Remarks on dynamical dark energy measured by the conformal age of the universe

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We elaborate on a model of conformal dark energy (dynamical dark energy measured by the conformal age of the universe) recently proposed in [H. Wei and R. G. Cai, arXiv:0708.0884] where the presentday dark energy density was taken to be $\rho_q \equiv 3\alpha^2 m_P^2/\eta^2$, where η is the conformal time and α is a numerical constant. In the absence of an interaction between the ordinary matter and dark energy field q, the model may be adjusted to the present values of the dark energy density fraction $\Omega_q \simeq 0.73$ and the equation of state parameter $w_q < -0.78$, if the numerical constant α takes a reasonably large value, $\alpha \gtrsim 2.6$. However, in the presence of a nontrivial gravitational coupling of q-field to matter, say $\tilde{Q}$, the model may be adjusted to the values $\Omega_q \simeq 0.73$ and $w_q \simeq -1$, even if $\alpha \sim \mathcal{O}(1)$, given that the present value of $\tilde{Q}$ is large. Unlike for the model in [R. G. Cai, arXiv:0707.4049], the bound $\Omega_q < 0.1$ during big bang nucleosynthesis (BBN) may be satisfied for almost any value of α. Here we discuss some other limitations of this proposal as a viable dark energy model. The model draws some parallels with the holographic dark energy; we also briefly comment on the latter model.

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I. INTRODUCTION

Inflation is an attractive paradigm for explaining small temperature fluctuations in the cosmic microwave background, the distribution of galaxies, the homogeneity and isotropy of the universe on scales of more than 100 Mpc and its spatial flatness, as inferred by recent WMAP data [1]. The current standard model of cosmology somehow combines the original hot big bang model and the primordial inflation [2], by virtue of the existence of a fundamental scalar field, called *inflaton*. However, the standard model of cosmology has some gaps and cracks; for instance, the recently observed accelerated expansion of the universe [3] appears to suggest in the fabric of the cosmos a self-repulsive dark energy component of magnitude about 73% of the total energy budget of the entire universe. Evidence in favor of this accelerated expansion has strengthened significantly as the result of further SNe Ia observations [4], surveys of large-scale structure (LSS) [5], and improved measurements of the cosmic microwave background [6]. The precise cause of this late-time acceleration and the nature of dark energy attributed to this effect, however, remain illusive.

The phenomenal role of a cosmological vacuum energy (or dark energy) has changed our vocabulary for describing the cosmological possibilities and the fate of our universe (see [7,8] for reviews). We do not understand whether the highly accelerated expansion shortly after the big bang—called inflation and the current accelerated expansion of the universe (caused by dark energy) are related. The understanding of dark energy's origin may be expected to provide some useful insights to many other puzzles in physics, including: What caused the early universe inflation? Why does dark energy/dark matter make up most of the universe?

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In a fundamental theory of gravity plus elementary particles and fields, it is quite plausible that the primordial inflation naturally led to have a dark energy effect in the conditions of the concurrent universe, i.e. when the universe became much larger than its size at the beginning. Such an effect can be explained through two somehow different mechanisms. In the first, and perhaps the most viable approach [9–11], the presentday dark energy effect could be realized as a remnant of the original inflaton field that went into a hide shortly after reheating (or even after inflation), but which started to play a new role during the matter-dominated epoch, especially, on large cosmological scales (> 100 Mpc), where gravity would almost fail to curve the spacetime, thereby leading to a spatially flat Friedmann-Lamaître-Robertson-Walker (FLRW) universe. In the second approach, the quantum fluctuations associated with an accelerating slice of a FRW metric (during the primordial inflation) could gradually overtake at late times the ambient matter distributions, tending to increase the rate of expansion of the universe on large cosmological scales. In this paper we discuss about the latter possibility, in the framework of a model of "conformal" dark energy (dynamical dark energy measured by the conformal age of the universe) recently proposed by Cai and Wei [12].

There has also been a fair amount of interest in the possibility that the dark energy is holographic [13,14]. The model of dynamical dark energy discussed in [12] has some similarities with the so-called holographic dark energy proposed earlier by Li [15]. We will briefly comment at the end on holographic dark energy models.

II. WHY SCALAR GRAVITY AFTER ALL?

The possibility remains that the cosmological constant (or the vacuum energy) is fundamentally variable. In order to give the idea a fair hearing, one should conceivably take some sort of dark energy potential. An appropriate

Lagrangian might be

$$\mathcal{L} = \sqrt{-g}\left(\frac{R}{2\kappa^2} - \frac{1}{2}(\partial q)^2 - V(q)\right) + \mathcal{L}_m, \qquad (1)$$

where κ is the inverse Planck mass $m_P^{-1} = (8\pi G_N)^{1/2}$, G_N is Newton's constant, q is a fundamental scalar (or dark energy) field, and $V(q)$ is its potential. Indeed, in the simplest dynamical dark energy models [9], dark energy is associated with the energy density of a scalar field with a canonical kinetic structure, as above. Most dynamical dark energy models, including the agegraphic and holographic dark energy, may be analyzed by maintaining the above structure of the theory.

For an analytic treatment it is necessary to evaluate the equations generated by variation of the action (1); thus a particular choice of a metric has to be made. In line with current observations, and because it greatly simplifies the calculations, we make the rather standard choice of a spatially flat, homogeneous metric: $ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$, where $a(t)$ is the scale factor of a spatially flat FRW universe. This is consistent with the measurements of the cosmic microwave background (CMB) anisotropies and large-scale structures of the universe, which indicate that the present universe is spatially flat and homogeneous on large scales.

