Quantum gravity corrections to the one loop scalar self-mass during inflation

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We compute the one loop corrections from quantum gravity to the self-mass-squared of a massless, minimally coupled scalar on a locally de Sitter background. The calculation was done using dimensional regularization and renormalized by subtracting fourth order BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmerman) counterterms. Our result should determine whether quantum gravitational loop corrections can significantly alter the dynamics of a scalar inflaton.

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I. INTRODUCTION

One can understand quantum loop effects as the reaction of classical field theory to virtual particles. Increasing the number density of these virtual particles strengthens quantum effects. The expansion of spacetime tends to do this by trapping virtual pairs in the Hubble flow and delaying their annihilation. During inflation the effect is so strong that long wavelength massless virtual particles can persist forever. On the other hand, most massless particles possess classical conformal invariance, which causes the rate at which they emerge from the vacuum to redshift so that the number density of virtual particles is not increased.

Massless, minimally coupled scalars and gravitons are unique in possessing zero mass without classical conformal invariance. Inflation results in a vast enhancement of quantum effects for these particles. That is the origin of the primordial scalar [1] and tensor [2] perturbations predicted by inflation [3,4]. Weinberg has recently shown that loop corrections to these perturbations are also enhanced, although not enough to make them observable [5,6].

Because the enhancement derives from long wavelength virtual particles, the strongest effects come from nonderivative interactions. A massless, minimally coupled scalar with a quartic self-interaction is pushed up its potential by inflationary particle production, thereby inducing a violation of the weak energy condition [7,8] and a nonzero scalar mass [9,10]. The vacuum polarization from a charged, massless, minimally coupled scalar induces a nonzero photon mass [11,12] and a small negative shift in the vacuum energy [13]. The inflationary creation of massless, minimally coupled scalars which are Yukawa-coupled to a massless fermion gives the fermion mass [14,15] and induces a negative vacuum energy that grows without bound [16]. And, more recently, a variety of other interesting quantum loop effects due to scalar particles have been investigated [17–21].

Gravitons have derivative interactions which weaken the enhancement they experience. At one loop order quantum gravity gives only a constant shift in the vacuum energy [22–24]. At two loops one finds a secular reduction [25] which might help explain why the observed cosmological constant is so much smaller than the natural scales of fundamental physics [26]. (But see [27] for a different view [28].) The inflationary production of gravitons also induces a growing fermion field strength [29].

It is natural to wonder about the result of combining a massless, minimally coupled scalar with gravity. If there are significant quantum corrections they might have important consequences for inflation, although we stress that our scalar is a spectator to Λ-driven inflation. In this paper we shall compute its self-mass-squared at one loop order. That is already a major task. In a subsequent work [30] we will use the result to solve for the quantum-corrected scalar mode functions to see if the inflationary production of gravitons has a significant impact on scalar propagation.

In the next section we derive those Feynman rules we shall require. The computation is done in Sec. III. In Sec. IV we first derive the necessary BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmerman) counterterms [31] and then use them to obtain a fully renormalized result. Because the effect we are seeking derives from infrared—indeed, cosmological scale—virtual particles, the ambiguity in the finite parts of these counterterms should not matter. It was possible to show this explicitly for the analogous impact of inflationary gravitons on massless fermions [29]. Our conclusions comprise Sec. V.

II. FEYNMAN RULES

To facilitate dimensional regularization we work in D spacetime dimensions. Our Lagrangian is

$$\mathcal{L} \equiv -\frac{1}{2}\partial_\mu \phi \partial_\mu \phi g^{\mu\nu}\sqrt{-g} + \frac{1}{16\pi G}(R-(D-2)\Lambda)\sqrt{-g}. \qquad (1)$$

Here G is Newton's constant and $\Lambda \equiv (D-1)H^2$ is the cosmological constant. Because our scalar is a spectator to Λ-driven inflation, its background value is zero. Our background geometry is the conformal coordinate patch of D-dimensional de Sitter space,

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$$ds^2 = a^2(-d\eta^2 + d\vec{x}\cdot d\vec{x}) \quad \text{where} \quad a(\eta) = -\frac{1}{H\eta}. \tag{2}$$

Perturbation theory is expressed using the graviton field $h_{\mu\nu}(x)$,

$$g_{\mu\nu}(x) \equiv a^2(\eta_{\mu\nu} + \kappa h_{\mu\nu}(x)) \quad \text{where} \quad \kappa^2 \equiv 16\pi G. \tag{3}$$

The inverse metric and the volume element have the following expansions:

$$g^{\mu\nu} = \frac{1}{a^2}(\eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu}{}_{\rho} h^{\rho\nu} - \ldots), \tag{4}$$

$$\sqrt{-g} = a^D(1 + \tfrac{1}{2}\kappa h + \tfrac{1}{8}\kappa^2 h^2 - \tfrac{1}{4}\kappa^2 h^{\rho\sigma}h_{\rho\sigma} + \ldots). \tag{5}$$

This computation requires the $\phi^2 h$ and $\phi^2 h^2$ interactions which derive from expanding the scalar kinetic term,

$$\begin{aligned} -\tfrac{1}{2}\partial_\mu\phi\partial_\nu g^{\mu\nu}\sqrt{-g} = {} & -\tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi a^{D-2} \\ & \times\{\eta^{\mu\nu} - \kappa h^{\mu\nu} + \tfrac{1}{2}\eta^{\mu\nu}\kappa h \\ & + \tfrac{1}{8}\eta^{\mu\nu}\kappa^2 h^2 - \tfrac{1}{4}\eta^{\mu\nu}\kappa^2 h^{\rho\sigma}h_{\rho\sigma} \\ & - \tfrac{1}{2}\kappa^2 h h^{\mu\nu} + \kappa^2 h^{\mu\rho}h_{\rho}^{\nu} + O(\kappa^3)\}. \end{aligned} \tag{6}$$

We represent the 3-point and 4-point interaction terms as vertex operators acting on the fields. For example, the first of the 3-point vertices is

$$-\tfrac{1}{2}\kappa a^{D-2}\partial_\alpha\phi\partial_\beta\phi h^{\alpha\beta} \Rightarrow V_1^{\alpha\beta} = i\kappa a^{D-2}\partial_1^\alpha\partial_2^\beta. \tag{7}$$

We number the fields "1", "2", "3", etc., starting with the two scalars and proceeding to the gravitons. Although we extract a factor of $\frac{1}{2}$ for the two identical scalars, it is more efficient, for our computation, *not* to extract a similar factor of $\frac{1}{2}$ for the identical gravitons of the 4-point vertices. Then we can dispense with the symmetry factor. So our first 4-point vertex is,

$$\begin{aligned} -\frac{\kappa^2}{16}a^{D-2}\partial^\mu\phi\partial_\mu\phi h^2 \Rightarrow U_1^{\alpha\beta\rho\sigma} \\ = -\frac{i}{8}\kappa^2 a^{D-2}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_1\cdot\partial_2. \end{aligned} \tag{8}$$

The 3-point vertices are listed in Table I; Table II gives the 4-point vertices.

TABLE I. 3-point vertex operators $V_I^{\alpha\beta}$ contracted into $\phi_1\phi_2 h_{\alpha\beta}$.

I	$V_I^{\alpha\beta}$
1	$i\kappa a^{D-2}\partial_1^\alpha\partial_2^\beta$
2	$-\frac{i}{2}\kappa a^{D-2}\eta^{\alpha\beta}\partial_1\cdot\partial_2$

TABLE II. 4-point vertex operators $U_I^{\alpha\beta\rho\sigma}$ contracted into $\phi_1\phi_2 h_{\alpha\beta}h_{\rho\sigma}$.

I	$U_I^{\alpha\beta\rho\sigma}$
1	$-\frac{i}{8}\kappa^2 a^{D-2}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_1\cdot\partial_2$
2	$-\frac{i}{4}\kappa^2 a^{D-2}\eta^{\alpha\rho}\eta^{\beta\sigma}\partial_1\cdot\partial_2$
3	$\frac{i}{2}\kappa^2 a^{D-2}\eta^{\alpha\beta}\partial_1^\rho\partial_2^\sigma$
4	$-i\kappa^2 a^{D-2}\partial_1^\alpha\eta^{\beta\rho}\partial_2^\sigma$

Three notational conventions will simplify our discussion of propagators. The first is to denote the background geometry with a hat,

$$\begin{aligned} \hat{g}_{\mu\nu} = a^2\eta_{\mu\nu}, \qquad \hat{g}^{\mu\nu} = \frac{1}{a^2}\eta^{\mu\nu}, \\ \sqrt{-\hat{g}} = a^D \quad \text{and} \quad \hat{R} = D(D-1)H^2. \end{aligned} \tag{9}$$

Second, because time and space are treated differently in the gauge we shall employ, it is useful to have expressions for the purely spatial parts of the Lorentz metric and the Kronecker delta,

$$\bar{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} + \delta_\mu^0\delta_\nu^0 \quad \text{and} \quad \bar{\delta}_\nu^\mu \equiv \delta_\nu^\mu - \delta_0^\mu\delta_\nu^0. \tag{10}$$

Finally, the various propagators have simple expressions in terms of $y(x;x')$, a function of the de Sitter invariant length $\ell(x;x')$ from x^μ to x'^μ,

$$\begin{aligned} y(x;x') &= 4\sin^2(\tfrac{1}{2}H\ell(x;x')) \\ &= aa'H^2\{\|\vec{x}-\vec{x}'\|^2 - (|\eta-\eta'| - i\delta)^2\}, \end{aligned} \tag{11}$$

where $a \equiv a(\eta)$ and $a' \equiv a(\eta')$.

The massless minimally coupled scalar propagator obeys

$$\partial_\mu(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu)i\Delta_A(x;x') = i\delta^D(x-x'). \tag{12}$$

It has long been known that there is no de Sitter invariant solution [32]. The de Sitter breaking solution which is relevant for cosmology is the one which preserves homogeneity and isotropy. This is known as the "E(3)" vacuum [33], and the minimal solution takes the form [7,8]

$$\begin{aligned} i\Delta_A(x;x') = A(y) + k\ln(aa') \\ \text{where } k \equiv \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}. \end{aligned} \tag{13}$$

The de Sitter invariant function $A(y)$ is [8]

$$\begin{aligned} A(y) \equiv \frac{H^{D-2}}{(4\pi)^{D/2}}\Bigg\{ & \frac{\Gamma(\frac{D}{2}-1)}{\frac{D}{2}-1}\left(\frac{4}{y}\right)^{(D/2)-1} + \frac{\Gamma(\frac{D}{2}+1)}{\frac{D}{2}-2}\left(\frac{4}{y}\right)^{(D/2)-2} \\ & - \pi\cot\left(\frac{\pi D}{2}\right)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} + \sum_{n=1}^{\infty}\Bigg[\frac{1}{n}\frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})}\left(\frac{y}{4}\right)^n \\ & - \frac{1}{n-\frac{D}{2}+2}\frac{\Gamma(n+\frac{D}{2}+1)}{\Gamma(n+2)}\left(\frac{y}{4}\right)^{n-(D/2)+2}\Bigg]\Bigg\}. \end{aligned} \tag{14}$$

To get the graviton propagator, we add the following gauge fixing term to the invariant Lagrangian [34]:

$$\mathcal{L}_{GF} = -\tfrac{1}{2}a^{D-2}\eta^{\mu\nu}F_\mu F_\nu,$$
$$F_\mu \equiv \eta^{\rho\sigma}(h_{\mu\rho,\sigma} - \tfrac{1}{2}h_{\rho\sigma,\mu} + (D-2)Hah_{\mu\rho}\delta^0_\sigma). \tag{15}$$

We can partially integrate the quadratic part of the gauge fixed Lagrangian to put it in the form $\frac{1}{2}h^{\mu\nu}D^{\rho\sigma}_{\mu\nu}h_{\rho\sigma}$, where the kinetic operator is

$$D_{\mu\nu}{}^{\rho\sigma} \equiv \left\{\frac{1}{2}\bar{\delta}_\mu{}^{(\rho}\bar{\delta}_\nu{}^{\sigma)} - \frac{1}{4}\eta_{\mu\nu}\eta^{\rho\sigma} - \frac{1}{2(D-3)}\delta^0_\mu\delta^0_\nu\delta^\rho_0\delta^\sigma_0\right\}D_A + \delta^0_{(\mu}\bar{\delta}^{(\rho}_{\nu)}\delta^{\sigma)}_0 D_B + \frac{1}{2}\left(\frac{D-2}{D-3}\right)\delta^0_\mu\delta^0_\nu\delta^\rho_0\delta^\sigma_0 D_C, \tag{16}$$

The three scalar differential operators are

$$D_A \equiv \partial_\mu(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu), \tag{17}$$

$$D_B \equiv \partial_\mu(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu) - \frac{1}{D}\left(\frac{D-2}{D-1}\right)\hat{R}\sqrt{-\hat{g}}, \tag{18}$$

$$D_C \equiv \partial_\mu(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu) - \frac{2}{D}\left(\frac{D-3}{D-1}\right)\hat{R}\sqrt{-\hat{g}}. \tag{19}$$

The graviton propagator in this gauge has the form of a sum of constant tensor factors times scalar propagators

$$i[{}_{\mu\nu}\Delta_{\rho\sigma}](x;x') = \sum_{I=A,B,C}[{}_{\mu\nu}T^I_{\rho\sigma}]i\Delta_I(x;x'). \tag{20}$$

We can get the scalar propagators by inverting the scalar kinetic operators,

$$D_I \times i\Delta_I(x;x') = i\delta^D(x-x') \quad \text{for } I = A, B, C. \tag{21}$$

The tensor factors are

$$[{}_{\mu\nu}T^A_{\rho\sigma}] = 2\bar{\eta}_{\mu(\rho}\bar{\eta}_{\sigma)\nu} - \frac{2}{D-3}\bar{\eta}_{\mu\nu}\bar{\eta}_{\rho\sigma}, \tag{22}$$

$$[{}_{\mu\nu}T^B_{\rho\sigma}] = -4\delta^0_{(\mu}\bar{\eta}_{\nu)(\rho}\delta^0_{\sigma)}, \tag{23}$$

$$[{}_{\mu\nu}T^C_{\rho\sigma}] = \frac{2}{(D-2)(D-3)}[(D-3)\delta^0_\mu\delta^0_\nu + \bar{\eta}_{\mu\nu}] \times [(D-3)\delta^0_\rho\delta^0_\sigma + \bar{\eta}_{\rho\sigma}]. \tag{24}$$

With these definitions and Eq. (21) we can see that the graviton propagator satisfies the following equation:

$$D_{\mu\nu}{}^{\rho\sigma} \times i[{}_{\rho\sigma}\Delta^{\alpha\beta}](x;x') = \delta^{(\alpha}_\mu\delta^{\beta)}_\nu i\delta^D(x-x'). \tag{25}$$

