternative derivation of the main results of this paper using tensorial notation. We then discuss the relationship between codimension one and codimension m branes in our setup.

1. Weyl tensor

From our brane Friedmann equation (75), we see that a dark radiation term is present, which is a bulk effect contributing to H. We might expect extra terms to be introduced into the Friedmann equation, since we are considering more than one transverse dimension. We now consider the Weyl tensor in detail, to determine the origin of this dark radiation term.

The Weyl tensor $W_{\mu\nu}{}^{\alpha\beta}$ is defined as

$$W_{\mu\nu}{}^{\rho\sigma} = R_{\mu\nu}{}^{\rho\sigma} - \frac{4}{n-2} g^{[\rho}{}_{[\mu} R^{\sigma]}{}_{\nu]} + \frac{2}{(n-1)(n-2)} R g^{[\rho}{}_{[\mu} g^{\sigma]}{}_{\nu]}. \quad (A1)$$

A certain contraction of this tensor determines the bulks effect on the brane. To see this, we consider the covariant braneworld equations of arbitrary codimension, given by Ref. [37]

$$R^{(p)}_{\mu\nu} = \frac{p-2}{n-2} \eta_\mu{}^\rho \eta_\nu{}^\sigma R_{\rho\sigma} + \frac{1}{n-2}\left(\eta^{\rho\sigma} R_{\rho\sigma} - \frac{p-1}{n-1} R\right)\eta_{\mu\nu} + \frac{p-1}{p^2} \bar{K}^\sigma \bar{K}_\sigma \eta_{\mu\nu} + \frac{p-2}{p} \bar{C}_{\mu\nu}{}^\sigma \bar{K}_\sigma - \bar{C}_\mu{}^{\rho\sigma} \bar{C}_{\nu\rho\sigma} + W_{\mu\nu}, \quad (A2)$$

where $\eta_{\mu\nu}$ is a tensor that projects other tensors tangentially to the brane. The Weyl tensor $W_{\mu\nu}$ is given by

$$W_{\mu\nu} = \eta_\mu{}^\sigma \eta_\nu{}^\kappa \eta_\tau{}^\rho W_{\rho\sigma}{}^\tau{}_\kappa, \tag{A3}$$

and $\bar{C}_{\mu\nu}{}^\rho$ by

$$\bar{C}_{\mu\nu}{}^\rho = \bar{K}_{\mu\nu}{}^\rho - \frac{1}{p}\eta_{\mu\nu}\bar{K}^\rho. \tag{A4}$$

The extrinsic curvature $\bar{K}_{\mu\nu}{}^\rho$ in this notation is

$$\bar{K}_{\mu\nu}{}^\rho = \eta_\nu{}^\sigma \eta_\mu{}^\alpha \nabla_\alpha \eta_\sigma{}^\rho. \tag{A5}$$

In this section we use $\bar{K}_{\mu\nu}{}^\rho$ as the extrinsic curvature, as opposed to K_{AB} which has been used in this paper. K_{AB} is actually a particular example of the more general $\bar{K}_{\mu\nu}{}^\rho$ above. To see this, we write (A5) as

$$\bar{K}_{\mu\nu}{}^\rho = \eta_\nu{}^\sigma \eta_\mu{}^\alpha \perp^\rho{}_\gamma \nabla_\alpha \eta_\sigma{}^\gamma \tag{A6}$$

$$= \eta_\nu{}^\sigma \eta_\mu{}^\alpha \perp^\rho{}_\gamma (\partial_\alpha \eta_\sigma{}^\gamma + \Gamma^\gamma{}_{\alpha\beta}\eta^\beta{}_\sigma - \Gamma^\beta{}_{\alpha\sigma}\eta_\beta{}^\gamma). \tag{A7}$$

