Acoustic perturbations on steady spherical accretion in Schwarzschild geometry

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The stationary background flow in the spherically symmetric infall of a compressible fluid, coupled to the space-time defined by the static Schwarzschild metric, has been subjected to linearized acoustic perturbations. The perturbative procedure is based on the continuity condition and it shows that the coupling of the flow with the geometry of space-time brings about greater stability for the flow, to the extent that the amplitude of the perturbation, treated as a standing wave, decays in time, as opposed to the amplitude remaining constant in the Newtonian limit. In qualitative terms this situation simulates the effect of a dissipative mechanism in the classical Bondi accretion flow, defined in the Newtonian construct of space and time. As a result of this approach it becomes impossible to define an acoustic metric for a conserved spherically symmetric flow, described within the framework of Schwarzschild geometry. In keeping with this view, the perturbation, considered separately as a high-frequency traveling wave, also has its amplitude reduced.

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I. INTRODUCTION

The spherically symmetric model of astrophysical accretion continues to enjoy an abiding appeal among researchers in accretion astrophysics, starting with the seminal paper written by Bondi [1] more than half a century ago. The basic simplicity of this model notwithstanding, it is actually quite appropriate for many realistic aspects of accretion processes, and certainly more than anything else, this model also allows a clear insight to be had into the related physics, much of which is often of quite an involved nature. As a result the spherically symmetric model is frequently the starting point from where it becomes possible to devise theoretical models of increasing physical complexity for accretion processes.

While general questions related to astrophysical accretion have been addressed and studied from various perspectives all along, it was not too long before Bondi's original treatment, carried out within the Newtonian construct of space and time, was extended to embrace a general relativistic description of spherically symmetric accretion. An early work in this regard was reported by Michel [2], which was followed by a spate of later works, some of which took up various issues ranging from the stability of solutions to their exact nature and critical aspects [3–9].

Stability of spherically symmetric accretion has been studied for a long time and in various possible ways [6,10–15]. With especial regard to the stability of spherical accretion on to a black hole, Moncrief's study [6] has shown no evidence of the development of any instability under the influence of a linearized potential perturbation (sound waves) on the standing background flow. The treatment presented in this paper returns to the same general theme, but the methods applied here, and the motivation that has prompted them, are different. First of all, as opposed to Moncrief's approach of perturbing a scalar potential, whose gradient is prescribed to be the velocity of the ideal fluid, in this paper the perturbation prescription that has been adopted is centered around the continuity condition. For the accreting system, the stationary solution of the continuity equation gives a first integral, which, within a constant geometric factor, is actually the matter inflow rate. The full description of the flow will imply that complete solutions of two coupled fields—the velocity field and the density field—will have to be obtained. Both these fields are connected to the matter accretion rate, and, therefore, perturbing the accretion rate about its constant stationary value will lead to a wave equation for a single perturbed field that will also convey enough useful information on the stability of both the velocity and the density fields. The perturbation itself will physically be an acoustic wave, following a Eulerian scheme that was originally employed by Petterson *et al.* [11] (and later also by Theuns and David [12]), in their study of the behavior of a traveling sound wave in a nonrelativistic spherically symmetric accreting system.

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While this is the general procedure that has been followed, the primary objective of the whole exercise has been to see whether or not, through this perturbative technique, it should be possible to establish an acoustic geometry for a flow that is, at the same time, described under fully general relativistic conditions. In recent years fluid analogue gravity has been a subject that has, extending over diverse types of fluid flows, called much attention upon itself [15–31]. In fluid dynamical processes a critical point is the point where the speed of the bulk motion matches the speed of information propagation through the fluid. Depending on the direction of the flow, information of any event occurring in either the supercritical region of the fluid (where the bulk flow is faster than the speed with which any information can travel) or its subcritical region (where information propagation overrides the bulk flow) will not percolate into the other region, through a surface determined by the critical condition. For an ideal fluid, therefore, the critical condition defines a barrier, which can be viewed as the fluid analogue of the event horizon of either a black hole or a white hole (the latter being simply time-reversed solutions of a black hole inflow), according to the direction in which the flow proceeds. It can be easily appreciated that for spherically symmetric accretion solutions the sonic surface defines the event horizon of an acoustic black hole. This feature has been studied extensively and understood well by now in both the Newtonian framework [15,27,30] and in the general relativistic framework [22,28]. The latter context is very interesting because it combines the metric properties of curved space-time with similar properties of the fluid itself flowing in the same space-time.

