1-brane sources for the light-cone worldsheet: Q-branion–$\bar{Q}$-branion scattering to one loop

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This paper extends the study, initiated by Rozowsky and Thorn, of gauge fields in interaction with Dirac fields living on separated parallel 1-branes. In a light-cone description, replacing static point sources by 1-brane sources allows p^+ conservation to be maintained in their presence, which simplifies the light-cone quantization procedure. Here we calculate on-shell branion scattering amplitudes through one loop in light-cone gauge, and thereby resolve a puzzling ambiguity encountered in the earlier off-shell calculations. We confirm that infrared divergences cancel in properly defined scattering probabilities. This work lays the groundwork for the incorporation of 1-brane sources in the light-cone worldsheet formalism.

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I. INTRODUCTION

The response of gauge fields to separated static quark antiquark ($Q\bar{Q}$) sources provides valuable information about a gauge theory. For example, in QCD this response is described by the expectation of a long rectangular Wilson loop $\langle W(L, T)\rangle$, with $T \gg L$, which provides among other things an elegant criterion for quark confinement: In pure Yang-Mills theory without quarks, $\langle W(L, T)\rangle \sim e^{-T_0 LT}[1 + O(\sum_n c_n e^{-\Delta_n T})]$ implies a constant confining force T_0. With quarks included, this criterion only works in the 't Hooft limit [1] $N_c \to \infty$ of QCD generalized from a 3 color to an N_c color gauge theory[2]. The confining force is due to the formation of a flux tube, whose excitations can be further studied by extracting the discrete (at $N_c = \infty$ [3]) excited energy levels Δ_n from an analysis of the large T behavior of $\langle W\rangle$.

Such a system of sources is also an insightful tool in the study of string/field duality as exemplified by the AdS/CFT correspondence [4]. On the field theory side the CFT is the conformally invariant $\mathcal{N} = 4$ super Yang-Mills theory. Its response to the $Q\bar{Q}$ source system can be calculated at weak 't Hooft coupling $\lambda \equiv N_c\alpha_s/\pi \ll 1$ by expanding $\langle W\rangle$ as a sum of planar Feynman diagrams. On the string side, $\langle W\rangle$ is given as a worldsheet path integral for an open string, moving on a manifold $\mathrm{AdS}_5 \times \mathrm{S}^5$, and whose ends are fixed to two points separated by a distance L on the boundary of AdS_5. At strong 't Hooft coupling the worldsheet dynamics can be treated semiclassically, enabling the calculation of the ground state energy $-c\sqrt{\lambda}/L$ of the flux tube [5] as well as its excited energy spectrum [3,6] when $\lambda \gg 1$.

Although the AdS/CFT correspondence asserts the equivalence of $\mathcal{N} = 4$ Yang-Mills to IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ at all couplings, very little is known about the physics, from either point of view, at intermediate coupling $\lambda = O(1)$. Some physical quantities [e.g. Bogomol'nyi-Prasad-Sommerfield (BPS) states] are protected by supersymmetry from dependence on the coupling, and so are "known" at all coupling. An example is the single straight Wilson line representing a static isolated quark. The circular Wilson loop, which is conformally related to the Wilson line and "almost" BPS, does depend on the coupling, but because all diagrams except rainbow graphs cancel, it is easily computable by graph summation at all coupling [7,8]. Unfortunately, the rectangular Wilson loop does not enjoy these cancellations. Nonetheless, some interesting qualitative insight into its behavior has been obtained by summing the ladder subset of planar diagrams [3,9,10].

In a separate line of development, the light-cone worldsheet formalism [11–13] has provided a way to map, in a generic way, the sum of all planar diagrams of a wide range of quantum field theories to a worldsheet dynamical system. Treated in mean field theory, a plausible approximation in the strong 't Hooft coupling limit, this worldsheet system resembles a string moving on an AdS-like manifold [14–16], encouraging the hope that a more exact treatment of it can help in understanding string/field duality at all coupling. Our aim in the present article is to take a first step toward including $Q\bar{Q}$ sources in this formalism, a task that involves complications which we briefly describe below.

