Inverting geometric transitions: Explicit Calabi-Yau metrics for the Maldacena-Nuñez solutions

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(Received 6 August 2007; published 16 November 2007)

Explicit Calabi-Yau metrics are derived that are argued to map to the Maldacena-Nuñez AdS solutions of M-theory and IIB under geometric transitions. In each case the metrics are singular where a H^2 Kähler two-cycle degenerates but are otherwise smooth. They are derived as the most general Calabi-Yau solutions of an ansatz for the supergravity description of branes wrapped on Kähler two-cycles. The ansatz is inspired by rewriting the AdS solutions, and the structure defined by half their Killing spinors, in this form. The world-volume theories of fractional branes wrapped at the singularities of these metrics are proposed as the duals of the AdS solutions. The existence of supergravity solutions interpolating between the AdS and Calabi-Yau metrics is conjectured and their boundary conditions briefly discussed.

DOI: 10.1103/PhysRevD.76.106010 PACS numbers: 04.65.+e, 11.25.Yb

I. INTRODUCTION AND MAIN IDEA

The AdS/CFT correspondence [1] is best understood for $D3$ branes at the apex of a Calabi-Yau cone. There are two ways in which we know how to think about this system. One is in terms of open string theory and probe $D3$ branes on the singular Calabi-Yau; at low energies, one gets a four-dimensional conformal field theory, at weak 't Hooft coupling, on the brane world volume. The other is in terms of closed string theory on the product of AdS_5 with a Sasaki-Einstein manifold; by the AdS/CFT correspondence, this is the same as the CFT at strong 't Hooft coupling. The classical link between the two geometries is a smooth supergravity solution, preserving half their supersymmetries, that interpolates between them; the Calabi-Yau singularity is excised and replaced with an AdS horizon at infinite proper distance. In this sense the branes are said to induce a geometric transition: they resolve (rather, remove to infinity) the singularity of the Calabi-Yau manifold. The geometrical data of both the Calabi-Yau and the Sasaki-Einstein manifold are encoded in the CFT (at weak and strong coupling, respectively), so interpolating the 't Hooft coupling in the CFT gives a quantum definition of the geometric transition. The dictionary—encoding and decoding Calabi-Yau and Sasaki-Einstein data in the CFT at weak and strong coupling, respectively—has been worked out in detail in beautiful work for $\mathbb{R}^6$, the conifold, and the $Y^{p,q}$ metrics [2–10].

Since the work of Maldacena and Nuñez [11], we know that there are many other ways in which anti-de Sitter geometries can be related to special holonomy manifolds and conformal quantum theories. In [11], three AdS solutions of M- and string theory were constructed: two AdS_5 solutions in 11 dimensions, with, respectively, sixteen and eight Killing spinors, and an AdS_3 solution admitting eight Killing spinors in IIB.[1] These were interpreted as arising, in the near-horizon limit, from branes wrapping H^2 Kähler two-cycles in, respectively, Calabi-Yau two-, three-, and threefolds. The dual conformal field theories are $\mathcal{N} = 2$ and $\mathcal{N} = 1$ in four dimensions, and $N = (2, 2)$ in three dimensions. Since this work, it has been found that there exist AdS solutions associated to all types of calibrated cycles in all types of special holonomy manifold of dimension ten or less; for example, [12–17]. The CFTs dual to AdS manifolds of this type define quantum gravity theories for calibrated geometries. In line with the intuition gained from branes at conical singularities, one would expect that the CFTs could be realized, at weak coupling, as the world-volume theories of fractional probe branes, wrapped on degenerating calibrated cycles in singular special holonomy manifolds. Such a system is likewise expected to undergo a geometric transition, with the singularity excised and replaced with an AdS region. Classically, there should be a supergravity solution interpolating between the Calabi-Yau and AdS geometries.

Our understanding of AdS/CFT for wrapped branes is much more rudimentary than for branes at conical singularities. Chief among the obstacles has been the inability to move beyond the near-horizon limit; typically, only the AdS geometries are known. The lesson from branes on cones is that in order to get a real handle on field theory *dynamics*—to write down the particle content and superpotential for a dual of a specific AdS solution—the associated Calabi-Yau geometry must be known. The main point of this paper is to give a way of associating a special holonomy metric to an AdS metric, illustrated for the Maldacena-Nuñez solutions. The main assumption of this paper is the existence of a supersymmetric supergravity solution interpolating between a special holonomy manifold and an AdS spacetime when there exists an AdS/CFT dual. Roughly, an interpolating solution should be a metric and a flux admitting two distinct limits in which the supersymmetry doubles, with the metric becoming Calabi-Yau in one limit and AdS in the other. More formally, we can think of the metric and flux of an interpolat-

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[1]These solutions will be denoted by MN(I), MN(II), and MN(III), respectively.

ing solution as providing a smooth and smoothly invertible map f:

$$f\colon \text{Special Holonomy} \longrightarrow \text{AdS}. \tag{1.1}$$

We take this as a definition of what is meant in this paper by a geometric transition. It is a purely classical definition; in more physical terms, such a map gives the full supergravity description of a wrapped brane. But if a CFT dual can be identified, the map can be promoted to the quantum level; the CFT itself provides the map, with the 't Hooft coupling the interpolating parameter.