This work purports to investigate the same analogue geometric properties of the flow, described in the Schwarzschild metric. A study along these very lines has been reported earlier by Bilić [20], whose work, however, is based on the usual practice of studying a perturbation on a scalar potential function. In contrast, as it must be emphasized once again, the present work approaches the whole question from the viewpoint of the continuity condition in the flow. This difference of approach turns out to be significant because the result deriving from the latter line of attack is negative, quite unlike what has been shown through the method used by Bilić [20]. This is rather curious because in the nonrelativistic framework both paths can be shown to lead to the same end [15,27,30]. It has been seen here that the coupling of the flow with the spherically symmetric geometry of space-time acts like a dissipative effect and breaks down the Lorentzian invariance that should be necessary to define an analogue metric. The invariance is restored in the Newtonian limit. By way of comparison and by following the same mathematical prescription, one could invoke a similar feature that arises because of viscosity in shallow-layer incompressible fluid flows where information propagates as gravity waves [21]. When viscosity is made to vanish, it becomes possible to establish an analogue black hole (or white hole) model for the flow with the equivalent event horizon being set down by the condition of the bulk flow speed matching the speed of gravity waves [24,26,29,31]. And so it is that when the conserved spherically symmetric accretion flow is decoupled from the space-time geometry in the Newtonian limit, one could easily define a metric for an acoustic event horizon, which would actually coincide with the sonic horizon of the flow.

This whole aspect of the flow is manifested in its stability as well. It has been seen that the stationary inflow solutions defined in the Schwarzschild metric are more stable under the effects of a linearized perturbation than what they should be in the Newtonian construct. Analogously mapped onto the properties of the Newtonian limit, this is like the effect of viscous dissipation lending greater stability to the flow than what it would have been for a perfect fluid.

Finally, as an interesting aside, it has been shown that the steady solution of the relativistic flow leads to the derivation of two standard pseudo-Newtonian potentials which are often applied to mimic general relativistic effects in the Newtonian framework of space and time.

II. GENERAL RELATIVISTIC EQUATIONS FOR SPHERICALLY SYMMETRIC ACCRETION

A complete description of a conserved spherically symmetric flow in Schwarzschild geometry will require some indispensible mathematical relations. The first of these is the equation for the spherically symmetric line element, which, in units of $c = 1$ and with the radial distance scaled by $2GM$, can be set down in one of its usually recognizable forms as

$$\mathrm{d}s^2 = -e^{-\lambda}\mathrm{d}t^2 + e^{\lambda}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{1}$$

where the metric coefficient $e^{-\lambda}$ can, in general, be understood to have a functional dependence on both r and t. For notational convenience it will henceforth be expressed as $e^{-\lambda} \equiv f(r, t)$.

This is to be followed by a relation for the energy-momentum tensor of a perfect fluid, given by [32]

$$T^{\mu\nu} = (\epsilon + p)v^{\mu}v^{\nu} + pg^{\mu\nu} \tag{2}$$

in which p is the pressure, ϵ is the energy density, and v^{μ} is the fluid four-velocity, which obeys the relation $v^{\mu}v_{\mu} = -1$. In tensorial notation the continuity condition can likewise be expressed as [33]

$$(\rho v^{\mu})_{;\mu} = 0 \tag{3}$$

with ρ being the particle number density of the perfect fluid.

Some algebra following the use of the two conditions given by Eqs. (1) and (3) will ultimately lead to a modified and explicit form of the continuity equation as

$$\frac{\partial}{\partial t}\left(\frac{\rho\sqrt{f+v^2}}{f}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho v r^2)=0. \tag{4}$$

This gives one relation connecting the radial flow velocity v and the local density ρ to each other. To solve for each of these two fields explicitly, it should be necessary to obtain another such relation. This can be derived from the momentum balance condition. To do so, it should be apt first to apply the requirement of energy-momentum conservation, $T^{\mu\nu}{}_{;\nu}=0$, on Eq. (2). This will give

$$(\epsilon+p)(v^{\mu}{}_{;\nu}v^{\mu}+v^{\nu}v^{\nu}{}_{;\nu})+(\epsilon+p)_{,\nu}v^{\mu}v^{\nu}+g^{\mu\nu}p_{,\nu}=0 \tag{5}$$

with ϵ having to be defined separately through the thermodynamic relation [33],

$$\frac{d\epsilon}{d\rho}=\frac{\epsilon+p}{\rho}+\rho T\frac{dS}{d\rho} \tag{6}$$

in which T is the temperature and S is the specific entropy.