An important ingredient of a cosmological model is matter Lagrangian, which may be given by [16]

$$\mathcal{L}_m \equiv \mathcal{L}(\beta^2(q)g_{\mu\nu}, \psi_m) = \sqrt{-g}\beta^4(q)\tilde{\rho}_i, \qquad (2)$$

where $\tilde{\rho}_{(i)} \propto \hat{a}^{-3(1+w_i)}$ $(i = m, r)$, $\hat{a} \equiv a\beta(q)$. Introduction of a fundamental scalar field q, its potential $V(q)$ and the coupling $\beta(q)$ between q and the ordinary matter (ρ_m) and radiation (ρ_r) may not be arbitrary rather a requirement for the presentday concordance model cosmology. These ingredients are strongly motivated by supergravity and superstring theories.

Einstein's equations following from Eqs. (1) and (2) are [17]

$$3H^2 = \kappa^2(\tfrac{1}{2}\dot{q}^2 + V(q) + \beta^4(\rho_m + \rho_r)), \qquad (3)$$

$$-2\dot{H} = \kappa^2(\dot{q}^2 + \beta^4(1 + w_m)\rho_m + \tfrac{4}{3}\beta^4\rho_r), \qquad (4)$$

where $w_i \equiv p_i/\rho_i$ and $\rho_i \propto (a\beta)^{-3(1+w_i)}$. The scalar field q couples to the trace of the matter stress tensor, $g^{\mu\mu}_{(i)}T^{(i)}_{\mu\nu}$, namely

$$-\nabla^2 q = \ddot{q} + 3H\dot{q} = -V_{,q} + \alpha_q T^{\mu}_{\mu(i)}, \qquad (5)$$

where $\alpha_q \equiv \frac{d\ln\beta(q)}{dq}$ and $H \equiv \dot{a}/a$ is the Hubble parameter (the dot denotes a derivative with respect to cosmic time t). Since $T^{\mu}_{\mu(m)} = -\rho_m + 3p_m \equiv -\rho_m(1 - 3w_m)$ and $T^{\mu}_{\mu(r)} = -\rho_r + 3p_r = 0$, the above equation of motion for q can be expressed in the following form[1]:

$$\dot{\rho}_q + 3H\rho_q(1 + w_q) = -\dot{q}\gamma\alpha_q\beta(q)\rho_m, \qquad (6)$$

where $\gamma \equiv (1 - 3w_m)$, $w_m \equiv p_m/\rho_m$, $\rho_q \equiv \frac{1}{2}\dot{q}^2 + V(q)$, $w_q \equiv p_q/\rho_q$. This equation, along with the equations of motion for ordinary fluids (matter and radiation):

$$\begin{aligned} \dot{\rho}_m + 3H\rho_m(1 + w_m) &= +\dot{q}\gamma\alpha_q\beta(q)\rho_m, \\ \dot{\rho}_r + 4H\rho_r &= 0, \end{aligned} \qquad (7)$$

guarantees the conservation of total energy, namely $\dot{\rho} + 3H(\rho + p) = 0$, where $\rho \equiv \rho_m + \rho_r + \rho_q$.

The set of autonomous equations of motion may be given by (see, e.g. [17,18])

$$\Omega_r + 3w_q\Omega_q + 3w_m\Omega_m + 2\varepsilon + 3 = 0, \qquad (8)$$

$$\Omega_q' + 2\varepsilon\Omega_q + 3(1 + w_q)\Omega_q = -\tilde{Q}, \qquad (9)$$

$$\Omega_m' + 2\varepsilon\Omega_m + 3(1 + w_m)\Omega_m = +\tilde{Q}, \qquad (10)$$

subject to the Friedmann constraint $\Omega_m + \Omega_r + \Omega_q = 1$, where the prime denotes the derivative with respect to $N \equiv \ln[a(t)] + \text{const}$, $\varepsilon = \dot{H}/H^2$, $\tilde{Q} \equiv \gamma q'\alpha_q\Omega_m$, $q' = \dot{q}/H$, $\Omega_i \equiv \kappa^2\beta^4\rho_i/(3H^2)$, and $\Omega_q = \kappa^2\rho_q/(3H^2)$. The fact that the radiation term ρ_r does not contribute to the scalar potential or the Klein-Gordon equation has an interesting implication: in the early universe, e.g. during or shortly after inflation, one can ignore the coupling $\beta(q)$, since $\rho_m \ll \rho_r$. During the matter-dominated universe, given that $\rho_m \propto 1/a^3$, $w_m \simeq 0$, and $a \propto t^{2/3}$ $(\varepsilon = -3/2)$, it is plausible that $\tilde{Q} \approx 0$. However, the coupling $\tilde{Q}$ may be relevant especially when $\rho_q \gtrsim \rho_m$, i.e., in the dark energy-dominated universe.

From Eq. (8), we find that the dark energy equation of state (EoS) is given by

$$w_{\rm DE} \equiv w_q = -\frac{2\varepsilon + 3 + 3\sum_i w_i\Omega_i + \Omega_r}{3\Omega_q}, \qquad (11)$$

where $i = m$ (matter) includes all forms of matter fields, such as, pressureless dust ($w = 0$), stiff fluid ($w = 1$), cosmic strings ($w = -1/3$), domain walls ($w = -2/3$), etc. One might also note the universe accelerates when the effective equation of state $w_{\rm eff}$ is less than $-1/3$ (where $w_{\rm eff} \equiv -1 - 2\varepsilon/3$), not when $w_q < -1/3$. In the particular case that $w_m = 0$ and $\Omega_r \approx 0$, so that the matter is approximated by a pressureless nonrelativistic perfect fluid, the universe accelerates for

$$w_q\Omega_q < -\tfrac{1}{3}. \qquad (12)$$

[1]The parameter α_q defined here corresponds to $-Q$ in Refs. [17,18], where it was assumed that $Q < 0$.

With the input $\Omega_q = 0.73$, we can see that the universe accelerates for $w_q < -0.46$.