Next, we note that in our coordinate system, $\eta_{\mu\nu} = g_{AB}\delta^A{}_\mu \delta^B{}_\nu$, $\eta_\mu{}^\nu = \delta_A{}^B \delta_B{}^\nu \delta_\mu{}^A$. Using this, as well as the relation $\eta_\mu{}^\nu \perp_\alpha{}^\mu = 0$, we find that the extrinsic curvature can be written as

$$\bar{K}_{\mu\nu}{}^\rho = \delta_\mu{}^A \delta_\nu{}^B \perp^\rho{}_\gamma \Gamma^\gamma{}_{AB} \tag{A8}$$

$$= -\tfrac{1}{2}\delta_\mu{}^A \delta_\nu{}^B \perp^{\rho\gamma} \partial_\gamma g_{AB}. \tag{A9}$$

Now, we can use the fact that due to the symmetry imposed on our metric, the only nonzero orthogonal derivative of g_{AB} is in the radial direction, which means we can write

$$\bar{K}_{\mu\nu}{}^\rho = \tfrac{1}{2}\delta_r{}^\rho \delta_\mu{}^A \delta_\nu{}^B \partial_r g_{AB}, \tag{A10}$$

where we used $\perp^{r\gamma} = g^{rr} = -1$. The only nonzero components of $\bar{K}_{\mu\nu}{}^\rho$ are $\rho = r$, and hence we can drop this index, and write

$$\bar{K}_{\mu\nu} = \tfrac{1}{2}\delta_\mu{}^A \delta_\nu{}^B \partial_r g_{AB}. \tag{A11}$$

Comparing $\bar{K}_{\mu\nu}$ in this section with the K_{AB} that we have been using;

$$K_{AB} = \partial_r g_{AB}, \tag{A12}$$

we see that they differ by a factor of $\frac{1}{2}$, which we must account for in what follows. In addition, we note that $\bar{K}_{\mu\nu\rho}$ will be used. This is given by $\perp_{\rho\alpha}\bar{K}_{\mu\nu}{}^\alpha$. Since $g_{rr} = -1$, when we lower the third index on the extrinsic curvature, we must introduce a factor of -1,

$$\bar{K}_{\mu\nu\rho} = -\delta^r_\rho \delta_\mu{}^A \delta_\nu{}^B K_{AB}. \tag{A13}$$

Returning to our codimension one calculation, Eq. (A2) can be used to calculate the brane evolution equation (75). Note the presence of the Weyl tensor in (A2); for codimension one objects it is $W_{\mu\nu}$ which gives the dark radiation term.

In five-dimensional, thin braneworld models, the Weyl tensor is singular at the position of the brane. For this reason, $W_{\mu\nu}$ in (A2) is not evaluated on the brane. This is the approach that we will take; we evaluate the Weyl tensor at some $r = \epsilon + \delta \gtrsim \epsilon$ outside the core. We look for a solution to the field equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$, since we have a cosmological constant only in the bulk. A solution to these field equations is given by

$$ds^2 = -h(a)dt^2 + \frac{da^2}{h(a)} + a^2[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)] + \alpha_0^2 \gamma_{ab} dy^a dy^b, \tag{A14}$$

where

$$h(a) = \frac{(n-1)}{4L^2}a^2 - \frac{\alpha}{a^2}, \tag{A15}$$

and the constant α_0^2 is given by $\alpha_0^2 = (m-2)L^2/(n-1)$.

From (A14) we can now calculate the Weyl tensor (A1). The relevant components that contribute to the brane evolution equations are $W_{DA}{}^{DB}$, where capital Latin indices run over the (3 + 1)-brane coordinates. To calculate the Weyl tensor contribution explicitly, we write $R_{DA}{}^{DB} = R_A{}^B - R_{aA}{}^{aB} - R_{rA}{}^{rB}$, and use the fact that for the metric (A14), we can set $R_{aA}{}^{aB} = 0$. Hence, using $R_{\mu\nu} = \Lambda_{\rm n} g_{\mu\nu}$ and $R = n\Lambda_{\rm n}$, we can write the relevant Weyl tensor components as

$$W_{Di}{}^{Dj} = \frac{(4-n)}{L^2}\delta_i{}^j + \frac{h'}{2a}\delta_i{}^j = \frac{15-3n}{4L^2}\delta_i{}^j + \frac{\alpha}{a^4}\delta_i{}^j, \tag{A16}$$

$$W_{Dt}{}^{Dt} = \frac{(4-n)}{L^2} + \frac{h''}{2} = \frac{15-3n}{4L^2} - \frac{3\alpha}{a^4}. \tag{A17}$$

We note that the standard dark radiation term is present in (A16) and (A17).