A further definition that is necessary is that of the speed of sound, a, which, under isentropic conditions, is expressed in terms of the thermodynamic quantities ϵ and p as [33]

$$a^2=\frac{\partial p}{\partial\epsilon}\bigg|_S. \tag{7}$$

Making use of the foregoing definition, along with the condition of constant entropy in Eq. (6), achieved by setting $dS=0$, and invoking the spherically symmetric line element from Eq. (1) once again, it should be a straightforward algebraic exercise to recast Eq. (5) in the form

$$\frac{\sqrt{f+v^2}}{f}\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}+\frac{1}{2}\frac{\partial f}{\partial r}-\frac{v\sqrt{f+v^2}}{f^2}\frac{\partial f}{\partial t}+\frac{a^2}{\rho}\left[\frac{v\sqrt{f+v^2}}{f}\frac{\partial\rho}{\partial t}+(f+v^2)\frac{\partial\rho}{\partial r}\right]=0 \tag{8}$$

which, incidentally, bears a close resemblance to a similar equation derived by Misner and Sharp [32] in their study of gravitational collapse.

The pressure p is connected to the density ρ through a polytropic equation of state, $p=k\rho^{\gamma}$, with k and γ being constants, the latter being the polytropic exponent. With the help of this equation of state, it becomes easy to show from Eqs. (6) and (7) that, when the fluid is isentropic, there is a relation between a and ρ that can be written as

$$\rho=\left[\frac{a^2}{\gamma k(1-na^2)}\right]^n \tag{9}$$

in which $n=(\gamma-1)^{-1}$, going by the usual definition of the polytropic index [34]. This connection between a and ρ now makes it possible to see Eq. (8) as the second relation, after Eq. (4), that can be written entirely and explicitly in terms of v and ρ. And so with the help of Eqs. (4), (8), and (9), it should now be possible to establish a complete quantitative description of the spherically symmetric flow defined within the Schwarzschild metric.

The stationary solutions of Eqs. (4) and (8) are written as

$$4\pi\bar{\mu}\rho v r^2=\dot{m} \tag{10}$$

and

$$\frac{1}{f+v^2}\frac{d}{dr}(f+v^2)=-\frac{2a^2}{\rho}\frac{d\rho}{dr}, \tag{11}$$

respectively [33]. In the former solution, the integration constant $\dot{m}$ is physically the matter flow rate, and $\bar{\mu}$ is the average density of particles in the flowing gas. While this solution is a direct first integral of the stationary continuity equation, the latter solution is not an integral solution. Nevertheless, it has certain interesting consequences which become very apparent in the nonrelativistic limit. In this limit, both $v^2\ll 1$ and $a^2\ll 1$, while $r\gg 2GM$.

Now the static spherically symmetric metric element will be given explicitly in Eq. (1) by

$$e^{-\lambda}=f(r)=1-\frac{2GM}{r}. \tag{12}$$

Therefore, in the nonrelativistic limit the leading order solution that can be obtained from Eq. (11) is

$$\frac{1}{2}\frac{d}{dr}(v^2)+\phi'(r)+\frac{a^2}{\rho}\frac{d\rho}{dr}=0 \tag{13}$$

in which $\phi'=f'/2f$. What Eq. (13) gives is the stationary Euler equation in the nonrelativistic limit, with ϕ being an effective potential driving the stationary flow. At large distances ϕ behaves like the classical Newtonian potential, but on length scales comparable to $2GM$, there will be a deviation from the Newtonian behavior.