The equations that interpolating solutions should satisfy are known, through various symmetry arguments. An important property of these solutions is that they should admit a global reduction of their frame bundle, to a subbundle of the appropriate structure [18,19]. For example, in the supergravity description of M5-branes wrapped on Kähler two-cycles in Calabi-Yau twofolds—maps $f\colon CY_2 \longrightarrow \text{AdS}_5$—the global structure of an interpolating solution is $SU(2)$. The structure is defined by eight Killing spinors, or alternatively, an almost complex structure J and a (2, 0) form Ω. The truncation of 11-dimensional supergravity to this frame bundle was first worked out by Fayyazuddin and Smith [20] (see also [18,21]). The metric and flux are

$$\begin{aligned} ds^2 &= L^{-1}ds^2(\mathbb{R}^{1,3}) + ds^2(\mathcal{M}_4) + L^2[dt^2 + t^2 ds^2(S^2)],\\ \star_7 F &= L^2 d(L^{-2}J). \end{aligned}\tag{1.2}$$

Here and throughout we follow all conventions and orientations of [18]. The Minkowski isometries are isometries of the full solution, and $\mathcal{M}_4$ admits a globally defined $SU(2)$ structure. The structure is constrained by the Fayyazuddin-Smith equations:

$$d(L^{-1/2}\Omega) = 0, \qquad dt \wedge \text{Vol}[S^2] \wedge d(LJ) = 0. \tag{1.3}$$

Eleven-dimensional supergravity, in this truncation, reduces to the torsion conditions (1.3) and the four-form Bianchi identity.

To the knowledge of the author, no interpolating solutions of these equations, or their analogues in other contexts, are known. However, in recent work [18,19,22], it has been shown how the supersymmetry conditions for general classes of supersymmetric AdS solutions of M-theory (including all known examples) can be derived from such equations. In particular, in [18] it was shown that the conditions of Lin, Lunin, and Maldacena (LLM) [23] for half–Bogomol'nyi-Prasad-Sommerfield (BPS) AdS_5 solutions can be derived from the Fayyazuddin-Smith equations. It follows that any solution of the LLM conditions can be rewritten as a solution of the Fayyazudin-Smith equations; and similarly for every other AdS solution covered by [18,19,22]. Applying this procedure to the MN(I) solution, we will see in the next section that it may be rewritten in the form

$$\begin{aligned} ds^2 &= L^{-1}\Big[ds^2(\mathbb{R}^{1,3}) + \frac{F}{2}ds^2(H^2)\Big]\\ &\quad + L^2[F^{-1}(du^2 + u^2(d\psi - P)^2) + dt^2 + t^2 ds^2(S^2)],\\ dP &= \text{Vol}[H^2], \end{aligned}\tag{1.4}$$

for particular determined functions $F(u, t)$, $L(u, t)$ and a particular choice of frame which will be discussed in detail. We use this form of the AdS solution as a guide to what the inverse geometric transition $f^{-1}\colon \text{MN(I)} \longrightarrow CY_2$ should be. Clearly, it should respect the topological structure of MN(I); the simplest choice, which we make, is that f is given by a solution $F(u, t)$, $L(u, t)$, of the Fayyazuddin-Smith equations. With this metric and the frame of Sec. II, they reduce to

$$\frac{1}{t^2}\partial_t(t^2\partial_t F) = -u\partial_u\Big(\frac{F}{u}\partial_u F\Big), \qquad L^3 = -\frac{1}{4u}\partial_u(F^2). \tag{1.5}$$

An interpolating solution of these equations has not been found. However, assuming one exists, the general Calabi-Yau solution of (1.5) is the image of MN(I) under f^{-1}. Up to an overall scale, the general Calabi-Yau solution is $L = 1$ and

$$d s_4^2 = \frac{dR^2}{(\frac{1}{R^4} - 1)} + \frac{R^2}{4}\Big[ds^2(H^2) + \Big(\frac{1}{R^4} - 1\Big)(d\psi - P)^2\Big]. \tag{1.6}$$

The range of R is $[-1, 0)$ or $(0, 1]$. As expected, the metric is singular, where the Kähler two-cycle H^2 degenerates. The singularity, at $R = 0$, is at finite proper distance. The metric is nonsingular at the H^2 bolt as $R^4 \longrightarrow 1$, if ψ has period 2π; we will see in the next section that this is precisely the periodicity that is inherited through f^{-1} from MN(I). Some additional evidence that this Calabi-Yau is a sensible candidate comes from the following. Every AdS_5 solution of the LLM conditions, including MN(I), is completely determined by a solution of the three-dimensional continuous Toda equation. There also exists a class of Calabi-Yau twofolds that is completely determined by a solution of the three-dimensional continuous Toda equation. This is such a Calabi-Yau metric, and furthermore it is given by the same solution of the Toda equation as MN(I). Toda-Calabi-Yau metrics have been obtained in this context before as scaling limits of the 1/2-BPS AdS_5 metrics [23,24]. Here this metric is obtained in a different way, as a solution of the 1/4-BPS Fayyazuddin-Smith equations. It will be interesting to see how these procedures are related.