The most singular part of the scalar propagator is the propagator for a massless, conformally coupled scalar [35],

$$i\Delta_{cf}(x;x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\Gamma\left(\frac{D}{2}-1\right)\left(\frac{4}{y}\right)^{(D/2)-1}. \tag{26}$$

The A-type propagator obeys the same equation as that of a massless, minimally coupled scalar. The de Sitter invariant B-type and C-type propagators are

$$i\Delta_B(x;x') = i\Delta_{cf}(x;x') - \frac{H^{D-2}}{(4\pi)^{D/2}}\sum_{n=0}^\infty\left\{\frac{\Gamma(n+D-2)}{\Gamma(n+\frac{D}{2})}\left(\frac{y}{4}\right)^n - \frac{\Gamma(n+\frac{D}{2})}{\Gamma(n+2)}\left(\frac{y}{4}\right)^{n-(D/2)+2}\right\}, \tag{27}$$

$$i\Delta_C(x;x') = i\Delta_{cf}(x;x') + \frac{H^{D-2}}{(4\pi)^{D/2}}\sum_{n=0}^\infty\left\{(n+1)\frac{\Gamma(n+D-3)}{\Gamma(n+\frac{D}{2})}\left(\frac{y}{4}\right)^n - \left(n-\frac{D}{2}+3\right)\frac{\Gamma(n+\frac{D}{2}-1)}{\Gamma(n+2)}\left(\frac{y}{4}\right)^{n-(D/2)+2}\right\}. \tag{28}$$

They can also be expressed as hypergeometric functions [36,37],

$$i\Delta_B(x;x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-2)\Gamma(1)}{\Gamma(\frac{D}{2})} \times {}_2F_1\left(D-2, 1; \frac{D}{2}; 1-\frac{y}{4}\right), \tag{29}$$

$$i\Delta_C(x;x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-3)\Gamma(2)}{\Gamma(\frac{D}{2})} \times {}_2F_1\left(D-3, 2; \frac{D}{2}; 1-\frac{y}{4}\right). \tag{30}$$

These propagators might look complicated but they are actually simple to use since the sums vanish in $D=4$, and every term in these sums goes like a positive power of $y(x;x')$. Therefore, only a small number of terms in the sums can contribute when multiplied by a fixed divergence.

III. ONE LOOP SELF-MASS-SQUARED

This is the heart of the paper. We first evaluate the contribution from the 4-point vertices of Table II. Then we compute the vastly more difficult contributions from products of two 3-point vertices from Table I. We do not renormalize at this stage, although we do take $D=4$ in finite terms. Renormalization is postponed until the next section.

A. Contributions from the 4-point vertices

The generic diagram topology is depicted in Fig. 1. The analytic form is

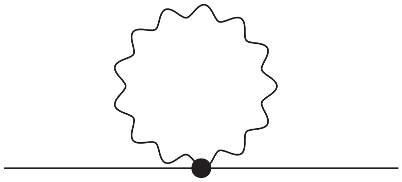

FIG. 1. Contribution from 4-point vertices.

$$-iM^2_{\rm 4pt}(x;x') = \sum_{I=1}^{4} U_I^{\alpha\beta\rho\sigma} i[{}_{\alpha\beta}\Delta_{\rho\sigma}](x;x)\delta^D(x-x'). \tag{31}$$

In reading off the various contributions from Table II one should note that, whereas "∂_2" acts upon x'^μ, the derivative operator "∂_1" must be partially integrated back onto the entire contribution. For example, the contribution from $U_1^{\alpha\beta\rho\sigma}$ is

$$-\frac{i}{8}\kappa^2 a^{D-2}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_1\cdot\partial_2 \times i[{}_{\alpha\beta}\Delta_{\rho\sigma}](x;x) \times \delta^D(x-x') \Rightarrow +\frac{i}{8}\kappa^2\partial^\mu\Big\{a^{D-2} i[{}^{\alpha}_{\alpha}\Delta^{\rho}{}_{\rho}](x;x)\partial'_\mu\delta^D(x-x')\Big\}. \tag{32}$$

Reading off the other terms from Table II gives

$$-iM^2_{\rm 4pt}(x;x') = -\frac{i}{8}\kappa^2\partial^\mu\{a^{D-2}i[{}^{\alpha}_{\alpha}\Delta^{\rho}{}_{\rho}](x;x)\partial_\mu \times\delta^D(x-x')\} + \frac{i}{4}\kappa^2\partial^\mu\{a^{D-2}i[{}^{\alpha\beta}\Delta_{\alpha\beta}] \times(x;x)\partial_\mu\delta^D(x-x')\} + \frac{i}{2}\kappa^2\partial_\rho\{a^{D-2}i[{}^{\alpha}_{\alpha}\Delta^{\rho\sigma}](x;x)\partial_\sigma\delta^D(x-x')\} - i\kappa^2\partial^\alpha\{a^{D-2}i[{}_{\alpha\rho}\Delta^{\rho\sigma}](x;x)\partial_\sigma\delta^D(x-x')\}. \tag{33}$$

It is apparent from expression (33) that we require the coincidence limits of each of the three scalar propagators [24],

$$\lim_{x'\to x} i\Delta_A(x;x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\Big\{-\pi\cot\Big(\frac{\pi}{2}D\Big) + 2\ln(a)\Big\}, \tag{34}$$

$$\lim_{x'\to x} i\Delta_B(x;x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\times -\frac{1}{D-2} \longrightarrow -\frac{H^2}{16\pi^2}, \tag{35}$$

$$\lim_{x'\to x} i\Delta_C(x;x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\times\frac{1}{(D-2)(D-3)} \longrightarrow \frac{H^2}{16\pi^2}. \tag{36}$$

Note that the B-type and C-type propagators are finite for $D=4$. The four contractions of the coincident graviton propagator we require are [24]

$$i[{}^{\alpha}_{\alpha}\Delta^{\rho}_{\rho}](x;x) \longrightarrow -4\Big(\frac{D-1}{D-3}\Big)i\Delta_A(x;x) + 4\frac{H^2}{16\pi^2}, \tag{37}$$

$$i[{}^{\alpha\beta}\Delta_{\alpha\beta}](x;x) \longrightarrow \frac{(D-1)(D^2-3D-2)}{D-3}i\Delta_A(x;x) - 2\frac{H^2}{16\pi^2}, \tag{38}$$

$$i[{}^{\alpha}_{\alpha}\Delta^{\rho\sigma}](x;x) \longrightarrow -\frac{4}{D-3}\bar{\eta}^{\rho\sigma}i\Delta_A(x;x) + [2\delta^\rho_0\delta^\sigma_0 + 2\bar{\eta}_{\rho\sigma}]\frac{H^2}{16\pi^2}, \tag{39}$$

$$i[{}_{\alpha\rho}\Delta^{\rho\sigma}](x;x) \longrightarrow \Big(\frac{D^2-3D-2}{D-3}\Big)\bar{\delta}^\sigma_\alpha i\Delta_A(x;x) + 2\delta^0_\alpha\delta^\sigma_0\frac{H^2}{16\pi^2}. \tag{40}$$

To save space we have taken $D=4$ in the finite contributions from the B-type and C-type propagators.

Substituting these relations into expression (33) and performing some trivial algebra gives the final result,

$$-iM^2_{\rm 4pt}(x;x') = \frac{i\kappa^2 H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} - \pi\cot\Big(\frac{\pi}{2}D\Big) \times\Big\{\frac{1}{4}D(D-1)\partial^\mu(a^{D-2}\partial_\mu\delta^D(x-x')) - Da^{D-2}\nabla^2\delta^D(x-x')\Big\} + \frac{i\kappa^2H^2}{4\pi^2}\{3\partial^\mu(a^2\ln(a)\partial_\mu\delta^4(x-x')) - 4\ln(a)a^2\nabla^2\delta^4(x-x') - \partial^\mu(a^2\partial_\mu\delta^4(x-x')) + a^2\nabla^2\delta^4(x-x')\} + O(D-4). \tag{41}$$

Note that each of these terms vanishes in the flat space limit of $H\to 0$ with the comoving time $t\equiv\ln(a)/H$ held fixed. The reason for this is that the coincidence limit of the flat space graviton propagator vanishes in dimensional regularization.

In order to combine $-iM^2_{\rm 4pt}$ with the 3-point contributions it is useful to introduce notation for the scalar d'Alembertian in de Sitter background,

$$\square \equiv \frac{1}{\sqrt{-\hat{g}}}\partial_\mu(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu) = \frac{1}{a^D}\partial^\mu(a^{D-2}\partial_\mu). \tag{42}$$

We also extract the logarithm from inside the d'Alembertian,

$$\partial^\mu(a^2\ln(a)\partial_\mu\delta^4(x-x')) = \tfrac{1}{2}\ln(aa')a^4\square\delta^4(x-x') + \tfrac{3}{2}H^2a^4\delta^4(x-x'). \tag{43}$$

With these conventions the final result takes the form,

$$-iM^2_{4\text{pt}}(x;x') = \frac{i\kappa^2H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \times\Big\{\Big[-\frac{1}{4}D(D-1)\pi\cot\Big(\frac{\pi}{2}D\Big) - 2 + 3\ln(aa')\Big]a^D\square + \Big[D\pi\cot\Big(\frac{\pi}{2}D\Big) + 2 - 4\ln(aa')\Big]a^{D-2}\nabla^2 + 9H^2a^D + O(D-4)\Big\}\delta^D(x-x'). \tag{44}$$

B. Contributions from the 3-point vertices

In this section we calculate the contributions from two 3-point vertex operators. It is diagrammatically represented in Fig. 2. Consulting Table I and remembering to partially integrate any derivative that acts upon an outer leg gives

$$-iM^2_{3\text{pt}}(x;x') = \sum_{I=1}^{2}V_I^{\alpha\beta}(x)\sum_{J=1}^{2}V_J^{\rho\sigma}(x')\times i[{}_{\alpha\beta}\Delta_{\rho\sigma}] \times(x;x')i\Delta_A(x;x') \tag{45}$$

$$= -\kappa^2\partial_\alpha\partial'_\rho\{(aa')^{D-2}i[^{\alpha\beta}\Delta^{\rho\sigma}]\partial_\beta\partial'_\sigma i\Delta_A\} + \frac{\kappa^2}{2}\partial^\mu\partial'_\rho\{(aa')^{D-2}i[^\alpha_\alpha\Delta^{\rho\sigma}]\partial_\mu\partial'_\sigma i\Delta_A\} + \frac{\kappa^2}{2}\partial_\alpha\partial'^\nu\{(aa')^{D-2}i[^{\alpha\beta}\Delta^\rho{}_\rho]\partial_\beta\partial'_\nu i\Delta_A\} - \frac{\kappa^2}{4}\partial^\mu\partial'^\nu\{(aa')^{D-2}i[^\alpha_\alpha\Delta^\rho{}_\rho]\partial_\mu\partial'_\nu i\Delta_A\}. \tag{46}$$

Upon substituting the graviton propagator, performing the contractions and segregating terms with the same scalar propagators, one finds three generic sorts of terms. The first are those which involve two A-type propagators,

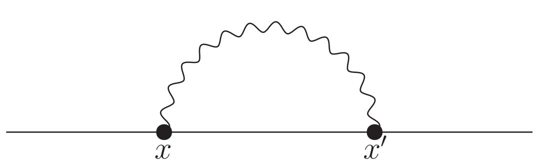

FIG. 2. Contribution from two 3-point vertices.

$$\kappa^2\nabla\cdot\nabla'[(aa')^{D-2}i\Delta_A\nabla\cdot\nabla' i\Delta_A] - \kappa^2\Big(\frac{D-1}{D-3}\Big)\partial_0\partial'_0[(aa')^{D-2}i\Delta_A\partial_0\partial'_0 i\Delta_A] + \kappa^2\partial_i\partial'_0[(aa')^{D-2}i\Delta_A\partial_i\partial'_0 i\Delta_A] + \kappa^2\partial_0\partial'_i[(aa')^{D-2}i\Delta_A\partial_0\partial'_i i\Delta_A]. \tag{47}$$

The second kind of term involves one A-type and one B-type propagator,

$$-\kappa^2\partial_0\partial'_0[(aa')^{D-2}i\Delta_B\nabla\cdot\nabla' i\Delta_A] - \kappa^2\partial_i\partial'_0[(aa')^{D-2}i\Delta_B\partial_0\partial'_i i\Delta_A] - \kappa^2\partial_0\partial'_i[(aa')^{D-2}i\Delta_B\partial_i\partial'_0 i\Delta_A] - \kappa^2\nabla \cdot\nabla'[(aa')^{D-2}i\Delta_B\partial_0\partial'_0 i\Delta_A]. \tag{48}$$

Finally, there is the case of one propagator of A-type and the other of C-type,

$$2\kappa^2\Big(\frac{D-2}{D-3}\Big)\partial_0\partial'_0[(aa')^{D-2}i\Delta_C\partial_0\partial'_0 i\Delta_A]. \tag{49}$$

Each of the nine terms in expressions (47)–(49) has the form

$$\kappa^2\partial_\mu\partial'_\nu[(aa')^{D-2}i\Delta_I(x;x')\partial_\rho\partial'_\sigma i\Delta_A(x;x')], \tag{50}$$

where "I" might be A, B, or C. Note that the three propagators can be written almost entirely as functions of $y(x;x')$ defined in (11),

$$i\Delta_A(x;x') = A(y) + k\ln(aa'), \quad i\Delta_B(x;x') = B(y), \quad \text{and} \quad i\Delta_C(x;x') = C(y). \tag{51}$$

The functions $A(y)$, $B(y)$, and $C(y)$ can be read off from expressions (14), (27), and (28), respectively. Note also that the inner derivatives eliminate the de Sitter breaking term of $i\Delta_A$,

$$\partial_\rho\partial'_\sigma i\Delta_A(x;x') = \delta^0_\rho\delta^0_\sigma\frac{i}{a^{D-2}}\delta^D(x-x') + A''(y)\frac{\partial y}{\partial x^\rho} \times\frac{\partial y}{\partial x'^\sigma} + A'(y)\frac{\partial^2 y}{\partial x^\rho\partial x'^\sigma}. \tag{52}$$

It follows that the analysis breaks up into three parts:
(i) *local contributions* from the delta function in (52);
(ii) *logarithm contributions* from the factor of $k\ln(aa')$ in the A-type propagator when $I = A$ in expression (50); and
(iii) *normal contributions* to expression (50) of the form,

$$\kappa^2\partial_\mu\partial'_\nu\Big\{(aa')^{D-2}I(y)\Big[A''\frac{\partial y}{\partial x^\rho}\frac{\partial y}{\partial x'^\sigma} + A'\frac{\partial^2 y}{\partial x^\rho\partial x'^\sigma}\Big]\Big\}. \tag{53}$$

We shall devote a separate part of this subsection to each.