Frequently in astrophysics it becomes convenient to dispense with the full mathematical rigor of general relativity and instead, in a Newtonian framework, employ a prescription in which general relativistic effects could be represented by an effective potential. In this "pseudo-Newtonian" approach, many such effective potentials have been suggested according to various specific requirements, of which two have been proposed by Artemova *et al.* [8,35]. Interestingly enough, the functional behavior of these two potentials could be derived from the form of ϕ implied in Eq. (13). Written together, they are

$$\phi=\frac{1}{2}\ln\left(1-\frac{2GM}{r}\right)\simeq -1+\sqrt{1-\frac{2GM}{r}}, \tag{14}$$

with the latter potential being seen to actually be a special case of the former. Both, of course, converge to the Newtonian limit on large length scales.

While all these results can be derived from the stationary background flow, to have any understanding of their stability under the effect of a linearized time-dependent perturbation, it will be necessary to go back to the two dynamic equations of the flow given by Eqs. (4) and (8).

III. LINEARIZED PERTURBATIONS ON STATIONARY SOLUTIONS

For the purpose of carrying out the stability analysis of stationary solutions, a standard assumption that is being imposed is that the metric is static, i.e. $\partial f/\partial t = 0$. This assumption is nothing unusual as far as accretion processes are concerned, in which the flow is driven by the gravitational field of an external accretor, with the self-gravity of the flow being ignored. One might account for the energy-momentum of the relativistic flow, but this is negligible in any likely accretion-related application. This will imply that the gravitational field will be unchanging in time [1,2], and so $f \equiv f(r)$, whose full expression can be read from Eq. (12).

The perturbation scheme itself will be set down as $v(r, t) = v_0(r) + v'(r, t)$ and $\rho(r, t) = \rho_0(r) + \rho'(r, t)$ with the subscript "0" indicating stationary values of v and ρ, and the primes indicating small time-dependent perturbations about the stationary values. At this point, following the method of Petterson *et al.* [11] and Theuns and David [12], it will be expedient for the perturbative analysis to define a new variable, $\psi = \rho v r^2$, which, as it is very obvious from Eq. (4), is closely associated with the matter flow rate, and whose stationary value, ψ_0, as it can be seen from Eq. (10), is a constant of the motion. The first-order fluctuations about this constant stationary value can be expressed as

$$\psi' = (v_0\rho' + \rho_0 v')r^2. \tag{15}$$

Another such relation connecting v', ρ', and ψ' can be derived from Eq. (4), and it will read as

$$\frac{\sqrt{f + v_0^2}}{f}\frac{\partial\rho'}{\partial t} + \frac{\rho_0 v_0}{f\sqrt{f + v_0^2}}\frac{\partial v'}{\partial t} = -\frac{1}{r^2}\frac{\partial\psi'}{\partial r}. \tag{16}$$

With the help of the two foregoing equations, it shall now be possible to express both ρ' and v' solely in terms of ψ'. These will be given as

$$\frac{\partial\rho'}{\partial t} = -\frac{1}{r^2}\left(\frac{v_0}{f}\frac{\partial\psi'}{\partial t} + \sqrt{f + v_0^2}\frac{\partial\psi'}{\partial r}\right) \tag{17}$$

and

$$\frac{\partial v'}{\partial t} = \frac{\sqrt{f + v_0^2}}{\rho_0 r^2}\left(\frac{\sqrt{f + v_0^2}}{f}\frac{\partial\psi'}{\partial t} + v_0\frac{\partial\psi'}{\partial r}\right), \tag{18}$$

respectively.

The speed of sound a is connected to ρ through Eq. (9), and so the perturbation in ρ has to affect a as well. This is to be written as $a^2 = a_0^2 + (\mathrm{d}a_0^2/\mathrm{d}\rho_0)\rho'$. Once this has been done, the linearized first-order fluctuations about the stationary momentum balance condition can be extracted from Eq. (8) and written as

$$\frac{\sqrt{f + v_0^2}}{f}\frac{\partial v'}{\partial t} + \frac{\partial}{\partial r}(v_0 v') + \frac{v_0\sqrt{f + v_0^2}}{f}\frac{a_0^2}{\rho_0}\frac{\partial\rho'}{\partial t} + 2v_0\frac{\partial\rho_0}{\partial r}\frac{a_0^2}{\rho_0}v' + (f + v_0^2)\frac{\partial}{\partial r}\left(\frac{a_0^2}{\rho_0}\rho'\right) = 0. \tag{19}$$

