

Seesaw mechanism in quark-lepton complementarityFlorian Plentinger,^{*} Gerhart Seidl,[†] and Walter Winter[‡]*Institut für Theoretische Physik und Astrophysik, Universität Würzburg, D-97074 Würzburg, Germany*

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We systematically construct realistic mass matrices for the type-I seesaw mechanism out of more than 20×10^{12} possibilities. We use only very generic assumptions from extended quark-lepton complementarity, i.e., the leptonic mixing angles between flavor and mass eigenstates are either maximal or parametrized by a single small quantity ϵ that is of the order of the Cabibbo angle $\epsilon \simeq \theta_C$. The small quantity ϵ also describes all fermion mass hierarchies. We show that special cases often considered in the literature, such as having a symmetric Dirac mass matrix or small mixing among charged leptons, constitute only a tiny fraction of our possibilities. Moreover, we find that in most cases the spectrum of right-handed neutrino masses is only mildly hierarchical. As a result, we provide for the charged leptons and neutrinos a selected list of 1981 qualitatively different Yukawa coupling matrices (or textures) that are parametrized by the Cabibbo angle and allow for a perfect fit to current data. In addition, we also briefly show how the textures could be generated in explicit models from flavor symmetries.

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I. INTRODUCTION

The impressive experimental advances that have been made during the past decade in solar [1,2], atmospheric [3], reactor [4,5], and accelerator [6] neutrino oscillation experiments, have very well established that neutrinos are massive. Since neutrinos are massless in the standard model (SM), the observation of neutrino masses provides evidence for new physics, such as an underlying grand unified theory (GUT) [7] (see also Ref. [8]). It is therefore believed that the smallness of the absolute neutrino mass scale $m_\nu \simeq 10^{-2}$ – 10^{-1} eV compared to the electroweak scale $\sim 10^2$ GeV gives us important information on the nature of the new physics. Today, the most widely accepted mechanism to generate small neutrino masses is the seesaw mechanism [9,10], in which the smallness of neutrino masses is linked to the hierarchy between the electroweak and the GUT scale $M_{\text{GUT}} \simeq 2 \times 10^{16}$ GeV [11].

In the type-I seesaw mechanism [9], the set of SM neutrinos ν_i ($i = 1, 2, 3$ is the generation index) is extended by three right-handed neutrinos ν_i^c , which are total singlets under the SM gauge group $G_{\text{SM}} = SU(3)_c \times SU(2)_L \times U(1)_Y$. In the basis $(\nu_1, \nu_2, \nu_3, \nu_1^c, \nu_2^c, \nu_3^c)$, this leads after electroweak symmetry breaking to a complex symmetric 6×6 matrix

$$M_\nu = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix}, \quad (1)$$

where 0 , M_D , and M_R are 3×3 matrices. The upper left matrix 0 has zero entries since there is no Higgs triplet that could have directly coupled to the ν_i . The entries in M_D are protected by electroweak gauge invariance and they are

therefore of the order $\sim 10^2$ GeV, while the matrix elements of M_R are of the order of the $B - L$ breaking scale $M_{B-L} \simeq 10^{14}$ GeV. After integrating out the right-handed neutrinos, we arrive at the effective low-energy 3×3 neutrino Majorana mass matrix

$$M_{\text{eff}} = -M_D M_R^{-1} M_D^T, \quad (2)$$

which gives rise to neutrino masses of the order $m_\nu \simeq 10^{-2}$ eV. The seesaw mechanism is attractive because M_{B-L} is very close to M_{GUT} , indicating a GUT origin of neutrino masses.

In GUT models, quarks and leptons are unified into multiplets, which is known as quark-lepton unification and one possibility to explore GUTs at present energies is to search for signatures of quark-lepton unification in the fermion mass and mixing parameters. Most notably, quark-lepton unification has to give an answer to the question why the quark mixing angles in the Cabibbo-Kobayashi-Maskawa (CKM) matrix V_{CKM} [12] and the leptonic mixing angles in the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix U_{PMNS} [13] are strikingly different. In the quark sector, all CKM mixing angles are small and can be approximately written as powers of the Cabibbo angle θ_C . In contrast to this, in the lepton sector, only the reactor angle θ_{13} is small, whereas the solar angle θ_{12} and the atmospheric angle θ_{23} are both large. In addition, while the quark and charged lepton mass ratios are strongly hierarchical, the neutrino masses exhibit, if any, only a mild hierarchy (for a recent global fit of neutrino data see, e.g., Ref. [14]).

Recently, quark-lepton complementarity (QLC) [15] (for an early approach see Ref. [16]) has been proposed as a possibility to account for the differences between the quark and lepton mixings. In QLC, the quark and lepton mixing angles are connected by the QLC relations

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$$\theta_{12} + \theta_C \approx \pi/4, \quad \theta_{23} + \theta_{cb} \approx \pi/4, \quad (3)$$

where $\theta_{cb} = \arcsin V_{cb}$. The crucial observation is that sum rules of the types shown in Eq. (3) can be easily obtained when the mixing among the neutrinos and among the charged leptons is described by maximal or CKM-like mixing angles. In this way, for example, the observed value of the solar angle $\theta_{12} \approx 33^\circ$ could be understood in terms of maximal ($\pi/4$) and Cabibbo-like (θ_C) mixing in the individual neutrino and charged lepton sectors. A complementary approach to the solar angle seems, on the other hand, to be suggested by the tribimaximal mixing scheme [17]. The properties of QLC have been studied in various respects: as a result of deviations from bimaximal mixing [18], in connection with sum rules [19], with emphasis on phenomenological implications [20], together with parametrizations of U_{PMNS} in terms of θ_C [21], in view of statistical arguments [22], in conjunction with renormalization group effects [23], and in model building realizations [24].

In Ref. [25], we have proposed an *extended* QLC, in which the mixing angles in both the charged lepton and the neutrino sector can take any value in the sequence $\pi/4, \epsilon, \epsilon^2, \dots$, where ϵ is of the order the Cabibbo angle $\epsilon \approx \theta_C$. In this paper, we will implement extended QLC in the type-I seesaw mechanism by assuming that all the mixing angles of charged leptons and left- and right-handed neutrinos take their values in this sequence. We also suggest that the mass eigenvalues of M_D and M_R are described by powers of ϵ as well. In this approach, the observed large mixing angles θ_{12} and θ_{23} can come from the charged leptons and/or neutrinos.¹ Moreover, in the neutrino sector, large mixing angles can originate from M_D and/or M_R . We systematically search for all mass matrices of charged leptons and neutrinos that satisfy the extended QLC assumptions, and extract all solutions that are consistent with current data in the CP conserving case.

The paper is organized as follows: We first motivate the assumptions underlying the extended QLC approach from the phenomenological and model building point of view in Sec. II. This section can be skipped by the reader already familiar with extended QLC. Next, in Sec. III, we describe the method for constructing all valid charged lepton and seesaw mass matrices that are compatible with data, demonstrate how to obtain the corresponding textures, and address further properties of our procedure. For the normal neutrino mass hierarchy, we first discuss the full sample of all valid mass matrices in Sec. IV, and then we show a selection of textures in Sec. V. A qualitative discussion of the inverted and degenerate neutrino mass schemes is included in Sec. VI, and a summary and conclusions can

¹For a recent study on the reconstruction of the seesaw mechanism from low-energy data see Ref. [26] and large mixing angles coming from the charged lepton sector were also considered, e.g., in Ref. [27].

be found in Sec. VII. Details of our method can be found in Appendices A and B.

II. MOTIVATION

In this section, we first present a brief review of the observed hierarchies of fermion masses and mixing angles and relate them to a small expansion parameter ϵ that is of the order of the Cabibbo angle. Then, we discuss two representative $SU(5)$ GUT examples that obtain the observed mass and mixing parameters from flavor symmetries. The observations made here will later, in Sec. III B, serve as a motivation for the hypotheses in extended QLC.

A. Masses and mixings of quarks and leptons

One of the most striking features of the fermion sector is that the mass and mixing parameters of quarks and charged leptons are strongly hierarchical. It is well known that these hierarchies can be approximately described by a small number $\epsilon \approx 0.2$. In the Wolfenstein parametrization [28], for example, the CKM matrix is given by

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \frac{1}{2}\epsilon^2 & \epsilon & A(\rho + i\eta)\epsilon^3 \\ -\epsilon & 1 - \frac{1}{2}\epsilon^2 & A\epsilon^3 \\ A(1 - \rho + i\eta)\epsilon^3 & -A\epsilon^2 & 1 \end{pmatrix}, \quad (4)$$

where ϵ is of the order of the Cabibbo angle $\theta_C \approx 0.2$, and A, ρ , and η are order unity parameters (for an update see Ref. [29]). Order of magnitude wise, the quark mixing angles can be written in terms of the parameter ϵ as

$$|V_{us}| \sim \epsilon, \quad |V_{cb}| \sim \epsilon^2, \quad |V_{ub}| \sim \epsilon^3. \quad (5)$$

An interesting connection between the Cabibbo angle and the quark masses is established by the Gatto-Sartori-Tonin-Oakes relation $\theta_C = \sqrt{m_d/m_s}$ [30], suggesting that also the fermion mass ratios arise from powers of ϵ . In fact, the mass ratios of the up and down quarks can, e.g., be crudely represented as powers of ϵ as²

$$m_u:m_c:m_t = \epsilon^6:\epsilon^4:1, \quad m_d:m_s:m_b = \epsilon^4:\epsilon^2:1, \quad (6)$$

where $m_b:m_t \sim \epsilon^2$, $m_\tau:m_b \sim 1$, and $m_t \approx 175$ GeV, whereas the mass ratios of the charged leptons are crudely given by

$$m_e:m_\mu:m_\tau = \epsilon^4:\epsilon^2:1. \quad (7)$$

These mass ratios have all to be understood as order of magnitude relations and depend on the energy scale. At one loop, in the minimal supersymmetric standard model (MSSM), the only changes in these relations are $m_c:m_t \sim \epsilon^4$ and $m_b:m_t \sim \epsilon^3$ at $\sim 10^{15}$ GeV (see, e.g., Ref. [31]). The changes due to renormalization group (RG) running are thus only comparatively small.

²We are interested here in an $SU(5)$ compatible fit.

In the neutrino sector, we have a situation that is substantially different from the charged fermion sectors. Experimentally, the PMNS matrix reads (cf. also Ref. [32])

$$U_{\text{PMNS}} = \begin{pmatrix} 0.82 - 0.85 & 0.52 - 0.57 & \leq 0.12 \\ 0.26 - 0.49 & 0.48 - 0.68 & 0.65 - 0.76 \\ 0.27 - 0.49 & 0.49 - 0.69 & 0.64 - 0.75 \end{pmatrix}, \quad (8)$$

which has, unlike V_{CKM} , large off-diagonal entries. In the standard parametrization, the 1σ ranges for the solar and the atmospheric mixing angles are then [14]

$$\sin^2\theta_{12} = 0.30_{-0.03}^{+0.02}, \quad \sin^2\theta_{23} = 0.5_{-0.07}^{+0.08}, \quad (9)$$

whereas we have a 3σ upper bound on the reactor angle that is $\sin^2\theta_{13} \leq 0.041$. The best-fit values of the mixing angles correspond to maximal atmospheric mixing $\theta_{23} \approx \pi/4$ and large, but not maximal, solar mixing $\theta_{12} \approx \pi/4 - \theta_C$, and a small reactor angle $\theta_{13} \lesssim \theta_C$. The 1σ bounds on the solar and atmospheric mass squared differences are [14]

$$\begin{aligned} \Delta m_{\odot}^2 &= (7.9_{-0.3}^{+0.3}) \times 10^{-5} \text{ eV}^2, \\ \Delta m_{\text{atm}}^2 &= (2.5_{-0.25}^{+0.2}) \times 10^{-3} \text{ eV}^2. \end{aligned} \quad (10)$$

The sign of Δm_{\odot}^2 is positive, whereas the sign of Δm_{atm}^2 can be either positive or negative, leading to currently three possible types of allowed neutrino mass spectra. Note that we roughly have

$$\Delta m_{\odot}^2 : \Delta m_{\text{atm}}^2 \sim \epsilon^2. \quad (11)$$

It is thus plausible that the neutrino sector, just like the quarks and charged leptons, is described by the same control parameter ϵ . The order of magnitude relations for the neutrino masses may thus be written as

$$\begin{aligned} m_1 : m_2 : m_3 &= \epsilon^2 : \epsilon : 1, & m_1 : m_2 : m_3 &= 1 : 1 : \epsilon, \\ m_1 : m_2 : m_3 &= 1 : 1 : 1, \end{aligned} \quad (12)$$

where m_1 , m_2 , and m_3 , denote the masses of the 1st, 2nd, and 3rd neutrino mass eigenstate. The 1st, 2nd, and 3rd equation in Eq. (12) describe a normal hierarchical (NH), inverse hierarchical (IH), and quasidegenerate (QD) spectrum, respectively.³

We thus see that the CKM angles and mass ratios of quarks and leptons are roughly given by some power $\sim \epsilon^n$ of the Cabibbo angle $\epsilon \approx \theta_C$. Besides that, the phenomenological QLC relations $\theta_{23} \approx \pi/4 - \epsilon^2$ and $\theta_{12} \approx \pi/4 - \epsilon$ involve maximal mixing angles. In the next section, we will discuss how these mass and mixing parameters may be reproduced in models.

B. Examples with quark-lepton unification

The common appearance of the masses and mixing angles in the quark and lepton sectors may point towards a quark-lepton unified theory. One might therefore wonder whether a description of the fermion mass and mixing parameters as given in Sec. II A can indeed be obtained in explicit models. We are interested in models in which the fermion mixing angles—prior to going to the mass eigenbasis—can be maximal or are given by some power of the Cabibbo angle $\epsilon \approx \theta_C$ which also describes all fermion mass ratios.

For this purpose, let us briefly review two GUT models [33,34] based on $SU(5)$, which show that in a quark-lepton unified theory it is (i) actually possible to generate realistic hierarchical fermion mass ratios and mixing angles that are described by powers $\sim \epsilon^n$ and that (ii) one can predict exact maximal mixing compatible with these hierarchies. We view the two GUT models as two representatives of a broad class of possible realistic models using Abelian (for early work see Ref. [35] and for more recent models see, e.g., Refs. [31,36]) or discrete non-Abelian (for recent studies including the quark sector see, e.g., Ref. [37] and in the context of GUTs see, e.g., Ref. [38]) flavor symmetries.⁴ Our observations can be viewed as a further motivation for the definition of extended QLC in Sec. III B, where we will, in particular, claim that both the neutrinos as well as the charged leptons can exhibit mixing angles that are maximal and/or $\sim \epsilon^n$. In this way, the size of θ_{12} will then be simply the result of a QLC-type sum rule.

Example 1: Cabibbo-type mass and mixing hierarchies.—Our first example is a supersymmetric $SU(5)' \times SU(5)''$ GUT with a $U(1)^N = \prod_{j=1}^N U(1)_j$ flavor symmetry group [33]. It yields, as a result of $U(1)^N$ flavor symmetry breaking, the masses and mixing angles of quarks and leptons roughly as powers of ϵ . The i th generation ($i = 1, 2, 3$) is charged under $SU(5)' \times U(1)_{k_i}$ [k_i labels a suitable subgroup $U(1)_{k_i} \subset \prod_{j=1}^N U(1)_j$] in an $SO(10)$ compatible way as $\mathbf{10}(-1)_i + \mathbf{\bar{5}}(3)_i + \mathbf{1}(-5)_i$, where the numbers in parentheses denote the $U(1)_{k_i}$ charges of the respective multiplets and the $SU(5)$ singlets are the right-handed neutrinos. Higgs superfields break the $U(1)^N$ gauge symmetry such that masses for quarks and leptons arise from higher-dimension terms via the Froggatt-Nielsen mechanism [40] illustrated in Fig. 1. In the presence of SM singlet scalar “flavons,” that break the flavor symmetry by acquiring universal vacuum expectation values (VEVs) v (crosses), and superheavy fermions with common mass M_F , which are charged under the flavor symmetry (internal solid lines), the mass terms of quarks and leptons become suppressed by integer powers of a small parameter $\epsilon = v/M_F$ that controls the flavor symmetry breaking. The integer power of ϵ is solely determined by the quantum

³For NH neutrinos, one can compute ϵ from the current best-fit values, which gives $0.15 \lesssim \epsilon \lesssim 0.22$ (3σ).

⁴For a more complete list of references on discrete non-Abelian flavor symmetries see Ref. [39].