1. Local contributions

These are the simplest contributions. They only come from the 2nd term of (47), the 4th term of (48) and from (49). To avoid overlap with the logarithm contributions of the next part we define the local contribution from the 4th term of (47) without the logarithm,

$$-\kappa^2\Big(\frac{D-1}{D-3}\Big)\partial_0\partial_0'\Big[(aa')^{D-2}A(y)\frac{i}{a^{D-2}}\delta^D(x-x')\Big] = \frac{i\kappa^2 H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D)}{(D-3)\Gamma(\frac{D}{2})} - \pi\cot\Big(\frac{D}{2}\pi\Big) \times\{-a^D\Box\delta^D(x-x') + a^{D-2}\nabla^2\delta^D(x-x')\}. \quad (54)$$

Note that we have chosen to convert primed derivatives into unprimed, and to absorb the temporal derivatives into a covariant d'Alembertian $\Box$,

$$-\partial_0(a^{D-2}\partial_0'\delta^D(x-x')) = -\partial^\mu(a^{D-2}\partial_\mu\delta^D(x-x')) + a^{D-2}\nabla^2\delta^D(x-x') \quad (55)$$

$$\equiv -a^D\Box\delta^D(x-x') + a^{D-2}\nabla^2\delta^D(x-x'). \quad (56)$$

This will facilitate renormalization.

The other two local contributions are finite. The 4th term of (48) gives

$$-\kappa^2\nabla\cdot\nabla'\Big[(aa')^{D-2}B(y)\times\frac{i}{a^{D-2}}\delta^D(x-x')\Big] = -\frac{i\kappa^2H^2}{16\pi^2}\times a^2\nabla^2\delta^4(x-x') + O(D-4). \quad (57)$$

And (49) gives

$$2\kappa^2\Big(\frac{D-2}{D-3}\Big)\partial_0\partial_0'\Big[(aa')^{D-2}C(y)\times\frac{i}{a^{D-2}}\delta^D(x-x')\Big] = \frac{i\kappa^2H^2}{4\pi^2}\{a^4\Box\delta^4(x-x') - a^2\nabla^2\delta^4(x-x')\} + O(D-4). \quad (58)$$

Summing the three local contributions gives

$$-iM^2_{\substack{\rm 3pt \\ \rm loc}}(x;x') = \frac{i\kappa^2H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \times\Big\{\Big[\Big(\frac{D-1}{D-3}\Big)\pi\cot\Big(\frac{\pi}{2}D\Big)+2\Big]a^D\Box + \Big[-\Big(\frac{D-1}{D-3}\Big)\pi\cot\Big(\frac{\pi}{2}D\Big)-\frac{7}{2}\Big]a^{D-2}\nabla^2 + O(D-4)\Big\}\delta^D(x-x'). \quad (59)$$

2. Logarithm contributions

These all come from expression (47). They can be simplified by using the propagator Eq. (12),

$$\partial_0(a^{D-2}\partial_0 i\Delta_A(x;x')) = -i\delta^D(x-x') + a^{D-2}\nabla^2A(y), \quad (60)$$

$$\partial_0'(a'^{D-2}\partial_0' i\Delta_A(x;x')) = -i\delta^D(x-x') + a'^{D-2}\nabla^2A(y). \quad (61)$$

One can also take the limit $D = 4$ because all the logarithm contributions are finite. For example, the function $A(y)$ is

$$A(y) = \frac{H^2}{16\pi^2}\Big\{\frac{4}{y} - 2\ln\Big(\frac{y}{4}\Big) - 1 + O(D-4)\Big\}. \quad (62)$$

The first term of (47) gives

$$\kappa^2\nabla\cdot\nabla'[(aa')^{D-2}k\ln(aa')\nabla\cdot\nabla' i\Delta_A(x;x')] = \frac{\kappa^2H^2}{8\pi^2}\ln(aa')(aa')^2\nabla^4A(y) + O(D-4). \quad (63)$$

The second term of (47) has the most complicated reduction,

$$-\kappa^2\Big(\frac{D-1}{D-3}\Big)\partial_0\partial_0'[(aa')^{D-2}k\ln(aa')\partial_0\partial_0' i\Delta_A(x;x')] = \frac{i3\kappa^2H^2}{8\pi^2}\ln(aa')\{-a^4\Box + 2a^2\nabla^2\}\delta^4(x-x') - \frac{3\kappa^2H^2}{8\pi^2}\ln(aa')(aa')^2\nabla^4A(y) - \frac{i9\kappa^2H^4}{8\pi^2}a^4\delta^4(x-x') - \frac{3\kappa^2H^3}{8\pi^2}(aa')^2(a\partial_0 + a'\partial_0')\nabla^2A(y) + O(D-4). \quad (64)$$

The third term of (47) gives

$$\kappa^2\partial_i\partial_0'[(aa')^{D-2}k\ln(aa')\partial_i\partial_0' i\Delta_A(x;x')] = -\frac{i\kappa^2H^2}{8\pi^2}\ln(aa')a^2\nabla^2\delta^4(x-x') + \frac{\kappa^2H^2}{8\pi^2}\ln(aa')(aa')^2\nabla^4A(y) + \frac{\kappa^2H^3}{8\pi^2}(aa')^2a'\partial_0'\nabla^2A(y) + O(D-4). \quad (65)$$

A very similar contribution derives from the final term of (47),

$$\kappa^2\partial_0\partial_i'[(aa')^{D-2}k\ln(aa')\partial_0\partial_i' i\Delta_A(x;x')] = -\frac{i\kappa^2H^2}{8\pi^2}\ln(aa')a^2\nabla^2\delta^4(x-x') + \frac{\kappa^2H^2}{8\pi^2}\ln(aa')(aa')^2\nabla^4A(y) + \frac{\kappa^2H^3}{8\pi^2}(aa')^2a\partial_0\nabla^2A(y) + O(D-4). \quad (66)$$

Combining all four terms results in some significant cancellations,

$$-iM^2_{\substack{3\text{pt}\\ \log}}(x;x') = \frac{\kappa^2 H^2}{8\pi^2}\{\ln(aa')[-3a^4\Box + 4a^2\nabla^2] \times i\delta^4(x-x') - 9H^2a^4 i\delta^4(x-x') - 2H(aa')^2(a\partial_0 + a'\partial_0')\nabla^2 A(y) + O(D-4)\}. \tag{67}$$

Each of the local terms in (67) cancels a similar finite, local 4-point contribution in (44), leaving only the nonlocal contribution involving derivatives of $A(y)$. It is possible to eliminate the temporal derivatives in this expression. However, the procedure is best explained in the final part of this subsection.

3. Normal contributions

These contributions are the most challenging. Our strategy for reducing them is to first extract the ∂_ρ and ∂'_σ derivatives from (53) *generically*, without exploiting the functional forms of $A(y)$, $B(y)$, and $C(y)$. We also convert all primed derivatives into unprimed ones and express the final result in terms of ten "external operators." This not only makes it possible to perceive general relations, it also reduces the superficial degree of divergence of the terms we must eventually expand. And it leaves functions of the de Sitter invariant variable $y(x;x')$ for which an improved expansion procedure is possible [38].

This step of extracting derivatives is still quite involved so we shall describe only the essentials in the body of the paper and consign the details to an appendix. The appendix also gives tabulated results for each of the ten external operators. The final reduction of these generic tabulated results is straightforward. This subsection closes with a description of the technique and with Tables IV and V giving the final potentially divergent and manifestly finite contributions, respectively. In the next section these results are processed through Tables VI, VII, VIII, and IX to give the final, unregulated result of Table X.

Our generic method for extracting derivatives requires one to carry out many indefinite integrations of functions of y. We define this operation by the symbol $I[f](y)$,

$$I[f](y) \equiv \int^y dy' f(y'). \tag{68}$$

If the function $F(y)$ is the product of two propagator functions, then acting two derivatives on it can never produce a delta function,

$$\partial_\rho\partial'_\sigma F(y) = F''(y)\frac{\partial y}{\partial x^\rho}\frac{\partial y}{\partial x'^\sigma} + F'(y)\frac{\partial^2 y}{\partial x^\rho \partial x'^\sigma}. \tag{69}$$

It follows that we can express the inner part of the basic normal contribution (53) in terms of integrals of such products,

$$f(y)\left\{A''(y)\frac{\partial y}{\partial x^\rho}\frac{\partial y}{\partial x'^\sigma} + A'(y)\frac{\partial^2 y}{\partial x^\rho\partial x'^\sigma}\right\} = \partial_\rho\partial'_\sigma I^2[fA''](y) + \frac{\partial^2 y}{\partial x^\rho \partial x'^\sigma} I[f'A'](y). \tag{70}$$

We must still deal with the final term of (70). In conformal coordinates the mixed second derivative of $y(x;x')$ is [39]

$$\frac{\partial^2 y}{\partial x^\rho \partial x'^\sigma} = H^2 aa'\{y\delta^0_\rho\delta^0_\sigma - 2a\delta^0_\rho H\Delta x_\sigma + 2a'H\Delta x_\rho \delta^0_\sigma - 2\eta_{\rho\sigma}\}. \tag{71}$$

Breaking this up into spatial and temporal components gives

$$\frac{\partial^2 y}{\partial x^0 \partial x'^0} = H^2 aa'[2 - y + 2aa'H^2\|\Delta\vec{x}\|^2], \qquad \frac{\partial^2 y}{\partial x^0 \partial x'^j} = H^2 aa' \times -2aH\Delta x_j, \tag{72}$$

$$\frac{\partial^2 y}{\partial x^i \partial x'^0} = H^2 aa' \times 2a'H\Delta x_i, \qquad \frac{\partial^2 y}{\partial x^i \partial x'^j} = H^2 aa' \times -2\eta_{ij}. \tag{73}$$

One consequence is

$$aa'H^2\|\Delta\vec{x}\|^2 f(y) = -\frac{1}{2}(D-1)I[f](y) - \frac{\nabla\cdot\nabla'}{4aa'H^2}I^2[f](y). \tag{74}$$

Another consequence is the relations[1]

$$f(y)\partial_0\partial'_0 A(y) = \partial_0\partial'_0 I^2[fA''](y) - \tfrac{1}{2}\nabla\cdot\nabla' I^3[f'A'](y) + H^2aa'\{(2-y)I[f'A'](y) - (D-1)I^2[f'A'](y)\}, \tag{75}$$

$$f(y)\partial_0\partial'_j A(y) = \partial_0\partial'_j I^2[fA''](y) + Ha\partial'_j I^2[f'A'](y), \tag{76}$$

$$f(y)\partial_i\partial'_0 A(y) = \partial_i\partial'_0 I^2[fA''](y) + Ha'\partial_i I^2[f'A'](y), \tag{77}$$

$$f(y)\partial_i\partial'_j A(y) = \partial_i\partial'_j I^2[fA''](y) - 2H^2aa'\eta_{ij}I[f'A'](y). \tag{78}$$

Using these identities it is possible to extract the derivatives from the first of the A terms,

[1] On the left-hand side of relation (75) we mean the naive second derivative, *without* the delta function.

$$\nabla\cdot\nabla'[(aa')^{D-2}A(y)\nabla\cdot\nabla' A(y)] = (aa')^{D-2}(\nabla\cdot\nabla')^2 I^2[AA''](y) - 2(D-1)H^2(aa')^{D-1}\nabla\cdot\nabla' I[A'^2](y) \quad (79)$$

$$= (aa')^{D-2}\nabla^4 I^2[AA''](y) + 2(D-1)H^2(aa')^{D-1}\nabla^2 I[A'^2](y). \quad (80)$$

Only the first term in the expansion of $I^2[AA''](y)$ contributes a divergence; we can set $D=4$ in the higher terms. Similarly, only the first two terms in the expansion of $I[A'^2](y)$ can diverge.

It is very simple to convert the primed spatial derivatives to unprimed ones,

$$\partial_i' f(y) = -\partial_i f(y). \quad (81)$$

We already used this relation in reducing the first of the A terms. For time derivatives it is useful to note

$$\frac{\partial y}{\partial x^0} = Ha(y - 2a'H\Delta\eta) = Ha\left(y - 2 + 2\frac{a'}{a}\right), \quad (82)$$

$$\frac{\partial y}{\partial x'^0} = Ha'(y + 2aH\Delta\eta) = Ha'\left(y - 2 + 2\frac{a}{a'}\right). \quad (83)$$

From this follow three important identities. The simple one is

$$(\partial_0 + \partial_0')f(y) = H(a+a')yf'(y). \quad (84)$$

Another result is

$$(a'\partial_0 + a\partial_0')f(y) = 2Haa' \times aa'H^2\|\Delta\vec{x}\|^2 f'(y) \quad (85)$$

$$= -(D-1)Haa'f(y) + \frac{\nabla^2}{2H}I[f](y). \quad (86)$$

The final identity results from combining (84) and (86),

$$(a\partial_0 + a'\partial_0')f(y) = (a+a')(\partial_0+\partial_0')f(y) - (a'\partial_0 + a\partial_0')f(y) \quad (87)$$

$$= H(a+a')^2 yf'(y) + (D-1)Haa'f(y) - \frac{\nabla^2}{2H}I[f](y). \quad (88)$$

We can now reduce the nonlocal logarithm contribution from Eq. (67). Applying (88) gives

$$\frac{\kappa^2H^2}{8\pi^2} - 2H(aa')^2(a\partial_0 + a'\partial_0')\nabla^2 A(y) = \frac{\kappa^2H^2}{16\pi^2}\{-12(aa')^3H^2\nabla^2 A - 4(aa')^2(a+a')^2\nabla^2(yA') + 2(aa')^2\nabla^4 I[A]\}. \quad (89)$$

The derivative and the integral are straightforward using the $D=4$ expansion for $A(y)$ given in (62). The final result is reported in Table III. Of course we have neglected terms which eventually vanish such as $\nabla^4 y$.