Partially differentiating Eq. (19) with respect to time, and making use of Eqs. (17) and (18) to eliminate ρ' and v', respectively, will ultimately deliver a linearized equation of motion for ψ' as

$$\frac{\partial}{\partial t}\left(\rho_0 h^{tt}\frac{\partial\psi'}{\partial t} + \rho_0 h^{tr}\frac{\partial\psi'}{\partial r}\right) + \frac{\partial}{\partial r}\left(\rho_0 h^{rt}\frac{\partial\psi'}{\partial t} + \rho_0 h^{rr}\frac{\partial\psi'}{\partial r}\right) = (1 - 2a_0^2)\frac{\mathrm{d}\rho_0}{\mathrm{d}r}\left(h^{rt}\frac{\partial\psi'}{\partial t} + h^{rr}\frac{\partial\psi'}{\partial r}\right) \tag{20}$$

with the coefficients $h^{\alpha\beta}$ having to be read from

$$h^{tt} = \frac{v_0\sqrt{f + v_0^2}}{f^2}(f + v_0^2 - v_0^2 a_0^2),$$

$$h^{tr} = h^{rt} = \frac{v_0^2(f + v_0^2)}{f}(1 - a_0^2)$$

and

$$h^{rr} = v_0\sqrt{f + v_0^2}[v_0^2 - (f + v_0^2)a_0^2].$$

At this stage it would be very instructive to examine the features of Eq. (20) in the nonrelativistic limit, where both v_0^2 and a_0^2 are vanishingly small compared to unity while f itself assumes the value of unity. As a consequence, in the nonrelativistic limit it will become eminently possible to reduce Eq. (20) to a compact form given by

$$\partial_\alpha(h^{\alpha\beta}\partial_\beta\psi') = 0 \tag{21}$$

with the Greek indices α and β running over t and r, as they did for the fully relativistic case earlier.

In Lorentzian geometry the d'Alembertian for a scalar field in curved space is given in terms of the metric $g_{\mu\nu}$ by [19]

$$\Delta\varphi \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) \tag{22}$$

with $g^{\mu\nu}$ being the inverse of the matrix implied by $g_{\mu\nu}$. Comparing Eq. (21) with Eq. (22), it would be tempting to look for an exact equivalence between $h^{\alpha\beta}$ (in the nonrelativistic limit) and $\sqrt{-g}g^{\mu\nu}$. This, however, cannot be done in a general sense. What can be appreciated, nevertheless, is that Eq. (21) gives a relation for ψ' which is of the type given by Eq. (22). The metrical part of Eq. (21), as given by the values of $h^{\alpha\beta}$, may then be extracted, and its inverse will incorporate the notion of a sonic horizon of an acoustic black hole when $v_0^2 = a_0^2$. This point of view has some features similar to the metric of a wave equation for a scalar field in curved space-time, obtained through a somewhat different approach, in which the velocity of an irrotational, inviscid, and barotropic fluid flow is first represented as the gradient of a scalar function, and then a perturbation is imposed on this scalar function [19,23].

While all this similarity is undoubtedly pleasing to note in the nonrelativistic limit, it is also quite obvious from Eq. (20) that in the fully general relativistic treatment there is a breakdown of the symmetry that leads to the devising of an analogue gravity model for the fluid flow. This is at striking variance with the conclusions arrived at by Bilić [20]. It is not very difficult to discern that the difference arises because of the way the perturbative studies have been prescribed in the two cases—the scalar potential approach of Bilić [20], and the continuity equation approach in the present case.

Having said so, it must nevertheless be pointed out that in some respects at least the fluid analogue approach is not entirely lost even for the general relativistic flow being studied here. Speaking by analogy, in the shallow-layer flow of a perfect liquid (without any viscosity) it has been shown that an analogue black hole or white hole model is very much a fact [21]. To expatiate on this particular issue, the white hole model is indeed quite an apposite model for the long-standing fluid dynamical problem of the hydraulic jump. This is a physical system in which a shallow layer of an outflowing liquid increases its flow height abruptly. Many works have connected this phenomenon to viscosity in the flow [24,31,36,37]. Viscous dissipation in the flow, on the other hand, adversely affects the invariance that makes the construction of an analogue model possible, even as it helps the hydraulic jump phenomenon itself to happen [24,31]. And so in a like fashion, while it is easy to recognize the notion of fluid analogue gravity for spherically symmetric accretion in the nonrelativistic situation, the coupling of the geometry of space-time with the perturbed field in the fully general relativistic scenario acts in the manner of a dissipative effect (just as viscosity does to a putatively inviscid shallow layer of flowing liquid). At least through the continuity equation approach, this precludes any hope of building an acoustic black hole model within the general relativistic flow, but, at the same time, this has a more favorable role to play as regards the stability of the stationary solutions.