The world-volume theory of fractional M5-branes wrapped at the singularity of this metric (whatever it might be) is proposed as the quantum dual of MN(I). Though the geometry is noncompact, this is not necessarily problematic, as the field theory should only encode oscillations in the directions transverse to the brane, purely in the fibre;

and the fibre has finite proper volume. The cycle may in any event be rendered compact by taking a freely acting quotient by a discrete subgroup of its isometry group. The Calabi-Yau will still be noncompact, because of the singularity.

In a similar vein, we obtain candidate Calabi-Yaus for inverse geometric transitions from MN(II) and MN(III). For MN(II), to be discussed in detail in Sec. III, the first step is to use the results of [4,18] to write it in the form

$$\begin{aligned} ds^2 &= L^{-1}\left[ds^2(\mathbb{R}^{1,3}) + \frac{F_1F_2}{3}ds^2(H^2)\right] \\ &\quad + L^2\left[F_1^{-1}\left(du^2 + \frac{u^2}{4}(d\psi + P - P')^2\right)\right. \\ &\quad \left. + F_2^{-1}\frac{u^2}{4}ds^2(S^2) + dt^2\right], \end{aligned} \tag{1.7}$$

with $dP = \mathrm{Vol}[S^2]$ and $dP' = \mathrm{Vol}[H^2]$. Then, letting L, F_1, F_2 be arbitrary functions of u, t, the general Calabi-Yau threefold solution[2] is, up to an overall scale,

$$\begin{aligned} ds^2 &= \frac{1}{2}(1 + \sin\xi)ds^2(H^2) + \frac{\cos^2\xi}{2(1+\sin\xi)}ds^2(S^2) \\ &\quad + \frac{1}{\cos^2\xi}(dR^2 + R^2(d\psi + P - P')^2), \end{aligned} \tag{1.8}$$

where $\sin\xi$ is a root of the cubic equation

$$-\tfrac{1}{3}\sin^3\xi + \sin\xi = \tfrac{2}{3} - R^2. \tag{1.9}$$

The metric is singular, as expected, at $\xi = -\pi/2$, $R = 2/\sqrt{3}$, where the H^2 cycle degenerates. The metric is smooth at $\xi = \pi/2$, which coincides with $R = 0$; there an S^3 smoothly degenerates, provided that ψ has the 4π periodicity it inherits from MN(II) under f^{-1}. The quantum dual of MN(II) is proposed to be the world-volume theory of fractional M5s wrapped at the singularity of this metric.

The MN(III) solution comes from $D3$ branes wrapped on a Kähler H^2 cycle in a Calabi-Yau threefold; the geometric transition is $CY_3 \to AdS_3$ in IIB. The discussion in this case proceeds along very much the same lines as for MN(II), and will be reported in detail elsewhere [25]. The image of MN(III) under the inverse transition is again the Calabi-Yau (1.8). The dual field theory is proposed to be the world-volume theory of $D3$ branes wrapped at the singularity. It seems that both M5 branes and $D3$ branes can probe the singularity of this manifold; the quantum descriptions are, respectively, four- and two-dimensional conformal theories. In the IIB description, given the metric, it might be possible to construct the field theory with existing techniques.

[2]This is the general metric with suitable regularity properties, that will be discussed in Sec. III.

The remainder of this paper is organized as follows. Section II is devoted to MN(I) and its Calabi-Yau image. Section III repeats the analysis of Sec. II for MN(II). Section IV contains conclusions and outlook, and also some discussion of the boundary conditions for interpolating solutions for the MN/Calabi-Yau pairs.

II. THE $\mathcal{N} = 2$ M-THEORY SOLUTION

To begin, we will review the $\mathcal{N} = 2$ AdS_5 geometry of [11] in some detail. The metric is given by

$$\begin{aligned} ds^2 &= \frac{1}{\lambda}\left[ds^2(AdS_5) + \frac{1}{2}ds^2(H^2) + (1 - \lambda^3\rho^2)(d\psi - P)^2\right. \\ &\quad \left. + \frac{\lambda^3}{4}\left(\frac{d\rho^2}{1-\lambda^3\rho^2} + \rho^2 ds^2(S^2)\right)\right], \end{aligned} \tag{2.1}$$

where

$$\lambda^3 = \frac{8}{1+4\rho^2}, \qquad dP = \mathrm{Vol}[H^2]. \tag{2.2}$$

Here and throughout we denote by $ds^2(\mathcal{M})$ the metric of unit radius of curvature on $\mathcal{M}$. The range of the coordinate ρ is either $\rho \in [-1/2, 0]$ or $\rho \in [0, 1/2]$. At $\rho = 0$, in either branch, the R-symmetry S^2 smoothly degenerates.[3] As $\rho \to \pm 1/2$, the R-symmetry $U(1)$, with coordinate ψ, smoothly degenerates, provided that ψ is identified with period 2π. Henceforth we will take ρ to be non-negative.