We eventually want to absorb all double time derivatives into covariant d'Alembertian's,

$$\square = -\frac{1}{a^2}\partial_0^2 - \frac{(D-2)H}{a}\partial_0 + \frac{1}{a^2}\nabla^2. \quad (90)$$

This is most effectively done with the internal factors of $(aa')^{D-2}$. For example, consider reducing one of the mixed A terms,

$$\partial_0\partial_i'[(aa')^{D-2}A(y)\partial_0\partial_i' A(y)] = \nabla'^2\partial_0[(aa')^{D-2}\partial_0 I^2[AA''](y)] + H\nabla'^2\partial_0[a^{D-1}a'^{D-2}I^2[A'^2](y)] \quad (91)$$

$$= -a^D a'^{D-2}\nabla^2\square I^2[AA''](y) + (aa')^{D-2}\nabla^4 I^2[AA''](y) + Ha^{D-1}a'^{D-2}\nabla^2\partial_0 I^2[A'^2](y) + (D-1)H^2a^Da'^{D-2}\nabla^2 I^2[A'^2](y). \quad (92)$$

Note also that we can convert a primed covariant d'Alembertian to an unprimed one if it acts on a function of just $y(x;x')$,

$$\square f(y) = H^2[(4y-y^2)f''(y) + D(2-y)f'(y)] = \square' f(y). \quad (93)$$

This is used in reducing the other mixed A term,

$$\partial_i\partial_0'[(aa')^{D-2}A(y)\partial_i\partial_0' A(y)] = \nabla^2\partial_0'[(aa')^{D-2}\partial_0' I^2[AA''](y)] + H\nabla^2\partial_0'[a^{D-2}a'^{D-1}I^2[A'^2](y)] \quad (94)$$

$$= -a^{D-2}a'^D\nabla^2\square' I^2[AA''](y) + (aa')^{D-2}\nabla^4 I^2[AA''](y) + Ha^{D-2}a'^{D-1}\nabla^2\partial_0' I^2[A'^2](y) + (D-1)H^2a^{D-2}a'^D\nabla^2 I^2[A'^2](y) \quad (95)$$

$$= -a^{D-2}a'^D\nabla^2\square I^2[AA''](y) + (aa')^{D-2}\nabla^4 I^2[AA''](y) - Ha^{D-3}a'^D\nabla^2\partial_0 I^2[A'^2](y) + \tfrac{1}{2}a^{D-3}a'^{D-1}\nabla^4 I^3[A'^2](y). \quad (96)$$

TABLE III. Nonlocal logarithm contributions from relation (67) with $x \equiv \frac{y}{4}$.

External operator	Coefficient of $\frac{\kappa^2H^4}{(4\pi)^4}$
$(aa')^3H^2\nabla^2$	$-\frac{12}{x} + 24\ln x$
$(aa')^2(a+a')^2H^2\nabla^2$	$\frac{4}{x}$
$(aa')^2\nabla^4$	$8\ln x - 16x\ln x$

The previous point can be summarized by the relations

$$\partial_0[(aa')^{D-2}\partial_0 f(y)] = -a^D a'^{D-2}\Box f(y) + (aa')^{D-2}\nabla^2 f(y), \quad (97)$$

$$\partial_0'[(aa')^{D-2}\partial_0' f(y)] = -a^{D-2}a'^{D}\Box f(y) + (aa')^{D-2}\nabla^2 f(y). \quad (98)$$

Another important point is that it is almost always best to write any single factor of the mixed product $\partial_0\partial_0'$ as follows:

$$\partial_0\partial_0' = \tfrac{1}{2}(\partial_0 + \partial_0')^2 - \tfrac{1}{2}\partial_0^2 - \tfrac{1}{2}\partial_0'^2. \quad (99)$$

So we find the ubiquitous reduction,

$$\begin{aligned}\partial_0\partial_0'[(aa')^{D-2}f(y)] &= (aa')^{D-2}\partial_0\partial_0' f(y) + (D-2)H(aa')^{D-2}(a'\partial_0 + a\partial_0')f(y) + (D-2)^2H^2(aa')^{D-1}f(y) \quad (100)\\ &= \tfrac{1}{2}(aa')^{D-2}(a^2 + a'^2)[\Box f(y) + H^2 y f'(y)] - (aa')^{D-2}\nabla^2 f(y) + \tfrac{1}{4}(D-2)(aa')^{D-2}\nabla^2 I[f](y) \\ &\quad + \tfrac{1}{2}(D-2)(D-3)H^2(aa')^{D-1}f(y) + \tfrac{1}{2}H^2(a+a')^2(aa')^{D-2}[(D-1)yf'(y) + y^2 f''(y)]. \end{aligned} \quad (101)$$

Another example is the two B-terms,

$$\begin{aligned}&\partial_i\partial_0'[(aa')^{D-2}B(y)\partial_0\partial_i'A(y)] + \partial_0\partial_i'[(aa')^{D-2}B(y)\partial_i\partial_0'A(y)] \\ &= \partial_i(\partial_0 + \partial_0')[(aa')^{D-2}B(y)(\partial_0 + \partial_0')\partial_i'A(y)] - \partial_i\partial_0[(aa')^{D-2}B(y)\partial_0\partial_i'A(y)] - \partial_i\partial_0'[(aa')^{D-2}B(y)\partial_0'\partial_i'A(y)] \quad (102)\\ &= -(a^2 + a'^2)(aa')^{D-2}\nabla^2\{\Box I^2[A''B](y) + H^2 I[A'B + yA''B](y)\} + (aa')^{D-2}\nabla^4\{2I^2[A''B](y) - \tfrac{1}{2}I^3[A'B'](y)\} \\ &\quad - H^2(a+a')^2(aa')^{D-2}\nabla^2 \times\{(D-2)I[A'B + yA''B](y) + yA'(y)B(y) + y^2A''(y)B(y) - yI[A'B'](y)\} \\ &\quad + (D-1)H^2(a^2 + aa' + a'^2)(aa')^{D-2}\nabla^2 I^2[A'B'](y). \end{aligned} \quad (103)$$

Extracting derivatives in this way from the various normal contributions results in functions of y which are acted upon by ten external operators,

$$\alpha \equiv (aa')^D\Box^2, \quad (104)$$

$$\beta \equiv (aa')^{D-1}(a^2 + a'^2)H^2\Box, \quad (105)$$

$$\gamma_1 \equiv (aa')^D H^4, \quad (106)$$

$$\gamma_2 \equiv (aa')^{D-1}(a^2 + a'^2)H^4, \quad (107)$$

$$\gamma_3 \equiv (aa')^{D-1}(a + a')^2H^4 = 2\gamma_1 + \gamma_2, \quad (108)$$

$$\delta \equiv (aa')^{D-2}(a^2 + a'^2)\nabla^2\Box, \quad (109)$$

$$\epsilon_1 \equiv (aa')^{D-1}H^2\nabla^2, \quad (110)$$

$$\epsilon_2 \equiv (aa')^{D-2}(a^2 + a'^2)H^2\nabla^2, \quad (111)$$

$$\epsilon_3 \equiv (aa')^{D-2}(a + a')^2H^2\nabla^2 = 2\epsilon_1 + \epsilon_2, \quad (112)$$

$$\zeta \equiv (aa')^{D-2}\nabla^4. \quad (113)$$

Tables XI, XII, XIII, XIV, XV, XVI, XVII, XVIII, XIX, and XX of the appendix give explicit results for each of these ten operators. Note that in addition to the three propagator functions $A(y)$, $B(y)$, and $C(y)$, we also employ the following less singular differences:

$$\Delta B \equiv B - A \quad \text{and} \quad \Delta C \equiv 2\Big(\frac{D-2}{D-3}\Big)(C - A). \quad (114)$$

The next step is substituting the explicit forms (14), (27), and (28) for the propagator functions into the results of Tables XI, XII, XIII, XIV, XV, XVI, XVII, XVIII, XIX, and XX and expanding to the required order. To understand what this is, note that we will be integrating the result with respect to x'^μ against a smooth function (the zeroth order mode solution) with the derivatives of the external operators acted *outside* the integrals. Because $y(x;x')$ vanishes like $(x - x')^2$ at coincidence, it is only necessary to retain the dimensional regularization for terms which would go like $1/y^2$ and higher for $D = 4$.

Although these tables involve a bewildering variety of different integrals and derivatives, careful examination of the results shows that they derive from just eight products of the propagator functions,

$$A'^2, \quad AA'', \quad A'B', \quad A''B, \quad A'\Delta B', \quad A''\Delta B, \quad A'\Delta C', \quad \text{and} \quad A''\Delta C. \quad (115)$$

The most singular products of A'^2 and AA'' always appear either doubly integrated—e.g., $I^2[AA'']$ in Table XI—or else integrated once and then multiplied by y—e.g., $-\frac{1}{2}yI[A'^2]$ in Table XII. Hence we need only retain the dimensional regularization for the $1/y^D$ terms of these expansions,

$$A'^2 = \frac{\Gamma^2(\frac{D}{2})}{16}\frac{H^{2D-4}}{(4\pi)^D}\Big\{\Big(\frac{4}{y}\Big)^D + 4\Big(\frac{4}{y}\Big)^3 + 4\Big(\frac{4}{y}\Big)^2 + O\Big(\frac{D-4}{y^3}\Big)\Big\}, \quad (116)$$

$$AA'' = \frac{\Gamma^2(\frac{D}{2})}{16} \frac{H^{2D-4}}{(4\pi)^D} \Big\{ \frac{D}{D-2} \Big(\frac{4}{y}\Big)^D - 4\Big(\frac{4}{y}\Big)^3 \ln\Big(\frac{y}{4}\Big) - 4\Big(\frac{4}{y}\Big)^2 \ln\Big(\frac{y}{4}\Big) - 2\Big(\frac{4}{y}\Big)^2 + O\Big(\frac{D-4}{y^3}\Big)\Big\}. \quad (117)$$

The product $A'B'$ can appear with only a single integration—e.g., $DI[A'B']$ in Table XII—or multiplied by a single factor of y—e.g., $DyA'B'$ in Table XIV. We must therefore retain the dimensional regularization for the $1/y^{D-1}$ term,

$$A'B' = \frac{\Gamma^2(\frac{D}{2})}{16} \frac{H^{2D-4}}{(4\pi)^D} \Big\{ \Big(\frac{4}{y}\Big)^D + (D-2)\Big(\frac{4}{y}\Big)^{D-1} + O\Big(\frac{D-4}{y^2}\Big)\Big\}. \quad (118)$$

However, the product $A''B$ is always shielded by two or more powers of y, so the expansion we require for it is

$$A''B = \frac{\Gamma^2(\frac{D}{2})}{16} \frac{H^{2D-4}}{(4\pi)^D} \Big\{ \frac{D}{D-2}\Big(\frac{4}{y}\Big)^D + 2\Big(\frac{4}{y}\Big)^3 + O\Big(\frac{D-4}{y^3}\Big)\Big\}. \quad (119)$$

The products involving ΔB and ΔC are less singular,

$$A'\Delta B' = \frac{\Gamma^2(\frac{D}{2})}{16} \frac{H^{2D-4}}{(4\pi)^D} \Big\{ -2\Big(\frac{4}{y}\Big)^{D-1} - 4\Big(\frac{4}{y}\Big)^2 + O\Big(\frac{D-4}{y^2}\Big)\Big\}, \quad (120)$$

$$A''\Delta B = \frac{\Gamma^2(\frac{D}{2})}{16} \frac{H^{2D-4}}{(4\pi)^D} \Big\{ 4\Big(\frac{4}{y}\Big)^3 \ln\Big(\frac{y}{4}\Big) + 2\Big(\frac{4}{y}\Big)^3 + 4\Big(\frac{4}{y}\Big)^2 \ln\Big(\frac{y}{4}\Big) + 2\Big(\frac{4}{y}\Big)^2 + O\Big(\frac{D-4}{y^3}\Big)\Big\}, \quad (121)$$

$$A'\Delta C' = \frac{\Gamma^2(\frac{D}{2})}{16} \frac{H^{2D-4}}{(4\pi)^D} \Big\{ -8\Big(\frac{4}{y}\Big)^{D-1} - 16\Big(\frac{4}{y}\Big)^2 + O\Big(\frac{D-4}{y^2}\Big)\Big\}, \quad (122)$$

$$A''\Delta C = \frac{\Gamma^2(\frac{D}{2})}{16} \frac{H^{2D-4}}{(4\pi)^D} \Big\{ 16\Big(\frac{4}{y}\Big)^3 \ln\Big(\frac{y}{4}\Big) + 8\Big(\frac{4}{y}\Big)^3 + 16\Big(\frac{4}{y}\Big)^2 \ln\Big(\frac{y}{4}\Big) + 8\Big(\frac{4}{y}\Big)^2 + O\Big(\frac{D-4}{y^3}\Big)\Big\}. \quad (123)$$

One next substitutes these expansions into the totals of Tables XI, XII, XIII, XIV, XV, XVI, XVII, XVIII, XIX, and XX and performs the necessary integrations, differentiations, multiplications, and summations. We must also multiply by the overall factor of κ^2. For example, the result for external operator α is

TABLE IV. Divergent normal contributions.

Ext. op.	Coef. of $\frac{\kappa^2 H^{2D-4}}{(4\pi)^D} \times \Gamma^2(\frac{D}{2})(\frac{4}{y})^{D-1}$	Coef. of $\frac{\kappa^2 H^{2D-4}}{(4\pi)^D} \times \Gamma^2(\frac{D}{2})(\frac{4}{y})^{D-2}$
α	0	$\frac{D}{(D-1)(D-2)^2}$
β	$-\frac{D}{4(D-1)}$	$-\frac{D^3-3D^2-4D+8}{4(D-1)(D-2)}$
γ_1	$-\frac{D(D-2)}{4}$	$-\frac{D^3-3D^2-4D+8}{4}$
γ_2	$\frac{D}{4}$	$-\frac{D^3-3D^2-4D+8}{4(D-1)}$
γ_3	$-\frac{D}{4}$	0
δ	0	0
ϵ_1	0	$\frac{(D^2-6D+4)}{2(D-1)(D-2)}$
ϵ_2	0	$\frac{(1-2D)}{(D-1)(D-2)}$
ϵ_3	0	0
ς	0	0

$$\kappa^2\{I^2[AA''] + I^2[A''\Delta C]\} = \frac{\Gamma^2(\frac{D}{2})}{16} \frac{\kappa^2 H^{2D-4}}{(4\pi)^D} I^2\Big[\frac{D}{D-2}\Big(\frac{4}{y}\Big)^D + 12\Big(\frac{4}{y}\Big)^3 \ln\Big(\frac{y}{4}\Big) + 8\Big(\frac{4}{y}\Big)^3 + 12\Big(\frac{4}{y}\Big)^2 \ln\Big(\frac{y}{4}\Big) + 6\Big(\frac{4}{y}\Big)^2 + O\Big(\frac{D-4}{y^3}\Big)\Big], \quad (124)$$

$$= \frac{\kappa^2 H^{2D-4}}{(4\pi)^D} \Gamma^2\Big(\frac{D}{2}\Big)\Big\{ \frac{D}{(D-1)(D-2)^2}\Big(\frac{4}{y}\Big)^{D-2} + 6\Big(\frac{4}{y}\Big)\ln\Big(\frac{y}{4}\Big) + 13\Big(\frac{4}{y}\Big) - 6\ln^2\Big(\frac{y}{4}\Big) - 18\ln\Big(\frac{y}{4}\Big) + O\Big(\frac{D-4}{y}\Big)\Big\}. \quad (125)$$

We have tabulated the results for each of the ten External Operators. Table IV gives the quadratically and logarithmically divergent terms; Table V gives the terms which are

TABLE V. Finite normal contributions in terms of $x \equiv \frac{y}{4}$.