III. WHAT IS DARK ENERGY?

We do not yet have any clue as to what dark energy is, and how to compute its present contribution from the first principles. A common lore is that "dark energy" is the Einstein's cosmological constant until proven otherwise, for the reason that it is the most economical interpretation of the data. The main observation that has led to this viewpoint is the following: the combination of WMAP3 and Supernova Legacy Survey data sets show a significant constraint on the dark energy equation of state, $w_{\rm DE} = -0.97^{+0.07}_{-0.09}$, on the ΛCDM model, i.e., in a flat universe, with a prior $w_m = 0$. Perhaps this observation is not yet sufficiently convincing to abandon other possibilities, at least, for two other reasons: first, no theoretical model, not even the most sophisticated, such as supersymmetry or string theory, is able to explain the presence of a small positive cosmological constant, in the amount that our observations require [7], $\rho_\Lambda \sim 5 \times 10^{-27}$ kg/m^3 or $\rho_\Lambda \sim 10^{-123}$ in Planck units; second, there are widespread claims that the analysis of the type Ia supernova data sets actually favor a time-varying dark energy equation of state at higher redshifts (see, e.g. [19], for a review).

Needless to emphasize, the possibility remains that dark energy is fundamentally variable. It is thus a fair approach to envisage for plausible phenomenological models and apply the observational results either to rule them all or select one of them. In order to give the idea a fair treatment, in this work we briefly review some recent attempts in this direction, namely, the models of conformal and "holographic" dark energy.

A. Dark energy measured by a cosmic time

In a recent proposal [20], Cai argued that the presentday dark energy density may be defined by the energy density of metric fluctuations in a Minkowski spacetime, namely

$$\rho_\Lambda \equiv \rho_q \propto \frac{1}{t_P^2 t^2} \equiv \frac{3n^2 m_P^2}{t^2}, \tag{13}$$

where the numerical coefficient $n \sim \mathcal{O}(1)$ and t_P is Planck's time. The above relation is somehow based on quantum kinematics or Heisenberg uncertainty type relations that put a limit on the accuracy of quantum measurements; we refer to the papers [21,22] and references therein, for further details. Without any reference to the field potential $V(q)$, by Eq. (13), one can perhaps understand that the quantum fluctuations in a Minkowski spacetime contribute to the expectation value of the stress tensor in a way that mimics the dark energy density at the present epoch. According to [20], the cosmic time

$$t = \int_0^a \frac{da}{Ha} = \int H^{-1} d\ln a \tag{14}$$

may be considered as the age of our universe. Differentiating this equation with respect to $\ln a$, we get

$$\frac{dt}{d\ln a} = \frac{1}{H}. \tag{15}$$

Further, from the definition

$$\Omega_q \equiv \frac{\rho_q}{3m_P^2 H^2} = \frac{n^2}{t^2 H^2}, \tag{16}$$

we get

$$tH = \pm \frac{n}{\sqrt{\Omega_q}}. \tag{17}$$

With $n > 0$, because of the requirement that $tH > 0$, we shall take the positive sign in (17). Then, differentiating Eq. (16) with respect to $\ln a$, we get

$$\Omega_q' + 2\varepsilon\Omega_q + \frac{2}{n}(\Omega_q)^{3/2} = 0. \tag{18}$$

In the absence of interaction between the q-field and matter, so that $\tilde{Q} = 0$, from Eq. (9), we find

$$w_q = -1 - \frac{1}{3}\frac{\Omega_q'}{\Omega_q} - \frac{2\varepsilon}{3}. \tag{19}$$

Comparing Eqs. (9) and (18) we get

$$w_q = -1 + \frac{2}{3n}\sqrt{\Omega_q}. \tag{20}$$

Obviously, with $\sqrt{\Omega_q}/n > 0$, or $tH > 0$, we get $w_q > -1$, in which case q behaves as a canonical scalar field or quintessence. From (20) it is easy to see that the q-field violates the strong energy condition, $w_q \geq -1/3$, for $\sqrt{\Omega_q} < n$, which is the minimal condition for a cosmic acceleration to occur in the absence of ordinary fluids (matter and radiation). With the input $\Omega_q = 0.73$, $w_q < -1/3$ for $n > 0.85$. The WMAP observations, which are sensitive to w_q over a redshift range of roughly 1100 (since decoupling), imply $w_q < -0.78$ (95% confidence level), which translates to the condition $n > 3\sqrt{\Omega_q}$. This last condition obviously leads to a result consistent with the discussion in [23], where the best-fit values were found to be $n = 3.4$ and $\Omega_q = 0.72$ in using the constraints from CMB and LSS observations.

Equating Eqs. (19) and (20) and then solving for Ω_q, we obtain

$$\frac{n}{\sqrt{\Omega_q}} = \begin{cases} \frac{1}{2}(1 + b_1 a^{-2}) & (\text{RD},\ a(t) = a_{\rm r,ini} t^{1/2}), \\ \frac{2}{3}(1 + b_2 a^{-3/2}) & (\text{MD},\ a(t) = a_{\rm m,ini} t^{2/3}), \end{cases} \tag{21}$$

where b_1 and b_2 are integration constants, and $a_{\mathrm{r,ini}}$ and $a_{\mathrm{m,ini}}$ are scale factors at the beginning of the radiation and matter-dominated epochs. In accordance with Eq. (17), the obvious choices are $b_1 = b_2 = 0$, since during both matter and radiation-dominated epochs $tH \approx$ const. The requirements $\Omega_q(1\ \mathrm{MeV}) < 0.1$ during big bang nucleosynthesis (BBN) and $\Omega_q < 1$ during the matter-dominated universe therefore imply that $n^2 < 1/40$ and $n < 2/3$, respectively. This result led us to conclude in [17] that the agegraphic dark energy with some fixed n in (13) is not a viable alternative to concordance cosmology.