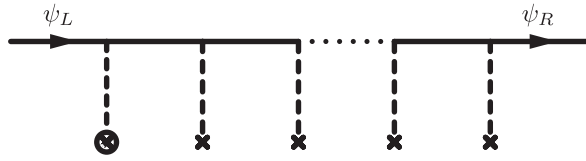


FIG. 1. Generation of higher-dimension fermion mass terms via the Froggatt-Nielsen mechanism. ψ_L and ψ_R are left- and right-handed SM fermions. Internal solid lines denote super-heavy fermions with common mass M_F , the circled cross is the usual SM Higgs vacuum expectation value (VEV), whereas the crosses without circle represent universal VEVs v of SM singlet scalars that break the flavor symmetry. After integrating out the heavy fermions, the resulting mass term is then given by the effective dimension- n operator $\langle H \rangle \epsilon^n \bar{\psi}_L \psi_R$, where $\epsilon = v/M_F$ serves as a small expansion parameter.

numbers of the left- and right-handed fermions ψ_L and ψ_R under the flavor symmetry. As a result, the mass matrices of the up quarks, down quarks, charged leptons, and neutrinos become

$$\begin{aligned}
 M_u &= m_t \begin{pmatrix} \epsilon^6 & \epsilon^5 & \epsilon^3 \\ \epsilon^5 & \epsilon^4 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}, \\
 M_d &= M_\ell^T = m_b \begin{pmatrix} \epsilon^4 & \epsilon^6 & \epsilon^3 \\ \epsilon^{10} & \epsilon^2 & \epsilon^2 \\ \epsilon^8 & \epsilon^3 & 1 \end{pmatrix}, \\
 M_{\text{eff}} &= m_\nu \begin{pmatrix} \epsilon^4 & \epsilon^2 & \epsilon \\ \epsilon^2 & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}.
 \end{aligned} \tag{13}$$

The representation of the mass matrices in Eq. (13) are examples of what we will call in the following *textures*: These are descriptions of the mass matrices showing only the order of magnitude (up to order one Yukawa couplings) of the entries in terms of powers of a small number ϵ that parametrizes the flavor symmetry breaking. At an order of magnitude level, the textures in Eq. (13) predict for the quarks and charged leptons the mass ratios and mixing angles of Eqs. (5)–(7), whereas the neutrino mass spectrum is of the type $m_1:m_2:m_3 = \epsilon:\epsilon:1$. The reactor mixing angle is small and of the order $\theta_{13} \sim \epsilon$, while the solar and the atmospheric mixing angles are large and of the orders $\theta_{12} \sim 1$ and $\theta_{23} \sim 1$. A proper choice of the order one Yukawa couplings then allows one to reproduce the solar and atmospheric mixing angles close to the current best-fit values.

Example 2: Maximal mixing.—Our second example is an $SU(5)$ GUT with a non-Abelian discrete flavor symmetry between the 2nd and 3rd generation [34]. It predicts a maximal atmospheric mixing angle $\theta_{23} = \pi/4$ and a zero reactor angle $\theta_{13} = 0$ as a consequence of the discrete symmetry. The basic flavor symmetry of the model is a

discrete Z_2 exchange symmetry that implements a maximal atmospheric mixing angle and acts on the $SU(5)$ multiplets of the 2nd and 3rd generation as $Z_2: \bar{\mathbf{5}}_2 \leftrightarrow \bar{\mathbf{5}}_3$, $\mathbf{10}_2 \leftrightarrow \mathbf{10}_3$, $\mathbf{1}_2 \leftrightarrow \mathbf{1}_3$, where the subscripts denote the generation indices. In addition, the model has a $U(1)$ family number symmetry that does not commute with the above Z_2 generator. As a consequence, the resulting down quark and charged lepton mass matrices can accommodate the hierarchical down quark and charged lepton masses and the CKM and PMNS mixing angles arise in the up quark and the neutrino sector, respectively. The total non-Abelian flavor symmetry enforces in M_{eff} a $\mu - \tau$ exchange symmetry. This predicts a maximal atmospheric mixing angle $\theta_{23} = \frac{\pi}{4}$ and a vanishing reactor angle $\theta_{13} = 0$. The solar angle θ_{12} , on the other hand, is large and can easily reproduce the current best-fit value.

These two examples show that hierarchical masses and mixings described by powers of ϵ as well as maximal mixing can be predicted in explicit GUT models from flavor symmetries. As a result of this motivation section, it is therefore plausible to assume that all mixing angles and mass hierarchies in the quark and lepton sectors are generated by a single small quantity $\epsilon \simeq \theta_C$ augmented by possibly maximal mixing in the up- and/or down-type sectors. This basic observation will be the basis for our hypotheses underlying extended QLC.

III. METHOD

In this section, we introduce our method for implementing extended QLC in the seesaw mechanism. For this purpose, we first briefly review the seesaw mechanism and discuss our notation for parametrizing the mass and mixing parameters in Sec. III A. Next, we define our QLC assumptions in Sec. III B, and outline our approach for generating and selecting the mass matrices of charged leptons and neutrinos in Sec. III C, i.e., we describe our general procedure. While Sec. III C is somewhat qualitative in some points, we give more details and a comment on the complexity in Appendix A. As the next step, we demonstrate how we produce textures in Sec. III D. Finally, we discuss the role of mass ratios, input parameters, and RG running in our routine in Sec. III E.

A. Neutrino mass and mixing nomenclature

In what follows, we assume that the left-handed SM neutrinos acquire their masses via the type-I seesaw mechanism [9].⁵ In the type-I seesaw mechanism, the Yukawa couplings generating the charged lepton and neutrino masses are

⁵The type-II seesaw mechanism [10] in extended QLC has already been covered in a previous analysis of the possible effective 3×3 Majorana neutrino mass matrices in Ref. [25] that can be viewed as being generated by the coupling to some Higgs triplet with small ($\sim 10^{-2}$ eV) VEV.

$$\mathcal{L}_Y = -(Y_\ell)_{ij} H^* \ell_i e_j^c - (Y_D)_{ij} i \sigma^2 H \ell_i \nu_j^c - \frac{1}{2} (M_R)_{ij} \nu_i^c \nu_j^c + \text{H.c.}, \quad (14a)$$

where $\ell_i = (\nu_i, e_i)^T$, e_i^c , and ν_i^c ($i = 1, 2, 3$ is the generation index), are the left-handed lepton doublets (ℓ_i), the right-handed charged leptons (e_i^c), and the right-handed SM singlet neutrinos (ν_i^c). In Eq. (14a), H is the Higgs doublet, Y_ℓ and Y_D are the 3×3 Dirac Yukawa coupling matrices of the charged leptons (Y_ℓ) and neutrinos (Y_D), M_R is the 3×3 Majorana mass matrix of the right-handed neutrinos, and $i\sigma^2$ is the 2×2 antisymmetric tensor. After electroweak symmetry breaking, H develops a vacuum expectation value $\langle H \rangle = (0, v/\sqrt{2})^T$, where $v \sim 10^2$ GeV, and the mass terms of the leptons become

$$\mathcal{L}_{\text{mass}} = -(M_\ell)_{ij} e_i e_j^c - (M_D)_{ij} \nu_i \nu_j^c - \frac{1}{2} (M_R)_{ij} \nu_i^c \nu_j^c + \text{H.c.}, \quad (14b)$$

where $M_\ell = \langle H \rangle Y_\ell$ is the charged lepton and $M_D =$

$\langle H \rangle Y_D$ the Dirac neutrino mass matrix. M_ℓ and M_D are complex 3×3 matrices that are described by 18 parameters and have entries of the order $\sim 10^2$ GeV. The matrix M_R is complex, symmetric, and described by 12 parameters. It has matrix elements of the order the $B - L$ breaking scale $M_{B-L} \sim 10^{15}$ GeV. The resulting complex symmetric 6×6 neutrino mass matrix is given in Eq. (1), and after integrating out the right-handed neutrinos, this gives the effective 3×3 neutrino mass matrix M_{eff} in Eq. (2) leading to masses $\sim 10^{-2}$ eV for the active neutrinos.

To analyze the origin of leptonic mixing, we consider the diagonalization of the mass terms in $\mathcal{L}_{\text{mass}}$ in Eq. (14b) by unitary matrices. Using the convention in Ref. [25], we can always write a general unitary 3×3 matrix U_{unitary} as

$$U_{\text{unitary}} = \text{diag}(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}) \cdot \hat{U} \cdot \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, 1), \quad (15a)$$

where the phases $\varphi_1, \varphi_2, \varphi_3, \alpha_1$, and α_2 , take their values in the interval $[0, 2\pi]$ and

$$\hat{U} = \begin{pmatrix} c_{12}c_{23} & s_{12}c_{23} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \quad (15b)$$

is a CKM-like matrix in the standard parametrization with $s_{ij} = \sin\hat{\theta}_{ij}$, $c_{ij} = \cos\hat{\theta}_{ij}$, where $\hat{\theta}_{ij} \in \{\hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{23}\}$ lie all in the first quadrant, i.e., $\hat{\theta}_{ij} \in [0, \frac{\pi}{2}]$, and $\hat{\delta} \in [0, 2\pi]$. The matrix \hat{U} is thus described by 3 mixing angles θ_{ij} and one phase δ , i.e., it has 4 parameters. The matrix U_{unitary} has five additional phases⁶ and contains therefore in total 9 parameters.

The leptonic Dirac mass matrices M_ℓ and M_D , and the Majorana mass matrices M_R and M_{eff} are diagonalized by

$$\begin{aligned} M_\ell &= U_\ell M_\ell^{\text{diag}} U_{\ell'}^\dagger, & M_D &= U_D M_D^{\text{diag}} U_{D'}^\dagger, \\ M_R &= U_R M_R^{\text{diag}} U_R^T, & M_{\text{eff}} &= U_\nu M_{\text{eff}}^{\text{diag}} U_\nu^T, \end{aligned} \quad (16)$$

where $U_\ell, U_{\ell'}, U_D, U_{D'}, U_R$, and U_ν are unitary mixing matrices, whereas $M_\ell^{\text{diag}}, M_D^{\text{diag}}, M_R^{\text{diag}}$, and $M_{\text{eff}}^{\text{diag}}$ are diagonal mass matrices with positive entries. We can always write the mixing matrices as the products

$$U_x = D_x \hat{U}_x K_x, \quad (17)$$

where \hat{U}_x are CKM-like matrices that are parametrized as in Eq. (15b), while D_x and K_x are given by $D_x = \text{diag}(e^{i\varphi_1^x}, e^{i\varphi_2^x}, e^{i\varphi_3^x})$ and $K_x = \text{diag}(e^{i\alpha_1^x}, e^{i\alpha_2^x}, 1)$, where the index x runs over $x = \ell, \ell', D, D', R, \nu$. The phases in D_x and K_x are all in the range $\varphi_1^x, \varphi_2^x, \varphi_3^x, \alpha_1^x, \alpha_2^x \in [0, 2\pi]$.

⁶For recent discussions of rephasing invariants in the lepton sector see, e.g., Ref. [41].

Each of the matrices \hat{U}_x in Eq. (17) contains four mixing parameters: three mixing angles and one phase. We denote the parameters of \hat{U}_x by $\theta_{12}^x, \theta_{13}^x, \theta_{23}^x$, and δ^x . For each of the matrices \hat{U}_x in Eq. (17), we define the mixing parameters by identifying in Eq. (15b) the mixing angles as $\hat{\theta}_{ij} \rightarrow \theta_{ij}^x$, and the phase as $\hat{\delta} \rightarrow \delta^x$.

The PMNS matrix is given by

$$U_{\text{PMNS}} = U_\ell^\dagger U_\nu = \hat{U}_{\text{PMNS}} K_{\text{Maj}}, \quad (18)$$

where \hat{U} is a CKM-like matrix parametrized as in Eq. (15b), and $K_{\text{Maj}} = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, 1)$ contains the Majorana phases ϕ_1 and ϕ_2 . The CKM-like matrix \hat{U}_{PMNS} in Eq. (18) is described by the solar angle θ_{12} , the reactor angle θ_{13} , the atmospheric angle θ_{23} , and the Dirac CP phase δ , which we identify in the standard parametrization of Eq. (15b) as $\hat{\theta}_{ij} \rightarrow \theta_{ij}$ and $\hat{\delta} \rightarrow \delta$. The PMNS matrix has thus 3 mixing angles and 3 phases and contains therefore 6 physical parameters.

Let us next express M_{eff} in terms of the mass eigenvalues and mixing angles introduced above. Inserting Eq. (17) into Eq. (16), we find

$$M_D = D_D \hat{U}_D K_D M_D^{\text{diag}} K_{D'}^* \hat{U}_{D'}^\dagger D_{D'}^*, \quad (19a)$$

$$M_R^{-1} = D_R^* \hat{U}_R^* K_R^* (M_R^{\text{diag}})^{-1} K_R \hat{U}_R^\dagger D_R^*. \quad (19b)$$

The effective neutrino mass matrix M_{eff} in Eq. (2) can thus be written as

$$M_{\text{eff}}^{\text{th}} = -D_D \hat{U}_D \tilde{K} M_D^{\text{diag}} \hat{U}_D^\dagger \tilde{D} \hat{U}_R^* (K_R^*)^2 \\ \times (M_R^{\text{diag}})^{-1} \hat{U}_R^\dagger \tilde{D} \hat{U}_R^* M_D^{\text{diag}} \tilde{K} \hat{U}_D^T D_D, \quad (20a)$$

where we have introduced $\tilde{K} = K_D^* K_{D'}$ and $\tilde{D} = D_{D'}^* D_R^*$. We have denoted the parametrization of M_{eff} in Eq. (20a) by an extra superscript “th” for “theoretical,” since none of the mass and mixing parameters on the right-hand side of Eq. (20a) are directly measurable in neutrino oscillations. Note that in the CP conserving case, the matrix $(K_R^*)^2$ drops out of the expression for M_{eff} in Eq. (20a). Equivalently to Eq. (20a), using Eqs. (17) and (18) in the expression for M_{eff} in Eq. (16), we can write M_{eff} also in the parametrization

$$M_{\text{eff}}^{\text{exp}} = D_\ell \hat{U}_\ell K_\ell \hat{U}_{\text{PMNS}} K_{\text{Maj}}^2 M_{\text{eff}}^{\text{diag}} \hat{U}_{\text{PMNS}}^T K_\ell \hat{U}_\ell^T D_\ell, \quad (20b)$$

where we have chosen the superscript “exp” for “experimental,” to label the representation of M_{eff} in Eq. (20b), since $M_{\text{eff}}^{\text{exp}}$ involves the matrices $M_{\text{eff}}^{\text{diag}}$ and U_{PMNS} containing the experimentally accessible mass and mixing parameters. Note that in the CP conserving case, $(K_{\text{Maj}})^2$ drops out of the expression for $M_{\text{eff}}^{\text{exp}}$. It is clear that $M_{\text{eff}}^{\text{th}} = M_{\text{eff}}^{\text{exp}} = M_{\text{eff}}$, since $M_{\text{eff}}^{\text{th}}$ and $M_{\text{eff}}^{\text{exp}}$ are just different parametrizations of M_{eff} .

B. Hypotheses for extended QLC

We will now formulate the assumptions underlying extended QLC in the type-I seesaw mechanism. Motivated by the discussion of the GUT examples in Sec. II B, extended QLC will in this context include assumptions on (i) the mixing parameters of U_ℓ , U_D , $U_{D'}$, and U_R , and assumptions on (ii) the eigenvalues of the mass matrices M_ℓ , M_{eff} , M_D and M_R .

Mixing angles.—Consider first the possible mixing angles in extended QLC. Following the idea of Ref. [25], we begin by assuming that all mixing angles parametrizing in Eq. (16) the mixing matrices U_ℓ , U_D , $U_{D'}$, and U_R can *a priori* take any of the values in the sequence $\pi/4, \epsilon, \epsilon^2, \dots$, where ϵ is a small number. Motivated by the quark sector, we will take $\epsilon \approx 0.2$, i.e., we assume that ϵ is of the order the Cabibbo angle. This applies the concept of extended quark-lepton complementarity in Ref. [25] to the mixing matrices U_D , $U_{D'}$, and U_R that diagonalize the renormalizable neutrino mass terms in Eq. (14b). Since the current 1σ error on the leptonic mixing angles is at most of the order ϵ^2 (see, e.g., Ref. [14]), we will truncate the sequence of mixing angles $\pi/4, \epsilon, \epsilon^2, \dots$ after the element ϵ^2 and identify there all terms of the order ϵ^n with $n \geq 3$ simply by “0.” In other words, in extended QLC, we restrict the possible range of mixing angles θ_{ij}^x ($x = \ell, D, D', R$) to the set of values $\theta_{ij}^x \in \{\pi/4, \epsilon, \epsilon^2, 0\}$, where 0 represents mixing angles $\sim \epsilon^n$ with $n \geq 3$. Since we want to compare with current neutrino data, the assumptions on the mixing angles are formulated at low energies ~ 1 GeV. The impact of RG

running when assuming these angles at a high scale will be discussed later in Sec. III E. Note that we cannot just rotate away U_ℓ , because this would make our mixing angle assumptions as powers of ϵ meaningless if induced by an underlying theory, such as a flavor symmetry.

Mass eigenvalues.—Next, let us specify the types of mass eigenvalues that we assume in extended QLC. We have to distinguish two types of mass eigenvalues—those of the lepton mass matrices M_ℓ and M_{eff} in the low-energy effective theory, and those of the Dirac and heavy Majorana mass matrices M_D and M_R of the neutrinos.

We assume that the mass spectra in the low energy effective theory are those of Eqs. (7) and (12). That means we assume $m_e:m_\mu:m_\tau = \epsilon^4:\epsilon^2:1$ for the charged leptons, whereas we have in the neutrino sector $m_1:m_2:m_3 = \epsilon^2:\epsilon:1$ for a NH, $m_1:m_2:m_3 = 1:1:\epsilon$ for an IH, and $m_1:m_2:m_3 = 1:1:1$ for a QD neutrino mass spectrum. As we will discuss later, our results are actually completely independent from the details of the charged lepton spectrum.

Let us now consider the mass eigenvalues of the Dirac and Majorana mass matrices M_D and M_R . We denote the mass eigenvalues of M_D by m_1^D, m_2^D , and m_3^D , and the mass eigenvalues of M_R by m_1^R, m_2^R , and m_3^R . Notice that these are not directly observable at low energies, but as before, we write the mass eigenvalues of M_D and M_R as powers of ϵ . Similar to Eqs. (7) and (12), we will therefore parametrize the mass eigenvalues M_D and M_R as

$$m_1^D:m_2^D:m_3^D = \epsilon^a:\epsilon^b:\epsilon^c \quad \text{and} \quad m_1^R:m_2^R:m_3^R = \epsilon^{a'}:\epsilon^{b'}:\epsilon^{c'}, \quad (21)$$

where a, b, c, a', b' , and c' are suitable non-negative integers ≤ 2 , and we define the absolute mass scales by $m_3^D = m_D \epsilon^c$ and $m_3^R = M_{B-L} \epsilon^{c'}$. As we will discuss in Appendix A 1, one can restrict the possible range of the integers in Eq. (21) to a fairly small set of numbers, such that it is sufficient to test all possible combinations up to second order in ϵ .