Ext. op.	Coef. of $\frac{\kappa^2 H^4}{(4\pi)^4}$
α	$\frac{6\ln x}{x} + \frac{13}{x} - 6\ln^2 x - 18\ln x$
β	$\frac{5}{x} - 18\ln x$
γ_1	$\frac{30}{x} - 108\ln x - 36$
γ_2	$-\frac{5}{x} - 18$
γ_3	$-\frac{5}{x} - 36$
δ	$-\frac{4\ln x}{x} - \frac{49}{6x} + 4\ln^2 x + 10\ln x + 12x\ln x$
ϵ_1	$-\frac{26}{3x} + 60\ln x - 120x\ln x + 72x$
ϵ_2	$\frac{13}{6x} - 12\ln x + 12x\ln x$
ϵ_3	$-\frac{11}{6x} + 36x\ln x + 12x$
ς	$\frac{10\ln x}{3} - 24x\ln x + 24x^2\ln x - 36x^2$

manifestly finite. In all cases the expressions were worked out by hand and then checked with Mathematica [40].

IV. RENORMALIZATION

In this section we obtain a completely finite result for the self-mass-squared by subtracting 4th-order BPHZ counterterms [31]. We first identify two invariant counterterms which can contribute to this 1PI (one particle irreducible) function at one loop. Because our gauge fixing functional (15) breaks de Sitter invariance [34], we must also consider noninvariant counterterms. We identify the only possible candidate based on a careful discussion of the residual symmetries of our gauge fixing functional. It remains to collect and compute the actual divergences. Contributions from the 4-point vertices are already local, as are the "local contributions" from the 3-point vertices. Using a now standard technique of partial integration [7] we segregate the divergences from the "normal contributions" of Table IV. In the end we identify the divergent parts of the three counterterms and report a completely finite result.

One renormalizes the scalar self-mass-squared by subtracting diagrams of the form depicted in Fig. 3. Because our scalar-graviton interactions have the form $\kappa^n h^n \partial\phi\partial\phi$, compared to the $\kappa^n h^n \partial h \partial h$ interactions of pure gravity, the superficial degree of divergence at one loop order is four, the same as that of pure quantum gravity. Of course the corresponding counterterms must contain two scalar fields, each of which has the dimension of a mass. Because we are dealing with one loop corrections from quantum gravity, all these counterterms must also carry a factor of the loop counting parameter $\kappa^2 = 16\pi G$, which has the dimension of an inverse mass-squared. Each counterterm must therefore have an additional mass dimension of four, either in the form of explicit masses or else as derivatives. The term with no derivatives is

$$\kappa^2 m^4 \phi^2 \sqrt{-g}. \tag{126}$$

There is no way to obtain an invariant with one derivative. Two derivatives can act either on the scalars or on the metric to produce a curvature. We can take the distinct terms to be

$$\kappa^2 m^2 \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} \sqrt{-g} \quad \text{and} \quad \kappa^2 m^2 \phi^2 R \sqrt{-g}. \tag{127}$$

There are no invariants with three derivatives. By judicious partial integration and use of the Bianchi identity we can take the distinct terms with four derivatives to be

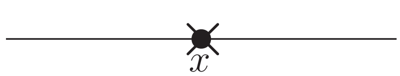

FIG. 3. Contribution from counterterms.

$$\begin{aligned} &\kappa^2 \phi_{;\mu\nu} \phi_{;\rho\sigma} g^{\mu\nu} g^{\rho\sigma} \sqrt{-g}, \\ &\kappa^2 \partial_\mu \phi \partial_\nu \phi R g^{\mu\nu} \sqrt{-g}, \\ &\kappa^2 \partial_\mu \phi \partial_\nu \phi R^{\mu\nu} \sqrt{-g}, \\ &\kappa^2 \phi^2 R^2 \sqrt{-g} \quad \text{and} \quad \kappa^2 \phi^2 R^{\mu\nu} R_{\mu\nu} \sqrt{-g}. \end{aligned} \tag{128}$$

Because our scalar is massless and mass is multiplicatively renormalized in dimensional regularization, we can dispense with (126) and (127). The last two counterterms of (128) cannot occur because the unrenormalized Lagragnian (1) is invariant under $\phi \to \phi + \text{const}$. The second and third terms of (128) become degenerate when one uses the background equation, $\hat{R}_{\mu\nu} = (D-1)H^2 \hat{g}_{\mu\nu}$. In the end just two independent invariant counterterms survive, each with its own coefficient,

$$\tfrac{1}{2}\alpha_1 \kappa^2 \Box\phi \Box\phi a^D \quad \text{and} \quad -\tfrac{1}{2}\alpha_2 \kappa^2 H^2 \partial_\mu \phi \partial^\mu \phi a^{D-2}. \tag{129}$$

The associated vertices are

$$\tfrac{1}{2}\alpha_1 \kappa^2 \Box\phi \Box\phi a^D \to i\alpha_1 \kappa^2 a^D \Box^2 \delta^D(x-x'), \tag{130}$$

$$-\tfrac{1}{2}\alpha_2 \kappa^2 H^2 \partial_\mu \phi \partial^\mu \phi a^{D-2} \to i\alpha_2 \kappa^2 H^2 a^D \Box \delta^D(x-x'). \tag{131}$$

Had our gauge condition respected de Sitter invariance, all the divergences in $-iM^2(x;x')$ could have been absorbed using (130) and (131) with appropriate choices for the divergent parts of the coefficients α_1 and α_2. Although the reasons for it are not completely understood, there seems to be an obstacle to adding a de Sitter invariant gauge fixing functional [34,41,42]. This is why we employed the noninvariant functional (15). We must therefore describe how de Sitter transformations act in our conformal coordinate system and which subgroup of them is respected by our gauge condition. The $\frac{1}{2}D(D+1)$ de Sitter transformations can be decomposed as follows:

(i) Spatial translations—$(D-1)$ distinct transformations.

$$\eta' = \eta, \qquad x'^i = x^i + \epsilon^i. \tag{132}$$

(ii) Rotations—$\frac{1}{2}(D-1)(D-2)$ distinct transformations.

$$\eta' = \eta, \qquad x'^i = R^{ij} x^j. \tag{133}$$

(iii) Dilatation—1 distinct transformation.

$$\eta' = k\eta, \qquad x'^i = kx^i. \tag{134}$$

(iv) Spatial special conformal transformations—$(D-1)$ distinct transformations.

$$\begin{aligned} \eta' &= \frac{\eta}{1 - 2\vec{\theta}\cdot\vec{x} + \|\vec{\theta}\|^2 x\cdot x}, \\ x'^i &= \frac{x^i - \theta^i x\cdot x}{1 - 2\vec{\theta}\cdot\vec{x} + \|\vec{\theta}\|^2 x\cdot x}. \end{aligned} \tag{135}$$

It turns out that our gauge choice breaks only spatial special conformal transformations (135) [29]. Hence we can use the other symmetries to restrict possible noninvariant counterterms. Spatial translational invariance means that there can be no dependence upon x^i except through the fields. Rotational invariance implies that spatial indices on derivatives must be contracted into one another. Dilatation invariance implies that derivatives and the conformal time η can only occur in the form $a^{-1}\partial_\mu$.

We can always use the invariant counterterms (130) and (131) to absorb a ∂_0^2 in favor of ∇^2 and a single ∂_0,

$$\square = \frac{1}{a^2}[-\partial_0^2 - (D-2)Ha\partial_0 + \nabla^2] \Rightarrow \frac{1}{a^2}\partial_0^2 = -\square - (D-2)\frac{H}{a}\partial_0 + \frac{\nabla^2}{a^2}. \tag{136}$$

We can also avoid $(\partial_0\varphi)^2$,

$$\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu} = -\frac{(\partial_0\varphi)^2}{a^2} + \frac{1}{a^2}\nabla\varphi\cdot\nabla\varphi \Rightarrow \frac{(\partial_0\varphi)^2}{a^2} = -\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu} + \frac{1}{a^2}\nabla\varphi\cdot\nabla\varphi. \tag{137}$$

One might think we need $Ha^{D-1}\partial_0\varphi\square\varphi$, but a partial integration allows it to be written in terms of an invariant counterterm and one with purely spatial derivatives,

$$Ha^{D-1}\partial_0\varphi\square\varphi \longrightarrow -Ha^{D-3}\partial_\mu\partial_0\varphi\partial_\nu\varphi\eta^{\mu\nu} - H^2a^{D-2}(\partial_0\varphi)^2 \tag{138}$$

$$= -\tfrac{1}{2}Ha^{D-3}\partial_0(\partial_\mu\varphi\partial_\nu\varphi)\eta^{\mu\nu} + H^2\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}\sqrt{-g} - H^2a^{D-2}\nabla\varphi\cdot\nabla\varphi, \tag{139}$$

$$\longrightarrow \tfrac{1}{2}(D-1)H^2\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}\sqrt{-g} - H^2a^{D-2}\nabla\varphi\cdot\nabla\varphi. \tag{140}$$

Another term one might consider is $Ha^{D-3}\partial_0\varphi\nabla^2\varphi$, but it can be partially integrated (twice) to give purely spatial deriviatives,

$$Ha^{D-3}\partial_0\varphi\nabla^2\varphi \longrightarrow -Ha^{D-3}\partial_0\nabla\varphi\cdot\nabla\varphi \tag{141}$$

$$= -\tfrac{1}{2}Ha^{D-3}\partial_0(\nabla\varphi\cdot\nabla\varphi), \tag{142}$$

$$\longrightarrow \tfrac{1}{2}(D-3)H^2a^{D-2}\nabla\varphi\cdot\nabla\varphi. \tag{143}$$

Based on these considerations we conclude that only three noninvariant counterterms might be needed in addition to the two invariant ones,

$$\tfrac{1}{2}\kappa^2a^{D-2}\square\varphi\nabla^2\varphi, \quad \tfrac{1}{2}\kappa^2a^{D-4}\nabla^2\varphi\nabla^2\varphi, \quad \text{and} \quad -\tfrac{1}{2}\kappa^2H^2a^{D-2}\nabla\varphi\cdot\nabla\varphi. \tag{144}$$

Because our gauge fixing term (15) becomes Poincaré invariant in the flat space limit of $H \to 0$ with the comoving time held fixed, any noninvariant counterterm must vanish in this limit. Hence we require only the final term of (144). The vertex it gives is

$$-\frac{1}{2}\alpha_3\kappa^2H^2a^D\frac{\nabla}{a}\varphi\cdot\frac{\nabla}{a}\varphi \longrightarrow i\alpha_3\kappa^2H^2a^{D-2}\nabla^2 \times \delta^D(x-x'). \tag{145}$$

The structure of the three possible counterterms serves to guide our further reduction of $-iM^2(x;x')$. First, we must convert all the factors of a' into a on the local terms. Second, we see that factors of $Ha^{D-3}\nabla^2\partial_0\delta^D(x-x')$ are not possible. Finally, it is not possible to get a divergence proportional to $H^3a^{D-1}\partial_0\delta^D(x-x')$ after using the delta function to convert all the factors of a' into factors of a.

It is now time to collect the divergent terms from the previous two sections. Those from the 4-point contributions and from the "local" 3-point contributions are already in a form which can be absorbed into the three counterterms. However, we must still bring the "normal" 3-point contributions of Table IV to this form. Recall that these terms involve powers of y that are not integrable for $D = 4$ dimensions,

$$\left(\frac{4}{y}\right)^{D-1} \quad \text{and} \quad \left(\frac{4}{y}\right)^{D-2}. \tag{146}$$

Our procedure is to extract d'Alembertians from these terms until they become integrable using the identity,

$$\square f(y) = H^2[(4y-y^2)f''(y) + D(2-y)f'(y)] + \text{Res}[y^{(D/2)-2}f]\frac{4\pi^{D/2}H^{2-D}}{\Gamma(\frac{D}{2}-1)}\frac{i}{\sqrt{-g}}\delta^D(x-x'). \tag{147}$$

Here Res$[F]$ stands for the residue of $F(y)$; that is, the coefficient of $1/y$ in the Laurent expansion of the function $F(y)$ around $y = 0$.

The key identity (147) allows us to extract a covariant d'Alembertian from each of the nonintegrable terms,

$$\left(\frac{4}{y}\right)^{D-1} = \frac{2}{(D-2)^2}\frac{\square}{H^2}\left(\frac{4}{y}\right)^{D-2} - \frac{2}{D-2}\left(\frac{4}{y}\right)^{D-2}, \tag{148}$$

$$\left(\frac{4}{y}\right)^{D-2} = \frac{2}{(D-3)(D-4)}\frac{\square}{H^2}\left(\frac{4}{y}\right)^{D-3} - \frac{4}{D-4}\left(\frac{4}{y}\right)^{D-3}. \tag{149}$$

We could use (149) on (148) to reduce them both to the power $1/y^{D-3}$. The power $1/y^{D-3}$ is integrable, so we could take $D = 4$ at this point were it not for the explicit factors of $1/(D-4)$.

To segregate the divergence on the local term we add zero in the form

$$0 = \frac{\Box}{H^2}\Big(\frac{4}{y}\Big)^{(D/2)-1} - \frac{D}{2}\Big(\frac{D}{2}-1\Big)\Big(\frac{4}{y}\Big)^{(D/2)-1} - \frac{(4\pi)^{D/2}H^{-D}}{\Gamma(\frac{D}{2}-1)}\frac{i}{a^D}\delta^D(x-x'). \tag{150}$$

Using (150) in (149) gives

$$\Big(\frac{4}{y}\Big)^{D-2} = \frac{2}{(D-3)(D-4)}\bigg\{\frac{(4\pi)^{D/2}H^{-D}}{\Gamma(\frac{D}{2}-1)}\frac{i\delta^D(x-x')}{a^D} + \frac{\Box}{H^2}\bigg[\Big(\frac{4}{y}\Big)^{D-3} - \Big(\frac{4}{y}\Big)^{(D/2)-1}\bigg]\bigg\} - \frac{4}{D-4}\bigg\{\Big(\frac{4}{y}\Big)^{D-3} - \frac{D(D-2)}{8(D-3)}\Big(\frac{4}{y}\Big)^{(D/2)-1}\bigg\} \tag{151}$$

$$= \frac{iH^{-D}(4\pi)^{D/2}}{(D-3)(D-4)\Gamma(\frac{D}{2})}\times(D-2)\frac{\delta^D(x-x')}{a^D} - \frac{\Box}{H^2}\bigg\{\frac{4}{y}\ln\Big(\frac{y}{4}\Big)\bigg\} + 2\Big(\frac{4}{y}\Big)\ln\Big(\frac{y}{4}\Big) - \Big(\frac{4}{y}\Big) + O(D-4). \tag{152}$$

The analogous result for the quadratically divergent term is

$$\Big(\frac{4}{y}\Big)^{D-1} = \frac{iH^{-D}(4\pi)^{D/2}}{(D-3)(D-4)\Gamma(\frac{D}{2})}\bigg\{\frac{2}{D-2}\frac{\Box}{H^2} - 2\bigg\}\times\frac{\delta^D(x-x')}{a^D} - \frac{1}{2}\frac{\Box^2}{H^4}\bigg\{\frac{4}{y}\ln\Big(\frac{y}{4}\Big)\bigg\} + \frac{\Box}{H^2}\bigg\{2\Big(\frac{4}{y}\Big)\ln\Big(\frac{y}{4}\Big) - \frac{1}{2}\Big(\frac{4}{y}\Big)\bigg\} - 2\Big(\frac{4}{y}\Big)\ln\Big(\frac{y}{4}\Big) + \Big(\frac{4}{y}\Big) + O(D-4). \tag{153}$$

The divergent local terms that result from applying (152) and (153) to Table IV are reported in Table VI. Table VII gives the corresponding finite terms. In each case we have eliminated the redundant external operators $\gamma_3 = 2\gamma_1 + \gamma_2$ and $\epsilon_3 = 2\epsilon_1 + \epsilon_2$.