It would be possible to modify this outcome only by dropping one or more premises of the standard model cosmology, such as, a matter-dominated flat universe did not exist, which then tells that the Einstein-de Sitter model is never realized truly. As an illustrative example, one may consider the following modification

$$\rho_q \propto \frac{1}{t_P^2 (t + t_1)^2} \equiv \frac{3n^2 m_P^2}{(t + t_1)^2}, \tag{22}$$

where t_1 is a constant with the dimension of time. In fact, a solution of the above structure arises in almost all scalar-tensor theories, e.g., with $V(q) \propto e^{-\lambda q/m_P}$ and $q(t) = (\lambda/2)\ln(t + t_1)$ (see e.g. [24]). In a standard approach, one normally sets $t_1 = 0$ using the coordinate parametrization freedom of t, with the assumption that such a shift in time only changes the position of the big bang singularity. However, let us assume here rather implicitly that no freedom was left so as to allow us to set $t_1 = 0$; therefore, $t_1 > 0$ henceforth.

Then, typically, we may assume that $t_1 > t_0$, where t_0 is the present age of the universe. From the definition $\Omega_q \equiv n^2/[(t + t_1)^2 H^2]$, we obtain

$$\frac{n}{\sqrt{\Omega_q}} = (t + t_1)H = tH\left(1 + \frac{t_1}{t}\right). \tag{23}$$

A comparison between Eqs. (21) and (23) shows that b_1 and b_2 are nonzero; more precisely,

$$t_1 = \frac{b_1}{a_{\mathrm{r,ini}}} = \frac{b_2}{a_{\mathrm{m,ini}}^{3/2}}.$$

The bound $\Omega_q(1\ \mathrm{MeV}) < 0.1$ during the BBN epoch may be satisfied for $40n^2 < (1 + t_1/t)^2$. Next, consider that, at present, $t_1 \equiv 2.33 t_0$, $t_0 \sim H_0^{-1}$, and $\Omega_q = 0.73$. This yields $w_q = -0.8$. That means, when the universe was half of the present age, $t/2 \sim \frac{3}{14} t_1 \sim 6.8$ Gyrs (approximately when $z \sim 1$), one had $w_q \simeq -0.82$ (assuming a matter-dominated universe with $tH = 2/3$), but $\Omega_q \simeq 0.57$. If such a variation in the dark energy density fraction is allowed by observations, then the agegraphic dark energy model, with the modification (23), may be consistent with the concordance cosmology.

On the other end, if $t_1 \ll t$ holds, then during the matter-dominated epoch to which the WMAP and supernovae measurements are sensitive, one finds $\Omega_q \simeq 9n^2/4$ with $a(t) \propto t^{2/3}$, in which case, one obviously requires $|n| < 2/3$ during the matter-dominated epoch.

B. Dark energy measured by a conformal time

Next, let us consider another model of dynamical dark energy proposed by Cai and Wei [12]. In this proposal, one takes the presentday dark energy density to be

$$\rho_q \equiv \rho_\Lambda \propto \frac{1}{l_P^2 \eta^2} \equiv \frac{3\alpha^2 m_P^2}{\eta^2}, \tag{24}$$

where the numerical factor $3\alpha^2$ is introduced for convenience and η is the conformal time

$$\eta = \int \frac{dt}{a} = \int (aH)^{-1} d\ln a. \tag{25}$$

Differentiating Eq. (25) with respect to $\ln a$, one finds

$$\frac{d\eta}{d\ln a} = \frac{1}{aH}. \tag{26}$$

Further, from the definition

$$\Omega_q \equiv \frac{\rho_q}{3m_P^2 H^2} = \frac{\alpha^2}{\eta^2 H^2}, \tag{27}$$

we find

$$\eta H = \frac{\alpha}{\sqrt{\Omega_q}}. \tag{28}$$

Differentiating Eq. (27) with respect to $\ln a$, we obtain

$$\Omega_q' + 2\varepsilon\Omega_q + \frac{2}{\eta H} e^{-\ln a}\Omega_q = 0. \tag{29}$$

Although α can take either sign, for a reason to be explained, we shall normally take $\alpha < 0$; the choice for the sign of α is actually linked to the choice of sign in $d\eta \equiv \pm a dt$.

By Eq. (24) one can perhaps understand that the universe starts out with zero vacuum energy, near the big bang, since $\eta \to -\infty$. This may not look very physical from the viewpoint that in almost all scalar field cosmologies the energy of the vacuum or potential energy might drop sharply during various phase transitions in the early universe. Nevertheless, the magnitude of the presentday dark energy density determined by Eq. (24) may be consistent with the cosmological observations, for $|\alpha| > 2.6$. In such a context, one should perhaps seek a dark energy that behaves very differently than the standard scalar field potential.

In fact, Eq. (24) draws some parallels with the known example of quintessential potential, $V(q) \propto q^{-2}$. It is generally expected that

$$\frac{\rho_q}{3} = \frac{1}{6}\dot{q}^2 + \frac{V(q)}{3} \equiv \frac{\alpha^2 m_P^2}{\eta^2}. \tag{30}$$

In the limit $\dot{q}^2 \ll V(q)$, or simply that $V(q) \propto \dot{q}^2$, we get

$$q^2 \propto \eta^2. \tag{31}$$

The limit of conformal time is $\eta \in (-\infty, 0)$; this then translates to the condition that $|q| \to \infty$ near the big bang, where $|\eta| \to \infty$, while $q \to 0$ in the asymptotic future, $\eta \to 0$.