The first complication is the awkwardness of describing a fixed point source on the light cone. Light-cone time is $\tau = (t + z)/\sqrt{2}$. This leaves $\boldsymbol{x} = (x, y)$ and $x^- = (t - z)/\sqrt{2}$ as spatial coordinates. We would like a point source to be at fixed x, y, z not fixed x, y, x^-. Fixed x^- would describe an object moving at the speed of light, and a point source at fixed $(x, y, \tau - x^-)$ would not be static with respect to light-cone time. Furthermore, either alternative would violate p^+ conservation, an essential ingredient of the natural light-cone symmetry of the bulk gauge theory, Galilei invariance in the transverse space. Since the momentum component $p^+ = (E + p^z)/\sqrt{2}$ plays the role of the Newtonian mass in Galilei boosts, Galilei invariance dictates its conservation. A way to maintain p^+ conserva-

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tion was proposed in [17]: fix the transverse coordinate of the source, but allow the source to move freely on a line parallel to the z axis. In the language of string theory the source is then not a point (0-brane) but a 1-brane. For brevity we have called particles living on a 1-brane branions. So in [17] we replaced the usual static $Q\bar{Q}$ system with a branion in color irrep N_c living on a 1-brane together with a branion in color irrep $\bar{N}_c$ living on a second 1-brane separated from the first by a distance L. Then p^+ conservation will be preserved at the price of having to solve the limited $1+1$ dynamics of the branions. This generalization of sources is also natural for the light-cone parametrization of string which relies on p^+ to label points on a string. In this parametrization the motion of the string in $x^-(\sigma, \tau)$ is completely constrained in terms of the transverse motion $\boldsymbol{x}(\sigma, \tau)$. Fixing $\boldsymbol{x}$ at the string ends allows no freedom to independently fix boundary conditions on x^-. Indeed x^- will have Neumann boundary conditions whether $\boldsymbol{x}$ has Dirichlet or Neumann boundary conditions.

The second complication, present even using 1-brane sources, is that the light-cone worldsheet formalism is constructed from the planar diagrams using transverse momentum space in the Feynman rules. As a consequence the worldsheet path integral is expressed in terms of $\boldsymbol{q}(\sigma, \tau)$ which is T-dual to the usual transverse coordinates $\boldsymbol{x}(\sigma, \tau)$: $\boldsymbol{q}' = \dot{\boldsymbol{x}}$. (The corresponding relation between $\dot{\boldsymbol{q}}$ and $\boldsymbol{x}'$ is complicated, depending on the detailed dynamics.) This is fundamental to the formalism, which is founded on the worldsheet representation of a gluon propagator:

$$\exp\left\{-i\frac{x^+}{2p^+}\boldsymbol{p}^2\right\} = \int_{\substack{\boldsymbol{q}(0,\tau)=0\\ \boldsymbol{q}(p^+,\tau)=\boldsymbol{p}}} DcDbD\boldsymbol{q} \times \exp\left\{\int_0^T d\tau \int_0^{p^+} d\sigma\left(b'c' - \frac{1}{2}\boldsymbol{q}'^2\right)\right\}. \tag{1}$$

Here σ, τ parameter space is a rectangle $0 \le \tau \le T \equiv ix^+$, $0 \le \sigma \le p^+$, and $\boldsymbol{q}(\sigma, \tau)$ is a worldsheet field satisfying Dirichlet boundary conditions such that $\boldsymbol{q}(p^+, \tau) - \boldsymbol{q}(0, \tau) = \boldsymbol{p}$. The Grassmann b, c ghost path integral cancels the determinant prefactor coming from the Gaussian $\boldsymbol{q}$ path integration. Replacing all the propagators in a planar Feynman diagram with this representation automatically constructs the diagram's worldsheet representation. The complication with introducing localized 1-brane sources at the worldsheet boundaries is that, in the string representation, the fixed 1-brane locations are $\boldsymbol{x}(0, \tau)$, $\boldsymbol{x}(p^+, \tau)$. The fact that they are static locations translates to simple Neumann conditions on the dual variables $\boldsymbol{q}' = 0$. This is not so bad. The complication comes in describing the separation between the branes,

$$\boldsymbol{L} = \boldsymbol{x}(p^+, \tau) - \boldsymbol{x}(0, \tau) = \int_0^{p^+} d\sigma \boldsymbol{x}'. \tag{2}$$

For a string in flat space $\boldsymbol{x}' = -\dot{\boldsymbol{q}}$, and the separation can be interpreted as a nonzero "momentum" associated with translational invariance in $\boldsymbol{q}$. But the worldsheet action derived from the sum of planar diagrams shows that $\boldsymbol{q}$ has, in general, very complicated interactions with other worldsheet degrees of freedom that are not necessarily interpreted as coordinates of a manifold, as they happily can be in the AdS/CFT case.