TABLE VI. Local normal contributions from Table IV.

External operator	Coef. of $\frac{i\kappa^2 H^{D-4}}{(4\pi)^{D/2}}\frac{\Gamma(\frac{D}{2})}{(D-3)(D-4)}\frac{\delta^D(x-x')}{a^D}$
α	$\frac{D}{(D-1)(D-2)}$
β	$-\frac{D}{2(D-1)(D-2)}\frac{\Box}{H^2} - \frac{(D+2)(D-4)}{4}$
γ_1	$-\frac{D^2}{2(D-2)}\frac{\Box}{H^2} - \frac{(D^4-5D^3+16D-16)}{4}$
γ_2	$\frac{(D-2)(D^3-3D^2-4D+8)}{4(D-1)}$
δ	0
ϵ_1	$\frac{(D^2-6D+4)}{2(D-1)}$
ϵ_2	$\frac{(1-2D)}{(D-1)}$
ς	0

TABLE VII. Finite normal contributions from Table IV with $x = \frac{y}{4}$.

External operator	Coefficient of $\frac{i\kappa^2 H^4}{(4\pi)^4}$
α	$\frac{\Box}{H^2}[-\frac{\ln x}{3x}] + \frac{2\ln x}{3x} - \frac{1}{3x}$
β	$\frac{\Box^2}{H^4}[\frac{\ln x}{6x}] + \frac{\Box}{H^2}[-\frac{\ln x}{3x} + \frac{1}{6x}]$
γ_1	$\frac{\Box^2}{H^4}[\frac{2\ln x}{x}] + \frac{\Box}{H^2}[-\frac{6\ln x}{x} + \frac{2}{x}] + \frac{4\ln x}{x} - \frac{2}{x}$
γ_2	$\frac{\Box}{H^2}[-\frac{2\ln x}{3x}] + \frac{4\ln x}{3x} - \frac{2}{3x}$
δ	0
ϵ_1	$\frac{\Box}{H^2}[\frac{\ln x}{3x}] - \frac{2\ln x}{3x} + \frac{1}{3x}$
ϵ_2	$\frac{\Box}{H^2}[\frac{7\ln x}{6x}] - \frac{7\ln x}{3x} + \frac{7}{6x}$
ς	0

The next step is to reexpress the local terms of Table VI as local counterterms. This is done by using the delta function to convert all factors of a' from the external operators into factors of a, and then passing all factors of a to the left. In most cases this is straightforward but $\beta\frac{\Box}{H^2}$ and β require the following identities:

$$(aa')^{D-1}(a^2+a'^2)\Box^2[a^{-D}\delta^D(x-x')] = [2a^D\Box^2 - 12H^2a^D\Box + 8a^{D-2}H^2\nabla^2 + 2(D^2-2D+2)H^4a^D]\delta^D(x-x'), \tag{154}$$

$$(aa')^{D-1}(a^2+a'^2)H^2\Box[a^{-D}\delta^D(x-x')] = 2a^D(H^2\Box - H^4)\delta^D(x-x'). \tag{155}$$

Our results for the three possible counterterms (130), (131), and (145) and are reported in Table VIII. Note that the contribution to (130) vanishes, as it must because this counterterm happens to be zero in flat space.

Another important consistency check comes from the local terms proportional to $i\kappa^2H^4a^D\delta^D(x-x')$, which are reported in Table IX. Recall that a counterterm of this form is forbidden by the symmetry $\phi \to \phi +$ const of the bare Lagrangian (1). Although three of the four contributions to

TABLE VIII. Normal contributions to counterterms from Table VI. All terms are multiplied by $\frac{i\kappa^2 H^{D-4}}{(4\pi)^{D/2}}\frac{\Gamma(\frac{D}{2})}{(D-3)(D-4)}$.

From	$a^D\Box^2\delta^D(x-x')$	$a^DH^2\Box\times$ $\delta^D(x-x')$	$a^{D-2}H^2\nabla^2\times$ $\delta^D(x-x')$
α	$\frac{D}{(D-1)(D-2)}$	0	0
$\beta\frac{\Box}{H^2}$	$-\frac{D}{(D-1)(D-2)}$	$\frac{6D}{(D-1)(D-2)}$	$-\frac{4D}{(D-1)(D-2)}$
β	0	$-\frac{(D+2)(D-4)}{2}$	0
$\gamma_1\frac{\Box}{H^2}$	0	$-\frac{D^2}{2(D-2)}$	0
ϵ_1	0	0	$\frac{(D^2-6D+4)}{2(D-1)}$
ϵ_2	0	0	$\frac{(2-4D)}{(D-1)}$
Total	0	$\frac{(D-4)(-D^3+D-4)}{2(D-1)(D-2)}$	$\frac{(D^3-16D^2+28D-16)}{2(D-1)(D-2)}$

TABLE IX. Other local normal contributions from Table VI.

Contrib. from	Coef. of $\frac{i\kappa^2 H^{D-4}}{(4\pi)^{D/2}} \frac{\Gamma(\frac{D}{2})}{(D-3)(D-4)} \times a^D H^4 \delta^D(x-x')$
$\beta \frac{\Box}{H^2}$	$-\frac{D(D^2-2D+2)}{(D-1)(D-2)}$
β	$\frac{(D+2)(D-4)}{2}$
γ_1	$-\frac{(D^4-5D^3+16D-16)}{4}$
γ_2	$\frac{(D-2)(D^3-3D^2-4D+8)}{2(D-1)}$
Total	$\frac{(D-4)(-D^4+5D^3-16D+16)}{4(D-2)}$

Table IX diverge, their sum is finite for $D = 4$. It does not vanish because the A-type propagator equation implies

$$ia^D \delta^D(x-x') = (aa')^D\{\Box A(y) - (D-1)kH^2\}. \quad (156)$$

Because the total for Table IX is finite one can take $D = 4$ and then use (156) to subsume the result into finite, nonlocal terms of the same form as have already been reported in Table V,

$$\text{Table 9} = \frac{i\kappa^2 H^{D-4}}{(4\pi)^{D/2}} \frac{(-D^4+5D^3-16D+16)\Gamma(\frac{D}{2})}{4(D-2)(D-3)} \times a^D H^4 \delta^D(x-x'), \quad (157)$$

$$\to \frac{\kappa^2 H^4}{8\pi^2} ia^4 \delta^4(x-x') \quad (158)$$

$$= \frac{\kappa^2 H^4}{(4\pi)^4}\left\{(aa')^4 H^2 \Box\left[2\times\frac{4}{y} - 4\ln\left(\frac{y}{4}\right)\right] - (aa')^4 H^4 \times 12\right\}. \quad (159)$$

TABLE X. All finite nonlocal contributions with $x \equiv \frac{y}{4}$, where $y(x;x')$ is defined in Eq. (11).

External operator	Coefficient of $\frac{\kappa^2 H^4}{(4\pi)^4}$
$(aa')^4\Box^3/H^2$	$-\frac{\ln x}{3x}$
$(aa')^4\Box^2$	$\frac{26\ln x}{3x} + \frac{38}{3x} - 6\ln^2 x - 18\ln x$
$(aa')^4 H^2\Box$	$-\frac{6\ln x}{x} + \frac{4}{x} - 4\ln x$
$(aa')^4 H^4$	$\frac{4\ln x}{x} + \frac{18}{x} - 120 - 108\ln x$
$(aa')^3(a^2+a'^2)\Box^3/H^2$	$\frac{\ln x}{6x}$
$(aa')^3(a^2+a'^2)\Box^2$	$-\frac{\ln x}{3x} + \frac{1}{6x}$
$(aa')^3(a^2+a'^2)H^2\Box$	$-\frac{2\ln x}{3x} + \frac{5}{x} - 18\ln x$
$(aa')^3(a^2+a'^2)H^4$	$\frac{4\ln x}{3x} - \frac{32}{3x} - 54$
$(aa')^3 H^2\nabla^2$	$-\frac{2\ln x}{3x} - \frac{16}{x} + 84\ln x - 48x\ln x + 96x$
$(aa')^3\nabla^2\Box$	$\frac{\ln x}{3x}$
$(aa')^2(a^2+a'^2)H^2\nabla^2$	$-\frac{7\ln x}{3x} + \frac{11}{2x} - 12\ln x + 48x\ln x + 12x$
$(aa')^2(a^2+a'^2)\nabla^2\Box$	$-\frac{17\ln x}{6x} - \frac{49}{6x} + 4\ln^2 x + 10\ln x + 12x\ln x$
$(aa')^2(a^2+a'^2)\nabla^2\Box$	$-\frac{17\ln x}{6x} - \frac{49}{6x} + 4\ln^2 x + 10\ln x + 12x\ln x$
$(aa')^2\nabla^4$	$\frac{10}{3}\ln x - 24x\ln x + 24x^2\ln x - 36x^2$

Table X includes this with the similarly finite results of Tables III, V, and VII.

Our final result for the regulated but unrenormalized, one loop self-mass-squared derives from combining expressions (44) and (59), and the local parts of (67), with Tables VIII and X. It takes the form

$$-iM^2_{\rm reg}(x;x') = i\kappa^2 a^D\left(\beta_1\Box^2 + \beta_2\Box + \beta_3\frac{\nabla^2}{a^2}\right) \times \delta^D(x-x') + \text{Table 10} + O(D-4). \quad (160)$$

The coefficients β_i are

$$\beta_1 = 0, \quad (161)$$

$$\beta_2 = \frac{H^{D-4}}{(4\pi)^{D/2}}\left\{\frac{(-D^3+D-4)\Gamma(\frac{D}{2}-1)}{4(D-1)(D-3)} - \frac{(D+1)(D-4)\Gamma(D)\pi\cot(\frac{\pi}{2}D)}{4(D-3)\Gamma(\frac{D}{2})}\right\} \quad (162)$$

$$= \frac{H^{D-4}}{(4\pi)^{D/2}}\left\{-\frac{61}{3} + O(D-4)\right\}, \quad (163)$$

$$\beta_3 = \frac{H^{D-4}}{(4\pi)^{D/2}}\left\{\frac{(D^3-16D^2+28D-16)\Gamma(\frac{D}{2})}{2(D-1)(D-2)(D-3)(D-4)} + \frac{(D^2-4D+1)\Gamma(D-1)\pi\cot(\frac{\pi}{2}D)}{(D-3)\Gamma(\frac{D}{2})} - 3\right\} \quad (164)$$

$$= \frac{H^{D-4}}{(4\pi)^{D/2}}\left\{-\frac{4}{D-4} + \frac{58}{3} + 2\gamma + O(D-4)\right\}. \quad (165)$$

(Here $\gamma \sim 0.577\,215$ is Euler's constant.) The obvious renormalization convention is to choose each of the three α_i's to absorb the corresponding β_i, leaving an arbitrary finite term $\Delta\alpha_i$,

$$\alpha_i = -\beta_i + \Delta\alpha_i. \quad (166)$$

We can now take the unregulated limit ($D = 4$) to obtain the final renormalized result,

$$-iM^2_{\rm ren}(x;x') = i\kappa^2 a^4\left(\Delta\alpha_1\Box^2 + \Delta\alpha_2\Box + \Delta\alpha_3\frac{\nabla^2}{a^2}\right)\delta^4(x-x') + \text{Table 10}. \quad (167)$$

V. DISCUSSION

We have computed one loop quantum gravitational corrections to the scalar self-mass-squared on a locally de Sitter background. The computation was done using dimensional regularization and renormalized by subtracting the three possible BPHZ counterterms. Because our gauge condition (15) breaks de Sitter invariance, one of these counterterms is noninvariant. Our final result, expression (167), consists of arbitrary finite contributions from the three counterterms plus the nonlocal contributions given in Table X.

The point of this exercise is to discover whether or not the inflationary production of gravitons has a significant effect upon minimally coupled scalars as it does on fermions [29]. We will check this in a subsequent paper [30] by computing one loop corrections to the scalar mode functions using the effective field equation,

$$\partial_\mu(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu\Phi(x)) - \int d^4x' M^2_{\rm ren}(x;x')\Phi(x') = 0. \tag{168}$$

Similar studies have already probed the effects of scalar self-interactions [9,10], fermions [43] and photons [44], but none has so far considered the effects of gravitons. Although our scalar is a spectator to Λ-driven inflation, the near flatness of inflaton potentials suggests that the result we shall obtain may apply as well to the inflaton of scalar-driven inflation.