IV. NONINTERACTING DARK ENERGY, $\tilde{Q} = 0$

Let us first consider the case $\tilde{Q} = 0$. From Eq. (9), we then get

$$w_q = -1 - \frac{1}{3}\frac{\Omega_q'}{\Omega_q} - \frac{2\varepsilon}{3}. \tag{32}$$

Comparing Eqs. (9) and (29) we get

$$w_q = -1 + \frac{2}{3\alpha} e^{-\ln a}\sqrt{\Omega_q}. \tag{33}$$

Equating Eqs. (32) and (33), and then solving for Ω_q, we find

$$\frac{1}{\sqrt{\Omega_q}} = \frac{c\alpha + \int e^{-\ln a}(e^{-\int \varepsilon d\ln a}) d\ln a}{\alpha e^{-\int \varepsilon \ln a}} = H\left(c + \frac{1}{\alpha}\int (a^2 H)^{-1} da\right), \tag{34}$$

where c is an integration constant. In the discussion below we often use the relation $e^{\ln a} = (1+z)^{-1}$, where z is the redshift parameter, so that $a(z=0) \equiv a_0 = 1$.

Equation (34) gives rise to

$$\frac{1}{\sqrt{\Omega_q}} = \begin{cases} (\alpha a)^{-1} + b_1 a^{-2} & (\text{RD}, \ a \propto t^{1/2}), \\ 2(\alpha a)^{-1} + b_2 a^{-3/2} & (\text{MD}, \ a \propto t^{2/3}), \end{cases} \tag{35}$$

where b_1, b_2 are integration constants. With the choice $b_1 = 0 = b_2$[2] one finds $w_q = -1/3$ (RD) or $w_q = -2/3$ (MD). Moreover, $\rho_q \propto 1/a^2$ (RD) or $\rho_q \propto 1/a$ (MD). However, especially, with $b_i > 0$,[3] one finds $-1 < w_q < -1/3$ (RD) or $-1 < w_q < -2/3$ (MD). If the integration constants b_1, b_2 can be large, namely $b_1 \gg a_{r,e}$ and $b_2 \gg a_{m,e}^{1/2}$, then during both the RD and MD epochs, $\rho_q \propto$ const, which mimics the case of a cosmological constant term.

Next, we consider a power-law expansion $a(t) \equiv [c_0 t + t_1]^m$, with an arbitrary m. We then find

$$\frac{1}{\sqrt{\Omega_q}} = \begin{cases} -\frac{m}{m-1}(\alpha a)^{-1} + c_1 a^{-1/m} & (m \neq 1), \\ \ln a(\alpha a)^{-1} + c_2 a^{-1} & (m = 1), \end{cases} \tag{36}$$

where c_1, c_2 are integration constants. Notice that, for the branch $m > 1$, a physical solution may require α to be negative, otherwise the quantity $\sqrt{\Omega_q}$ diverges at some stage of cosmic evolution, for $c_1 > 0$. Of course, the choice $\alpha > 0$ and $c_1 < 0$ is also allowed. In either case, $w_q < -1$, since $\frac{1}{\alpha}\sqrt{\Omega_q} < 0$.

A somewhat amusing result is, however, that one can adjust the parameters c_1 and α such that $\Omega_q \simeq 0.73$ and $w_q < -1$ even for $m < 1$ (or $\varepsilon < -1$), in which case the universe would be decelerating (cf. Fig. 1).

For solving the system of equations (8)–(10), analytically, one should perhaps make one or more simplifying assumptions. It is worth noting that most of the radiation energy in the present universe is in the cosmic microwave background, which makes up a fraction of roughly 5×10^{-5} of the total density of the universe. For this reason, let us make the assumption that the matter is described by a pressureless (nonrelativistic) perfect fluid, i.e. $w_m \simeq 0$. We then get

$$\varepsilon = -\frac{\Omega_m'}{2\Omega_m} - \frac{3}{2} = -\frac{\Omega_r'}{2\Omega_r} - 2, \qquad \Omega_q = 1 - (1 + Ce^{\ln a})\Omega_r, \qquad \Omega_m = \Omega_r C e^{\ln a}. \tag{37}$$

The numerical constant C may be fixed using observational inputs. Ideally, $\Omega_q \simeq 0.73$ and $\Omega_m \simeq 0.27$ at the present epoch ($a \simeq 1$) imply that $C \simeq 5400$. The matter-radiation equality epoch, $\Omega_r \simeq \Omega_m$, then corresponds to the scale factor $a \simeq 1.85 \times 10^{-4} a_0$ (where a_0 is the present value of a). That means the universe may have experienced about 8.6 e-folds of expansion since the epoch of matter-radiation equality. This result is almost a model independent outcome, as long as $w_m \simeq 0$ holds during the matter dominance.

Let us choose the integration constant c_1 in (36) such that $\Omega_q \simeq 0.73$ at present, $a = a_0 = 1$. This yields

$$c_1 = 1.17 + \frac{1}{\alpha(1+\varepsilon)} \quad (\alpha < 0), \quad \text{or} \qquad c_1 = -1.17 + \frac{1}{\alpha(1+\varepsilon)} \quad (\alpha > 0). \tag{38}$$

By satisfying either of these conditions one gets $\Omega_q = 0.73$ at $a = 1$, for any value of α. Figures 2 and 3 show the evolution of density parameters Ω_r, Ω_m, Ω_q, and the equation of state w_q.

The model of dark energy in [12] possesses some distinct features as compared to a simpler model in [20]. Notably, due to the presence of the factor $e^{-\ln a}$ in Eq. (33), the dark energy equation of state parameter w_q does not behave, even in the limit $\Omega_q \to 0$, as that for a

[2] Or simply that $b_1 \ll a_{r,e}$ and $b_2 \ll a_{m,e}$, where $a_{r,e}$ and $a_{m,e}$ are the scale factors at the end of radiation and matter-dominated epochs.

[3] Unlike for the model in [20], Ω_q can be varying even deep into the matter-dominated universe ($tH = 2/3$) since $\eta H \neq$ const.