Thus we have confirmed the presence of the dark radiation term in our setup. To understand why we obtain this term, we return to the metric (A14). We see that we can split this metric into two components; a five-dimensional part given by

$$ds_{[5]}^2 = -h(a)dt^2 + \frac{da^2}{h(a)} + a^2[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{A18}$$

which is the standard five-dimensional metric considered in the literature [2], that is the standard five-dimensional Schwarzschild AdS line element, and a second component

$$ds_{[m-1]}^2 = \alpha_0^2 \gamma_{ab} dy^a dy^b, \tag{A19}$$

which is simply pure AdS. This second component is the

$(m-1)$ codimensions. From this split the origin of the dark radiation term becomes a little clearer; we might expect to obtain a dark radiation term from the five-dimensional part (A18) of our metric. The remaining $(m-1)$ codimensions in (A19) are pure AdS only, and hence will contribute only terms like $\sim 1/L^2$ to $W_{\mu\nu}$. The fact that we obtain a dark radiation term in the brane Friedmann equation is a consequence of our choice of metric ansatz. If we dropped our assumption of spherical symmetry in the extra dimensions, or introduced additional bulk fields, then we would obtain more complicated bulk effects on the brane scale factor. In other words, in this particular model we have over-constrained the bulk, and assumed that it is simply pure AdS away from the brane.

2. Covariant braneworld equations

Finally, we verify that we can obtain our equations using the covariant braneworld equations given in, for example, [37]. We will see that we can consider our brane as either a codimension one or codimension m object, and still obtain the same brane equation. We will explain why this is so at the end of the section.

We begin with the generalized Gauss equation for a p-brane of arbitrary codimension, as found in [37]. This equation relates the Ricci tensor of the brane, $R^{(p)}$, to the full n-dimensional Ricci tensor $R_{\mu\nu}$, the extrinsic curvature $\bar{K}_{\mu\nu}{}^{\rho}$ and the appropriately contracted Weyl tensor $W_{\mu\nu}$. We will show that we can obtain our brane Friedmann equation using two approaches. In the first method, we consider a codimension one object, with the $(m-1)$ "codimensions" not as transverse dimensions but rather as brane parallel dimensions. In this approach the radial coordinate r acts as the orthogonal coordinate, and we find our Friedmann equation arises from the Gauss equation.

In the second approach, we consider our brane as a codimension m object, and calculate the Weyl tensor and background Ricci tensor in terms of the extrinsic curvatures K_A^B and L_a^b. With this approach, we obtain the same Friedmann equation. We will show this, and then explain why we obtain the same result regardless of whether we consider the $(m-1)$ codimensions as brane parallel or brane-orthogonal directions.

In [37], the Ricci tensor $R^{(p)}_{\mu\nu}$ and scalar $R^{(p)}$ of a p-brane embedded in an n-dimensional background space have been calculated, and are given by

$$\begin{aligned} R^{(p)}_{\mu\nu} = {}& \frac{p-2}{n-2}\eta_\mu{}^\rho\eta_\nu{}^\sigma R_{\rho\sigma} \\ &+ \frac{1}{n-2}\left(\eta^{\rho\sigma}R_{\rho\sigma} - \frac{p-1}{n-1}R\right)\eta_{\mu\nu} \\ &+ \frac{p-1}{p^2}\bar{K}^\sigma\bar{K}_\sigma\eta_{\mu\nu} + \frac{p-2}{p}\bar{C}_{\mu\nu}{}^\sigma\bar{K}_\sigma \\ &- \bar{C}_\mu{}^{\rho\sigma}\bar{C}_{\nu\rho\sigma} + W_{\mu\nu}, \end{aligned} \tag{A20}$$