In total, we see that in our hypotheses for extended QLC all mass hierarchies and small mixing angles become described by powers of the hierarchy parameter ϵ . In Sec. III E, we will discuss the validity and precision of our assumptions in extended QLC when taking, e.g., GUT relations among fermion masses and RG effects into account.

C. Procedure: Systematic construction of the parameter space

Let us now describe our three-step procedure for generating all textures M_ℓ , M_D , and M_R , which satisfy extended QLC for the seesaw mechanism (cf. Fig. 2 for illustration):

First step.—We generate all possibilities for the effective neutrino mass matrix $M_{\text{eff}}^{\text{th}}$ in Eq. (20a). Here, we assume that the mixing angles entering U_D , $U_{D'}$, and U_R

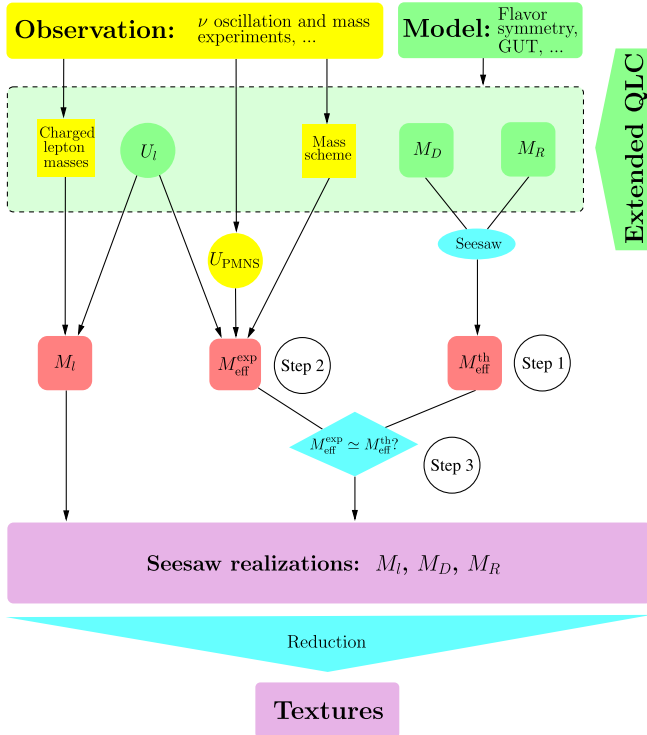


FIG. 2 (color online). Procedure for obtaining the seesaw realizations and texture sets in extended QLC.

can take any values in the set

$$\theta_{ij}^x \in \{\pi/4, \epsilon, \epsilon^2, 0\}, \quad (22)$$

where $x = D, D', R$. Moreover, we suppose in Eq. (20a) that M_D^{diag} and M_R^{diag} are on the general forms as given in Eq. (21) with eigenvalues 1, ϵ , or ϵ^2 . For simplicity, we will confine ourselves to the CP conserving case⁷ where all phases are taken from the set $\delta^x, \varphi_1^x, \varphi_2^x, \varphi_3^x, \alpha_1^x, \alpha_2^x \in \{0, \pi\}$.

Second step.—We generate all possibilities for the neutrino mass matrix $M_{\text{eff}}^{\text{exp}}$ in Eq. (20b). For U_{PMNS} , we use values motivated by the current best-fit values. In particular, we use the following input, which could be experimentally confirmed or rejected in the coming years⁸

$$\theta_{12} = \pi/4 - \epsilon, \quad \theta_{13} = 0, \quad \theta_{23} = \pi/4 \quad (23)$$

⁷Note that in the case of CP violation, some textures may change due to cancellations, so the number of textures will increase, like one would expect. Nevertheless, a complete systematic analysis (all phases between 0 and 2π are allowed) is up to now not possible because of lack of computing power. This may change in some years.

⁸See, e.g., Refs. [42,43] for long-baseline experiments on a scale of the coming ten years, Refs. [44,45] for an up scale reactor θ_{12} measurement, Ref. [46] for the potential of various different superbeam upgrades, and Ref. [47] for a neutrino factory measurement.

with $\epsilon = 0.2$. These values represent the current best-fit values [14] very closely.⁹ A different choice for these parameters can be equally well applied, but it will change the final results.

For U_ℓ , we follow Eq. (22), and for the phases, we test all real possibilities. Furthermore, we insert the neutrino mass spectra given in Eq. (12) into $M_{\text{eff}}^{\text{diag}}$, and again test all possibilities.

Third step.—Next, we match all possibilities from step 1 and step 2, i.e., we select all parameter combinations for which

$$M_{\text{eff}}^{\text{th}}|_{\epsilon=0.2} \simeq M_{\text{eff}}^{\text{exp}}|_{\epsilon=0.2} \quad (24)$$

at $\mathcal{O}(\epsilon^3)$. For details (and the exact numerical implementation) of the matching procedure, see Appendix A 2; in particular, note that our procedure automatically factors out the overall neutrino mass scale, i.e., $m_\nu = m_D^2/M_R$ will be automatically satisfied. In the following, we will call a (seesaw) *realization* a valid set $\{M_D, M_R, U_\ell\}$, or more precisely, a combination of all involved mixing parameters, phases, and mass hierarchies, for which Eq. (24) is fulfilled. In other words, a realization is thus a set of input parameters compatible with current experimental best-fit values which describes $M_{\text{eff}}^{\text{th}}$ and the 6×6 matrix M_ν completely, and it contains the left-handed charged lepton mixing. In total, our procedure requires that we systematically scan 20×10^{12} different possible realizations. For details on the complexity, see Appendix A 3.

Note that in our procedure the observed large leptonic mixing angles can be generated either in the charged lepton sector and/or the neutrino sector. Furthermore, in the neutrino and/or the Majorana mass matrix of the right-handed neutrinos. This means that we do not make any special assumptions simplifying the structure of the seesaw mechanism, such as taking M_ℓ or M_R to be diagonal, or M_D symmetric.

D. Texture extraction and order unity couplings

Let us now describe how we extract the textures for the charged leptons and neutrinos from the seesaw realizations that satisfy Eq. (24). In the course of applying the three-step procedure described in Sec. III C, we have already produced for each realization the pair of matrices M_D and M_R . In the same way, we determine for each valid realization the charged lepton mass matrix¹⁰ M_ℓ by rotating to the left-handed flavor basis as $M_\ell = U_\ell M_\ell^{\text{diag}}$, where M_ℓ^{diag}

⁹Note that these values correspond not only to the best-fit values, but are also often considered as an interesting symmetry limit in “exceptional” [39] neutrino mass models.

¹⁰Note that we choose, for simplicity, the right-handed charged lepton mixing matrix to be the unit matrix $U_{\ell'} = 1$. This choice is, however, not limiting our procedure since $U_{\ell'}$ does not enter into U_{PMNS} .

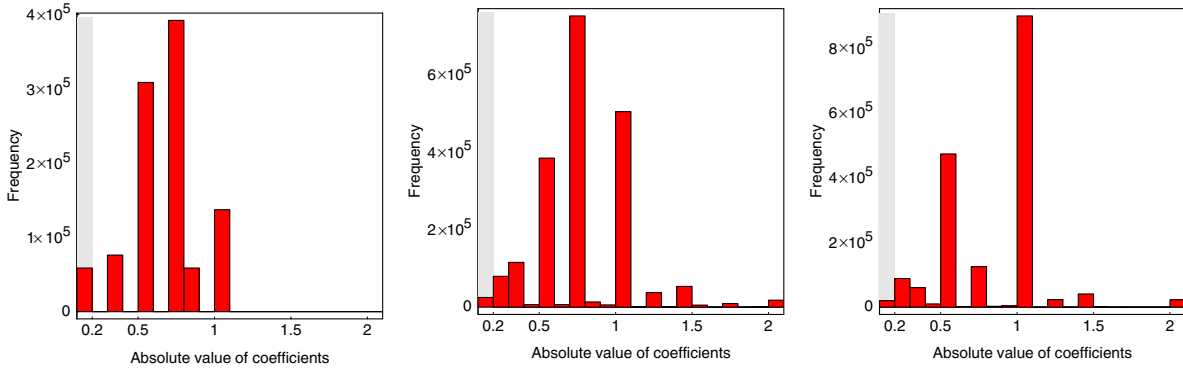


FIG. 3 (color online). Distribution of the order one couplings in M_ℓ (left), M_D (center), and M_R (right) for all valid seesaw realizations of NH neutrinos (for all orders and for all matrix elements). The gray-shaded region marks the area in which the order one couplings are smaller than ϵ .

contains the masses given in Eq. (7). Next, we analytically expand the mass matrices in ϵ as

$$M_x = M_x^{(0)} + M_x^{(1)}\epsilon + M_x^{(2)}\epsilon^2 + \mathcal{O}(\epsilon^3), \quad (25)$$

where $x = \ell, D, R$, and identify for each matrix element the leading contribution as the lowest order in ϵ . The texture for M_x is then found by substituting each matrix element of M_x by its leading power ϵ^k . For $k \geq 3$, we take $\epsilon^k \rightarrow 0$. In doing so, we always “drop” the order one coefficients that multiply the leading powers ϵ^k and round them to one (unless they are zero). For a given realization, we call the collection of textures for M_ℓ , M_D , and M_R a *texture set*. For any texture set obtained in this way, we argue that one can always adjust the order unity coefficients such that the Yukawa couplings are brought in perfect agreement with data. An important prerequisite for this statement is that the different orders in the expansion in Eq. (25) do not “interfere” with each other, i.e., the involved coefficients are really of order unity. This property for the coefficients will be checked below. It is important to keep in mind that, in general, more than one realization may lead to the same texture set. Because of the reduction from in general several realizations to one texture set, we will call our texture set producing technique “texture reduction.” Note also the important fact that the texture reduction is based on an *analytic expansion* in ϵ as opposed to just using powers of 0.2 for a purely numerical fit of the mass matrix elements.

As an example for texture reduction, consider the following mass matrix M_D (cf. texture/realization No. 1 in Table I):

$$M_D = m_D \begin{pmatrix} -\frac{\epsilon^2}{\sqrt{2}} & -\epsilon & 0 \\ -\epsilon^2 & -\frac{1}{\sqrt{2}} + \frac{\epsilon^2}{2\sqrt{2}} & \frac{1}{\sqrt{2}} - \frac{\epsilon^2}{2\sqrt{2}} \\ \frac{\epsilon^2}{\sqrt{2}} & -\frac{\epsilon^2}{\sqrt{2}} & -\epsilon + \frac{\epsilon^2}{\sqrt{2}} \end{pmatrix} \\ \rightarrow \begin{pmatrix} \epsilon^2 & \epsilon & 0 \\ \epsilon^2 & 1 & 1 \\ \epsilon^2 & \epsilon^2 & \epsilon \end{pmatrix}.$$

Here, “ \rightarrow ” symbolizes, up to an overall mass scale, the identification of the leading order terms in the expansion in ϵ that contribute to the mass matrix elements in M_D . The matrix on the right-hand side of \rightarrow then represents the texture corresponding to the mass matrix M_D on the left-hand side.

In order for the expansion in Eq. (25) to be useful, the Yukawa couplings should be of order unity (at least if one wants not to rely on some sort of fine-tuning). The expansion suggests a criterion for what “order unity” actually means: the coefficients should lie in the interval $[\epsilon, \dots, \epsilon^{-1}] \simeq [0.2, \dots, 5]$, such that they would not imitate different orders in ϵ . In Fig. 3, we show to which extent this condition is indeed satisfied for the valid realizations.

Figure 3 depicts the coefficients of the $M_x^{(i)}$ in Eq. (25) for M_ℓ (left), M_D (center), and M_R (right) for all valid seesaw realizations and NH neutrinos (including all orders and for all matrix elements). Interestingly, it turns out that in 99.9% of all cases, the order unity Yukawa couplings lie in the range $\epsilon, \dots, \epsilon^{-1}$, which justifies the expansion of the textures in ϵ . For example, for M_ℓ and M_D , the peaks at around $0.7 \simeq 1/\sqrt{2}$ are predominant. Note that, though in Fig. 3 the first bin corresponds to coefficients smaller than ϵ , our mapping of textures is unambiguous. This is because in the cases where the leading order coefficient of a matrix element becomes very small (< 0.2), the other coefficients become also very small—with the exception of 0.10% of all cases for the matrix elements.¹¹ As a consequence, the leading order term indeed rarely numerically interferes with the higher orders. This picture would change if ϵ was larger. For instance, for $\epsilon = 0.5$, the leading order identification for M_D fails in 0.43% of all cases, and for M_R it fails in 0.01% of all cases, which is, however, still only a remarkably small fraction of all cases.

¹¹All these exceptions appear only in M_D .

TABLE I. Complete set of selected seesaw textures/realizations for the NH case where $\xi \in \{0, \epsilon^2\}$ (see text).

No.	M_ℓ	M_D	M_R	m_i^D/m_D m_i^R/M_{B-L}	$(\theta_{12}^\ell, \theta_{13}^\ell, \theta_{23}^\ell)$ $(\theta_{12}^D, \theta_{13}^D, \theta_{23}^D)$ $(\theta_{12}^{D'}, \theta_{13}^{D'}, \theta_{23}^{D'})$ $(\theta_{12}^R, \theta_{13}^R, \theta_{23}^R)$
1	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon & 0 \\ \xi & 1 & 1 \\ \epsilon^2 & \xi & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$(\epsilon^2, 1, \epsilon)$ $(\epsilon^2, \epsilon, 1)$	$(\xi, \epsilon, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \xi)$ $(\xi, \xi, \frac{\pi}{4})$ $(\xi, 0, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \epsilon)$
2	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon^2 & \epsilon \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 1 & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & \epsilon & 1 \\ \epsilon & 1 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon^2)$ $(\epsilon^2, \epsilon, 0)$ $(\epsilon^2, \frac{\pi}{4}, \epsilon)$
3	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & \epsilon \\ \epsilon^2 & 1 & \epsilon \\ \epsilon & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & \epsilon & 1 \\ \epsilon & 1 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ $(0, \frac{\pi}{4}, \epsilon)$ $(\epsilon^2, \epsilon^2, \epsilon)$ $(\epsilon, \frac{\pi}{4}, \epsilon^2)$
4	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ \xi & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & \epsilon^2 & 1 \end{pmatrix}$	$(\epsilon, \epsilon, 1)$ $(\epsilon, 1, 1)$	$(\epsilon, \epsilon, \frac{\pi}{4})$ $(\frac{\pi}{4}, \xi, \xi)$ $(\epsilon, \xi, \frac{\pi}{4})$ $(\epsilon, \epsilon, \xi)$
5	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \epsilon & \epsilon \\ 1 & 1 & 1 \\ 0 & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & \epsilon & 1 \\ \epsilon & 1 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(0, \epsilon, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, 0)$ $(\frac{\pi}{4}, \epsilon^2, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \epsilon^2)$
6	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & \epsilon^2 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, \xi, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \epsilon)$ $(\epsilon, \epsilon, \xi)$ $(\epsilon, \epsilon, \epsilon)$
7	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon \\ \epsilon & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & \epsilon^2 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, \xi, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \epsilon)$ $(\epsilon, \epsilon, \epsilon)$ $(\epsilon, \epsilon, \xi)$
8	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \xi \\ \epsilon & \epsilon & \xi \\ \xi & \xi & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$(\epsilon, \epsilon, 1)$ $(\epsilon, 1, 1)$	$(\epsilon, \epsilon, \frac{\pi}{4})$ $(\frac{\pi}{4}, \xi, \xi)$ (ϵ, ξ, ξ) $(\epsilon, \epsilon, \frac{\pi}{4})$
9	$\begin{pmatrix} 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ \epsilon^2 & \epsilon & \epsilon \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \xi & 1 \\ \xi & 1 & \xi \\ 1 & \xi & 1 \end{pmatrix}$	$(1, \epsilon, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \epsilon, \frac{\pi}{4})$ $(\xi, \frac{\pi}{4}, \xi)$ $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\xi, \frac{\pi}{4}, \xi)$
10	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon^2 & 1 & \epsilon \\ \epsilon & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \xi \\ 0 & 1 & 0 \\ \xi & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, \xi, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \xi)$ $(0, 0, \epsilon)$ $(0, \xi, \xi)$
11	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon^2 & 1 & \xi \\ \epsilon & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \xi \\ 0 & 1 & 0 \\ \xi & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, \xi, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \xi)$ $(0, 0, \xi)$ $(0, \xi, \epsilon)$
12	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \xi \\ 0 & \epsilon & \epsilon \\ \xi & \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \xi & \xi \\ \xi & 1 & 0 \\ \xi & 0 & 1 \end{pmatrix}$	$(\epsilon, \epsilon, 1)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ (ξ, ξ, ϵ) (ξ, ξ, ϵ) (ξ, ξ, ϵ)

TABLE I. (Continued)