This manifold, as a solution of 11-dimensional supergravity, admits 16 Killing spinors. The Killing spinors may be used to define an identity structure—a preferred frame associated to them. The structure is discussed in detail in [18]. We choose coordinates for the preferred frame according to

$$\begin{aligned} e^1 + ie^2 &= \frac{1}{\sqrt{2\lambda}}e^{i\psi}(d\mu + i\sinh\mu\, d\beta), \\ e^3 &= \sqrt{\frac{1-\lambda^3\rho^2}{\lambda}}(d\psi - \cosh\mu\, d\beta), \\ \hat{\rho} &= \frac{\lambda d\rho}{2\sqrt{1-\lambda^3\rho^2}}, \qquad \hat{r} = \lambda^{-1/2}dr, \end{aligned} \tag{2.3}$$

where we have chosen Poincaré coordinates on AdS,

$$ds^2(AdS_5) = e^{-2r}ds^2(\mathbb{R}^{1,3}) + dr^2. \tag{2.4}$$

The remaining directions play no role in the rest of the discussion.

The MN(I) solution is a particular case of a broader class of half-BPS AdS_5 solutions which are completely determined by a solution of the three-dimensional continuous Toda equation.[4] The Toda equation,

[3]The R-symmetry of the dual theory is $SU(2) \times U(1)$.

[4]It is strongly believed, at least by the author, that all half-BPS AdS_5 solutions of M-theory are of this form.

$$\nabla^2_{\mathbb{R}^2} D + \partial^2_\rho e^D = 0, \tag{2.5}$$

may be viewed as a three-dimensional Laplace equation,

$$\nabla^2_3 D = 0, \tag{2.6}$$

on a three-manifold with metric

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + e^D \mathrm{d}s^2(\mathbb{R}^2). \tag{2.7}$$

The metric on every half-BPS AdS_5 solution of 11-dimensional supergravity determined by the Toda equation may be written as follows [23]:

$$\mathrm{d}s^2 = \frac{1}{\lambda}\bigg[\mathrm{d}s^2(AdS_5) + (1-\lambda^3\rho^2)(\mathrm{d}\psi + V)^2 + \frac{\lambda^3}{4} \times \bigg(\frac{1}{1-\lambda^3\rho^2}[\mathrm{d}\rho^2 + e^D \mathrm{d}s^2(\mathbb{R}^2)] + \rho^2 \mathrm{d}s^2(S^2)\bigg)\bigg], \tag{2.8}$$

where

$$\lambda^3 = \frac{-\partial_\rho D}{\rho(1-\rho\partial_\rho D)}, \tag{2.9}$$

$$V = \tfrac{1}{2}\star_2 \mathrm{d}_2 D, \tag{2.10}$$

where d_2 is the exterior derivative restricted to $\mathbb{R}^2$, and D solves (2.5). The MN(I) solution is given by

$$e^D = \frac{1}{4x_1^2}(1-4\rho^2). \tag{2.11}$$

In order to make the relationship between the MN(I) geometry and wrapped branes more concrete, we now want to exhibit it as a solution of the Fayazzuddin-Smith equations. The essential point is that, in addition to its identity structure, MN(I) also admits an $SU(2)$ structure, defined by *half* its Killing spinors, which indeed solves the Fayyazuddin-Smith equations. In [18] it was shown how to obtain an arbitrary Toda-AdS_5 manifold as a 1/4 BPS solution of the wrapped-brane conditions by constructing its $SU(2)$ structure. Here we will apply these general results to the specific case of interest.

The canonical frame of the identity structure is related to the canonical frame of the $SU(2)$ structure by a local rotation. If we define the Minkowski frame e^a, e^4, $\hat{t} = L\mathrm{d}t$, $a = 1, 2, 3$, with e^a, e^4 a basis for $\mathcal{M}_4$, the relationship between the "AdS" and "Minkowski" frames is given by

$$e^a_{\mathrm{Mink}} = e^a_{\mathrm{AdS}}, \qquad e^4 = \cos\theta\hat{\rho} + \sin\theta\hat{r}, \qquad \hat{t} = -\sin\theta\hat{\rho} + \cos\theta\hat{r}. \tag{2.12}$$

For more details of this procedure, which seems to be a universally applicable way of writing AdS manifolds in a wrapped-brane form, the reader is referred to [18,19,22].[5]

[5]The frame rotation, as a way of deriving warped AdS_{d+2} supersymmetry conditions from warped $\mathbb{R}^{1,d}$ supersymmetry conditions, was first employed in [4].