IV. STABILITY ANALYSIS: STANDING WAVES

In treating the perturbation as a standing wave, it will be essential to identify proper boundary conditions to constrain the wave at two spatial points. Between these two points the properties of the standing wave could be studied. One boundary condition for the perturbation is conveniently fixed at the outer boundary of the stationary flow itself, where any solution naturally decays out. So should any perturbation imposed on it.

In determining an appropriate inner boundary condition, on the other hand, one encounters much greater difficulties. The nature of the accretor itself has an influential role to play in this matter. If it is a black hole, then the infalling matter has to cross the event horizon at the maximum possible rate [38], and the only feasible solution should be a transonic one, which will smoothly pass through a singular point in the flow [39]. On the other hand, if the accretor is a compact object with a physical surface, then the inner boundary condition becomes subject to many complications, which in turn will leave its imprint on the character of an inflow solution in the vicinity of the stellar boundary. The solution could either be transonic or shocked (discontinuous) subsonic or continuously subsonic. These distinct aspects have been discussed at length by Petterson *et al.* [11] and Theuns and David [12]. In this context Moncrief [6] has pointed out that, as long as the stellar surface is enclosed within the sonic surface, his perturbative arguments hold both for a black hole and a compact star.

The standing wave analysis that has been pursued here will, of necessity, require the background solution to be continuous everywhere and globally well behaved. Besides this, the wave will also have to die out at the two chosen boundaries. Now the only solutions that will meet all these requirements in a general sense are the purely subsonic solutions. While these may not entirely be representative of the precise manner of infall in the general relativistic scenario, a mathematical study of their stability will reveal the true extent of the influence that the coupling of the flow with the geometry of space-time will have *vis-à-vis* what it is in the Newtonian flat space-time limit. Stability of subsonic flows in the latter situation has been studied thoroughly by Petterson *et al.* [11] and Theuns and David [12].

The solution $\psi'(r,t) = p(r)\exp(-i\omega t)$ is to be applied in Eq. (20) and the resulting expression multiplied throughout by p. This will lead to

$$h^{tt}p^2\omega^2 + i\left\{\frac{d}{dr}(h^{tr}p^2) - h^{rt}p^2\frac{d}{dr}[\ln(f+v_0^2)]\right\}\omega + \frac{h^{rr}}{f+v_0^2}\frac{dp}{dr}\frac{d}{dr}[p(f+v_0^2)] - \frac{d}{dr}\left(h^{rr}p\frac{dp}{dr}\right) = 0, \quad (23)$$

which will then have to be integrated over the entire spatial range within which the standing wave is continuously distributed. At the two boundaries of this range the amplitude of the wave is required to vanish. So the "surface" terms obtained from integrating Eq. (23) will also have to vanish at the boundaries. This will leave a residual quadratic equation which will be a dispersion relation for ω. This relation will be in the form

$$A\omega^2 - 2iB\omega + C = 0 \quad (24)$$

with the three coefficients above being read as

$$A = \int h^{tt} p^2 \mathrm{d}r, \qquad B = \int \frac{h^{rt} p^2}{2} \frac{\mathrm{d}}{\mathrm{d}r}[\ln(f + v_0^2)]\mathrm{d}r$$

and

$$C = \int \frac{h^{rr}}{f + v_0^2} \frac{\mathrm{d}p}{\mathrm{d}r} \frac{\mathrm{d}}{\mathrm{d}r}[p(f + v_0^2)]\mathrm{d}r,$$

respectively.