No.	M_ℓ	M_D	M_R	m_i^D/m_D m_i^R/M_{B-L}	$(\theta_{12}^\ell, \theta_{13}^\ell, \theta_{23}^\ell)$ $(\theta_{12}^D, \theta_{13}^D, \theta_{23}^D)$ $(\theta_{12}^R, \theta_{13}^R, \theta_{23}^R)$ $(\theta_{12}^R, \theta_{13}^R, \theta_{23}^R)$
13	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon^2 - \xi & 1 & \xi \\ \epsilon & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \epsilon \\ 0 & 1 & 0 \\ \epsilon & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\xi, \epsilon, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \xi)$ (ξ, ξ, ξ) $(\epsilon, \epsilon, \frac{\pi}{4})$
14	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & \epsilon^2 \\ \epsilon & 1 & \epsilon^2 - \xi \\ \epsilon^2 & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & \epsilon^2 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ $(\xi, \epsilon, \epsilon)$ (ϵ, ξ, ξ) $(\epsilon, \epsilon, \xi)$
15	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \epsilon \\ \epsilon^2 & 1 & \epsilon^2 \\ \epsilon & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & \epsilon & 1 \\ \epsilon & 1 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ $(0, \frac{\pi}{4}, \epsilon)$ $(\epsilon^2, \epsilon^2, \epsilon^2)$ $(\epsilon, \frac{\pi}{4}, \epsilon)$
16	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon^2 \\ 1 & 1 & \epsilon^2 \\ \epsilon & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon^2, \epsilon, \epsilon)$ $(\frac{\pi}{4}, \epsilon^2, 0)$ $(\frac{\pi}{4}, \epsilon, \epsilon^2)$
17	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & \epsilon \\ 1 & \epsilon & 1 \\ 1 & \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \epsilon & 1 \\ \epsilon & 1 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$	$(\epsilon, \epsilon, 1)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(\epsilon, \epsilon^2, \frac{\pi}{4})$ $(0, \frac{\pi}{4}, \epsilon^2)$ $(\epsilon, \frac{\pi}{4}, \epsilon^2)$
18	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & 0 \\ \epsilon & \epsilon & \epsilon \\ \epsilon^2 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$(\epsilon, \epsilon, 1)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{4}, 0, \epsilon)$ $(\epsilon^2, \epsilon^2, \epsilon^2)$ $(\frac{\pi}{4}, \epsilon, \epsilon)$
19	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & \epsilon \\ \epsilon & \epsilon & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$(\epsilon, \epsilon, 1)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon, \epsilon^2, \epsilon)$ $(0, \frac{\pi}{4}, \epsilon)$ $(0, \frac{\pi}{4}, 0)$ (ξ, ϵ, ξ)
20	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & \epsilon^2 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(0, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon, \epsilon, \epsilon)$
21	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & \epsilon^2 \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & \epsilon^2 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(0, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon^2, \epsilon, \frac{\pi}{4})$
22	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & 0 \\ \xi & 1 & \epsilon \\ \xi & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \xi \\ 0 & 1 & 0 \\ \xi & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(0, \epsilon, \frac{\pi}{4})$ $(\xi, \epsilon, \epsilon)$ $(\xi, \xi, \frac{\pi}{4})$
23	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & 0 \\ \xi & 1 & 1 \\ \xi & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \xi & \xi \\ \xi & 1 & 0 \\ \xi & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(0, \epsilon, \frac{\pi}{4})$ $(\xi, \epsilon, \frac{\pi}{4})$ (ξ, ξ, ϵ)
24	$\begin{pmatrix} 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \xi \\ 1 & 1 & \xi \\ \xi & \xi & 1 \end{pmatrix}$	$(1, \epsilon, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \epsilon, \xi)$ $(\xi, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{4}, \xi, \xi)$

TABLE I. (Continued)

No.	M_ℓ	M_D	M_R	m_i^D/m_D m_i^R/M_{B-L}	$(\theta_{12}^\ell, \theta_{13}^\ell, \theta_{23}^\ell)$ $(\theta_{12}^D, \theta_{13}^D, \theta_{23}^D)$ $(\theta_{12}^{D'}, \theta_{13}^{D'}, \theta_{23}^{D'})$ $(\theta_{12}^R, \theta_{13}^R, \theta_{23}^R)$
25	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & 0 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon^2 \\ \epsilon & 1 & 0 \\ \epsilon^2 & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ $(0, \frac{\pi}{4}, \epsilon)$ $(\epsilon, \frac{\pi}{4}, \epsilon)$ $(\epsilon, 0, \epsilon)$
26	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & 0 \\ \epsilon & 1 & \epsilon \\ \epsilon & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon^2 - \xi \\ \epsilon & 1 & 0 \\ \epsilon^2 - \xi & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(0, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \epsilon)$ $(\epsilon, \xi, \epsilon)$
27	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \xi \\ \epsilon & \epsilon & 1 \\ \epsilon & \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$(\epsilon, \epsilon, 1)$ $(\epsilon, 1, 1)$	$(\epsilon, \epsilon, \xi)$ $(\frac{\pi}{4}, \xi, \frac{\pi}{4})$ (ϵ, ξ, ξ) $(\epsilon, \epsilon, \frac{\pi}{4})$
28	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & \epsilon^2 \\ \epsilon & 1 & 1 \\ \epsilon^2 & 0 & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon^2 - \xi & \epsilon \\ \epsilon^2 - \xi & 1 & 0 & 0 \\ \epsilon & 0 & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ $(0, \xi, \epsilon)$ $(\epsilon, \xi, \frac{\pi}{4})$ $(\xi, \epsilon, \epsilon)$
29	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon^2 & \epsilon \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 1 & \epsilon \\ 0 & 1 & \xi \\ \epsilon & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & \xi & 1 \\ \xi & 1 & \xi \\ 1 & \xi & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, \frac{\pi}{4}, \epsilon)$ $(\frac{\pi}{4}, \frac{\pi}{4}, \xi)$ $(\epsilon, \epsilon, \xi)$ $(\xi, \frac{\pi}{4}, \xi)$ $(\epsilon, \frac{\pi}{4}, \xi)$
30	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 1 & \epsilon \\ \epsilon^2 & 1 & \epsilon^2 \\ \epsilon & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & \xi & 1 \\ \xi & 1 & \xi \\ 1 & \xi & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(\epsilon, \epsilon, \xi)$ $(\xi, \frac{\pi}{4}, \xi)$ $(\epsilon, \frac{\pi}{4}, \xi)$
31	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 1 & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & \epsilon & 1 \\ \epsilon & 1 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(\xi, \epsilon, 0)$ $(\xi, \frac{\pi}{4}, \epsilon)$ $(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$
32	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & 0 \\ 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \xi \\ 1 & 1 & \xi \\ \xi & \xi & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\xi, 0, \frac{\pi}{4})$ $(\frac{\pi}{4}, \xi, \xi)$ $(\frac{\pi}{4}, \xi, \xi)$ (ξ, ξ, ξ)
33	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \xi & \xi \\ \xi & 1 & 0 \\ \xi & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\xi, \frac{\pi}{4}, \frac{\pi}{4})$ $(0, \epsilon, \frac{\pi}{4})$ (ξ, ξ, ξ) (ξ, ξ, ξ)
34	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \xi & \epsilon \\ \epsilon & 1 & \epsilon \\ \epsilon & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & \xi & 0 \\ \xi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\xi, \frac{\pi}{4}, \frac{\pi}{4})$ $(0, \epsilon, \xi)$ $(\xi, \xi, \frac{\pi}{4})$ $(0, \epsilon, \xi)$
35	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \epsilon & 1 \\ \epsilon & 1 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{4}, \frac{\pi}{4}, 0)$ $(\epsilon, \frac{\pi}{4}, \epsilon)$ $(0, \frac{\pi}{4}, \xi)$
36	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 1 & \epsilon \\ 0 & 1 & \epsilon \\ \epsilon & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & \epsilon^2 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \xi)$ $(\epsilon, \frac{\pi}{4}, \epsilon)$ $(\epsilon, \epsilon, 0)$

TABLE I (Continued)

No.	M_ℓ	M_D	M_R	m_i^D/m_D m_i^R/M_{B-L}	$(\theta_{12}^\ell, \theta_{13}^\ell, \theta_{23}^\ell)$ $(\theta_{12}^D, \theta_{13}^D, \theta_{23}^D)$ $(\theta_{12}^{D'}, \theta_{13}^{D'}, \theta_{23}^{D'})$ $(\theta_{12}^R, \theta_{13}^R, \theta_{23}^R)$
37	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 1 & \epsilon \\ 0 & 1 & 0 \\ \epsilon & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & \epsilon^2 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(0, \frac{\pi}{4}, \xi)$ $(\frac{\pi}{4}, \frac{\pi}{4}, \xi)$ $(\epsilon, \frac{\pi}{4}, 0)$ $(\epsilon, \epsilon, \epsilon)$
38	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \epsilon^2 \\ \epsilon^2 - \xi & 1 & \epsilon \\ \epsilon^2 & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & \epsilon \\ \epsilon^2 & 1 & 0 \\ \epsilon & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(0, \epsilon, \frac{\pi}{4})$ (ξ, ξ, ξ) $(\xi, \epsilon, \epsilon)$
39	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \epsilon^2 \\ \epsilon^2 - \xi & 1 & 0 \\ \epsilon^2 & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & \xi & \epsilon \\ \xi & 1 & 0 \\ \epsilon & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(0, \epsilon, \frac{\pi}{4})$ (ξ, ξ, ϵ) (ξ, ϵ, ξ)
40	$\begin{pmatrix} 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & 0 \\ \epsilon^2 - \xi & 0 & 1 \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \xi \\ \epsilon & 1 & 0 \\ \xi & 0 & 1 \end{pmatrix}$	$(\epsilon, \epsilon, 1)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(\epsilon, 0, \frac{\pi}{4})$ (ξ, ξ, ϵ) (ϵ, ξ, ξ)
41	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & \epsilon \\ \epsilon^2 & 1 & 0 \\ \epsilon & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(0, \frac{\pi}{4}, \xi)$ $(\frac{\pi}{4}, \frac{\pi}{4}, \xi)$ $(\xi, \frac{\pi}{4}, \frac{\pi}{4})$ $(0, \epsilon, \epsilon)$
42	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon^2 \\ \epsilon & 1 & 0 \\ \epsilon^2 & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(0, \frac{\pi}{4}, \xi)$ $(\frac{\pi}{4}, \frac{\pi}{4}, \xi)$ $(\xi, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon, 0, \epsilon)$
43	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon^2 & \epsilon^2 - \xi \\ \epsilon^2 - \xi & \epsilon & \epsilon^2 \\ 1 & \xi & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \xi & 1 \\ \xi & \epsilon & \xi \\ 1 & \xi & 1 \end{pmatrix}$	$(\epsilon^2, \epsilon, 1)$ $(\epsilon^2, \epsilon, 1)$	$(\epsilon, \epsilon, \frac{\pi}{4})$ (ϵ, ξ, ξ) $(\epsilon, \frac{\pi}{4}, \xi)$ $(\xi, \frac{\pi}{4}, \xi)$
44	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon^2 & \epsilon^2 - \xi \\ 1 & \epsilon & 1 \\ 1 & \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \xi & 1 \\ \xi & \epsilon & \xi \\ 1 & \xi & 1 \end{pmatrix}$	$(\epsilon^2, \epsilon, 1)$ $(\epsilon^2, \epsilon, 1)$	$(\epsilon, \epsilon, \xi)$ $(\epsilon, \xi, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \xi)$ $(\xi, \frac{\pi}{4}, \xi)$
45	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon^2 & \epsilon \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & \epsilon^2 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, \frac{\pi}{4}, \epsilon)$ $(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon^2)$ $(\epsilon^2, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon^2, \epsilon, \frac{\pi}{4})$
46	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \epsilon \\ 1 & 1 & \epsilon^2 \\ 1 & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \xi \\ 1 & 1 & \xi \\ \xi & \xi & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, 0, \xi)$ $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{4}, \epsilon, \xi)$ $(\frac{\pi}{4}, \xi, \xi)$
47	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \epsilon \\ 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, 0, \xi)$ $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{4}, \xi, \xi)$ $(\frac{\pi}{4}, \xi, \epsilon)$
48	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon & \epsilon \\ \epsilon^2 & \epsilon & \epsilon \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon^2 & \xi \\ \epsilon^2 & \epsilon & 0 \\ \xi & 0 & 1 \end{pmatrix}$	$(\epsilon^2, \epsilon, 1)$ $(\epsilon^2, \epsilon, 1)$	$(\epsilon, \xi, \frac{\pi}{4})$ $(\frac{\pi}{4}, \epsilon, \epsilon)$ $(\xi, 0, \xi)$ $(\epsilon, \xi, 0)$

TABLE I. (Continued)

No.	M_ℓ	M_D	M_R	m_i^D/m_D m_i^R/M_{B-L}	$(\theta_{12}^\ell, \theta_{13}^\ell, \theta_{23}^\ell)$ $(\theta_{12}^{D'}, \theta_{13}^{D'}, \theta_{23}^{D'})$ $(\theta_{12}^{D''}, \theta_{13}^{D''}, \theta_{23}^{D''})$ $(\theta_{12}^R, \theta_{13}^R, \theta_{23}^R)$
49	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon & 0 \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$(\epsilon^2, 1, \epsilon)$ $(\epsilon^2, \epsilon, 1)$	$(\epsilon^2, \epsilon, 0)$ $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon, \epsilon^2, \frac{\pi}{4})$ $(\epsilon^2, \epsilon, \frac{\pi}{4})$
50	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & 0 & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$(\epsilon^2, 1, \epsilon)$ $(\epsilon^2, \epsilon, 1)$	$(\epsilon^2, \epsilon, 0)$ $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon, \epsilon^2, \frac{\pi}{4})$ $(\epsilon^2, \epsilon, \frac{\pi}{4})$
51	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & \epsilon \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 - \xi & \epsilon \\ \epsilon & 1 & \epsilon \\ \epsilon & 1 & \epsilon \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon^2 \\ \epsilon & \epsilon^2 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	(ξ, ξ, ϵ) $(\xi, \frac{\pi}{4}, \frac{\pi}{4})$ (ϵ, ξ, ξ) $(\epsilon, 0, \frac{\pi}{4})$
52	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & \epsilon \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	(ξ, ξ, ϵ) $(\xi, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon, \xi, \frac{\pi}{4})$ $(\epsilon, 0, \xi)$
53	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & \epsilon \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \epsilon \\ 0 & 1 & 0 \\ \epsilon & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	(ξ, ξ, ϵ) $(\xi, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon, \xi, \frac{\pi}{4})$ $(0, \epsilon, \xi)$
54	$\begin{pmatrix} 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & \epsilon \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix}$	$(1, \epsilon, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \epsilon, \epsilon)$ $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon^2, \frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{4}, \epsilon, 0)$
55	$\begin{pmatrix} 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ \epsilon & \epsilon & \epsilon \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix}$	$(1, \epsilon, \epsilon)$ $(\epsilon, 1, 1)$	$(\frac{\pi}{4}, \epsilon, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \epsilon)$ $(\epsilon^2, \frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{4}, \epsilon, 0)$
56	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon & \epsilon \\ 1 & 1 & \epsilon \\ \epsilon^2 & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon^2, \epsilon, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \epsilon^2)$ $(\frac{\pi}{4}, \epsilon, \epsilon^2)$ $(\frac{\pi}{4}, \epsilon, \epsilon^2)$
57	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon & \epsilon \\ 1 & 1 & \epsilon \\ 1 & 1 & \xi \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	(ξ, ϵ, ξ) $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{4}, \epsilon, \xi)$ $(\frac{\pi}{4}, \epsilon, \xi)$
58	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon & \xi \\ \epsilon & \epsilon & 1 \\ \epsilon & \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & 0 & \epsilon \\ 0 & \epsilon & \xi \\ \epsilon & \xi & 1 \end{pmatrix}$	$(\epsilon^2, \epsilon, 1)$ $(\epsilon^2, \epsilon, 1)$	$(\epsilon, \epsilon, \xi)$ $(\frac{\pi}{4}, \xi, \frac{\pi}{4})$ (ξ, ϵ, ξ) (ξ, ϵ, ξ)
59	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \epsilon & 1 \\ 1 & \epsilon^2 & 1 \\ 1 & \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \epsilon & 1 \\ \epsilon & 1 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$	$(1, \epsilon, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(\epsilon, \frac{\pi}{4}, \epsilon^2)$ $(\epsilon, \frac{\pi}{4}, \epsilon^2)$
60	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \epsilon & \epsilon \\ 1 & 1 & 0 \\ \xi & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, 0, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \xi)$ $(\frac{\pi}{4}, \xi, \xi)$ $(\frac{\pi}{4}, \epsilon, \xi)$

TABLE I. (Continued)