In the case at hand, the rotation angle is related to the AdS warp factor and the coordinate ρ by

$$\cos\theta = \lambda^{3/2}\rho, \tag{2.13}$$

and also the warp factors are related by

$$L = \lambda e^{2r}. \tag{2.14}$$

Near $\rho = 0$, the AdS radial direction aligns with $\pm e^4$. Near $\rho = 1/2$, it aligns with $\hat{t}$. Since we know everything on the right-hand side of (2.12), we can see that

$$e^4 = Le^{-r}\mathrm{d}\bigg(-\sqrt{\frac{1-4\rho^2}{8}}e^{-r}\bigg), \qquad \hat{t} = L\mathrm{d}\bigg(-\frac{\rho}{2}e^{-2r}\bigg). \tag{2.15}$$

Therefore defining the Minkowski-frame coordinates

$$u = -\sqrt{\frac{1-4\rho^2}{8}}e^{-r}, \qquad t = -\frac{\rho}{2}e^{-2r}, \tag{2.16}$$

we can rewrite the AdS_5 solution as

$$\mathrm{d}s^2 = L^{-1}\bigg[\mathrm{d}s^2(\mathbb{R}^{1,3}) + \frac{F}{2}\mathrm{d}s^2(H^2)\bigg] + L^2[F^{-1}(\mathrm{d}u^2 + u^2(\mathrm{d}\psi - P)^2) + \mathrm{d}t^2 + t^2\mathrm{d}s^2(S^2)], \tag{2.17}$$

where $F = e^{2r}$ is determined by a root of the quadratic

$$2t^2e^{4r} + u^2e^{2r} - \tfrac{1}{8} = 0. \tag{2.18}$$

One of these roots is always negative, so we choose the other, which is always positive:

$$F = \frac{u^2}{4t^2}(-1 + \sqrt{1 + t^2/u^4}). \tag{2.19}$$

The warp factor in the Minkowski frame is

$$L^3 = \frac{u^2}{\sqrt{1+t^2/u^4}}\bigg(\frac{-1+\sqrt{1+t^2/u^4}}{4t^2}\bigg)^2. \tag{2.20}$$

The canonical frame for the $SU(2)$ structure, rewritten in terms of the new coordinates, is

$$e^1 + ie^2 = \sqrt{\frac{F}{2L}}e^{i\psi}(\mathrm{d}\mu + i\sinh\mu\mathrm{d}\beta), \quad e^3 = -\frac{Lu}{\sqrt{F}}(\mathrm{d}\psi - \cosh\mu\mathrm{d}\beta), \quad e^4 = \frac{L}{\sqrt{F}}\mathrm{d}u, \qquad \hat{t} = L\mathrm{d}t, \tag{2.21}$$

with a minus sign in the second equation because of the definition of u. The $SU(2)$ structure then takes the standard form,

$$J = e^{12} + e^{34}, \qquad \Omega = (e^1 + ie^2)(e^3 + ie^4), \tag{2.22}$$

and it may now be verified by explicit computation that it satisfies the Fayyazudin-Smith equations. This was, of course, guaranteed by the construction, but it serves as a consistency check. Having obtained the $SU(2)$ structure of MN(I), it is now an obvious thing to use it as an ansatz for further, topologically related, solutions of the

Fayyazuddin-Smith equations. To this end, we let F and L be arbitrary functions of u t, and insert the frame (2.21) into the $SU(2)$ torsion conditions and Bianchi identity. They reduce to the single nonlinear second order pde for F:

$$\frac{1}{t^2}\partial_t(t^2\partial_t F) = -u\partial_u\left(\frac{F}{u}\partial_u F\right). \tag{2.23}$$

Given a solution of this equation, L is then determined by

$$L^3 = -\frac{1}{4u}\partial_u(F^2). \tag{2.24}$$

As a purely mathematical aside, we observe that the other root of the quadratic (2.18) is also a solution of these equations. But of particular interest is the most general Calabi-Yau solution of this system. It may be most easily determined by imposing $L =$ constant and closure of J. The (2, 0) form $L^{-1/2}\Omega$ is always closed with this ansatz. The general Calabi-Yau solution is

$$F^2 = a + bu^2. \tag{2.25}$$

For a metric of the right signature, we must have $a > 0$, $b < 0$. By rescaling, we can set $b = -2$, so that $L = 1$ (up to an overall scale in the 11-dimensional metric). This Calabi-Yau is diffeomorphic to a Toda-Calabi-Yau, as may be seen by performing the coordinate transformation

$$16U^2 = a - 2u^2. \tag{2.26}$$

Defining $A^2 = a/4$, the metric becomes

$$ds^2 = \frac{4}{\partial_u D}(d\alpha + V)^2 + \partial_u D(du^2 + e^D ds^2(\mathbb{R}^2)), \tag{2.27}$$

where

$$e^D = \frac{1}{4x_1^2}(A^2 - 4U^2), \tag{2.28}$$

which, modulo the constant, is the same solution of the Toda equation as that determining MN(I). An alternative form of the metric, reminiscent of Eguchi-Hanson, is given by choosing the coordinate

$$R^2 = \frac{1}{a^{1/4}}\sqrt{2a - 4u^2}. \tag{2.29}$$

Up to an overall scale the metric becomes

$$ds^2 = \frac{dR^2}{(\frac{1}{R^4} - 1)} + \frac{R^2}{4}\left[ds^2(H^2) + \left(\frac{1}{R^4} - 1\right)(d\psi - P)^2\right], \tag{2.30}$$

which is the form given in the introduction.