Under the condition that for inflow solutions $v < 0$ and $(\mathrm{d}\rho_0/\mathrm{d}r) < 0$, it shall be easy to verify that $(B/A) < 0$ by referring to Eq. (11). Therefore, the discriminant of the solution of Eq. (24) will hold the key regarding the stability of the standing wave. Once again, for subsonic solutions, it can be argued that $(C/A) < 0$. And so, if $|C/A| > (B/A)^2$, it will imply that the time-dependent part of the standing wave, given by $\exp(\mathrm{i}\omega t)$, will have an oscillatory nature, with the amplitude of the oscillation being damped in time. Hence the background solution will be stable. If, on the other hand, $|C/A| < (B/A)^2$, then there will be two real roots of ω, both negative, indicating that the perturbation will be overdamped. So, one way or the other, the amplitude of the standing wave will be damped, lending stability to the stationary background solution in strong measure.

This is a rather intriguing state of affairs indeed. The decay of the amplitude of the standing wave would imply that something in the nature of a dissipative effect is active in what is otherwise a conservative system. For a conserved flow in the Newtonian regime, Petterson *et al.* [11] have shown that the perturbation will have a constant amplitude. Any decay of the amplitude of the standing wave could only be reproduced when one accounts for viscosity in the flow [13]. Coming back to the conserved general relativistic case, the only possible explanation for the decaying behavior can be that the coupling of the flow with the geometry of space-time acts in the manner of an "effective" dissipation. And, consistent with this line of thinking, it can also be shown that in the Newtonian limit one does indeed regain the expected constancy of the amplitude of the standing waves.

V. STABILITY ANALYSIS: TRAVELING WAVES

The manner in which the stability of the flow is influenced by its coupling with the space-time metric could also be examined by fashioning the perturbation to be a high-frequency traveling wave. A comparison could then be made with the corresponding analysis carried out in the Newtonian structure of space and time by Petterson *et al.* [11] who argued that the traveling waves could cause a growth in the fluctuations on the flow rate, but would not drive the background flow from it stationary profile. To this extent the stability of the flow should be preserved.

High-frequency traveling waves are to be first defined precisely in the present context by the fact that their wavelength should be much smaller than the Schwarzschild radius of the black hole. This will imply that the frequency ω should be correspondingly large. With this restriction on ω, the spatial part of the perturbation can then be prescribed in terms of a power series as

$$p_\omega(r) = \exp\left[\sum_{l=-1}^{\infty} \omega^{-l} k_l(r)\right], \tag{25}$$

and this is then to be applied to a slightly modified rendering of Eq. (23) which goes as

$$h^{rr}\frac{\mathrm{d}^2 p}{\mathrm{d}r^2} + \left\{\frac{\mathrm{d}h^{rr}}{\mathrm{d}r} - 2\mathrm{i}\omega h^{tr} - h^{rr}\frac{\mathrm{d}}{\mathrm{d}r}[\ln(f + v_0^2)]\right\}\frac{\mathrm{d}p}{\mathrm{d}r} - \left\{\omega^2 h^{tt} + \mathrm{i}\omega\frac{\mathrm{d}h^{rt}}{\mathrm{d}r} - \mathrm{i}\omega h^{rt}\frac{\mathrm{d}}{\mathrm{d}r}[\ln(f + v_0^2)]\right\}p = 0. \tag{26}$$

From the result of this extended algebraic exercise, all coefficients of ω^2 are to be collected and their sum is to be set to zero. This will give a solution for k_{-1}, which will look like

$$k_{-1} = \mathrm{i}\int (h^{rr})^{-1}[h^{tr} \pm \sqrt{(h^{tr})^2 - h^{rr}h^{tt}}]\mathrm{d}r. \tag{27}$$

Similarly summing up all the coefficients of ω to be zero, and applying the value of k_{-1}, as Eq. (27) gives it, will deliver a solution for k_0 as

$$k_0 = \ln\{(f + v_0^2)[\sqrt{(h^{tr})^2 - h^{rr}h^{tt}}]^{-1}\}^{1/2}. \tag{28}$$

Likewise, the solution of k_1 could be found by setting the sum of the coefficients of ω^0 to be zero. In terms of k_{-1} and k_0 this can be expressed as

$$2\left(h^{rr}\frac{\mathrm{d}k_{-1}}{\mathrm{d}r} - \mathrm{i}h^{tr}\right)\frac{\mathrm{d}k_1}{\mathrm{d}r} + \frac{\mathrm{d}}{\mathrm{d}r}\left(h^{rr}\frac{\mathrm{d}k_0}{\mathrm{d}r}\right) + h^{rr}\frac{\mathrm{d}k_0}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r}[k_0 - \ln(f + v_0^2)] = 0. \tag{29}$$