No.	M_ℓ	M_D	M_R	m_i^D/m_D m_i^R/M_{B-L}	$(\theta_{12}^\ell, \theta_{13}^\ell, \theta_{23}^\ell)$ $(\theta_{12}^D, \theta_{13}^D, \theta_{23}^D)$ $(\theta_{12}^{D'}, \theta_{13}^{D'}, \theta_{23}^{D'})$ $(\theta_{12}^R, \theta_{13}^R, \theta_{23}^R)$
61	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & 0 & \epsilon \\ \xi & 1 & 1 \\ \epsilon^2 & \epsilon & \xi \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$(\epsilon^2, 1, \epsilon)$ $(\epsilon^2, \epsilon, 1)$	$(\xi, \epsilon, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \xi)$ $(\xi, \xi, \frac{\pi}{4})$ $(\xi, 0, \frac{\pi}{4})$
62	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon & 0 \\ \epsilon^2 & 1 & 1 \\ \epsilon^2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$(\epsilon^2, 1, \epsilon)$ $(\epsilon^2, \epsilon, 1)$	(ξ, ϵ, ξ) $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\xi, \xi, \frac{\pi}{4})$ $(\xi, 0, \frac{\pi}{4})$
63	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & 0 & \epsilon \\ \epsilon^2 & 1 & 1 \\ \epsilon^2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$(\epsilon^2, 1, \epsilon)$ $(\epsilon^2, \epsilon, 1)$	(ξ, ϵ, ξ) $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\xi, \xi, \frac{\pi}{4})$ $(\xi, 0, \frac{\pi}{4})$
64	$\begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon & 0 \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \epsilon \\ 0 & 1 & 0 \\ \epsilon & 0 & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(0, \epsilon, \xi)$ $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\epsilon, \epsilon, \frac{\pi}{4})$ $(0, \epsilon, \xi)$
65	$\begin{pmatrix} 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \xi & 1 \\ 0 & \epsilon & \xi \\ \epsilon^2 & \xi & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon^2 & \xi \\ \epsilon^2 & \epsilon & 0 \\ \xi & 0 & 1 \end{pmatrix}$	$(\epsilon^2, \epsilon, 1)$ $(\epsilon^2, \epsilon, 1)$	$(\frac{\pi}{4}, \epsilon, \frac{\pi}{4})$ $(\xi, \frac{\pi}{4}, \xi)$ $(\xi, 0, \xi)$ $(\epsilon, \xi, 0)$
66	$\begin{pmatrix} 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon^2 & \xi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \xi & 1 \\ \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon^2 & \xi \\ \epsilon^2 & \epsilon & 0 \\ \xi & 0 & 1 \end{pmatrix}$	$(\epsilon^2, \epsilon, 1)$ $(\epsilon^2, \epsilon, 1)$	$(\frac{\pi}{4}, \epsilon, \xi)$ $(\xi, \frac{\pi}{4}, \frac{\pi}{4})$ $(\xi, 0, \xi)$ $(\epsilon, \xi, 0)$
67	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & \epsilon \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & \epsilon \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \xi & 1 \\ \xi & 1 & \xi \\ 1 & \xi & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, \xi, \epsilon)$ $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(0, \frac{\pi}{4}, \xi)$
68	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & \epsilon \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \epsilon \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \epsilon & 1 \\ \epsilon & 1 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, \xi, \epsilon)$ $(\epsilon, \frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{4}, \frac{\pi}{4}, \xi)$ $(0, \frac{\pi}{4}, \epsilon)$
69	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & \epsilon^2 & \epsilon \\ 1 & 1 & 1 \\ \epsilon & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & \xi & 1 \\ \xi & 1 & \xi \\ 1 & \xi & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, \xi, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \epsilon)$ $(\frac{\pi}{4}, \frac{\pi}{4}, \epsilon)$ $(0, \frac{\pi}{4}, \xi)$
70	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & 1 \\ 0 & \epsilon^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 & \epsilon \\ 1 & 1 & 1 \\ \epsilon & \epsilon & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & \epsilon & 1 \\ \epsilon & 1 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$	$(\epsilon, 1, \epsilon)$ $(\epsilon, 1, 1)$	$(\epsilon, \xi, \frac{\pi}{4})$ $(\epsilon, \frac{\pi}{4}, \epsilon)$ $(\frac{\pi}{4}, \frac{\pi}{4}, \xi)$ $(0, \frac{\pi}{4}, \epsilon)$
71	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon & \epsilon^2 \\ 1 & \epsilon & 1 \\ 1 & \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \epsilon & 1 \\ \epsilon & \epsilon & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$	$(\epsilon^2, \epsilon, 1)$ $(\epsilon^2, \epsilon, 1)$	$(\xi, \xi, 0)$ $(\frac{\pi}{4}, \xi, \frac{\pi}{4})$ $(\xi, \frac{\pi}{4}, 0)$ $(\xi, \frac{\pi}{4}, \epsilon)$
72	$\begin{pmatrix} 0 & 0 & \xi \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon^2 & \epsilon & \epsilon^2 \\ 1 & \epsilon & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$(\epsilon^2, \epsilon, 1)$ $(\epsilon^2, \epsilon, 1)$	$(\xi, \xi, 0)$ $(\frac{\pi}{4}, \xi, \frac{\pi}{4})$ $(\xi, \frac{\pi}{4}, \epsilon)$ $(\xi, \frac{\pi}{4}, 0)$

E. Dependence on input and renormalization group effects

Let us now discuss how the choice of the input parameters from measurement affects our results. First of all, we use the current best-fit values as an input [cf. Eq. (23)]. Any other choice of experimental input parameters will give different results, but our procedure can be equally well applied to any other preferred choice of input values. We have chosen these input values because these results can be expected to remain valid for the next ten years or so, unless the best-fit values change (such as if $\sin^2 2\theta_{13} \sim 0.1$ were indeed found). We have also computed the data set for different values of θ_{13} , but a presentation of these results would clearly exceed the scope of this paper.

As far as the charged lepton mass spectra in Eq. (7) are concerned, a variation of the mass ratios will not have any effect at all on the selection and extraction of the texture sets: Any change in the spectrum drops completely out of our routine. The particular choice of the charged lepton mass hierarchies in Eq. (7) has been motivated by comparison with the down quark spectrum in $SU(5)$. Choosing a different parametrization of the charged lepton masses, for example, with different powers of ϵ , will of course have an effect on the form of the extracted texture sets for M_ℓ , but it does not affect the selection of the realizations in any way. In particular, a modification of the charged lepton mass spectra in Eq. (7), e.g., to implement the Georgi-Jarlskog relation $m_\mu:m_\tau = 3m_s:m_b$ [48], would not change any of our results for the textures. In fact, one can easily obtain the textures of the charged leptons for any other choice of the hierarchy by using U_ℓ directly.

The extended QLC hypotheses in Sec. III B should hold at high energies such as M_{GUT} , but they are compared in our method with low-energy data. We therefore have to address the stability of the extended QLC assumptions under RG running from the high scale down to, say, around ~ 1 GeV. Generally, the empirical QLC sum rule $\theta_{12} = \pi/4 - \theta_C$ is satisfied up to a precision of about $\sim 1^\circ$. It is therefore reasonable to take in our routine nonzero mixing angles θ_{ij}^x into account that can be as small as $\epsilon^2 \sim 2^\circ$. Moreover, it is known that the Cabibbo angle θ_C does practically not run [49], and $V_{cb} \sim \epsilon^2$ changes typically only by a factor smaller than 2 when running from ~ 1 GeV up to the Planck scale $\sim 10^{19}$ GeV [50].

Let us have a more precise look at the RG evolution of neutrino masses and mixings [51]. First, note that, due to the smallness of the charged lepton Yukawa couplings, the running of a possibly maximal atmospheric mixing angle θ_{23} is negligible, unless one works in the MSSM with large $\tan\beta$ [43]. Simple expressions for the running of lepton mixing angles have been recently presented in Ref. [52]: When running M_{eff} from the GUT scale down to low energies, the corrections to the leptonic mixing angle θ_{ij} are smaller than $\sim |m_i + m_j|^2 (|m_i|^2 - |m_j|^2)^{-1} \times 10^{-2}$, where m_i and m_j are the eigenvalues of the i th and j th

neutrino mass eigenstates of M_{eff} at the GUT scale. An appreciable running of leptonic mixing angles can thus only be expected in the IH or QD case. For NH neutrinos, however, the corrections are $\lesssim 1^\circ$ and, thus, negligible. Moreover, a tuning of phases always allows to switch off completely any RG effect on neutrino mixing angles—even in the case of inverse hierarchical and degenerate neutrinos [52]. A similar result has been obtained in the bottom-up approach in Ref. [53], where the starting point are the fixed low-energy observables. In addition, while the overall neutrino mass scale is affected by RG running, the neutrino mass ratios are hardly changed. Since our results should be very stable under RG running for the NH case (irrespective of the phases), we will focus in this paper on this type of hierarchy.

IV. CURRENTLY ALLOWED REALIZATIONS FOR THE NH CASE

In this section, we focus on the constructed set of currently allowed seesaw realizations for a normal neutrino mass hierarchy (NH case). We obtain 173 084 different realizations for the NH case, which reduce to 8030 cases if one does not count different phase combinations as different cases. This leads to 1981 different texture sets, i.e., different combinations of M_ℓ , M_D , and M_R . Naturally, we cannot show all of these possibilities in this paper. In Sec. V, we will therefore apply some selection criteria to reduce this data set further and present the texture sets which seem to be most interesting to us. In this section, however, we discuss general features and some statistics of the constructed seesaw realizations for NH neutrinos.

We concentrate on NH also because RG effects on neutrino mass ratios and mixing angles are expected to be small in this case (see Sec. V). This means that our generic assumptions, which may hold at some high energy scale (say at $\sim 10^{16}$ GeV), do not change significantly when running down to low energies where we match to experiment. In the NH case, one can easily diagonalize M_{eff} for the allowed realizations in order to check that Eq. (24) produces observables in agreement with current data. We have done this exercise: We have determined U_ν by diagonalizing M_{eff} and computed $U_{\text{PMNS}} = U_\ell^\dagger U_\nu$, employing the corresponding matrix U_ℓ . From U_{PMNS} , one can read off the mixing angles just as described in Ref. [25].¹² Note that we do not expect to reproduce exactly our input values in Eq. (23), since we do not require exact matching precision in Eq. (24) (see also Appendix A 2).

To describe the compatibility of a realization with current data, we use the performance indicator

¹²See, e.g., also Ref. [54] for a discussion of neutrino mass matrix diagonalization.

$$\chi^2 \equiv \left(\frac{\sin^2 \theta_{12} - 0.3}{0.3 \times \sigma_{12}} \right)^2 + \left(\frac{\sin^2 \theta_{23} - 0.5}{0.5 \times \sigma_{23}} \right)^2, \quad (26)$$

(e.g., $\chi^2 = 11.83$ corresponds to a 3σ C.L. exclusion for 2 d.o.f.). This corresponds to a Gaussian χ^2 approximation in $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$ with the current best-fit values. For the relative 1σ errors, we use $\sigma_{12} \simeq 9\%$ (for $\sin^2 \theta_{12}$) and $\sigma_{23} \simeq 16\%$ (for $\sin^2 \theta_{23}$) [14]. Note that we only find $\sin^2 \theta_{13} \ll 0.04$ below the current bound, i.e., we do not have to impose an additional selection criterion. Figure 4 shows the distribution of the valid seesaw realizations as a function of χ^2 defined in Eq. (26). Obviously, Eq. (24) already ensures that the neutrino mixing angles of each realization are compatible with current bounds. It turns out that in all valid cases $\theta_{13} \ll 1^\circ$ and only 6.5% of the realizations lead to $11.83 \lesssim \chi^2 \lesssim 17$ (which corresponds to a C.L. between 3 and 4σ for 2 d.o.f.). Therefore, the selected realizations are all in perfect agreement with current data. Note that one might naively expect to find realizations with θ_{13} around the best-fit value of 0° since this value has been used as an input for $M_{\text{eff}}^{\text{exp}}$. However, in almost all cases one obtains θ_{23} to be around 50° despite the best-fit input value of 45° .

Figure 5 shows the distribution of mass spectra or hierarchies proportional to (m_1^D, m_2^D, m_3^D) and (m_1^R, m_2^R, m_3^R) for M_D and M_R , respectively (each normalized to the corresponding heaviest mass eigenvalue of M_D and M_R), for the NH case. These distributions are obtained by simply counting the number of realizations with a certain mass spectrum. Observe that M_R has as a mass spectrum only $(\epsilon^2, \epsilon, 1)$ and $(\epsilon, 1, 1)$. Note that the spectra labeled as $(\epsilon^2, \epsilon, 1)$ include also strongly hierarchical cases where the right-handed Majorana neutrino mass spectrum can actually be $(\epsilon^n, \epsilon, 1)$, with some suitable $n \geq 3$ (see Appendix A 1). Figure 5 shows that, for M_R , in all valid seesaw realizations, the mildly hierarchical mass spectrum $(\epsilon, 1, 1)$ clearly dominates the (strongly) hierarchical spectrum $(\epsilon^n, \epsilon, 1)$ with $n \geq 2$. As it will turn out later in Sec. V, this observation is also supported at the texture level: more

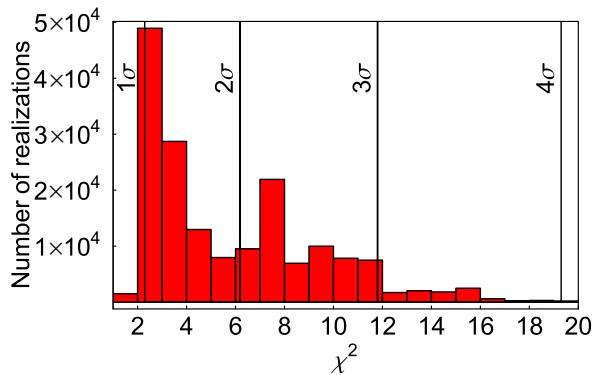


FIG. 4 (color online). Distribution of valid seesaw realizations for the NH case as a function of χ^2 as defined in Eq. (26). The values of θ_{13} are all in agreement with current data.

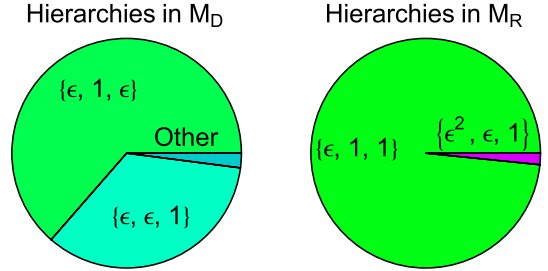


FIG. 5 (color online). Distributions of hierarchies in M_D (left) and M_R (right) leading to a NH neutrino masses for all valid seesaw realizations.

than 80% of the extracted textures lead to a mild mass hierarchy $(\epsilon, 1, 1)$ for the right-handed neutrino masses. We will come back to this point in the next paragraph. The absence of a degenerate mass spectrum for M_R in the NH (and IH) case is a simple consequence and selection effect of Eq. (A1c) in combination with the assumption $M_{\text{eff}}^{\text{exp}} \sim \text{diag}(\epsilon^2, \epsilon, 1)$. There are many more possibilities for the mass spectra of M_D but they are dominated by the types $(\epsilon, 1, \epsilon)$ and $(\epsilon, \epsilon, 1)$. In Fig. 5, the pie piece “Other” also contains the hierarchy $(\epsilon^2, \epsilon, 1)$, which implies that we have the same hierarchy in M_R (but not vice versa). In our method, no charged-lepton or quark-type hierarchy is produced. We find from Fig. 5 that there are many possibilities to obtain a normal neutrino mass hierarchy, but one cannot claim that this hierarchy appears *typically* in M_D or M_R and then translates into M_{eff} .

The distributions of mass spectra in Fig. 5 may have immediate relevance for leptogenesis [55] when crudely extrapolating our results to the CP nonconserving case. In at least 80% of the cases that we found (cf. Sec. V), the right-handed neutrino mass spectrum is of the mildly hierarchical form $(\epsilon, 1, 1)$. Thus, if the mass m_1^R of the lightest right-handed neutrino is in the range $m_1^R \lesssim 10^{12}$ GeV, the seesaw scale set by the mass of the heaviest right-handed neutrino m_3^R would have to be significantly lower than the usual $B - L$ breaking scale $\sim 10^{14}$ GeV. For the mildly hierarchical right-handed neutrino mass spectrum $(\epsilon, 1, 1)$, successful leptogenesis might be achieved in two ways: (i) via resonant leptogenesis [56] (for recent models see, e.g., Ref. [57]) or (ii) by taking flavor effects into account [58] (for a connection with low-energy CP violation see, e.g., Ref. [59]). In the resonant limit, m_3^R could be as low as several TeV, thereby making this scenario testable at a collider. Strongly hierarchical right-handed neutrino masses, which is the standard case considered in the literature for leptogenesis, are in our analysis found to be by about a factor of 5 less abundant than the mild hierarchy. The possible strongly hierarchical right-handed neutrino mass spectra are all of the type $(\epsilon^n, \epsilon, 1)$, where $n \geq 2$. Allowing n to be sufficiently large (say $n = 8$), the strongly hierarchical case can fit into a scheme with a seesaw scale of the order $m_3^R \sim 10^{14}$ GeV and sufficient

Mixing angles for normal hierarchy

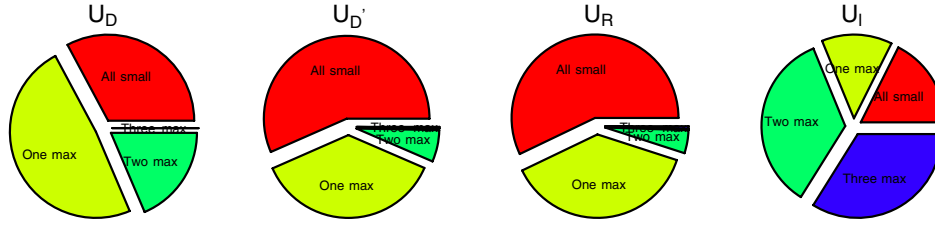


FIG. 6 (color online). Distributions of mixings in U_D , $U_{D'}$, U_R , and U_ℓ (in columns) leading to a normal neutrino mass hierarchy. The different pie labels refer to the number of maximal mixing angles, where “All small” corresponds to all mixing angles $\leq \epsilon$.