III. THE $\mathcal{N} = 1$ M-THEORY SOLUTION

Again, we begin with a review of the AdS geometry. The MN(II) metric is

$$ds^2 = \frac{1}{\lambda}\left[ds^2(\mathrm{AdS}_5) + \frac{1}{3}ds^2(H^2) + \frac{1}{9}(1 - \lambda^3\rho^2)(ds^2(S^2) + (d\psi + P - P')^2) + \frac{\lambda^3}{4(1 - \lambda^3\rho^2)}d\rho^2\right], \tag{3.1}$$

where now

$$\lambda = \frac{4}{4 + \rho^2}, \qquad dP = \mathrm{Vol}[S^2], \qquad dP' = \mathrm{Vol}[H^2]. \tag{3.2}$$

This time, the range of ρ is $[-2/\sqrt{3}, 2/\sqrt{3}]$; at $\rho = \pm 2/\sqrt{3}$, an S^3 smoothly degenerates. This manifold admits eight Killing spinors, which collectively define an $SU(2)$ structure. If we define the frame

$$\begin{aligned} e^1 + ie^2 &= \frac{1}{\sqrt{3\lambda}}e^{i\gamma\psi}(d\mu + i\sinh\mu d\beta),\\ e^3 + ie^4 &= \frac{1}{3}\sqrt{\frac{1 - \lambda^3\rho^2}{\lambda}}e^{i\delta\psi}(d\theta + i\sin\theta d\phi),\\ e^5 &= \frac{1}{3}\sqrt{\frac{1 - \lambda^3\rho^2}{\lambda}}(d\psi + P - P'),\\ \hat{\rho} &= \frac{\lambda}{2\sqrt{1 - \lambda^3\rho^2}}d\rho, \end{aligned} \tag{3.3}$$

where the constant phases γ, δ sum to unity, then the $SU(2)$ structure forms are given by

$$J_4 = e^{12} + e^{34}, \qquad \Omega_4 = (e^1 + ie^2)(e^3 + ie^4). \tag{3.4}$$

It may be explicitly verified that this six-dimensional $SU(2)$ structure satisfies the conditions of [4].

The MN(II) solution is interpreted as coming from M5 branes wrapping a H^2 Kähler two-cycle in a Calabi-Yau threefold. Again, we will make this more precise, by exhibiting MN(II) as a solution of the 1/8 BPS $SU(3)$ analogue of the Fayazzuddin-Smith equations. In this case, half the Killing spinors of the AdS manifold define an $SU(3)$ structure, with structure forms J_6, Ω_6. Then the supergravity description of 1/8 BPS M5 branes wrapping a Kähler two-cycle in a Calabi-Yau threefold [18,26] is as follows. The metric and flux are

$$\begin{aligned} ds^2 &= L^{-1}ds^2(\mathbb{R}^{1,3}) + ds^2(\mathcal{M}_6) + L^2dt^2,\\ \star_7 F &= -L^2 d(L^{-2}J_6), \end{aligned} \tag{3.5}$$

where $\mathcal{M}_6$ admits a globally defined $SU(3)$ structure, and again, all conventions and orientations follow [18]. The torsion conditions for the structure are

$$dt \wedge d(L^{-1}J \wedge J) = 0, \qquad d(L^{-3/2}\Omega) = 0. \tag{3.6}$$

These, together with the Bianchi identity, are sufficient to guarantee a solution of 11-dimensional supergravity.

We now perform the frame rotation exactly as in the previous section. The relationship between the Minkowski and AdS frames is

$$\begin{aligned} e^a_{\mathrm{Mink}} &= e^a_{\mathrm{AdS}}, \qquad e^6 = \cos\theta\hat{\rho} + \sin\theta\hat{r},\\ \hat{t} &= -\sin\theta\hat{\rho} + \cos\theta\hat{r}, \end{aligned} \tag{3.7}$$

where now $a = 1, \ldots, 5$. Again, $\lambda^{3/2}\rho = \cos\theta$. Therefore, $\hat{r}$ is antialigned with $\hat{t}$ at $\rho = -2/\sqrt{3}$. It then rotates through an angle of π as ρ spans its range, so that it is aligned with $\hat{t}$ at $\rho = 2/\sqrt{3}$. We find that e^6, $\hat{t}$ are given by

$$e^6 = Le^{-r/2}\mathrm{d}\Big(-\frac{1}{3}e^{-3r/2}\sqrt{4-3\rho^2}\Big),\qquad \hat{t} = L\mathrm{d}\Big(-\frac{\rho}{2}e^{-2r}\Big). \tag{3.8}$$