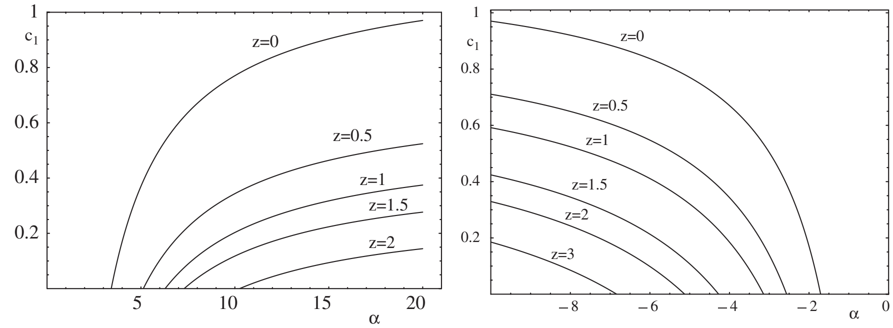

FIG. 1. The contour lines that give rise to dark energy density parameter $\Omega_q = 0.73$, with $m = 0.8$ (left plot) and $m = 2$ (right plot). z is the redshift parameter defined via $z = e^{-\ln a} - 1$.

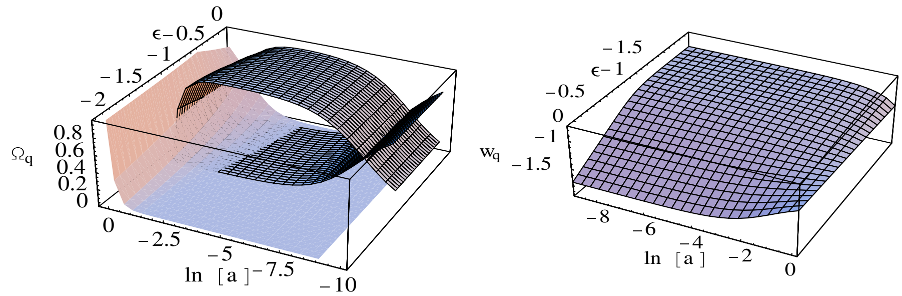

FIG. 2 (color online). The density parameters Ω_m, Ω_r, Ω_q (from top to bottom) and the dark energy EoS parameter w_q. We have taken $\alpha = -2.7$.

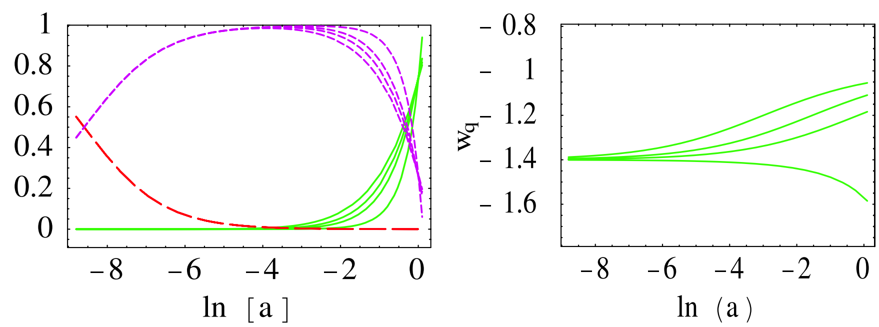

FIG. 3 (color online). The density parameters Ω_m (short-dashed, pink line), Ω_r (long-dashed, red line), and Ω_q (solid, green line) with $\varepsilon = -0.4$ and $|\alpha| = 1, 3, 5, 10$ (top to bottom for Ω_m and w_q, while opposite for Ω_q).

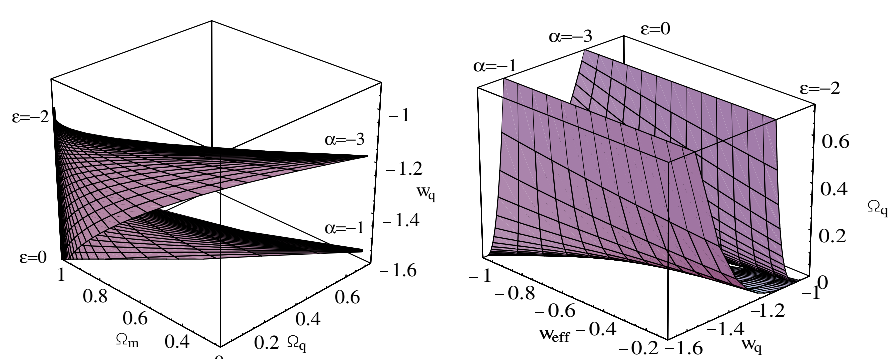

FIG. 4 (color online). A parametric 3D plot in the range $-4 \leq \ln a \leq 0$, where $\Omega_m + \Omega_q \simeq 1$, with $|\alpha| = 1, 3$. As seen in the plots, the dark energy equation of state w_q depends in the past on the acceleration parameter ε, but for $\Omega_q \gtrsim \Omega_m + \Omega_r$, it is highly dependent on the choice of the parameter α.