Special cases for normal hierarchy

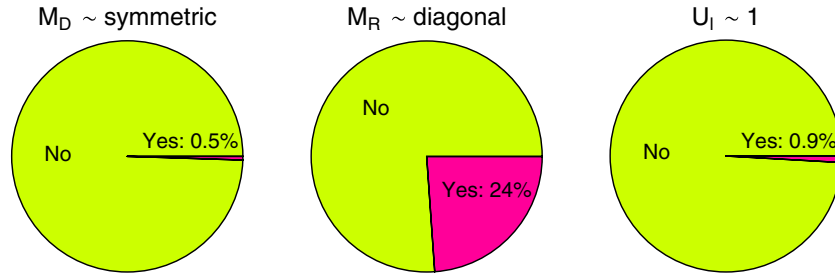


FIG. 7 (color online). Fraction of special cases for normal mass hierarchy. The different pies show the fractions of the cases with symmetric M_D , diagonal M_R , and $U_\ell \simeq \mathbb{1}$ of all allowed realizations (not texture sets). This figure is based on the mixing matrices, where $U_D \simeq U_{D'}$ in the first case, $U_R \simeq \mathbb{1}$ and $U_\ell \simeq \mathbb{1}$ in the second and third case, respectively. For the similarity condition “ \simeq ,” we allow ϵ^2 deviations in the mixing angles. For instance, for an exact $U_R = \mathbb{1}$, one would have only 2% of all realizations.

baryon asymmetry could again be generated through flavored leptogenesis.

Figure 6 shows the distributions of mixing angles, where we concentrate on the number of maximal (“max”) mixing angles $\theta_{ij}^x = \pi/4$ appearing in U_D , $U_{D'}$, U_R , and U_ℓ . If there is no maximal mixing angle, we call the scenario “All small.” Let us first note that the distributions of the mixing angles in M_ν are very different from U_ℓ . In U_ℓ , we often find large mixings, which means that the large lepton mixing angles are not necessarily created in the neutrino sector, but can also come very often from the charged lepton sector. Note that the pie slice “All small” in U_ℓ represents more or less CKM-like mixings in U_ℓ . There are also many possibilities with “trimaximal” mixing¹³ (i.e., all three mixing angles θ_{ij}^x are maximal for some given sector x) in U_ℓ . This is different from the other mixing matrices, where trimaximal mixing hardly occurs, and either one maximal mixing angle or only small mixing angles are preferred. In particular, in $U_{D'}$ and U_R , only small mixings are typical.

Figure 7 shows the fraction of realizations that exhibit symmetric M_D , and/or diagonal M_R , and/or $U_\ell \simeq \mathbb{1}$. In this figure, “ $M_D \sim$ symmetric” means that M_D is symmetric

up to possible corrections of the order ϵ^2 . It is evident that there are only very few realizations with symmetric M_D or $U_\ell \simeq \mathbb{1}$, and none with diagonal M_D , which is not surprising from what we have learned above. One may conclude from this result that there are plenty of possibilities to implement the seesaw mechanism without these constraints, which have, however, often been imposed in existing literature.

Generally, note that, while one may argue that one can construct more possibilities in a more general and sophisticated framework, our realizations result from very generic and simple assumptions without adding another level of complexity. In this sense, our generic assumptions are much simpler than *ad hoc* constraints, such as requiring that U_ℓ be diagonal. It is interesting to note that we find for M_R many realizations that are close to a diagonal form.

V. SELECTION OF TEXTURES FOR THE NH CASE

In this section, we present a selection of texture sets for the NH case. These texture sets satisfy certain selection criteria listed in Appendix B 1. The full set of the texture sets is available in Ref. [60]. Table I shows these 72 texture sets for M_ℓ , M_D , and M_R , together with associated example realizations leading to these texture sets. For each texture set, we always choose the realization with the lowest χ^2 ,

¹³Not to be confused with tribimaximal mixing.

i.e., the realization which fits data best. For this realization, the table lists the mass spectra of M_D and M_R , as well as the mixing angles θ_{ij}^x in the different sectors. For each case in Table I, we have collected in Table V (in Appendix B 2) all the corresponding phases (0 or π) in order to allow for a complete reconstruction of the realizations and Yukawa coupling matrices. In addition, one can find there the PMNS mixing angles, as well as the number of realizations that become identified with each texture set through the texture reduction. In Table I, the parameter ξ can take the values $\xi \in \{0, \epsilon^2\}$.

In Table II, we divide some of the interesting textures/realizations from Table I into certain classes, such as lopsided M_ℓ (only θ_{23}^ℓ is maximal), anarchic¹⁴ M_D ($\theta_{13}^D = \theta_{23}^D = \theta_{12}^D = \theta_{23}^{D'} = \theta_{12}^{D'} = \frac{\pi}{4}$), lopsided M_D (at least one of the mixing angles θ_{12}^x and θ_{23}^x , $x = D, D'$, is maximal—anarchic cases are excluded), hierarchical M_R [hierarchical mass spectrum but no maximal mixing angle for M_R , i.e., $M_R^{\text{diag}} \propto \text{diag}(\epsilon^2, \epsilon, 1)$], semianarchic M_R (one of the angles θ_{12}^R or θ_{23}^R is maximal), “diamond-type” M_R ($\theta_{13}^R = \pi/4$ and the corresponding texture has a diamond shape [25]), and presence of a “dead angle ξ ” (a mixing angle $\xi \leq \epsilon^2$ that does not affect the corresponding matrix in the texture¹⁵). Note that, as already mentioned in Sec. IV, about 80% of all textures in Table I have a mildly hierarchical spectrum for the right-handed neutrino masses that is of the form $M_R^{\text{diag}} \propto \text{diag}(\epsilon, 1, 1)$, whereas a strongly hierarchical spectrum $M_R^{\text{diag}} \propto \text{diag}(\epsilon^n, \epsilon, 1)$ ($n \geq 2$) occurs in only roughly 20% of the cases. Although this classification may to a certain extent be incomplete, it could nevertheless serve to characterize significant features of the textures. Moreover, all realizations in Table I have in common that $\theta_{12} \simeq 33^\circ$, $\theta_{13} \simeq 0^\circ$, and $\theta_{23} \simeq 51^\circ$ (see Table V). Therefore, if future experiments measure θ_{23} smaller than 45° , the presented realizations could be tested.

Let us now illustrate how the textures in Table I could be generated in explicit models by considering two examples. For this purpose, assume an M -fold Z_N product flavor symmetry group $G_F = \prod_{i=1}^M Z_N^{(i)} = Z_N \times Z_N \times \cdots \times Z_N$. We suppose that for each individual group $Z_N^{(i)}$ there are two types of SM singlet flavon fields f_i and f'_i that carry different $Z_N^{(i)}$ charges but which are singlets under transformations of all the other groups $Z_N^{(j)}$, with $j \neq i$. All flavons shall acquire universal VEVs: $\langle f_i \rangle \simeq \langle f'_i \rangle \simeq v$, for $i = 1, 2, \dots, M$, where $v = \epsilon \cdot M_F$ and M_F is some fundamental Froggatt-Nielsen (see Sec. II B) messenger scale. Now, let us specialize to the case $M = 7$ and $N = 4$, and

¹⁴Cases of anarchic M_R cannot appear in Table I due to the selection criteria in Appendix B 1.

¹⁵For example, if a matrix element reads $A\epsilon^2 + B\xi\epsilon + \mathcal{O}(\epsilon^4)$, where A and B are order one coefficients, then the choice of $\xi \in \{0, \epsilon^2\}$ has no impact on the texture, i.e., $A\epsilon^2 + \mathcal{O}(\epsilon^4) \rightarrow \epsilon^2$ and $A\epsilon^2 + B\epsilon^3 + \mathcal{O}(\epsilon^4) \rightarrow \epsilon^2$.

TABLE II. Classification of textures and realizations which may be of special interest. Here, “ \simeq ” means up to phases. The case $M_D^{\text{diag}} \propto M_R^{\text{diag}}$ allows, in combination with Eq. (A1c), only for the mass spectra $(\epsilon^2, \epsilon, 1)$. Note that the relative fraction of hierarchical M_R^{diag} in Table I is much higher than prior to applying the selection criteria, as it is evident from Fig. 5.

Class	Texture No.
$U_\ell \simeq V_{\text{CKM}}$	46, 47
$U_\ell \simeq \mathbb{1}$	33, 34, 71, 72
$\theta_{ij}^\ell \neq \frac{\pi}{4}$	20, 27, 33–35, 44, 46, 47, 49–53, 57, 58, 62–64, 67, 68, 71, 72
Lopsided M_ℓ	1, 4–8, 10, 11, 13, 43, 48, 56, 60, 61, 69, 70
Bimaximal M_ℓ	9, 17, 21–23, 26, 32, 38–40, 55, 59, 65,
Trimaximal M_ℓ	3, 12, 14, 15, 16, 18, 19, 25, 28
$U_D \simeq U_{D'}$	12, 23
Anarchic M_D	24, 54
Lopsided M_D	8, 20, 21, 23, 27, 32, 33, 35, 41, 42, 45, 49, 50, 52, 53, 62–64, 67, 68
$U_R \simeq \mathbb{1}$	10, 33
Hierarchical M_R	48, 58, 65, 66
Semianarchic M_R	1, 16, 18, 24, 32, 46, 47, 49, 50, 54–57, 60–63
Diamond M_R	2, 3, 5, 9, 15, 17, 19, 29–31, 35, 43, 44, 67–72
$M_D^{\text{diag}} \propto M_R^{\text{diag}}$	43, 44, 48, 58, 65, 66, 71, 72
$M_R^{\text{diag}} \propto \text{diag}(\epsilon^2, \epsilon, 1)$	1, 43, 44, 48, 49, 50, 58, 61, 62, 63, 65, 66, 71, 72
Dead angle ξ	1, 6, 7, 9, 10, 13, 14, 24, 28, 30–33, 36–38, 41, 42, 46–48, 52, 53, 57, 61–64, 68, 70–72

assume that each pair of flavons f_i and f'_i carries the $Z_4^{(i)}$ charges $f_i \sim 1$ and $f'_i \sim 2$, i.e., f_i is singly and f'_i doubly charged under $Z_4^{(i)}$. Technically, this amounts to realizing fractional charges for the Z_2 subgroups of $Z_4^{(i)}$. We assign the leptons G_F quantum numbers as shown in Table III for two example models. In both models, we have assigned in Table III each lepton a row vector (q_1, q_2, \dots, q_7) , where q_i denotes the Z_4 charge of the lepton under the group $Z_4^{(i)}$ ($i = 1, 2, \dots, 7$). Models 1 and 2 in Table III, respectively, produce the texture sets 17 and 18 in Table I via the Froggatt-Nielsen mechanism. Since we have already found in Tables I and V valid realizations for these textures, we know, without any further calculation, that the order unity Yukawa couplings in model 1 and model 2 can be chosen such that they reproduce the lepton mass and mixing parameters in perfect agreement with data. Indeed, the explicit realizations in Tables I and V allow, if one wishes, for a complete reconstruction of such a valid set of order one Yukawa couplings.

The two models above represent only very specific examples of how one could directly apply Table I to

TABLE III. Assignment of G_F charges to the leptons. Models 1 and 2 lead, respectively, to the textures No. 17 and No. 18 in Table I (see text).

Field	Model 1	Model 2
ν_1^c	(0, 0, 0, 1, 0, 1, 1)	(2, 0, 0, 2, 0, 0, 1)
ν_2^c	(2, 0, 0, 1, 0, 1, 1)	(2, 0, 0, 2, 0, 0, 1)
ν_3^c	(0, 0, 0, 1, 0, 1, 1)	(0, 2, 0, 0, 2, 1, 0)
ℓ_1	(0, 2, 0, 0, 1, 0, 1)	(2, 0, 2, 2, 2, 1, 0)
ℓ_2	(0, 0, 0, 0, 1, 1, 0)	(2, 2, 0, 2, 2, 1, 0)
ℓ_3	(0, 0, 2, 1, 0, 0, 1)	(0, 2, 2, 2, 2, 0, 1)
e_1^c	(2, 2, 2, 1, 1, 1, 1)	(0, 0, 0, 0, 0, 1, 1)
e_2^c	(2, 2, 2, 0, 0, 1, 0)	(2, 2, 2, 0, 0, 3, 3)
e_3^c	(0, 2, 2, 0, 0, 0, 0)	(2, 2, 2, 2, 2, 3, 3)

identify the possible flavor symmetries and their breaking in a model. However, there are certainly many more possibilities. For example, we assumed here, for simplicity, copies of a single discrete Abelian flavor symmetry group, but new possibilities arise when considering product groups with different Z_N subgroups or also non-Abelian flavor symmetries. Moreover, note that, since $U_{\ell'}$ has been set equal to the unit matrix, Table I lists only a few percent of the actual total number of texture sets in extended QLC. A systematized scan for models generating the textures in extended QLC should also include these extra cases. In addition, it would be interesting to explore the compatibility with further constraints from GUT relations or cancellation of anomalies (see, e.g., Ref. [61]).

VI. IH AND QD CASE

Even though the main focus of our study is the case of NH neutrino masses, we have also calculated the valid realizations for IH and QD neutrinos using Eq. (12) for the respective mass hierarchies. These cases are qualitatively different from the NH case because RG effects may be relevant here. In this section, we present a qualitative discussion of the IH and QD cases.

A. IH spectrum

In the IH case, we find 797 928 different realizations, which corresponds to 10 196 different qualitative cases ignoring phases. This is about a factor of 5 more realizations than in the NH case. After texture reduction, one obtains 4268 different texture sets, i.e., different combinations of M_ℓ , M_D , and M_R , which is about twice the number as in the NH case. This indicates a higher redundancy at the level of the realizations as compared to the NH case, which we will comment on more explicitly in the QD case in Sec. VIB. Since some of the selection criteria for NH neutrino masses are based on the χ^2 selector, which is not defined in the IH case, the texture reduction of the texture sets for the IH case has to be carried out in a way

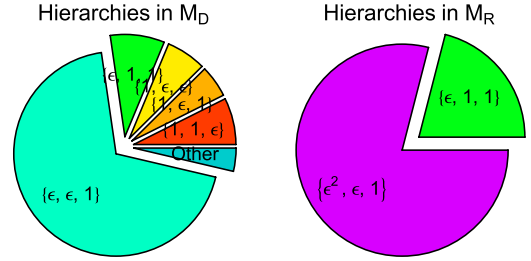


FIG. 8 (color online). Distributions of mass hierarchies for M_D (left) and M_R (right) leading to IH neutrino masses for all seesaw realizations.

that differs from the procedure described in Appendix B 1. However, by arguments similar to those in Appendix B 1, we find a reduction of the number of cases that is comparable to that in the NH case.

The mass spectra of M_D and M_R for the IH case are shown in Fig. 8. In contrast to the case of NH neutrino masses (cf. Fig. 5), M_D shows a larger variation of possible mass hierarchies. It turns out that most realizations for M_D have a mass hierarchy of the type $(\epsilon, \epsilon, 1)$. In M_R , we obtain the same mass spectra as for the NH case, but with a nearly opposite weighing. For M_D , one may have an inverse mass hierarchy, whereas M_R^{diag} has by construction a “normal” ordering (cf. Appendix A 1).

Figure 9 shows the distribution of maximal mixing angles for the IH case. Note that one observes a similar distribution for NH neutrinos (cf. Fig. 6), namely, that in most cases large leptonic mixing angles come from U_ℓ . However, for IH neutrino masses, there is a slight tendency to larger mixing angles in U_D , $U_{D'}$, and U_R , and smaller mixing angles in U_ℓ than in the NH case. The distribution of the special cases in Fig. 7 resembles that for the normal hierarchy. For example, M_R is approximately diagonal in about 15% of all cases.

B. QD spectrum

The case of QD neutrino masses is trickier from the computational point of view because of redundancy. As a simple example, consider Eq. (20b) with $M_{\text{eff}}^{\text{diag}} = \mathbb{1}$. In the real case, one can use any \hat{U}_{PMNS} and \hat{U}_ℓ in this equation in order to obtain $M_{\text{eff}}^{\text{exp}} = \mathbb{1}$, i.e., the different cases for \hat{U}_ℓ are not really qualitatively different and a statistical analysis, such as above, does not make much sense. In addition, the matching to experiment (or the experimental data as a selection criterion) is, in this way, rather meaningless. From Eq. (24), one can read off that we require $M_{\text{eff}}^{\text{th}} \propto \mathbb{1} + \mathcal{O}(\epsilon^3)$. This implies that any realization producing $M_{\text{eff}}^{\text{th}} \simeq c \cdot \mathbb{1}$ will be accepted by the algorithm, which are many. An interesting observation in the degenerate case is that the realizations divide into two qualitatively different cases: the trivial case, i.e., $M_D = M_R = \mathbb{1}$, and a large number of nontrivial cases. There is no such trivial case for IH neutrinos.

Mixing angles for inverted hierarchy

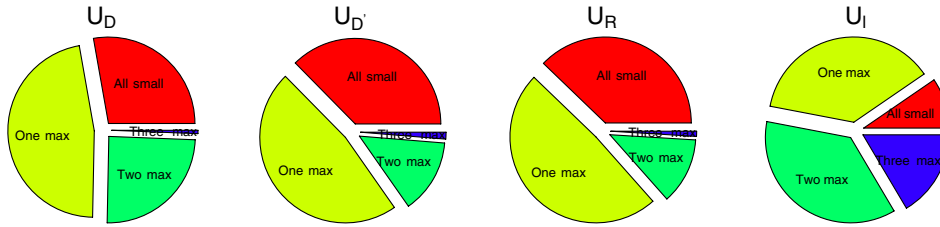


FIG. 9 (color online). Distributions of mixing angles in U_D , U'_D , U_R , and U_ℓ (in columns) that lead to IH neutrino masses for all seesaw realizations. The different pie labels refer to the number of maximal mixing angles, where “All small” corresponds to all mixing angles $\leq \epsilon$.