where

$$W_{\mu\nu} = \eta_\mu{}^\sigma\eta_\nu{}^\kappa\eta_\tau{}^\rho W_{\rho\sigma}{}^\tau{}_\kappa, \tag{A21}$$

and

$$\begin{aligned} R^{(p)} = {}& \frac{p-1}{n-2}\left(2\eta^{\rho\sigma}R_{\rho\sigma} - \frac{p}{n-1}R\right) + \frac{p-1}{p}\bar{K}^\sigma\bar{K}_\sigma \\ &- \bar{C}_{\lambda\mu}{}^\nu\bar{C}^{\lambda\mu}{}_\nu + W, \end{aligned} \tag{A22}$$

where

$$\bar{C}_{\mu\nu}{}^\rho = \bar{K}_{\mu\nu}{}^\rho - \frac{1}{p}\eta_{\mu\nu}\bar{K}^\rho. \tag{A23}$$

To begin, we consider a codimension one object of dimension $p = m+3$. If we consider our analysis as describing a codimension one object, with r being the codimension, then $R^{(p)} = R^{(g)} + R^{(\gamma)}$, and we can calculate our brane Friedmann equation from (A22).

To do so, we will need to evaluate the projected Weyl tensor, specifically the trace of (A21),

$$W = \eta^{\kappa\sigma}W_{\rho\sigma}{}^\tau{}_\kappa\eta^\rho{}_\tau. \tag{A24}$$

Remembering that the brane tangential projection $\eta_{\mu\nu}$ now runs over the standard four dimensions as well as the $(m-1)$ spherically symmetric dimensions, the relevant components of (A24) are given by

$$\begin{aligned} W_{Di}{}^{Dj} &= \frac{(4-n)}{L^2}\delta_i{}^j + \frac{h'}{2a}\delta_i{}^j \\ &= -\frac{15-3n}{4(n-1)}\Lambda_{\rm n}\delta_i{}^j + \frac{\alpha}{a^4}\delta_i{}^j, \end{aligned} \tag{A25}$$

$$W_{Dt}{}^{Dt} = \frac{(4-n)}{L^2} + \frac{h''}{2} = -\frac{15-3n}{4(n-1)}\Lambda_{\rm n} - \frac{3\alpha}{a^4}, \tag{A26}$$

$$W_{ab}{}^{ab} = \frac{5(m-1)}{(m+3)}\Lambda_{\rm n}, \tag{A27}$$

$$W_{aB}{}^{aB} + W_{Ba}{}^{Ba} = -\frac{8(m-1)}{m+3}\Lambda_{\rm n}. \tag{A28}$$

Now taking the trace of (A25) and summing over all of the above Weyl tensor contributions to W gives the result $W = 0$, as it should be; the trace of the Weyl tensor is zero for codimension one objects.

Using all of the above in (A22), as well as the fact that $p = m+3$, $n = m+4$, gives us

$$\begin{aligned} R^{(g)} + R^{(\gamma)} = {}& 2(R^a{}_a + R^A{}_A) - R_\alpha{}^\alpha - \frac{(m+2)}{4(m+3)} \\ &\times (K^2 + L^2 + 2KL) + \frac{1}{4}(K_A{}^B K_B{}^A + L_a{}^b L_b{}^a) \\ &- \frac{1}{4(m+3)}(K^2 + L^2 + 2KL). \end{aligned} \tag{A29}$$

Finally, we use the fact that

$$R^{(g)} = -6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right), \quad \text{(A30)}$$

as well as

$$2R_r{}^r - R_\alpha{}^\alpha = R_\alpha{}^\alpha - 2(R_a{}^a + R_A{}^A) = 2\frac{T_r{}^r}{M^{N-2}}, \quad \text{(A31)}$$

to write (A29) as

$$-6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) = -2\frac{T_r{}^r}{M^{n-2}} - R^{(\gamma)} - \frac{1}{4}(K^2 + L^2 + 2KL) + \frac{1}{4}(K_C{}^D K_D{}^C + L_a{}^b L_b{}^a). \quad \text{(A32)}$$