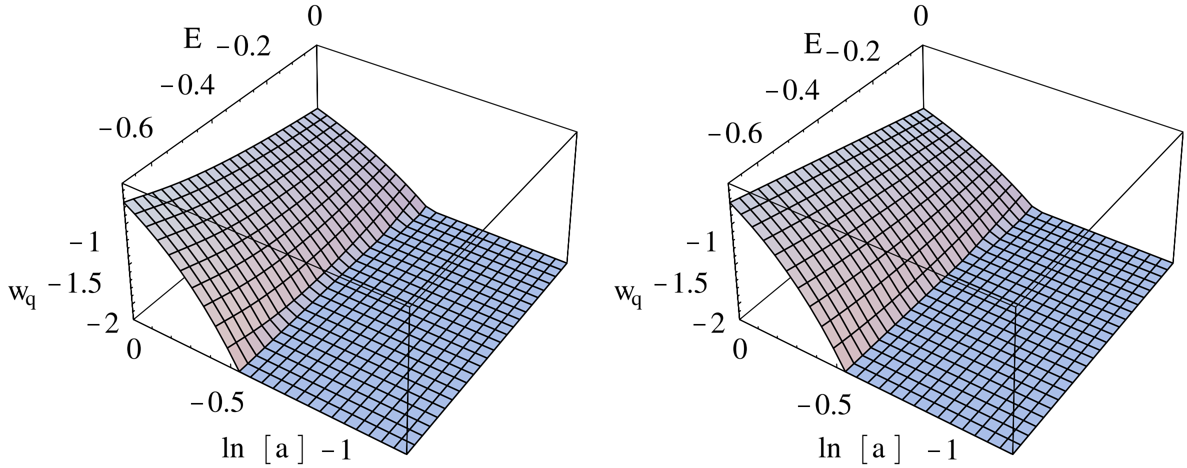

FIG. 5 (color online). The EoS parameter w_q as in Eq. (40) (left plot) and in Eq. (43) (right plot). We have taken here $|\alpha| = 2.7$ as suggested in [28].

cosmological constant term, for which $w_\Lambda = -1$; the EoS w_q rather depends on the acceleration parameter ε, as is clearly seen from Fig. 4. Anyhow, in the case $\tilde{Q} = 0$, there is no solution for which $\Omega_q \simeq 0.73$ and $w_q \sim -1$ unless $|\alpha| \to \infty$.

In order to get a cosmological evolution with $w_q \sim -1$, as required for the best-fit concordance model cosmology, one should perhaps consider the case $\tilde{Q} \neq 0$. A mechanism that works only for $\tilde{Q} = 0$ solves nobody's problem; it perhaps only represents our ignorance about a universal coupling between a fundamental scalar (or dark energy) field and the ordinary (baryonic and dark) matter.

V. INTERACTING DARK ENERGY, $\tilde{Q} \neq 0$

The cosmological observations have provided a strong evidence that the current expansion of the universe is accelerating. In the following discussion, we therefore assume that $\varepsilon > -1$. Especially, in the case $\varepsilon \simeq$ const, because of the constraint (24), the following particular solution

$$\frac{1}{\alpha}\sqrt{\Omega_q} = (1+\varepsilon)[\alpha c_1(1+\varepsilon)e^{\varepsilon \ln a} - \mathrm{e}^{-\ln a}]^{-1}, \qquad (39)$$

where c_1 is an integration constant, is also a viable solution to the system of equations (8)–(10), with $\tilde{Q} \neq 0$. However, as a notable difference, the dark energy equation of state is now given by

$$w_q = -1 + \frac{2}{3\alpha}\mathrm{e}^{-\ln a}\sqrt{\Omega_q} - \frac{\tilde{Q}}{3\Omega_q}. \qquad (40)$$

We shall normally take $\alpha c_1 < 0$, otherwise the quantity Ω_q diverges at some stage of evolution. As a consequence, the quantity $\alpha^{-1}\sqrt{\Omega_q}$ remains negative. Equation (40) then shows that it is possible to get $w_q \simeq -1$, given that $\tilde{Q} < 0$. This is a viable scenario.

To proceed analytically, let us assume that $w_m = 0$. Then, the coupling $\tilde{Q}$ is given by[4]

$$\tilde{Q} \equiv q'\alpha_q\Omega_m = \Omega_m' + 2\varepsilon\Omega_m + 3\Omega_m. \qquad (41)$$

One also notes that, with $w_m = 0$, the Friedmann equation $\Omega_{\mathrm{tot}} = 1$ gives rise to

$$\Omega_q = 1 - (1 + C\mathrm{e}^{\ln a})\Omega_r, \qquad \Omega_m = \Omega_r C\mathrm{e}^{\ln a}, \qquad (42)$$

where C is an integration constant. From Eq. (11), we then get

$$w_q = \frac{2\varepsilon + 3}{3\Omega_q}, \qquad (43)$$

which is a valid approximation as long as $\Omega_r \ll 1$ and $w_m \simeq 0$. As one would expect, the results coming from the above two expressions for w_q, i.e. Eqs. (40) and (43), agree at low redshifts, that is, for $\ln a \lesssim 0$, see Fig. 5. This agreement is better for $|\alpha| \gg 1$, in which case Ω_q overtakes Ω_m only at a slow rate.

One of the undesirable features of the model in [12] is that, as we go to higher redshifts, $\ln a \ll 0$, the coupling $|\tilde{Q}|$ decreases at a slower rate than the dark energy density fraction Ω_q, thereby leading to a divergent $w_q \equiv p_q/\rho_q$ and/or a negative value for the squared speed of sound, $v^2 \equiv dp_q/d\rho_q$.[5] A possible resolution of this problem is to allow a much larger value for α in the past, i.e. $|\alpha| \gg 1$.[6]

[4]The form of matter-scalar coupling that we consider in this paper precisely follows a canonical kinetic structure of the theory determined by the actions (1) and (2). A different functional form for the scalar-matter coupling used, for example, in [25], namely $\tilde{Q} \propto \Omega_q$, may lead to a somewhat different result than found here.

[5]This result is perhaps consistent with the findings in a recent paper [26].

[6]This is opposite of that in the agegraphic dark energy model [20], since $\eta \to -\infty$ in the infinite past, whereas $t \to 0$ in the early universe.

That means, a dark energy model with some fixed value of α can hardly explain most cosmological properties of our universe that we observe. This is analogous to a situation in a standard scalar field cosmology with a simple exponential potential $V \propto e^{-\lambda(\phi/m_P)}$, having a constant slope parameter, $\lambda = \text{const}$ (see, e.g. [27] for a related discussion).