VII. SUMMARY AND CONCLUSIONS

Quark-lepton unification generally leads to predictions in the patterns of the Yukawa coupling matrices of the fermions, which are often called “textures.” Because of the high dimension of the parameter space, a direct systematic study or scan of all valid textures at the level of the Yukawa couplings is difficult to perform. Therefore, we suggest a “bottom-up” search strategy: We construct the parameter space of all valid textures from very generic (low-level) assumptions in the relevant leptonic mixing matrices and mass spectra. In this approach, we use the context of extended quark-lepton complementarity, which means that all mixing angles are either maximal or described by powers of $\epsilon \simeq \theta_C$. In addition, all mass ratios (of the neutrinos and charged leptons, as well as the mass hierarchies in M_D and M_R) are also parametrized by powers ϵ . This expansion parameter may be the remnant of a flavor symmetry describing the masses and mixing angles in both the quark and lepton sectors within a quark-lepton unified context. Note that in this setting, the solar neutrino mixing angle can only emerge as a combination of maximal mixing and θ_C , i.e., our assumptions are somewhat more general than the usual QLC relations [see, e.g., Eq. (3)].

Our procedure, in short terms, is as follows (cf. Fig. 2): First, we systematically construct all combinatorial possibilities for all mixing matrices and mass spectra for the type-I seesaw mechanism up to the order ϵ^2 . In each case, we compute the effective neutrino mass matrix $M_{\text{eff}}^{\text{th}}$ from that. In the second step, we determine all possibilities for effective neutrino mass matrix $M_{\text{eff}}^{\text{exp}}$ containing the experimental information from U_{PMNS} , the neutrino mass hierarchy, and U_ℓ . In the third step, the two matrices are matched, i.e., the realizations compatible with current data are selected. Finally, we identify the leading order entries in the matrices, which lead to the sets of textures. We scan about 20×10^{12} possibilities focusing on the real, CP conserving case, and we mainly discuss a normal neutrino mass hierarchy. As an experimentally motivated input value for $\sin^2 2\theta_{13}$, we take the current best-fit value $\sin^2 2\theta_{13} = 0$ as an example.

Compared to individual models in the literature, our assumptions are too generic to be able to predict the outcome *a priori*, which may reduce the bias. This key feature allows for the interpretation of the valid result statistics with respect to observables, constructed hierarchies, etc. For example, we find many different cases with a very mild hierarchy in M_R (with an abundance of about 80% in the textures), a result which does not involve any bias from the point of view of the assumptions. In addition, we find only very few cases with a symmetric Dirac neutrino mass matrix M_D or small mixing ($U_\ell \simeq \mathbb{1}$) in the charged lepton sector. However, we find a roughly diagonal right-handed neutrino mass matrix M_R in relatively many cases, i.e., in about 24% of all realizations for the normal neutrino mass hierarchy. It would be interesting to investigate signals of lepton flavor violation for our list of matrices, and connect our results to statistical studies along the lines of Ref. [62] or to recent attempts of scanning the SM parameters including quark and lepton masses [63].

We have presented a complete selection of 72 examples for Yukawa coupling textures satisfying specific selection criteria. A more complete list of 1981 texture sets (obtained by relaxing the selection criteria) can be found in Ref. [60]. We have shown in two examples that the textures are very useful for a direct search for flavor symmetries predicting current data. Therefore, our list(s) of textures can be understood as an intermediate result from the point of view of model building. They might be used to systematically test and find flavor symmetries and allow for a more systematic approach to constructing models.

We conclude that systematic, machine-supported searches of parameter spaces related to model building may, in fact, open up new possibilities. While conventional approaches focus on individual models in greater depth, our approach produces *all* valid lepton mass matrices with the only bias of the (often very generic) input assumptions. Therefore, they allow for the identification of new possibilities, as well as for more general studies of the discussed parameter space. Hence, our study should not only be interpreted with respect to the input assumptions, but also with respect to the procedure itself.

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APPENDIX A: DETAILS OF THE PROCEDURE, HEURISTICS, AND COMPLEXITY

In this Appendix, we give details of the algorithm, heuristics (also physically relevant ones, everything which makes the algorithm faster but does not lead to a loss of generality), and complexity.

1. Parameter space for the mass hierarchies

In the procedure described in Sec. III C, we have considered all possible diagonal mass matrices $M_D^{\text{diag}} = m_D \text{diag}(\epsilon^a, \epsilon^b, \epsilon^c)$ and $M_R^{\text{diag}} = m_D \text{diag}(\epsilon^{a'}, \epsilon^{b'}, \epsilon^{c'})$, with suitable non-negative integers a, b, c, a', b', c' that are compatible with the observed neutrino mass squared differences and leptonic mixing angles. As we will now see, the relevant range of these integers can actually be restricted considerably if we take the parametrization of the neutrino mass spectrum in Eq. (12) into account.

By factoring out common powers in ϵ , we see that M_D^{diag} and M_R^{diag} can be written as

$$\begin{aligned} M_D^{\text{diag}} &= m_D \text{diag}(\epsilon^m, \epsilon^n, 1), \\ M_D^{\text{diag}} &= m_D \text{diag}(\epsilon^m, 1, \epsilon^n), \\ M_D^{\text{diag}} &= m_D \text{diag}(1, \epsilon^m, \epsilon^n), \end{aligned} \quad (\text{A1a})$$

and

$$M_R^{\text{diag}} = M_{B-L} \text{diag}(\epsilon^p, \epsilon^q, 1) \quad \text{with } 0 \leq q \leq p, \quad (\text{A1b})$$

where m, n, p , and q , are non-negative integers, and $m_D \sim 10^2$ GeV. In Eq. (A1b), we have made use of the possibility to bring M_R^{diag} to a strictly hierarchical form, which is expressed by the condition $0 \leq q \leq p$. One then has, however, no longer the freedom to choose the order of the mass eigenvalues of M_D^{diag} , and that is why Eq. (A1a) includes all permutations of M_D^{diag} without any specific ordering of m and n , i.e., we allow in Eq. (A1a) for both the cases $n \leq m$ and $n > m$. In the following, we set $m_\nu \equiv m_D^2/M_{B-L}$. Although $m_D \sim 10^2$ GeV and $M_{B-L} \sim$

10^{14} GeV would be preferred values, we can, of course, always rescale $m_D \rightarrow a \cdot m_D$ and $M_{B-L} \rightarrow a^2 \cdot M_{B-L}$ by some factor a , thereby leaving the absolute neutrino mass scale $m_\nu \sim 10^{-2}$ eV unchanged. From $\det M_{\text{eff}}^{\text{diag}} = (\det M_D^{\text{diag}})^2 \det(M_R^{\text{diag}})^{-1}$ we find the relation

$$2(m+n) - p - q = k, \quad (\text{A1c})$$

where k is a non-negative integer that depends on the type of neutrino mass spectrum in Eq. (12): We have $k = 3$ for a normal, $k = 1$ for an inverted, and $k = 0$ for a degenerate neutrino mass spectrum.¹⁶

We are now interested in determining the allowed ranges for the integers m, n, p , and q in Eqs. (A1). Later, we will apply Eq. (24) up to order ϵ^2 , i.e., we will require a numerical matching precision $\mathcal{O}(\epsilon^3)$ between $M_{\text{eff}}^{\text{th}}$ and $M_{\text{eff}}^{\text{exp}}$. From that matching precision it follows that only powers up to ϵ^2 are relevant in the individual factors in the product Eq. (20a), because higher order terms will be absorbed by this matching uncertainty. The only nontrivial aspect in this argument is the factor M_R^{-1} . Let us write $(M_R^{\text{diag}})^{-1} = \epsilon^{-p} M_{B-L}^{-1} \text{diag}(1, \epsilon^{p-q}, \epsilon^p)$, and redefine $m_D \rightarrow \epsilon^{p/2} m_D$ and $M_R \rightarrow \epsilon^p M_R$, which leaves m_ν unaffected. Consider now the entries $\epsilon^m, \epsilon^n, \epsilon^{p-q}$, and ϵ^p , in M_D^{diag} and $(M_R^{\text{diag}})^{-1}$. If $m, n, p-q$, or p are larger than 2, then the corresponding contribution to M_{eff} will be absorbed in the matching precision. It follows that it is sufficient to restrict the maximum values of m, n, p , and q to 2, i.e., we need to consider for the powers of ϵ only

$$0 \leq m, n, p, q \leq 2, \quad (\text{A2})$$

where we have used that $0 \leq q \leq p$. The matrices M_D^{diag} and M_R^{diag} with higher powers of ϵ fall then into one of the classes already covered by a combination of powers satisfying Eq. (A1c).

2. Matching of matrices and the absolute neutrino mass scale

In this section, we describe the numerical implementation of Eq. (24): $M_{\text{eff}}^{\text{th}}|_{\epsilon=0.2} \simeq M_{\text{eff}}^{\text{exp}}|_{\epsilon=0.2}$. For our procedure, it makes sense to require a matching precision of $\mathcal{O}(\epsilon^3)$ since ϵ^2 is the maximum power that is used for the mass hierarchies and mixing angles (which was chosen because higher orders would be absorbed by the current measurement precision). This implies that a higher order matching precision will be too precise/restrictive because ϵ^3 powers are not directly produced, and a lower order matching precision would not be able to distinguish cases differing by ϵ^2 terms. Numerically, the $\mathcal{O}(\epsilon^3)$ term can have any order one coefficient. We require a precision of $P = \epsilon^3$ for Eq. (24), which turns out to be reasonably

¹⁶From Eq. (A1c) and $p \geq q$, it also follows that $m, n \leq \frac{k}{2} + p$.

restrictive by using actual tests, i.e., we require¹⁷

$$(M_{\text{eff}}^{\text{exp}})_{ij} - P \leq c \cdot (M_{\text{eff}}^{\text{th}})_{ij} \leq (M_{\text{eff}}^{\text{exp}})_{ij} + P, \quad (\text{A3a})$$

with $P = \epsilon^3$ and c is a constant which can be absorbed into (or can come from) the absolute mass scale, mixing angles, and the highest hierarchy power in M_R .

In order to evaluate Eq. (A3a), we have to determine c in order to check whether $M_{\text{eff}}^{\text{th}}$ falls into the proper interval. Let us rewrite Eq. (A3a) as

$$(M_{\text{eff}}^{\text{exp}})_{ij}^{\text{low}} \leq c(M_{\text{eff}}^{\text{th}})_{ij} \leq (M_{\text{eff}}^{\text{exp}})_{ij}^{\text{up}}, \quad (\text{A3b})$$

where $(M_{\text{eff}}^{\text{exp}})_{ij}^{\text{low}} = (M_{\text{eff}}^{\text{exp}})_{ij} - P$ and $(M_{\text{eff}}^{\text{exp}})_{ij}^{\text{up}} = (M_{\text{eff}}^{\text{exp}})_{ij} + P$. This corresponds to 6×2 independent inequalities. Dividing now these inequalities by $(M_{\text{eff}}^{\text{th}})_{ij}$ gives

$$\frac{(M_{\text{eff}}^{\text{exp}})_{ij}^{\text{low}}}{(M_{\text{eff}}^{\text{th}})_{ij}} \leq c \leq \frac{(M_{\text{eff}}^{\text{exp}})_{ij}^{\text{up}}}{(M_{\text{eff}}^{\text{th}})_{ij}} \quad \text{for } (M_{\text{eff}}^{\text{th}})_{ij} > 0, \quad (\text{A3c})$$

$$\frac{(M_{\text{eff}}^{\text{exp}})_{ij}^{\text{up}}}{(M_{\text{eff}}^{\text{th}})_{ij}} \leq c \leq \frac{(M_{\text{eff}}^{\text{exp}})_{ij}^{\text{low}}}{(M_{\text{eff}}^{\text{th}})_{ij}} \quad \text{for } (M_{\text{eff}}^{\text{th}})_{ij} < 0.$$

Watch for the special case $(M_{\text{eff}}^{\text{th}})_{ij} = 0$. The intersection of these intervals is then obtained as

$$\underbrace{\max\left(\frac{(M_{\text{eff}}^{\text{exp}})_{ij}^{\text{L}}}{(M_{\text{eff}}^{\text{th}})_{ij}}\right)}_{c^{\text{L}}} \leq c \leq \underbrace{\min\left(\frac{(M_{\text{eff}}^{\text{exp}})_{ij}^{\text{U}}}{(M_{\text{eff}}^{\text{th}})_{ij}}\right)}_{c^{\text{U}}} \quad (\text{A3d})$$

with $(M_{\text{eff}}^{\text{exp}})_{ij}^{\text{L}}$ and $(M_{\text{eff}}^{\text{exp}})_{ij}^{\text{U}}$ chosen according to the case selection in Eq. (A3c) for i and j individually. Only if $c^{\text{L}} \leq c^{\text{U}}$, then we have an allowed range for c . If, however, $c^{\text{L}} > c^{\text{U}}$, the realization is refused. Note that the constant c is not used any further on since it may have different origins. However, the appropriate absolute mass scale will be implicitly determined in order to satisfy Eq. (24).

3. Complexity and counting of cases

In order to avoid in our procedure the generation of equivalent realizations and double counting of cases, several heuristics can be used. As far as the phases in Eq. (20a) are concerned, we only have the matrices D_D , \tilde{K} , \tilde{D} for the real case, which leads to $2^{(3+2+3)} = 2^8 = 256$ phases. For

$\theta_{13}^x = 0$, the phase δ^x is unphysical, and we only test one case. This leads to $4^6 \times (3 \times 2 + 1)^3$ different angle combinations in $M_{\text{eff}}^{\text{th}}$. For $M_{\text{eff}}^{\text{exp}}$ in Eq. (20b), we do not generate D_ℓ in Eq. (20b) because the phases in the corresponding D_D will match a valid case of $M_{\text{eff}}^{\text{exp}}$ since both matrices appear on the outside (a valid phase in D_D then actually corresponds to two cases for D_D and D_ℓ). In the real case, we therefore have only $2^2 = 4$ phases from K_ℓ , and $4^2 \times (3 \times 2 + 1) = 112$ cases for the angles in \hat{U}_ℓ . This has to be multiplied with the corresponding allowed hierarchies in M_D and M_R . It is easy to derive that there are 12 possible cases for a normal neutrino mass hierarchy, 9 possible cases for an inverted hierarchy, and 13 cases for the degenerate case (cf. Appendix A 1). In addition, we test six cases for the true values in \hat{U}_{PMNS} . Since in the degenerate case any \hat{U}_ℓ and any \hat{U}_{PMNS} leads to a diagonal $M_{\text{eff}}^{\text{exp}}$, it is sufficient to test one case for \hat{U}_ℓ and K_ℓ . This means that the degenerate case hardly contributes to the complexity. Further simplifications are possible for $M_D^{\text{diag}} = \mathbb{1}$, etc. In this case, one has to watch that the relative counting of different hierarchy/mixing angle cases is affected. In total, we test

$$N = 256 \times 4^6 \times (3 \times 2 + 1)^3 \times (112 \times 4 \times (12 + 9) + 13) \times 6 \simeq 20 \times 10^{12} \quad (\text{A4})$$

different combinatorial possibilities, which require about 2 months of running time on a modern computer using a C-based software.

The number of allowed combinations may be used as a statistical measure in some cases. The interpretation is then related to the number of different possibilities.

APPENDIX B: DETAILS OF THE RESULTS FOR THE NH CASE

This section contains supplementary material for Sec. V, such as the selection criteria used and the phases necessary for a complete reconstruction of the charged lepton and neutrino Yukawa couplings.

TABLE IV. Total number of texture sets, and the numbers of distinct textures for M_ℓ , M_D , and M_R . Each row corresponds to the application of the selection criteria specified in the first column (see text).

Selection criterion	No. of textures	No. of M_ℓ	No. of M_D	No. of M_R
None	1981	20	621	35
1–2	1048	20	475	31
1–3	447	20	270	22
1–5	72	17	65	21

¹⁷The data set of valid models is very sensitive to this P . If it is much smaller, we do not find any valid realizations because ϵ^3 terms are not directly produced by our procedure. If it is somewhat larger, the realizations found fail consistency with data. For example, for a normal neutrino mass hierarchy, one can diagonalize $M_{\text{eff}}^{\text{th}}$ and read off the mixing angles as in Ref. [25]. These mixing angles can be compared with the actual ones which have been used as input values for $M_{\text{eff}}^{\text{exp}}$. We find that our matching precision is sufficient such that only realizations compatible with current data at least at the 5σ confidence level are allowed for the NH case (most of them actually fit much better); cf. Fig. 4. A weaker constraint on P allows more realizations which provide, however, too bad a fit.

1. Texture set selection criteria

Let us now describe the selection criteria leading to the texture sets presented in Sec. V. In producing Table I, we have used the following criteria:

- (i) The seesaw realizations should resist an increased experimental pressure provided that the current best-fit values are unchanged, i.e., $\sin^2 2\theta_{13}$ will not be found. We impose an extrapolated experimental limit by using $\sigma_{12} \simeq 4.6\%$ [44] and $\sigma_{23} \simeq 10\%$ [43] in Eq. (26), which corresponds to about a decade from now. Since θ_{13} is in any case smaller than 1° and would anyway resist an increased experimental pressure, we do not have to consider it for a further selection.
- (ii) Textures that differ by entries of the order ϵ^2 should be included with only one example. We choose the realization with the best χ^2 .
- (iii) The selection of realizations should be stable under ϵ^2 variations of the mixing angles, i.e., if we find mixing angles ϵ^2 , the realization has to be valid for $\epsilon^2 \rightarrow 0$ as well. We include the newly obtained texture sets in Table I by introducing the angle $\xi \in \{0, \epsilon^2\}$. Note that these cases are initially generated as well, but they might be filtered out later by Eq. (24).
- (iv) We omit texture sets with anarchical M_R (matrices just filled with entries “1”), since such structureless

textures do not, in general, yield much useful information.