Although we will leave definitive resolution of these difficulties to future work, we can catch a glimpse of the issues involved by considering the light-cone description of an AdS string [18]. We choose coordinates so that the line element in AdS is $ds^2 = R^2(dx_\mu dx^\mu + dz^2)/z^2$. Then the worldsheet action for a string moving on AdS_5 is

$$S_{ws} \equiv \int d^2\xi \mathcal{L} = -\frac{T_0}{2}\int d^2\xi\sqrt{g}g^{\alpha\beta}\frac{R^2}{z^2}(\partial_\alpha x\cdot\partial_\beta x + \partial_\alpha z\partial_\beta z).$$

For $\mathcal{N} = 4$ super Yang-Mills theory, $T_0R^2 = \sqrt{g^2N_c/4\pi^2} = \sqrt{\lambda}$. Light-cone parametrization of the string means $x^+ = \tau$ and $\mathcal{P}^+ = 1$, where $\mathcal{P}^+$ is the momentum conjugate to x^-. Then in this parametrization

$$S_{ws} \to \int d\tau \int_0^{p^+} d\sigma \frac{1}{2}\left[\dot{\boldsymbol{x}}^2 + \dot{z}^2 - \frac{R^4T_0^2}{z^4}(\boldsymbol{x}'^2 + z'^2)\right].$$

For a closed string one must also impose the constraint $\int_0^{p^+} d\sigma(\boldsymbol{x}'\cdot\boldsymbol{\mathcal{P}} + z'\Pi) = 0$. The equation of motion for $\boldsymbol{x}$ following from this action is

$$\ddot{\boldsymbol{x}} = \left(\frac{R^4T_0^2\boldsymbol{x}'}{z^4}\right)'.$$

To put the AdS string action in a form similar to the light-cone worldsheet action read off from graph summation, we do the T-dual transformation

$$\boldsymbol{q}' = \dot{\boldsymbol{x}}, \qquad \dot{\boldsymbol{q}} = \frac{R^4T_0^2}{z^4}\boldsymbol{x}'.$$

The integrability condition for these equations implies the equation of motion for $\boldsymbol{x}$. Expressing the worldsheet Lagrangian in terms of $\boldsymbol{q}$ gives

$$\mathcal{L} \to \frac{1}{2}\left(-\boldsymbol{q}'^2 + \frac{z^4}{R^4T_0^2}\dot{\boldsymbol{q}}^2 + \dot{z}^2 - \frac{R^4T_0^2}{z^4}z'^2\right).$$

We recognize in the $\boldsymbol{q}'^2$ term the part of the quantum field theory (QFT) worldsheet action coming from the propagator representation (1). The rest of the AdS worldsheet action must simulate the sum over planar loop corrections. Notice that the $\dot{\boldsymbol{q}}$ dependence is negligible near the boundary of AdS ($z = 0$). The intuitive origin of such terms is explained in the foundational papers on the light-cone worldsheet [11–13]. We just mention here that a loop is represented on the QFT worldsheet by a line segment at fixed σ on which $\dot{\boldsymbol{q}} = 0$. Thus terms in the action that energetically favor this condition will be gradually brought into the worldsheet action as one includes more and more loops. It is very plausible that in the strong 't Hooft coupling limit a mean field treatment of the sum over loops can be represented by a bulk term in the action similar to the $\dot{\boldsymbol{q}}^2$ term in the AdS string action. Our purpose here is to give an indication of how to describe separated 1-branes using the light-cone worldsheet formalism. From the T-duality transform we see that a string ending on two 1-branes separated by $\boldsymbol{L}$ must satisfy the constraint

$$\boldsymbol{L} = \int_0^{p^+} d\sigma \frac{z^4}{T_0^2 R^4} \dot{\boldsymbol{q}}. \tag{3}$$