Next, we consider the field equations "junction conditions" for our thick brane approach. As before, we must integrate them over the brane-orthogonal coordinates. However, since we are now simply considering a codimension one object, we only integrate over the r coordinate. Doing so, we obtain

$$K_i{}^j|_{r=\epsilon} = \frac{M_{\rm b}^{m-1}}{M^{n-2}}\left(\tilde{T}_i{}^j - \delta_i{}^j\frac{\tilde{T}}{n-2}\right), \quad \text{(A33)}$$

$$K_t{}^t|_{r=\epsilon} = \frac{M_{\rm b}^{m-1}}{M^{n-2}}\left(\tilde{T}_t{}^t - \delta_t{}^t\frac{\tilde{T}}{n-2}\right), \quad \text{(A34)}$$

$$L_a{}^b|_{r=\epsilon} = \frac{M_{\rm b}^{m-1}}{M^{n-2}}\left(\tilde{T}_a{}^b - \delta_a{}^b\frac{\tilde{T}}{n-2}\right) - \frac{2}{\epsilon}\delta_a{}^b. \quad \text{(A35)}$$

The final step is to use (A33)–(A35) to write (A32) as

$$3\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) + \frac{M_{\rm b}^{2m-2}}{8(n-2)^2M^{2n-4}}((n-2)^2\tilde{T}_A{}^B\tilde{T}_B{}^A + B_m(\tilde{T}_A{}^A)^2 + (n-2)^2\tilde{T}_a{}^b\tilde{T}_b{}^a + B_m(\tilde{T}_a{}^a)^2) + \frac{M_{\rm b}^{2m-2}}{8(n-2)^2M^{2n-4}} \times (2B_m(\tilde{T}_a{}^a + \tilde{T}_r{}^r)\tilde{T}_A{}^A + 2B_m\tilde{T}_a{}^a\tilde{T}_r{}^r + C_m(\tilde{T}_r{}^r)^2) - \frac{M_{\rm b}^{m-1}}{2\epsilon M^{n-2}}\tilde{T}_a{}^a - \frac{(m-1)M_{\rm b}^{m-1}}{2\epsilon M^{n-2}}\tilde{T}^r{}_r - \frac{(m-1)(m-2)}{\epsilon^2} = -\frac{\Lambda}{M^{n-2}}, \quad \text{(A36)}$$

which is the same equation as (56) for a codimension m brane. Note that since we have assumed that our space contains a codimension one object, the trace of the Weyl tensor vanishes. However, $W_{\mu\nu}$ does not vanish; as we have shown above we obtain a dark radiationlike term.

It may seem quite unnatural that we obtain the same Friedmann equation whether we consider a codimension one or m object. The reason we do so is because we have assumed spherical symmetry. In our paper, we have considered surfaces of constant $r = \epsilon$ only; since the other codimensions are spherically symmetric we do not have to set them to a particular fixed value (since our final result will not depend on our choice). Hence the $(m-1)$ spherically symmetric dimensions could equally well be brane tangential or brane-orthogonal coordinates. The only difference between the two would be the form of the $(m-1)$ components of the energy momentum tensor $\tilde{T}_a{}^b$, which would be small for a codimension m brane, but potentially large for a codimension one brane.

We now proceed with our second approach, that is to consider our brane as a four-dimensional object of codimension m. Now, we will no longer have the trivial result $W = 0$, however we find that we still obtain the same equation relating the brane scale factor $a(t)$ with $\tilde{T}_A{}^B$, $\tilde{T}_r{}^r$, and $\tilde{T}_a{}^b$.