Defining the Minkowski-frame coordinates,

$$u = -\frac{1}{3}e^{-3r/2}\sqrt{4-3\rho^2}, \qquad t = -\frac{\rho}{2}e^{-2r}, \tag{3.9}$$

the metric in the Minkowski frame is given by

$$\mathrm{d}s^2 = L^{-1}\Big[\mathrm{d}s^2(\mathbb{R}^{1,3}) + \frac{F^2}{3}\mathrm{d}s^2(H^2)\Big] + L^2\Big[F^{-1}(\mathrm{d}u^2 + \frac{u^2}{4}[\mathrm{d}s^2(S^2) + (\mathrm{d}\psi + P - P')^2]) + \mathrm{d}t^2\Big], \tag{3.10}$$

where $F = e^r$. This time, in order to determine F in terms of the Minkowski-frame coordinates, we must find the roots of a *quartic* polynomial. The polynomial is

$$12t^2e^{4r} + 9u^2e^{3r} - 4 = 0. \tag{3.11}$$

The Minkowski frame is given by

$$\begin{aligned} e^1 + ie^2 &= \frac{F}{\sqrt{3L}}e^{i\gamma\psi}(\mathrm{d}\mu + i\sinh\mu\mathrm{d}\beta),\\ e^3 + ie^4 &= -\frac{Lu}{2\sqrt{F}}e^{i\delta\psi}(\mathrm{d}\theta + i\sin\theta\mathrm{d}\phi),\\ e^5 &= -\frac{Lu}{2\sqrt{F}}(\mathrm{d}\psi + P - P'),\\ e^6 &= \frac{L}{\sqrt{F}}\mathrm{d}u, \qquad \hat{t} = Ldt. \end{aligned} \tag{3.12}$$

Again, the minus signs come from the definition of u. Then the $SU(3)$ structure of MN(II) is given by

$$J_6 = e^{12} + e^{34} + e^{56},\qquad \Omega_6 = (e^1 + ie^2)(e^3 + ie^4)(e^5 + ie^6). \tag{3.13}$$

At this point, repeating the analysis of the previous section directly, we would let L, F become arbitrary functions of u, t, and then find the general Calabi-Yau solution. The torsion conditions and Bianchi identity reduce to

$$\partial_t^2F + \frac{1}{u}\partial_u(uF\partial_uF) = 0, \qquad L^3 + \frac{2F^2}{3u}\partial_uF = 0. \tag{3.14}$$

It follows from the construction of [4,18] that the root of the quartic corresponding to the MN(II) solution solves these equations. It seems very likely that so do all the roots, though this has not been verified. However, it turns out that there is no Calabi-Yau solution. To find one, we must extend the ansatz, to

$$\mathrm{d}s^2 = L^{-1}\Big[\mathrm{d}s^2(\mathbb{R}^{1,3}) + \frac{F_1F_2}{3}\mathrm{d}s^2(H^2)\Big] + L^2\Big[F_1^{-1}\Big(\mathrm{d}u^2 + \frac{u^2}{4}(\mathrm{d}\psi + P - P')^2\Big) + F_2^{-1}\frac{u^2}{4}\mathrm{d}s^2(S^2) + \mathrm{d}t^2\Big]. \tag{3.15}$$

This extension of the ansatz is not unnatural as it clearly contains MN(II) as the special case $F_1 = F_2$. Furthermore it leaves the (3, 0) form Ω invariant;[6] it is a purely Kähler deformation of the $SU(3)$ structure. We also make the obvious modification of the frame ansatz. In general, the torsion conditions and Bianchi identity are rather complicated. However, it is easy to determine the most general Calabi-Yau solution with this ansatz, imposing closure of J and constancy of L. The Calabi-Yau condition reads

$$\partial_tF_1 = \partial_tF_2 = 0, \qquad \frac{1}{3}\partial_u(F_1F_2) + \frac{u}{2F_1} = 0,\qquad \partial_u\Big(\frac{u^2}{4F_2}\Big) - \frac{u}{2F_1} = 0, \tag{3.16}$$

with general (positive signature) solution

$$F_1 = \frac{3a^4}{u^2}\cos^2\xi, \qquad F_2 = \frac{u^2}{2a^2}\frac{(1+\sin\xi)}{\cos^2\xi}, \tag{3.17}$$

where a^2, b are constants and $\sin\xi$ is a root of the cubic equation

$$-\frac{1}{3}\sin^3\xi + \sin\xi = b - \frac{u^4}{12a^6}. \tag{3.18}$$

This Calabi-Yau has two moduli. One, as usual, is just the overall scale. Defining

$$R = \frac{u^2}{2\sqrt{3}a^3}, \tag{3.19}$$

the metric is

$$\mathrm{d}s^2 = a^2\Big[\frac{1}{2}(1+\sin\xi)\mathrm{d}s^2(H^2) + \frac{\cos^2\xi}{2(1+\sin\xi)}\mathrm{d}s^2(S^2) + \frac{1}{\cos^2\xi}(\mathrm{d}R^2 + R^2(\mathrm{d}\psi + P - P')^2)\Big],\qquad -\frac{1}{3}\sin^3\xi + \sin\xi = b - R^2. \tag{3.20}$$

The modulus b parametrizes inequivalent metrics. The generic metric of this form has three degeneration points: $R = 0$ and $\xi = \pm\pi/2$. The point $\xi = -\pi/2$ (where the

[6] Observe that $L^{-3/2}\Omega$ is always closed with this frame ansatz.