A significant difference between this and previous scalar studies [9,10,43,44] is that quantum gravity is not renormalizable. Although we could absorb divergences with quartic, BPHZ counterterms, no physical principle fixes the finite coefficients $\Delta\alpha_i$ of these counterterms. That ambiguity is one way of expressing the problem of quantum gravity. However, a little thought reveals that we will be able to get unambiguous results for late time corrections to the mode functions. The reason is that the scalar d'Alembertian annihilates the tree order mode solution,

$$\Phi_0(x;\vec{k}) = u(\eta,k)e^{i\vec{k}\cdot\vec{x}} \quad \text{where } u(\eta,k) = \frac{H}{\sqrt{2k^3}}\Big[1 - \frac{ik}{Ha}\Big]\exp\Big[\frac{ik}{Ha}\Big]. \tag{169}$$

Hence only the third counterterm makes a nonzero contribution, and its effect rapidly redshifts away,

$$\int d^4x'\kappa^2a^4\Big(\Delta\alpha_1\Box^2 + \Delta\alpha_2\Box + \Delta\alpha_3\frac{\nabla^2}{a^2}\Big)\delta^4(x-x') \times \Phi_0(x';\vec{k}) = -\kappa^2a^4\times\Delta\alpha_3\frac{k^2}{a^2}\Phi_0(x;\vec{k}). \tag{170}$$

It is instructive to compare our de Sitter background result (167) with its flat space analogue. In the flat space limit of $H \rightarrow 0$ with fixed comoving time, the scalar and graviton propagators become

$$i\Delta_A^{\rm flat}(x;x') = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}}\frac{1}{\Delta x^{D-2}}, \tag{171}$$

$$i[{}_{\alpha\beta}\Delta^{\rm flat}_{\rho\sigma}](x;x') = \Big[2\eta_{\alpha(\rho}\eta_{\sigma)\beta} - \frac{2}{D-2}\eta_{\alpha\beta}\eta_{\rho\sigma}\Big] \times \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}}\frac{1}{\Delta x^{D-2}}. \tag{172}$$

Here Δx^2 is the Poincaré length function analogous to $y(x;x')$,

$$\Delta x^2(x;x') \equiv \|\vec{x}-\vec{x}'\|^2 - (|\eta-\eta'| - i\delta)^2. \tag{173}$$

Two features of the flat space propagators deserve comment:

(1) both propagators are manifestly Poincaré invariant; and

(2) the coincidence limits of both propagators vanish in dimensional regularization.

We exploited the first property to drop all but the final noninvariant counterterm on the list (144). And the second property explains why our 4-point contribution (44) vanishes in the flat space limit,

$$\begin{aligned} -iM^2_{\substack{\rm flat\\ \rm 4pt}}(x;x') = &-\frac{i\kappa^2}{8}\partial^\mu\{i[{}^\alpha_\alpha\Delta^{\rm flat}_{\rho\sigma}](x;x)\eta^{\rho\sigma}\partial_\mu\delta^D(x-x')\} \\ &+\frac{i\kappa^2}{4}\partial^\mu\{i[{}^{\alpha\beta}\Delta^{\rm flat}_{\alpha\beta}](x;x)\partial_\mu\delta^D(x-x')\} \\ &+\frac{i\kappa^2}{2}\partial^\rho\{i[{}^\alpha_\alpha\Delta^{\rm flat}_{\rho\sigma}](x;x)\partial^\sigma\delta^D(x-x')\} \\ &- i\kappa^2\partial_\alpha\{i[{}^{\alpha\rho}\Delta^{\rm flat}_{\rho\sigma}](x;x)\partial^\sigma\delta^D(x-x')\} = 0. \end{aligned} \tag{174}$$

An only slightly less trivial computation reveals that the flat space limit of our total 3-point contribution should also vanish,

$$\begin{aligned} -iM^2_{\substack{\rm flat\\ \rm 3pt}}(x;x') = &-\kappa^2\partial^\alpha\partial'^\rho\{i[{}_{\alpha\beta}\Delta^{\rm flat}_{\rho\sigma}](x;x')\partial^\beta\partial'^\sigma i\Delta^{\rm flat}_A(x;x')\} \\ &+\frac{\kappa^2}{2}\partial^\mu\partial'^\rho\{i[{}^\alpha_\alpha\Delta^{\rm flat}_{\rho\sigma}]\partial_\mu\partial'^\sigma i\Delta^{\rm flat}_A\} \\ &+\frac{\kappa^2}{2}\partial^\alpha\partial'^\nu\{i[{}_{\alpha\beta}\Delta^{\rm flat}_{\rho\sigma}]\eta^{\rho\sigma}\partial^\beta\partial'_\nu i\Delta^{\rm flat}_A\} \\ &-\frac{\kappa^2}{4}\partial^\mu\partial'^\nu\{i[{}^\alpha_\alpha\Delta^{\rm flat}_{\rho\sigma}](x;x') \\ &\times\eta^{\rho\sigma}\partial_\mu\partial'_\nu i\Delta^{\rm flat}_A(x;x')\} \end{aligned} \tag{175}$$

$$= -\frac{\kappa^2\Gamma^2(\frac{D}{2}-1)}{16\pi^D}\partial^\alpha\partial^\rho\Big\{\frac{1}{\Delta x^{D-2}}\partial^\beta\partial^\sigma\frac{1}{\Delta x^{D-2}}\Big\} \times[2\eta_{\alpha(\rho}\eta_{\sigma)\beta} - \eta_{\alpha\beta}\eta_{\rho\sigma}] \tag{176}$$

$$= -\frac{\kappa^2\Gamma^2(\frac{D}{2}-1)}{16\pi^D}\partial^2\Big\{\frac{1}{\Delta x^{D-2}}\partial^2\frac{1}{\Delta x^{D-2}}\Big\} \tag{177}$$

$$= 0. \tag{178}$$

This result has a number of consequences:

(1) It explains why the highest dimension counterterm

(130) fails to appear;

(2) it explains why all the entries of Table X vanish in the flat space limit except the first line,

$$\frac{\kappa^2 H^2}{(4\pi)^4}(aa')^4\Box^3\Big\{-\frac{1}{3}\times\frac{4}{y}\ln\Big(\frac{y}{4}\Big)\Big\} \longrightarrow -\frac{1}{3}\frac{\kappa^2}{(4\pi)^4}\partial^6\Big\{\frac{4}{\Delta x^2}\ln\Big(\frac{1}{4}H^2\Delta x^2\Big)\Big\}, \quad (179)$$

and the fifth line,

$$\frac{\kappa^2 H^2}{(4\pi)^4}(aa')^3(a^2+a'^2)\Box^3\Big\{\frac{1}{6}\times\frac{4}{y}\ln\Big(\frac{y}{4}\Big)\Big\} \longrightarrow +\frac{1}{3}\frac{\kappa^2}{(4\pi)^4}\partial^6\Big\{\frac{4}{\Delta x^2}\ln\Big(\frac{1}{4}H^2\Delta x^2\Big)\Big\}; \quad (180)$$

(3) it explains why (179) and (180) cancel in the flat space limit; and

(4) it means that any physical effect we find must derive entirely from the nonzero Hubble constant.

This is the right point to comment on accuracy. This has been a long and tedious computation, involving the combination of many distinct pieces. It is significant when these pieces join together to produce results that can be checked independently, such as the vanishing of the flat space limit. One sees that in the way the α and $\beta\frac{\Box}{H^2}$ contributions to the α_1 counterterm cancel in Table VIII. Another example is the way three individually divergent terms combine in Table IX to produce a finite result for a counterterm that is forbidden by the shift symmetry of the bare Lagrangian (1).

Although the α_1 counterterm had to vanish by the flat space limit, we do not yet understand why the coefficient the α_2 counterterm is finite. The contribution of this term to the scalar self-mass-squared vanishes in flat space, but it would seem to affect the $\phi + h \rightarrow \phi + h$ scattering amplitude. The divergences on this were explored in the classic paper of 't Hooft and Veltman [45]. Unfortunately, their on-shell analysis makes no distinction between $R(\partial\phi)^2$—which we have—and $(\partial\phi)^4$—which we do not have.

Finally, we should comment on what subset of the full de Sitter group is respected by our result (167). Recall that our gauge fixing term breaks spatial special conformal transformation (135). This is why the noninvariant counterterm (145) occurs. It is also responsible for the noninvariant factors of ∇^2 and $a^2 + a'^2$ in Table X. Because these breakings derive entirely from the gauge condition, we expect them to have no physical consequence.

The graviton and scalar propagators also break the dilatation symmetry (134). Unlike the violation of spatial special conformal transformations, the breaking of dilatation invariance is a physical manifestation of inflationary particle production and can have important consequences. Dilatation breaking comes in the $\ln(aa')$ term of the A-type propagator (14). These logarithms were responsible for the secular growth that was found in the fermion field strength [29], so one might expect them to drive any effect on scalars as well. However, it turns out that the factors of $\ln(aa')$ all drop out. For the scalar propagator this is a trivial consequence of the fact that it always carries one primed and one unprimed derivative. Logarithms from the graviton propagator do appear in the 4-point contributions (44), and in the 3-point logarithm contributions (67). But all factors of $\ln(aa')$ cancel in the final result (167), which turns out to respect dilatation invariance. Because of this we suspect that there will be no significant late time corrections to the mode functions at one loop order.

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APPENDIX: EXTRACTING DERIVATIVES

We group the various normal contributions into seven parts:

$$P_1 \equiv \nabla\cdot\nabla'[(aa')^{D-2}A\nabla\cdot\nabla' A], \quad (A1)$$

$$P_2 \equiv \partial_i\partial_0'[(aa')^{D-2}A\partial_i\partial_0' A] + \partial_0\partial_i'[(aa')^{D-2}A\partial_0\partial_i' A], \quad (A2)$$

$$P_3 \equiv \partial_0\partial_0'[(aa')^{D-2}A\partial_0\partial_0' A], \quad (A3)$$

$$P_4 \equiv -\partial_0\partial_0'[(aa')^{D-2}B\nabla\cdot\nabla' A], \quad (A4)$$

$$P_5 \equiv -\partial_i\partial_0'[(aa')^{D-2}B\partial_0\partial_i' A] - \partial_0\partial_i'[(aa')^{D-2}B\partial_i\partial_0' A], \quad (A5)$$

$$P_6 \equiv -\nabla\cdot\nabla'[(aa')^{D-2}B\partial_0\partial_0' A], \quad (A6)$$

$$P_7 \equiv \partial_0\partial_0'[(aa')^{D-2}\Delta C\partial_0\partial_0' A]. \quad (A7)$$

In these definitions the exprssion "$\partial_0\partial_0' A(y)$" means the naive derivative, *without* the delta function. Also note that we have suppressed the unbiquitous factors of κ^2.

An important simplification in reducing P_2 is to achieve a symmetric form which has no ∂_0. This can be done by adding Eqs. (92) and (95) and then using Eq. (88),

$$P_2 = (-\delta + 2\zeta)I^2[AA''] + (D-1)\epsilon_2 I^2[A'^2] + H(aa')^{D-2}(a\partial_0 + a'\partial_0')\nabla^2 I^2[A'^2] \quad (A8)$$

$$= (-\delta + 2\zeta)I^2[AA''] + (D-1)\epsilon_2 I^2[A'^2] + \epsilon_3 yI[A'^2] + (D-1)\epsilon_1 I^2[A'^2] - \frac{\zeta}{2}I^3[A'^2]. \quad (A9)$$

Another organizational point concerns removing factors of y from inside integrals. This is desirable because it reduces the number of distinct integrals which appear. It can always be accomplished by partial integration. We will illustrate using the function acted upon by $-\epsilon_3$ in Eq. (103),

$$F(y) \equiv (D-2)I[A'B + yA''B] + yA'B + y^2A''B - yI[A'B']. \tag{A10}$$

Note the relations

$$A'B + yA''B = A'B + y\frac{\partial}{\partial y}I[A''B] \tag{A11}$$

$$= \frac{\partial}{\partial y}\{yI[A''B]\} + A'B - I[A''B] \tag{A12}$$

$$= \frac{\partial}{\partial y}\{yI[A''B]\} + I[A'B']. \tag{A13}$$

We can therefore write

$$F(y) = y^2A''B + (D-1)yI[A''B] + (D-2)I^2[A'B']. \tag{A14}$$

With (A13) and (A14) we can read off the following result for P_5 from Eq. (103):

$$\begin{aligned} P_5 = {} & \delta I^2[BA''] - (D-1)\epsilon_1 I^2[B'A'] \\ & - (D-1)\epsilon_2 I^2[B'A'] + \zeta\{-2I^2[BA''] \\ & + \frac{1}{2}I^3[B'A']\} + \epsilon_2\{yI[A''B] + I^2[A'B']\} \\ & + \epsilon_3\{y^2A''B + (D-1)yI[A''B] \\ & + (D-2)I^2[A'B']\}. \end{aligned} \tag{A15}$$

Many terms involving A in P_2 combine with cognate terms involving B in P_5 to produce the less singular propagator function $\Delta B = B - A$. Summing expressions (A9) and (A15) gives

$$\begin{aligned} P_{2+5} = {} & \delta I^2[\Delta BA''] - (D-1)\epsilon_1 I^2[\Delta B'A'] \\ & - (D-1)\epsilon_2 I^2[\Delta B'A'] + \epsilon_3 yI[A'^2] \\ & + \zeta\{-2I^2[\Delta BA''] + \frac{1}{2}I^3[\Delta B'A']\} + \epsilon_2\{yI[A''B] \\ & + I^2[A'B']\} + \epsilon_3\{y^2A''B + (D-1)yI[A''B] \\ & + (D-2)I^2[A'B']\}. \end{aligned} \tag{A16}$$

In contradistinction to P_2 and P_5, the reduction of the other parts is facilitated by further subdivision immediately after employing identities (75) and (78),

$$\begin{aligned} f(y)\partial_0\partial_0' A(y) = {} & \partial_0\partial_0' I^2[fA''] + 2aa'H^2I[f'A'] \\ & - aa'H^2\Big\{(D-1) + y\frac{\partial}{\partial y}\Big\}I^2[f'A'] \\ & - \frac{1}{2}\nabla\cdot\nabla' I^3[f'A'], \end{aligned} \tag{A17}$$

$$f(y)\nabla\cdot\nabla' A(y) = \nabla\cdot\nabla' I^2[fA''] - 2(D-1)aa'H^2I[f'A]. \tag{A18}$$

One employs (A17) on P_3 (from which we can read off the result for P_7) and P_6 to give the subparts,

$$P_{3a} \equiv \partial_0\partial_0'[(aa')^{D-2}\partial_0\partial_0' I^2[AA'']], \tag{A19}$$

$$P_{3b} \equiv 2\partial_0\partial_0'[(aa')^{D-1}H^2I[A'^2]], \tag{A20}$$

$$P_{3c} \equiv -\partial_0\partial_0'[(aa')^{D-1}H^2\Big\{(D-1) + y\frac{\partial}{\partial y}\Big\}I^2[A'^2]], \tag{A21}$$

$$P_{3d} \equiv -\tfrac{1}{2}\partial_0\partial_0'[(aa')^{D-2}\nabla\cdot\nabla' I^3[A'^2]], \tag{A22}$$

$$P_{7a} \equiv \partial_0\partial_0'[(aa')^{D-2}\partial_0\partial_0' I^2[\Delta CA'']], \tag{A23}$$

$$P_{7b} \equiv 2\partial_0\partial_0'[(aa')^{D-1}H^2I[A'\Delta C']], \tag{A24}$$

$$P_{7c} \equiv -\partial_0\partial_0'[(aa')^{D-1}H^2\Big\{(D-1) + y\frac{\partial}{\partial y}\Big\}I^2[A'\Delta C']], \tag{A25}$$

$$P_{7d} \equiv -\tfrac{1}{2}\partial_0\partial_0'[(aa')^{D-2}\nabla\cdot\nabla' I^3[A'\Delta C']], \tag{A26}$$

$$P_{6a} \equiv -\nabla\cdot\nabla'[(aa')^{D-2}\partial_0\partial_0' I^2[BA'']], \tag{A27}$$

$$P_{6b} \equiv -2\nabla\cdot\nabla'[(aa')^{D-1}H^2I[A'B']], \tag{A28}$$

$$P_{6c} \equiv \nabla\cdot\nabla'[(aa')^{D-1}H^2\Big\{(D-1) + y\frac{\partial}{\partial y}\Big\}I^2[A'B']], \tag{A29}$$

$$P_{6d} \equiv \tfrac{1}{2}\nabla\cdot\nabla'[(aa')^{D-2}\nabla\cdot\nabla' I^3[A'B']]. \tag{A30}$$