For reasons of self-consistency it shall be necessary at this stage to show that successive terms in the power series given by Eq. (25) will obey the requirement that $\omega^{-l}|k_l(r)| \gg \omega^{-(l+1)}|k_{l+1}(r)|$, which will also imply that the power series will converge rapidly with increasing l, and so it can be truncated after the first few terms. Mindful of the fact that ω is large, the first three terms, involving k_{-1}, k_0, and k_1, respectively, conform to this self-consistency requirement. A simple asymptotic check suffices to show that $k_{-1} \sim r$, $k_0 \sim \ln r$, and $k_1 \sim r^{-1}$. In any case both k_{-1} and k_1 make contributions to the phase of the traveling wave. In this linearized treatment, therefore, the most conspicuous contribution to the amplitude comes from k_0, and any impression of the stability of the flow can be unambiguously derived from this term only.

Using only the solutions of k_{-1} and k_0, the dominant properties of the perturbation could be set down as

$$\psi'(r, t) \simeq \xi_{\pm}\left[\frac{f + v_0^2}{\sqrt{(h^{tr})^2 - h^{rr}h^{tt}}}\right]^{1/2} \times \exp\left\{i\omega \int (h^{rr})^{-1}[h^{tr} \pm \sqrt{(h^{tr})^2 - h^{rr}h^{tt}}]dr\right\} \times e^{i\omega t} \tag{30}$$

with ξ being a constant, and with the positive and negative signs, placed together, indicating a superposition of incoming (corresponding to the negative sign) and outgoing (corresponding to the positive sign) traveling waves, respectively.

It will now be very much worthwhile to scrutinize the expression for k_0 more closely and see how the geometry of space-time makes its contribution to stability. Using the derived values of h^{tt}, h^{tr}, and h^{rr}, it can be shown that k_0 picks up a term that goes as $\ln f$. By virtue of the fact that $f < 1$, the logarithm of f will be negative, and consequently its effect on k_0 would be to detract from its value in the Newtonian limit (where $f = 1$). And so, where general relativistic effects will have to be accounted for, the amplitude of the traveling waves will become subdued. Once again it will not be difficult to see that this effect is owed entirely to the coupling between the flow and the curvature of space-time.

To delve into some more details, the amplitude of the perturbation could be recast very simply as

$$|\psi'| \sim \left(\frac{f + v_0^2}{a_0^2 v_0^2}\right)^{1/4}, \tag{31}$$

a form which is quite helpful in shedding a clear light on the asymptotic behavior. In the Newtonian limit, it is obvious that $|\psi'| \sim (a_0 v_0)^{-1/2}$, a state of affairs whose stability has been cogently argued for by Petterson *et al.* [11]. An equal measure of stability is to be seen near the event horizon as well, with a smooth passage for the traveling wave through the sonic region somewhere in between. After this it becomes easy to argue for the asymptotic stability of the steady solutions.

VI. CONCLUDING REMARKS

Some parting comments would well be in order. It has been shown already that the manner in which the perturbative study has proceeded, beginning with the continuity condition of the flow, has led to a failure in establishing an acoustic geometry for general relativistic spherically symmetric accretion. On the other hand, as far as stability is concerned, this approach has been in perfect qualitative conformity with earlier studies which have dwelt on the question of the stability of the flow solutions. Indeed as regards stability, in particular, it has been argued and shown that general relativistic effects enhance the stability of the stationary solutions. This fact actually opens up an interesting possibility.

Schwarzschild space-time defines a geometry of closed curvature. A standing wave in this geometry exhibits a damping of its amplitude, and so a stable behavior is implied. In the Newtonian limit, the standing waves continue to have a constant amplitude. In this respect the behavior may once again be viewed to be stable. However, the situation could become radically different in the geometry of open curvature. One might conjecture that in this case, simply because of the nature of the geometry, there will be an unstable behavior, manifested through a growth in the amplitude of a standing wave. Indeed, this speculation is not without its foundations. In a different context, but with good consonance, similar features are to be seen in cosmic microwave background anisotropy in compact hyperbolic spaces [40,41].

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