- (v) We also show only one example for each texture set $(M_1, M_2, M_3) \sim (M_\ell, M_D, M_R)$, where M_i ($i = 1, 2, 3$) are the corresponding textures obtained after texture reduction, which have one pair $\{M_j, M_k\}$ appearing together with more than one possible M_i ($i \neq j \neq k$). In this case, we keep the texture set associated with a representation that has the lowest χ^2 .

By using the above selection criteria, we obtain the list of 72 texture sets shown in Tables I and V. The effects of the different selection criteria on the number of texture sets is shown in Table IV. In this table, we also show the numbers of distinct textures for M_ℓ , M_D , and M_R .

2. Supplementary information for the texture sets

Table V shows extra details of the realizations listed in Table I. It contains the complete set of phases, the PMNS mixing angles, χ^2 (in the 10 yr limit, cf. Appendix B 1), and the number of realizations leading to each texture set (for $\xi = \epsilon^2$ in ambiguous cases). Note that we chose $\varphi_1^{D'} = \varphi_2^{D'} = \varphi_3^{D'} = \alpha_1^D = \alpha_2^D = 0$ since these phases appear in $M_{\text{eff}}^{\text{th}}$ only in combination with other phases [see Eq. (20)], and can be absorbed into the other phases. Tables I and V provide together the complete information sufficient to fully reconstruct the Yukawa coupling matrices of the 72 realizations.

TABLE V. Supplementary information for Table I ($\varphi_{1,2,3}^{D'} = \alpha_{1,2}^D = 0$) (see text).

No.	$(\delta^l, \alpha_1^l, \alpha_2^l)$	$(\delta^D, \varphi_1^D, \varphi_2^D, \varphi_3^D)$	$(\delta^{D'}, \alpha_1^{D'}, \alpha_2^{D'})$	$(\delta^R, \varphi_1^R, \varphi_2^R, \varphi_3^R)$	$(\theta_{12}, \theta_{13}, \theta_{23})$	χ^2	Cases
1	(0, π , 0)	(π , 0, 0, π)	(π , π , π)	(0, 0, 0, π)	(34.0°, 0.2°, 52.2°)	7.12	18
2	(π , 0, π)	(π , 0, 0, π)	(0, 0, π)	(π , 0, π , π)	(33.6°, 0.2°, 51.5°)	5.29	38
3	(π , 0, 0)	(0, 0, π , 0)	(0, 0, 0)	(π , 0, π , 0)	(33.5°, 0.2°, 51.3°)	4.9	26
4	(0, 0, π)	(0, 0, 0, π)	(π , 0, π)	(π , 0, π , π)	(33.5°, 0.1°, 51.2°)	4.71	17
5	(0, 0, 0)	(0, 0, 0, 0)	(0, π , 0)	(0, 0, 0, π)	(33.0°, 0.4°, 51.2°)	4.7	17
6	(π , π , 0)	(π , 0, 0, 0)	(π , π , π)	(0, 0, 0, 0)	(33.3°, 0.4°, 51.2°)	4.7	177
7	(π , π , 0)	(π , 0, 0, 0)	(0, 0, 0)	(π , 0, 0, 0)	(33.3°, 0.4°, 51.2°)	4.7	63
8	(0, 0, π)	(0, 0, 0, π)	(0, π , 0)	(0, 0, π , π)	(33.5°, 0.1°, 51.2°)	4.71	17
9	(π , π , π)	(0, 0, π , 0)	(π , 0, 0)	(0, 0, π , 0)	(32.9°, 0.2°, 51.2°)	4.76	23
10	(π , 0, 0)	(π , 0, 0, π)	(0, 0, 0)	(0, 0, 0, π)	(33.2°, 0.2°, 51.3°)	4.78	597
11	(π , 0, 0)	(π , 0, 0, π)	(0, 0, 0)	(0, 0, 0, π)	(33.2°, 0.2°, 51.3°)	4.78	835
12	(0, π , π)	(0, 0, 0, 0)	(0, π , 0)	(π , 0, 0, π)	(33.4°, 0.0°, 51.3°)	4.81	475
13	(0, 0, 0)	(π , 0, 0, π)	(0, 0, 0)	(π , 0, 0, 0)	(33.4°, 0.2°, 51.3°)	4.84	104
14	(π , 0, 0)	(π , 0, π , 0)	(π , π , 0)	(0, 0, 0, π)	(33.5°, 0.6°, 51.3°)	4.89	14
15	(π , 0, 0)	(0, 0, π , 0)	(0, 0, 0)	(π , 0, π , 0)	(33.5°, 0.2°, 51.3°)	4.9	17
16	(π , 0, 0)	(π , 0, π , 0)	(0, π , 0)	(0, 0, 0, π)	(33.5°, 0.6°, 51.3°)	4.9	120
17	(π , 0, 0)	(π , 0, 0, 0)	(0, 0, π)	(π , 0, 0, π)	(33.3°, 0.0°, 51.4°)	4.96	1138
18	(0, π , π)	(0, 0, π , π)	(0, π , π)	(0, 0, π , π)	(33.5°, 0.2°, 51.4°)	4.97	9
19	(0, π , π)	(0, 0, 0, 0)	(0, 0, 0)	(0, 0, 0, 0)	(33.4°, 0.0°, 51.4°)	4.99	387
20	(π , 0, π)	(π , 0, 0, 0)	(0, π , 0)	(0, 0, π , π)	(33.7°, 0.2°, 51.2°)	5.02	26
21	(π , 0, 0)	(0, 0, 0, π)	(0, 0, 0)	(0, 0, 0, π)	(33.4°, 0.1°, 51.5°)	5.03	776
22	(π , 0, 0)	(0, 0, 0, π)	(0, 0, π)	(π , 0, π , π)	(33.2°, 0.1°, 51.5°)	5.03	876
23	(π , 0, 0)	(0, 0, 0, π)	(π , π , 0)	(0, 0, π , π)	(33.2°, 0.1°, 51.5°)	5.03	1351
24	(π , π , π)	(0, 0, π , 0)	(0, π , π)	(π , 0, π , π)	(33.5°, 0.3°, 51.4°)	5.05	27

TABLE V. (Continued)

No.	$(\delta^l, \alpha_1^l, \alpha_2^l)$	$(\delta^D, \varphi_1^D, \varphi_2^D, \varphi_3^D)$	$(\delta^{D'}, \alpha_1^{D'}, \alpha_2^{D'})$	$(\delta^R, \varphi_1^R, \varphi_2^R, \varphi_3^R)$	$(\theta_{12}, \theta_{13}, \theta_{23})$	χ^2	Cases
25	$(\pi, 0, 0)$	$(0, 0, \pi, 0)$	$(\pi, \pi, 0)$	$(0, 0, 0, \pi)$	$(33.5^\circ, 0.1^\circ, 51.4^\circ)$	5.06	392
26	$(\pi, 0, 0)$	$(0, 0, 0, \pi)$	$(0, 0, 0)$	$(0, 0, 0, \pi)$	$(33.4^\circ, 0.1^\circ, 51.5^\circ)$	5.09	307
27	$(0, 0, \pi)$	$(0, 0, 0, \pi)$	$(0, \pi, 0)$	$(0, 0, \pi, 0)$	$(33.5^\circ, 0.2^\circ, 51.4^\circ)$	5.09	26
28	$(\pi, 0, 0)$	$(0, 0, \pi, 0)$	$(\pi, 0, 0)$	$(\pi, 0, \pi, 0)$	$(33.7^\circ, 0.2^\circ, 51.3^\circ)$	5.11	296
29	$(\pi, 0, \pi)$	$(\pi, 0, 0, \pi)$	(π, π, π)	$(0, 0, 0, \pi)$	$(33.6^\circ, 0.2^\circ, 51.4^\circ)$	5.16	5
30	$(\pi, 0, \pi)$	$(\pi, 0, 0, \pi)$	(π, π, π)	$(0, 0, 0, \pi)$	$(33.6^\circ, 0.2^\circ, 51.4^\circ)$	5.16	5
31	$(\pi, 0, \pi)$	$(\pi, 0, 0, \pi)$	$(0, 0, \pi)$	$(\pi, 0, \pi, \pi)$	$(33.6^\circ, 0.2^\circ, 51.5^\circ)$	5.29	38
32	$(\pi, 0, 0)$	$(0, 0, \pi, 0)$	$(0, \pi, \pi)$	$(\pi, 0, 0, \pi)$	$(33.3^\circ, 0.2^\circ, 51.7^\circ)$	5.31	343
33	(π, π, π)	$(\pi, 0, 0, 0)$	$(0, 0, 0)$	$(0, 0, \pi, 0)$	$(33.6^\circ, 0.1^\circ, 51.5^\circ)$	5.31	83
34	(π, π, π)	$(\pi, 0, 0, 0)$	$(0, 0, 0)$	$(0, 0, \pi, 0)$	$(33.6^\circ, 0.1^\circ, 51.5^\circ)$	5.31	81
35	$(0, \pi, 0)$	$(\pi, 0, 0, \pi)$	$(0, \pi, 0)$	$(\pi, 0, \pi, 0)$	$(33.7^\circ, 0.1^\circ, 51.5^\circ)$	5.32	143
36	$(\pi, 0, \pi)$	$(\pi, 0, 0, \pi)$	$(\pi, 0, 0)$	$(0, 0, 0, \pi)$	$(32.9^\circ, 0.2^\circ, 51.6^\circ)$	5.33	17
37	$(\pi, 0, \pi)$	$(\pi, 0, 0, \pi)$	$(0, \pi, \pi)$	$(0, 0, 0, 0)$	$(32.9^\circ, 0.2^\circ, 51.6^\circ)$	5.33	17
38	$(\pi, 0, 0)$	$(0, 0, 0, \pi)$	$(\pi, 0, 0)$	$(0, 0, 0, \pi)$	$(33.2^\circ, 0.1^\circ, 51.7^\circ)$	5.33	17
39	$(\pi, 0, 0)$	$(0, 0, 0, \pi)$	$(\pi, 0, 0)$	$(0, 0, 0, \pi)$	$(33.2^\circ, 0.1^\circ, 51.7^\circ)$	5.33	33
40	$(\pi, 0, 0)$	$(0, 0, 0, 0)$	$(0, 0, \pi)$	$(\pi, 0, 0, \pi)$	$(33.2^\circ, 0.1^\circ, 51.7^\circ)$	5.37	17
41	$(\pi, 0, \pi)$	$(\pi, 0, 0, \pi)$	$(\pi, 0, 0)$	$(0, 0, \pi, \pi)$	$(33.1^\circ, 0.0^\circ, 51.8^\circ)$	5.47	26
42	$(\pi, 0, \pi)$	$(\pi, 0, 0, \pi)$	$(0, \pi, \pi)$	$(0, 0, 0, \pi)$	$(33.1^\circ, 0.0^\circ, 51.8^\circ)$	5.47	17
43	$(0, 0, \pi)$	$(\pi, 0, \pi, 0)$	$(\pi, 0, 0)$	$(0, 0, 0, \pi)$	$(33.3^\circ, 0.2^\circ, 51.8^\circ)$	5.48	5
44	$(0, 0, \pi)$	$(\pi, 0, \pi, 0)$	$(\pi, 0, 0)$	$(0, 0, 0, \pi)$	$(33.3^\circ, 0.2^\circ, 51.8^\circ)$	5.48	14
45	$(\pi, 0, \pi)$	$(\pi, 0, 0, \pi)$	$(0, \pi, \pi)$	$(\pi, 0, \pi, \pi)$	$(34.0^\circ, 0.3^\circ, 51.3^\circ)$	5.66	18
46	$(0, 0, 0)$	$(\pi, 0, 0, \pi)$	$(0, 0, 0)$	$(\pi, 0, 0, \pi)$	$(33.9^\circ, 0.7^\circ, 51.5^\circ)$	5.84	5
47	$(0, 0, 0)$	$(\pi, 0, 0, \pi)$	$(\pi, 0, 0)$	$(0, 0, 0, 0)$	$(34.0^\circ, 0.7^\circ, 51.5^\circ)$	5.96	5
48	(π, π, π)	$(0, 0, 0, \pi)$	$(0, \pi, \pi)$	$(0, 0, 0, 0)$	$(33.7^\circ, 0.1^\circ, 51.9^\circ)$	5.98	9
49	$(\pi, 0, \pi)$	$(\pi, 0, 0, 0)$	(π, π, π)	$(0, 0, 0, \pi)$	$(33.5^\circ, 0.4^\circ, 52.0^\circ)$	6.02	34
50	$(\pi, 0, \pi)$	$(\pi, 0, 0, 0)$	$(0, 0, 0)$	$(\pi, 0, \pi, 0)$	$(33.5^\circ, 0.4^\circ, 52.0^\circ)$	6.02	26
51	$(\pi, 0, \pi)$	$(0, 0, 0, 0)$	$(\pi, 0, 0)$	$(0, 0, 0, \pi)$	$(33.8^\circ, 0.3^\circ, 51.9^\circ)$	6.19	86
52	$(\pi, 0, \pi)$	$(0, 0, 0, 0)$	$(\pi, 0, 0)$	$(0, 0, 0, \pi)$	$(33.8^\circ, 0.3^\circ, 51.9^\circ)$	6.19	87
53	$(\pi, 0, \pi)$	$(0, 0, 0, 0)$	$(0, \pi, \pi)$	$(0, 0, \pi, \pi)$	$(33.8^\circ, 0.3^\circ, 51.9^\circ)$	6.19	108
54	$(0, 0, 0)$	$(0, 0, \pi, \pi)$	$(0, \pi, \pi)$	$(0, 0, \pi, \pi)$	$(34.4^\circ, 0.1^\circ, 51.1^\circ)$	6.26	14
55	(π, π, π)	$(\pi, 0, \pi, \pi)$	$(0, 0, 0)$	$(0, 0, \pi, \pi)$	$(34.4^\circ, 0.1^\circ, 51.1^\circ)$	6.26	14
56	$(0, 0, 0)$	$(\pi, 0, 0, \pi)$	$(\pi, \pi, 0)$	$(\pi, 0, 0, \pi)$	$(32.5^\circ, 0.2^\circ, 52.0^\circ)$	6.44	79
57	$(0, 0, 0)$	$(\pi, 0, 0, \pi)$	$(\pi, \pi, 0)$	$(\pi, 0, 0, \pi)$	$(32.5^\circ, 0.2^\circ, 52.0^\circ)$	6.44	34
58	$(0, \pi, \pi)$	$(\pi, 0, 0, \pi)$	$(0, 0, 0)$	$(0, 0, \pi, 0)$	$(34.2^\circ, 0.7^\circ, 51.5^\circ)$	6.46	5
59	$(0, \pi, 0)$	$(0, 0, \pi, 0)$	$(0, 0, \pi)$	$(0, 0, 0, \pi)$	$(34.0^\circ, 0.2^\circ, 52.0^\circ)$	6.59	5
60	$(0, 0, 0)$	$(\pi, 0, 0, \pi)$	$(\pi, 0, \pi)$	$(0, 0, 0, \pi)$	$(34.0^\circ, 0.4^\circ, 52.0^\circ)$	6.77	5
61	$(0, \pi, 0)$	$(\pi, 0, 0, \pi)$	$(0, 0, 0)$	$(0, 0, \pi, 0)$	$(34.0^\circ, 0.2^\circ, 52.2^\circ)$	7.12	18
62	$(0, \pi, 0)$	$(\pi, 0, 0, \pi)$	(π, π, π)	$(0, 0, 0, \pi)$	$(34.0^\circ, 0.2^\circ, 52.2^\circ)$	7.12	18
63	$(0, \pi, 0)$	$(\pi, 0, 0, \pi)$	$(0, 0, 0)$	$(0, 0, \pi, 0)$	$(34.0^\circ, 0.2^\circ, 52.2^\circ)$	7.12	18
64	(π, π, π)	$(\pi, 0, 0, 0)$	$(0, 0, \pi)$	$(0, 0, \pi, \pi)$	$(34.1^\circ, 0.2^\circ, 52.2^\circ)$	7.25	17
65	$(\pi, 0, \pi)$	$(0, 0, \pi, \pi)$	$(0, 0, \pi)$	$(0, 0, 0, 0)$	$(34.5^\circ, 0.3^\circ, 51.8^\circ)$	7.84	9
66	$(0, \pi, 0)$	$(\pi, 0, \pi, \pi)$	$(0, \pi, 0)$	$(0, 0, 0, 0)$	$(34.5^\circ, 0.3^\circ, 51.8^\circ)$	7.84	9
67	(π, π, π)	$(0, 0, 0, 0)$	$(0, \pi, \pi)$	$(0, 0, \pi, \pi)$	$(34.9^\circ, 0.4^\circ, 51.8^\circ)$	9.31	26
68	(π, π, π)	$(0, 0, 0, 0)$	$(\pi, 0, 0)$	$(0, 0, 0, 0)$	$(34.9^\circ, 0.4^\circ, 51.8^\circ)$	9.31	31
69	$(0, 0, 0)$	$(\pi, 0, 0, 0)$	$(0, 0, 0)$	$(0, 0, \pi, \pi)$	$(34.9^\circ, 0.4^\circ, 51.8^\circ)$	9.31	26
70	$(0, 0, 0)$	$(\pi, 0, 0, 0)$	(π, π, π)	$(0, 0, 0, 0)$	$(34.9^\circ, 0.4^\circ, 51.8^\circ)$	9.31	31
71	$(0, 0, 0)$	$(\pi, 0, 0, 0)$	$(0, \pi, 0)$	$(0, 0, \pi, 0)$	$(35.3^\circ, 0.3^\circ, 51.3^\circ)$	10.73	17
72	$(0, 0, 0)$	$(\pi, 0, 0, 0)$	$(0, \pi, 0)$	$(0, 0, \pi, 0)$	$(35.3^\circ, 0.3^\circ, 51.3^\circ)$	10.73	17

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