Applying the second identity (A18) to P_1 and P_6 gives their subparts,

$$P_{1a} \equiv \nabla\cdot\nabla'[(aa')^{D-2}\nabla\cdot\nabla' I^2[AA'']], \tag{A31}$$

$$P_{1b} \equiv -2(D-1)\nabla\cdot\nabla'[(aa')^{D-1}H^2I[A'^2]], \tag{A32}$$

$$P_{4a} \equiv -\partial_0\partial_0'[(aa')^{D-2}\nabla\cdot\nabla' I^2[BA'']], \tag{A33}$$

$$P_{4b} \equiv 2(D-1)\partial_0\partial_0'[(aa')^{D-1}H^2I[A'B']]. \tag{A34}$$

Of course there is no problem further reducing the spatial derivatives. The following generic reductions serve to reduce terms involving the operator $\partial_0\partial_0'$,

$$\partial_0\partial_0'[(aa')^{D-2}\partial_0\partial_0' f(y)] = \{\alpha - \delta + \zeta\}f(y), \tag{A35}$$

$$\partial_0\partial_0'[(aa')^{D-1}H^2 f(y)] = \Big\{\frac{\beta}{2} + \frac{1}{2}(D-1)(D-2)\gamma_1 + \frac{\gamma_2}{2}y\frac{\partial}{\partial y} + \frac{\gamma_3}{2}\Big[(D-1)y\frac{\partial}{\partial y} + y^2\frac{\partial^2}{\partial y^2}\Big] + \epsilon_1\Big[-1 + \frac{D}{4}I\Big]\Big\}f(y), \tag{A36}$$

$$\partial_0\partial_0'[(aa')^{D-2}\nabla\cdot\nabla' f(y)] = \Big\{-\frac{\delta}{2} - \frac{1}{2}(D-2)(D-3)\epsilon_1 - \frac{\epsilon_2}{2}y\frac{\partial}{\partial y} - \frac{\epsilon_3}{2}\Big[(D-1)y\frac{\partial}{\partial y} + y^2\frac{\partial^2}{\partial y^2}\Big] + \zeta\Big[1 - \frac{1}{4}(D-2)I\Big]\Big\}f(y), \tag{A37}$$

$$\nabla\cdot\nabla'[(aa')^{D-2}\partial_0\partial_0' f(y)] = \Big\{-\frac{\delta}{2} - \frac{1}{2}(D-1)(D-2)\epsilon_1 - \frac{\epsilon_2}{2}y\frac{\partial}{\partial y} - \frac{\epsilon_3}{2}\Big[(D-1)y\frac{\partial}{\partial y} + y^2\frac{\partial^2}{\partial y^2}\Big] + \zeta\Big[1 + \frac{1}{4}(D-2)I\Big]\Big\}f(y). \tag{A38}$$

Tables XI, XII, XIII, XIV, XV, XVI, XVII, XVIII, XIX, and XX give the results for each of the ten external operators.

TABLE XI. Contributions acted upon by $\alpha = (aa')^D \Box^2$.

Part	Contribution acted upon by α
P_{3a}	$I^2[AA'']$
P_{7a}	$I^2[\Delta CA'']$
Total	$I^2[AA''] + I^2[\Delta CA'']$

TABLE XII. Contributions acted upon by $\beta = (aa')^{D-2}(a^2 + a'^2)H^2\Box$.

Part	Contribution acted upon by β
P_{3b}	$I[A'^2]$
P_{3c}	$-\frac{1}{2}yI[A'^2] - (\frac{D-1}{2})I^2[A'^2]$
P_{4b}	$(D-1)I[A'B']$
P_{7b}	$I[A'\Delta C']$
P_{7c}	$-\frac{1}{2}yI[A'\Delta C'] - (\frac{D-1}{2})I^2[A'\Delta C']$
Total	$DI[A'B'] - I[A'\Delta B'] - \frac{1}{2}yI[A'^2] - (\frac{D-1}{2})I^2[A'^2] + I[A'\Delta C'] - \frac{1}{2}yI[A'\Delta C'] - (\frac{D-1}{2})I^2[A'\Delta C']$

TABLE XIII. Contributions acted upon by $\gamma_1 = (aa')^D H^4$.

Part	Contribution acted upon by γ_1
P_{3b}	$(D-1)(D-2)I[A'^2]$
P_{3c}	$-\frac{1}{2}(D-1)(D-2)yI[A'^2] - \frac{1}{2}(D-1)^2(D-2)I^2[A'^2]$
P_{4b}	$(D-1)^2(D-2)I[A'B']$
P_{7b}	$(D-1)(D-2)I[A'\Delta C']$
P_{7c}	$-\frac{1}{2}(D-1)(D-2)I[A'B'] - (D-1)(D-2)I[A'\Delta B'] - \frac{1}{2}(D-1)(D-2)yI[A'^2] - \frac{1}{2}(D-1)^2(D-2)I^2[A'^2] + (D-1)(D-2)I[A'\Delta C'] - \frac{1}{2}(D-1)(D-2)yI[A'\Delta C'] - \frac{1}{2}(D-1)^2(D-2)I^2[A'\Delta C']$
Total	$D(D-1)(D-2)I[A'B'] - (D-1)(D-2)I[A'\Delta B'] - \frac{1}{2}(D-1)(D-2)yI[A'^2] - \frac{1}{2}(D-1)^2(D-2)I^2[A'^2] + (D-1)(D-2)I[A'\Delta C'] - \frac{1}{2}(D-1)(D-2)yI[A'\Delta C'] - \frac{1}{2}(D-1)^2(D-2)I^2[A'\Delta C']$

TABLE XIV. Contributions acted upon by $\gamma_2 = (aa')^{D-1} \times (a^2 + a'^2)H^4$.

Part	Contribution Acted upon by γ_2
P_{3b}	yA'^2
P_{3c}	$-\frac{1}{2}y^2A'^2 - \frac{D}{2}yI[A'^2]$
P_{4b}	$(D-1)yA'B'$
P_{7b}	$yA'\Delta C'$
P_{7c}	$-\frac{1}{2}y^2A'\Delta C' - \frac{D}{2}yI[A'\Delta C']$
Total	$DyA'B' - yA'\Delta B' - \frac{1}{2}y^2A'^2 - \frac{D}{2}yI[A'^2] + yA'\Delta C' - \frac{D}{2}yI[A'\Delta C'] - \frac{1}{2}y^2A'\Delta C'$

TABLE XV. Contributions acted upon by $\gamma_3 = (aa')^{D-1}(a + a')^2H^4$.

Part	Contribution acted upon by γ_3
P_{3b}	$(D-1)yA'^2 + y^2(A'^2)'$
P_{3c}	$-Dy^2A'^2 - \frac{1}{2}y^3(A'^2)' - \frac{1}{2}(D-1)DyI[A'^2]$
P_{4b}	$(D-1)^2yA'B' + (D-1)y'(A'B')'$
P_{7b}	$(D-1)yA'\Delta C' + y^2(A'\Delta C')'$
P_{7c}	$-Dy^2A'\Delta C' - \frac{1}{2}y^3(A'\Delta C')' - \frac{1}{2}(D-1)DyI[A'\Delta C']$
Total	$(D-1)DyA'B' + Dy^2(A'B')' - (D-1)yA'\Delta B' - y^2(A'\Delta B')' - Dy^2A'^2 - \frac{1}{2}y^3(A'^2)' - \frac{1}{2}(D-1)DyI[A'^2] + (D-1)yA'\Delta C' + y^2(A'\Delta C')' - Dy^2A'\Delta C' - \frac{1}{2}y^3(A'\Delta C')' - \frac{1}{2}(D-1)DyI[A'\Delta C']$

TABLE XVI. Contributions acted upon by $\delta = (aa')^{D-2}(a^2 + a'^2)\nabla^2\Box$.

Part	Contribution acted upon by δ
P_{2+5}	$I^2[\Delta BA'']$
P_{3a}	$-I^2[AA'']$
P_{3d}	$\frac{1}{4}I^3[A'^2]$
P_{4a}	$\frac{1}{2}I^2[BA'']$
P_{6a}	$\frac{1}{2}I^2[BA'']$
P_{7a}	$-I^2[\Delta CA'']$
P_{7d}	$\frac{1}{4}I^3[A'\Delta C']$
Total	$2I^2[\Delta BA''] + \frac{1}{4}I^3[A'^2] - I^2[\Delta CA''] + \frac{1}{4}I^3[A'\Delta C']$

TABLE XVII. Contributions acted upon by $\epsilon_1 = (aa')^{D-1}H^2\nabla^2$.

Part	Contribution acted upon by ϵ_1
P_{1b}	$2(D-1)I[A'^2]$
P_{2+5}	$-(D-1)I^2[A'\Delta B']$
P_{3b}	$-2I[A'^2] + \frac{D}{2}I^2[A'^2]$
P_{3c}	$yI[A'^2] + (D-1)I^2[A'^2] - \frac{D}{4}yI^2[A'^2]$ $-\frac{1}{4}D(D-2)I^3[A'^2]$
P_{3d}	$\frac{1}{4}(D-2)(D-3)I^3[A'^2]$
P_{4a}	$\frac{1}{2}(D-2)(D-3)I^2[BA'']$
P_{4b}	$-2(D-1)I[A'B'] + \frac{1}{2}(D-1)DI^2[A'B']$
P_{6a}	$\frac{1}{2}(D-1)(D-2)I^2[BA'']$
P_{6b}	$2I[A'B']$
P_{6c}	$-yI[A'B'] - (D-1)I^2[A'B']$
P_{7b}	$-2I[A'\Delta C'] + \frac{D}{2}I^2[A'\Delta C']$
P_{7c}	$yI[A'\Delta C'] + (D-1)I^2[A'\Delta C'] - \frac{D}{4}yI^2[A'\Delta C']$ $-\frac{1}{4}D(D-2)I^3[A'\Delta C']$
P_{7d}	$\frac{1}{4}(D-2)(D-3)I^3[A'\Delta C']$
Total	$-2(D-2)I[A'\Delta B'] - yI[A'\Delta B'] - (\frac{5D-4}{2})I^2[A'\Delta B']$ $+\frac{D^2}{2}I^2[A'B'] - \frac{D}{4}yI^2[A'^2] + (D-2)^2I^2[A''B]$ $-\frac{3}{4}(D-2)I^3[A'^2] - 2I[A'\Delta C'] + yI[A'\Delta C']$ $+(\frac{3D-2}{2})I^2[A'\Delta C'] - \frac{D}{4}yI^2[A'\Delta C'] - \frac{3}{4}(D-2)I^3[A'\Delta C']$

TABLE XVIII. Contributions acted upon by $\epsilon_2 = (aa')^{D-2} \times (a^2 + a'^2)H^2\nabla^2$.

Part	Contribution acted upon by ϵ_2
P_{2+5}	$-(D-1)I^2[A'\Delta B'] + yI[BA''] + I^2[A'B']$
P_{3d}	$\frac{1}{4}yI^2[A'^2]$
P_{4a}	$\frac{1}{2}yI[BA'']$
P_{6a}	$\frac{1}{2}yI[BA'']$
P_{7d}	$\frac{1}{4}yI^2[A'\Delta C']$
Total	$2yI[BA''] + I^2[A'B'] - (D-1)I^2[A'\Delta B']$ $+\frac{1}{4}yI^2[A'^2] + \frac{1}{4}yI^2[A'^2\Delta C']$

TABLE XIX. Contributions acted upon by $\epsilon_3 = (aa')^{D-2} \times (a + a')^2H^2\nabla^2$.

Part	Contribution acted upon by ϵ_3
P_{2+5}	$y^2A''B + (D-1)yI[A''B] + (D-2)I^2[A'B'] + yI[A'^2]$
P_{3d}	$(\frac{D-1}{4})yI^2[A'^2] + \frac{1}{4}y^2I[A'^2]$
P_{4a}	$(\frac{D-1}{2})yI[A''B] + \frac{1}{2}y^2A''B$
P_{6a}	$(\frac{D-1}{2})yI[A''B] + \frac{1}{2}y^2A''B$
P_{7d}	$(\frac{D-1}{4})yI^2[A'\Delta C'] + \frac{1}{4}y^2I[A'\Delta C']$
Total	$2y^2A''B + 2(D-1)yI[A''B] + (D-2)I^2[A'B']$ $+yI[A'^2] + (\frac{D-1}{4})yI^2[A'^2] + \frac{1}{4}y^2I[A'^2]$ $+(\frac{D-1}{4})yI^2[A'\Delta C'] + \frac{1}{4}y^2I[A'\Delta C']$

TABLE XX. Contributions acted upon by $\zeta = (aa')^{D-2}\nabla^4$.

Part	Contribution acted upon by ς
P_{1a}	$I^2[AA'']$
P_{2+5}	$-2I^2[\Delta BA''] + \frac{1}{2}I^3[A'\Delta B']$
P_{3a}	$I^2[AA'']$
P_{3d}	$-\frac{1}{2}I^3[A'^2] + (\frac{D-2}{8})I^4[A'^2]$
P_{4a}	$-I^2[A''B] + (\frac{D-2}{4})I^3[A''B]$
P_{6a}	$-I^2[A''B] - (\frac{D-2}{4})I^3[A''B]$
P_{6d}	$\frac{1}{2}I^3[A'B']$
P_{7a}	$I^2[\Delta CA'']$
P_{7d}	$-\frac{1}{2}I^3[A'\Delta C'] + (\frac{D-2}{8})I^4[A'\Delta C']$
Total	$-4I^2[A''\Delta B] + I^3[A'\Delta B'] + (\frac{D-2}{8})I^4[A'^2]$ $+I^2[\Delta CA''] - \frac{1}{2}I^3[A'\Delta C'] + (\frac{D-2}{8})I^4[A'\Delta C']$

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