In fact, this quantity is conserved by the AdS dynamics and so it is a constraint that imposed initially will hold for all times thereafter. However, its analogue in the QFT worldsheet will depend in detail on the outcome of the sum over loops, and is not expected in a generic theory to have at all the simplicity of this formula. On the other hand, from the point of view of Feynman diagrams in quantum field theory, there is no doubt about how the 1-brane separation enters. Quantum fields, representing branions, in $1 + 1$ space-time dimensions will live on each 1-brane, and these fields will interact with the gauge fields in the bulk. A gluon propagator that ends on a branion will have that end localized on the corresponding brane. If the gluon propagator is expressed in transverse momentum space, this means the gluon branion vertex will be associated with a factor $e^{i\boldsymbol{Q}\cdot\boldsymbol{r}}$ where $\boldsymbol{r}$ is the transverse location of the 1-brane and $\boldsymbol{Q}$ is the gluon momentum. We expect it to be a significant challenge, beyond the scope of this article, to figure out in detail how this simple prescription turns into a constraint like (3) in the light-cone worldsheet formalism.

We devote the remainder of this article to a study of the branion-gauge field interactions at weak coupling, i.e. to the evaluation of the corresponding Feynman diagrams through one loop. Although we shall not attempt to give a definitive interpretation of our results in terms of the light-cone worldsheet here, we shall take a step in that direction by regularizing the loop integrals in a worldsheet-friendly way. We employ dual momentum variables, and the ultraviolet cutoff $e^{-\delta \boldsymbol{q}^2}$ as in [14,15,19–21]. We shall extend the results of [17] in important ways. In the latter work four-branion one-loop Feynman diagrams were evaluated in light-cone gauge, but with the branions off shell. A simple ultraviolet cutoff on transverse momentum was imposed, and infrared divergences were regulated by discretizing p^+. This is natural from the point of view of light-cone worldsheet path integrals, because it is nothing more nor less than defining the path integrals on a lattice [22–24]—a very standard thing to do. However, in [17] we examined the continuous p^+ limit for the off-shell amplitudes we computed, and found some residual artificial $p^+ = 0$ divergences. These are artificial because true infrared divergences are not present off shell. On the other hand, off-shell amplitudes are gauge noninvariant and unphysical, so such artificial divergences are not ruled out. Unfortunately, in [17] it was found that these divergences did not disappear unambiguously in the on-shell limit: this limit involved quantities of the form $0/0$, with values that depended on exactly how the on-shell limit was taken. This issue was left unresolved in [17], because it was also tangled up with conventional infrared divergences which were beyond the scope of that paper.

In the work described here we calculate on-shell scattering amplitudes, using discretized p^+ as an infrared cutoff that makes these on-shell quantities finite. Then we do the standard Lee-Nauenberg analysis of infrared divergences and show that they cancel as they should, allowing an unambiguous continuum limit of the p^+ sums. The resolution of the ambiguity found in [17] is that the on-shell limit and continuous p^+ limit do not commute: one must only take p^+ continuous for physical on-shell quantities. In the course of these calculations, we identify all of the counterterms that are needed to remove gauge violating artifacts that crop up because of ultraviolet divergences. We identify these counterterms by comparing the results of our δ regularization to the results given by dimensional regularization in the transverse dimensionality. The assumption here is that dimensional regularization gives the correct gauge invariant results. In fact, one of these inferred counterterms is essential for the cancellation of infrared divergences, giving some independent support for this assumption. This last counterterm shows worldsheet nonlocal features when directly interpreted. However, as in the case of some of the counterterms needed in the gluon scattering calculations of [21], it is possible to realize them locally if additional worldsheet fields are introduced.

The rest of the paper is organized as follows. In Sec. II we summarize the Feynman rules for branions in interaction with bulk gauge fields and give the tree level branion scattering amplitude. In the next three sections we evaluate the branion and gluon self-energy diagrams, triangle diagrams, and box diagrams, respectively. In Sec. VI we show that the residual infrared divergences in the one-loop elastic scattering amplitudes cancel in their contribution to scattering probabilities against divergences in the probability for the emission of extra soft gluons. Concluding remarks are in Sec. VII. Finally, there are two appendixes in which needed loop integrals are evaluated.