To proceed, we consider (A22) again. Now we have $p = 4$ and $R^{(p)} = R^{(g)}$. We begin by calculating the projected Weyl tensor, using (A1). The relevant components in our coordinate system are $W_{AB}{}^{AB}$, which are explicitly

$$W_{AB}{}^{AB} = R_{AB}{}^{AB} - \frac{6}{(n-2)}R^A{}_A + \frac{12}{(n-1)(n-2)}R_\alpha{}^\alpha. \quad \text{(A37)}$$

Using (A37) in (A22), we find that the brane Ricci tensor R^p may be written as

$$R^{(p)} = R_{AB}{}^{AB} - \frac{3}{16}K^2 + \frac{1}{4}K^A_B K_A{}^B - \frac{1}{16}K^2. \quad \text{(A38)}$$

Next, we use the relation (A31), as well as

$$R_{AB}{}^{AB} = R_A{}^A - R_{aA}{}^{aA} - R_{rA}{}^{rA} \quad \text{(A39)}$$

to write (A38) as

$$R^{(p)} = R^\alpha{}_\alpha - 2R^a{}_a - R_A{}^A - R_{aA}{}^{aA} - R_{rA}{}^{rA} + \frac{1}{4}K^A{}_B K^B{}_A - \frac{1}{4}K^2 - 2\frac{T_r{}^r}{M^{N-2}} \quad \text{(A40)}$$

$$= R_r{}^r - R_a{}^a - R_{aA}{}^{aA} - R_{rA}{}^{rA} + \frac{1}{4}K^A{}_B K^B{}_A - \frac{1}{4}K^2 - 2\frac{T_r{}^r}{M^{n-2}} \quad \text{(A41)}$$

we now evaluate the terms in (A41). We find

$$R_r{}^r - R_a{}^a = \frac{1}{2}K' + \frac{1}{4}(K_A{}^B K_B{}^A + L_a{}^b L_b{}^a) - R^{(\gamma)} - \frac{1}{4}L^2 - \frac{1}{4}KL, \quad \text{(A42)}$$

$$-R_{aA}{}^{aA} = -\frac{1}{4}KL, \tag{A43}$$

$$-R_{rA}{}^{rA} = -\frac{1}{2}K' - \frac{1}{4}K_A{}^B K_B{}^A. \tag{A44}$$

Using these in (41) gives us

$$R^{(p)} = \frac{1}{4}(K_A{}^B K_B{}^A + L_a{}^b L_b{}^a) - \frac{1}{4}(K^2 + L^2) - \frac{1}{2}KL - 2\frac{T_r{}^r}{M^{n-2}} - R^{(\gamma)}. \tag{A45}$$

Once again, if we then proceed to write $K_A{}^B$ and $L_a{}^b$ in terms of $\tilde{T}_\mu{}^\nu$, we would find the same equation as we found in both the previous sections and above (when we considered the brane as a codimension one object). We stress that we obtain the same result because of the symmetry imposed on the $(m-1)$ codimensions. This means that our choice of the angular coordinates will not affect our Ricci scalar $R^{(p)}$, and hence we can consider the spherically symmetric coordinates either as being brane orthogonal or brane tangential.

Finally, we note that our result that codimension one and m branes are equivalent for a metric such as ours is only valid for a thick brane of codimension one or m. If we consider thin branes, then in the codimension m case we would obtain δ-function singularities in $K_A{}^B$ and hence $R^{(p)}$, whereas for codimension one branes there would be no divergent behavior. The singular behavior is removed in the codimension m case since we have smeared the brane energy momentum tensor over a finite region of space.

The brane Friedmann equation that we have derived is written in terms of the brane energy momentum tensor, which has the standard four-dimensional components $\tilde{T}_{AB}$, and in addition nonzero components in the (a, b) and (r, r) directions, $\tilde{T}_{ab}$ and $\tilde{T}_{rr}$. Above, we have shown that we can write down an evolution equation for $a(t)$ in terms of $\tilde{T}_{AB}$, $\tilde{T}_{ab}$, and $\tilde{T}_{rr}$, regardless of whether the object is codimension one or m. The only difference between the two setups will be the form of the $\tilde{T}_{ab}$ components of the brane energy momentum tensor. When we have m codimensions, we expect that $\tilde{T}_{ab}$ will be some small quantity, which asymptotes to zero in the thin brane limit. For a codimension one brane, however, the $\tilde{T}_{ab}$ will have the same form as the $\tilde{T}_{AB}$ components. This is the principle difference between the two treatments. However in this paper we do not explicitly specify the brane energy momentum tensor.

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