While the assumption of power-law expansion of the scale factor can be relaxed, e.g., during a transition from matter dominance to dark energy dominance, we do not expect it to greatly alter our results.

VI. THE HOLOGRAPHIC DARK ENERGY AT A GLANCE

Some of the above difficulties may not arise in the model of "holographic" dark energy proposed by Li and others. In this model, the vacuum energy density is given by

$$\rho_q \equiv \rho_\Lambda = \frac{3c^2 m_P^2}{R_h^2}, \tag{44}$$

where

$$R_h \equiv a\int_t^\infty \frac{dt_*}{a(t_*)} = a\int_x^\infty \frac{dx}{Ha} = \pm\frac{1}{H}\frac{c}{\sqrt{\Omega_q}} \tag{45}$$

is the proper size of the future event horizon and $x \equiv \ln a$. The last term in (45) follows from the definition $\Omega_q \equiv c^2/H^2R_h^2$. The analogue of the constraint equation (29) is

$$\Omega_q' + 2\varepsilon\Omega_q + 2\Omega_q\left(1 - \frac{1}{R_h H}\right) = 0. \tag{46}$$

For $c > 0$, one takes the positive sign in Eq. (45), so that $R_h H > 0$.

In the absence of interaction between the q-field and matter, so $\tilde{Q} = 0$, from Eqs. (9) and (46), we find

$$w_q = -\frac{1}{3} - \frac{2}{3}\frac{1}{R_h H} = -\frac{1}{3} - \frac{2}{3c}\sqrt{\Omega_q}. \tag{47}$$

In particular, for the power-law expansion $a = [c_0 t + t_1]^m$, the explicit solution is given by

$$\frac{1}{\sqrt{\Omega_q}} = \begin{cases} \frac{m}{(m-1)c} + c_1 a^{(m-1)/m} & (m \neq 1), \\ -\frac{1}{c}\ln a + c_2 & (m = 1), \end{cases} \tag{48}$$

where c_1 and c_2 are integration constants. Therefore, by choosing

$$c_1 \gg \begin{cases} \frac{1}{c} a_{r,e} & (a \propto t^{1/2},\ \text{RD}), \\ \frac{2}{c} a_{m,e}^{1/2} & (a \propto t^{2/3},\ \text{MD}), \end{cases} \tag{49}$$

where $a_{r,e}$ and $a_{m,e}$ are the scale factors at the end of radiation and matter-dominated epochs, one finds $\Omega_q \propto a$ (MD) and $\Omega_q \propto a^2$ (RD). This then implies that during both the MD and RD epochs, the holographic dark energy density scales as $\rho_q \propto 1/a^2$. It is thus conceivable that the dark energy density overtakes both the radiation and matter energy densities at some stage of cosmic evolution since $\rho_r \propto 1/a^4$ and $\rho_m \propto 1/a^3$. The nucleosynthesis bound $\Omega_q(1\ \text{MeV}) < 0.1$ may also be satisfied for almost any value of c, although $c < 1.17$ may be required to get $w_q < -0.82$ with the input $\Omega_q \simeq 0.73$ at present. The constraint $c \geq \sqrt{\Omega_q}$ may also be imposed by demanding that the de Sitter entropy, $S \equiv A/4G_N = \pi m_P^2 R_h^2$ does not decrease, that is $\dot{R}_h = -1 + c/\sqrt{\Omega_q} > 0$. A detailed analysis with $\tilde{Q} \neq 0$ appears elsewhere.

VII. DISCUSSIONS

Dynamical dark energy models with the vacuum energy density $\rho_q \propto 1/t^2$ may lead to some undesirable features, especially, during the matter and radiation-dominated epochs, since $\rho_q \propto 1/a^3$ (MD) and $\rho_q \propto 1/a^4$ (RD). This rules out, for instance, a transition from matter dominance to dark energy dominance, unless the late-time acceleration arises due to some other dynamics, e.g., a nontrivial growing interaction between the q-field and matter. This situation is improved by assuming that $\rho_q \propto 1/(t + t_1)^2$, with $t_1 \gtrsim t_{\text{present}} \equiv t_0$, as we discussed above.

The model of conformal dark energy proposed by Cai and Wei [12] may be consistent with quantum kinematics, in the sense that the uncertainty relation (or the second law of thermodynamics, in an equivalent form) is obeyed. Also, the model does not suffer from the problem of causality, unlike the holographic dark energy model, with $c < 1$. Nevertheless, the conformal dark energy model in [12] has some undesirable features, such as, in the presence of a nontrivial coupling between the q-field and ordinary matter, the dark energy equation of state parameter w_q may diverge as higher redshifts, thereby leading to a negative value for the squared speed of sound, $v^2 \equiv dp_q/d\rho_q$. The main reason for this odd behavior is that the dark energy density fraction Ω_q varies (actually decreases) too fast in the past, unless $|\alpha|$ takes a value significantly larger than unity, which is, however, not compatible with the epoch of matter dominance, where $\Omega_q < 0.2$.

The other obvious drawback of the conformal dark energy model is that it only provides a kinematic approach to dark energy, by outlining a possible time decay of dark energy component, but the model does not explain much about the dynamics, that is, the origin or nature of dark energy. Both the conformal and holographic dark energy models are interesting in the sense that they satisfy some holographic entropy bounds (or laws of thermodynamics, in equivalent forms). But they still raise some other important concerns: Why quantum corrections to the vacuum energy contribute to the presentday dark energy density

($\sim 10^{-12}$ eV^4) dominantly, whereas many known contributions to ρ_Λ, including the classical effects of quantum fields, do not? and why it is comparable to the energy density of matter today?

The holographic dark energy model is perhaps a step forward among the recent attempts in probing a time variation of dark energy within the framework of quantum gravity, even though the model has some pitfalls, such as, a semiclassical instability due to a negative value for the squared of sound speed.

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