II. FEYNMAN RULES FOR BRANIONS AND 4 BRANION TREES

The light-cone setup and light-cone gauge Feynman rules for branions, taken to be $1 + 1$ Dirac fermions,

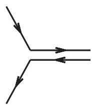
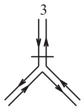
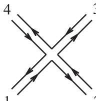
Light-Cone Feynman Rules	
	$-\frac{i}{\gamma^\alpha p_\alpha + m}$
	$-\frac{i}{K^2}\left(\eta^{\mu_1\mu_2} - \frac{K^{\mu_1}\eta^{\mu_2 +} + K^{\mu_2}\eta^{\mu_1 +}}{K^+}\right)$
	$ig\gamma^\alpha$
	$-ig\,\eta^{\mu_1\mu_2}(Q_1 - Q_2)^{\mu_3}$
	$ig^2\left[2\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} - \eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}\right]$

FIG. 1. Light-cone Feynman rules using "double line" notation. All momenta in vertices are taken to be incoming, and the line in the three-gluon vertex distinguishes the three cyclic orderings. Index α only includes brane coordinates, while indices μ_i run over all coordinates.

were obtained in [17] and summarized in a table of that reference reproduced here in Fig. 1 for the reader's convenience. The physical process we analyze in this article is on-shell branion-branion scattering, where the two incoming branions as well as the two outgoing branions are on different 1-branes separated by a distance $\boldsymbol{L}$. We write the amplitude for this process as a Fourier transform:

$$\tilde{\Gamma}(p, q, Q_\parallel, \boldsymbol{L}) \equiv \int \frac{d\boldsymbol{Q}}{(2\pi)^2} e^{i\boldsymbol{Q}\cdot\boldsymbol{L}} \Gamma(p, q, Q_\parallel, \boldsymbol{Q}). \qquad (4)$$

Here p, q are the 2-vector momenta of the two incoming branions and $Q_\parallel$ is the momentum transfer of the process, the final 2-vector momenta being $p + Q_\parallel$, $q - Q_\parallel$, respectively. The integrand Γ is evaluated by the usual momentum space Feynman rules, with the understanding that the branions can absorb or give up any amount of transverse momentum with no change of state. The gluons attached to the right branion carry away a total transverse momentum of $\boldsymbol{Q}$ which is absorbed by the left branion from the gluons attached to it. We calculate Γ by fixing this total transverse momentum and integrating over all the other momenta as loop momenta.

With this understanding, we find for the lowest order (tree) contribution to this process,

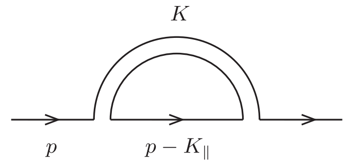


FIG. 2. Branion self-energy diagram. Only the gluon line carries transverse momentum.

$$\Gamma^{\rm Tree} = \frac{2ig^2\gamma_1^+\gamma_2^+ Q^-}{Q^+ Q^2} = \frac{2ig^2\gamma_1^+\gamma_2^+ Q^-}{Q^+(\boldsymbol{Q}^2 - 2Q^+Q^-)}. \qquad (5)$$

Because the branions are free to move in only one dimension the on-shell condition is very restrictive: there is only the option of forward and backward scattering: $Q_\parallel = 0$, $q - p$, respectively. To avoid $Q^+ = 0$ issues we restrict consideration in the rest of the paper to on-shell backward scattering, $Q^+ = q^+ - p^+$, $Q^- = -m^2(q^+ - p^+)/2q^+p^+$. Then

$$\Gamma^{\rm Tree} = -\frac{ig^2m^2\gamma_1^+\gamma_2^+}{p^+q^+\boldsymbol{Q}^2 + m^2(q^+ - p^+)^2}, \qquad (6)$$

$$\tilde{\Gamma}^{\rm Tree} = -\frac{ig^2m^2\gamma_1^+\gamma_2^+}{2\pi p^+q^+} K_0\left(Lm\frac{q^+ - p^+}{\sqrt{p^+q^+}}\right). \qquad (7)$$

It is easy enough to obtain $\tilde{\Gamma}$ for the tree amplitude in terms of the Kelvin function K_0. But for the one-loop calculations that follow we calculate Γ and do not carry out the final Fourier transformation that would convert it to $\tilde{\Gamma}$.