H^2 cycle degenerates) is necessarily and expectedly singular. This is where fractional branes are wrapped, in the probe picture. If $R = 0$ and $\xi = \pi/2$ do not coincide, the point $R = 0$ is also singular, since there the $U(1)$ degenerates with a 4π periodicity inherited from the AdS frame. To analyze what happens near $\xi = \pi/2$, we expand the cubic to fourth order in ξ in the vicinity of this point to find

$$R^2 = \left(b - \frac{2}{3}\right) + \frac{\xi^4}{4}. \tag{3.21}$$

Clearly we require $b \geq 2/3$ (otherwise $\xi = \pi/2$ is not part of the space). If $b > 2/3$, then R^2 goes to a fixed positive value at $\xi = \pi/2$; the metric there is clearly singular when written in terms of ξ. However, the point in moduli space $b = 2/3$ where the $R = 0$ and $\xi = \pi/2$ degeneration points of the metric coincide is special. With $b = 2/3$ the metric near $\xi = \pi/2$ becomes

$$\mathrm{d}s^2 = a^2\bigg[\mathrm{d}s^2(H^2) + \mathrm{d}\xi^2 + \frac{\xi^2}{4}(\mathrm{d}s^2(S^2) + (\mathrm{d}\psi + P - P')^2)\bigg]. \tag{3.22}$$

With the periodicity of ψ as inherited from the AdS frame, an S^3 smoothly degenerates. The Calabi-Yau (3.20), with $b = 2/3$, is interpreted as the image of MN(II) under an inverse geometric transition. Analyzing the relationship between R and ξ near $\xi = -\pi/2$, we can deduce that the range of R is either $[-2/\sqrt{3}, 0)$ or $(0, 2/\sqrt{3}]$. In either branch, the singularity is at finite proper distance.

IV. CONCLUSIONS AND OUTLOOK

In this paper a way of mapping a supersymmetric AdS manifold to a special holonomy manifold has been proposed. The main conclusion is that this procedure should be applicable to all known wrapped-brane AdS solutions of string and M-theory; it will be very interesting to explore the special holonomy metrics in each case. Given the metrics for the string theory solutions, it should be possible to make progress towards constructing the dual field theories.

The construction relies in an essential way on the existence of an interpolating solution. For MN(I) and (II), we can say a little about what the boundary conditions for an interpolation should be. For MN(I), the interpolating solution should be globally smooth, and should contain a neighborhood where the metric is diffeomorphic to the limit of the metric (2.30) as $R \to 1$. It should also contain a neighborhood where the metric is diffeomorphic to the limit as $\rho \to 0$ of (3.1). For MN(II), an interpolating solution should contain a neighborhood diffeomorphic to the Calabi-Yau metric (3.20) near $\xi = \pi/2$. It should also contain a neighborhood where the metric is diffeomorphic to (3.1) as $\rho \to \pm 2/\sqrt{3}$. Global topological considerations will be important in trying to construct an interpolation; for example, by a careful analysis it should be possible to fix the relative scales of the Maldacena-Nuñez/Calabi-Yau metrics. This might be done, for example, by comparing the sizes of the H^2 bolts in the AdS and Calabi-Yau metrics, at the point in each where the $U(1)$ or S^3 degenerates, for MN(I) and MN(II), respectively. However, since an interpolating solution would necessarily be cohomogeneity two, and the governing equations are nonlinear, finding one explicitly will be challenging. It might be worthwhile to perform a numerical analysis, if a better handle can be obtained on the boundary conditions.

There might be other, more complicated, Calabi-Yau manifolds that could be related to the Maldacena-Nuñez solutions. This would be analogous to the way in which conical Calabi-Yaus can be thought of as generic local models for a particular sort of singularity, in a manifold whose global structure could be much more complicated. Placing D-branes at the singularity is usually argued to produce an AdS throat, which is insensitive to the global structure. It would be interesting to know if the Calabi-Yau metrics obtained here can be thought of in a similar way—as local models of a more generic type of singularity. The topology of the manifolds in this paper is more complicated, so it is not obvious yet whether or not this is true. In any event, for the purposes of constructing the dual, in the conical case only the geometry near the singularity—the conical metric—is required. Analogously, for the purposes of constructing the duals of the Maldacena-Nuñez solutions, the metrics of this paper are interpreted as the appropriate backgrounds.

There appears to be an intriguing link between solutions of the various nonlinear equations we have encountered and roots of polynomials. This seems to suggest some underlying algebraic geometry which has not been properly appreciated. It will be interesting to explore this in more detail; it appears to be a generic feature of how AdS manifolds solve wrapped-brane structure equations.

ACKNOWLEDGMENTS

I would like to express my gratitude to Pau Figueras and Jerome Gauntlett for collaboration during different stages of this project. This work was supported by EPSRC.

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