III. SELF-ENERGY DIAGRAMS

A. Branion self-energy

The branion self-energy diagram is shown in Fig. 2. Notice that the gluon propagates in the bulk whereas the fermion resides on the 1-brane. Applying the Feynman rules (Fig. 1) we find

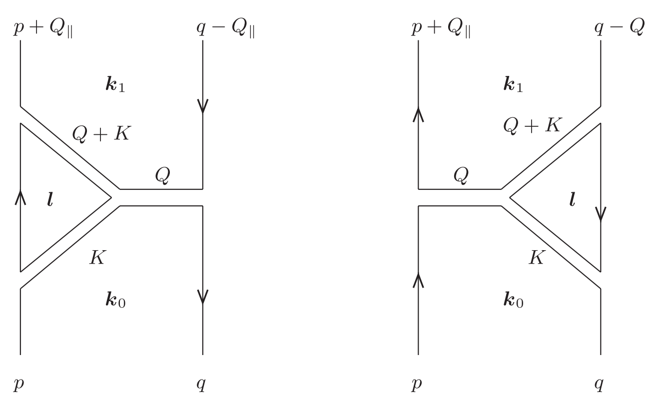


FIG. 3. Triangle Feynman diagrams contributing to the four-point amplitude. The arrows show the direction of color flow, and p, q are incoming momenta.

$$
\begin{aligned}
-i\Sigma(p) &= g^2 N_c \int \frac{d^4K}{(2\pi)^4} e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2} \frac{\gamma^+(m-\gamma\cdot(p-K_\parallel))\gamma^+}{m^2+(p-K_\parallel)^2}\frac{2K^-}{K^+K^2} \\
&= g^2N_c\gamma^+\int\frac{d^4K}{(2\pi)^4}e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2}\frac{2(p^+-K^+)}{m^2+(p-K_\parallel)^2}\frac{2K^-}{K^+K^2}.
\end{aligned} \tag{8}
$$

To evaluate the K^- integral by residues we must add a semicircle at infinity that gives a finite contribution, since the integrand only falls as $1/K^-$ at large K^-:

$$
\frac{2(p^+-K^+)}{m^2+(p-K_\parallel)^2}\frac{2K^-}{K^+K^2}\sim\frac{2(p^+-K^+)}{2(p^+-K^+)K^-}\frac{2K^-}{(-2)K^{+2}K^-}\sim-\frac{1}{K^{+2}K^-}. \tag{9}
$$

Thus the added semicircular contour will contribute $-i\pi/K^{+2}$ if it closes the contour in the upper half plane and $+i\pi/K^{+2}$ if it closes in the lower half plane. In evaluating the K^- integral by residues it is convenient to close in the upper half plane when $K^+>p^+$ and in the lower half plane when $K^+<p^+$. The integral over K^- will be given by the residues of any poles inside the closed contour minus the contributions of the added semicircular contours:

$$
\begin{aligned}
-i\Sigma(p) &= g^2N_c\gamma^+\int\frac{d\boldsymbol{K}}{(2\pi)^4}e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2}\bigg[\sum_{K^+}\text{Residue}\frac{2(p^+-K^+)}{m^2+(p-K_\parallel)^2}\frac{2K^-}{K^+K^2}+i\pi\sum_{K^+>p^+}\frac{1}{K^{+2}}-i\pi\sum_{K^+<p^+}\frac{1}{K^{+2}}\bigg] \\
&= g^2N_c\gamma^+\int\frac{d\boldsymbol{K}}{(2\pi)^4}e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2}\bigg[\sum_{K^+}\text{Residue}\frac{2(p^+-K^+)}{m^2+(p-K_\parallel)^2}\frac{2K^-}{K^+K^2}-2i\pi\sum_{0<K^+<p^+}\frac{1}{K^{+2}}\bigg].
\end{aligned} \tag{10}
$$

Because of our choice of closed contours, only the pole at $K^2-i\epsilon=0$ contributes to the integral and then only when $0<K^+<p^+$:

$$
\begin{aligned}
-i\Sigma(p) &= \frac{ig^2N_c\gamma^+}{8\pi^3}\sum_{0<K^+<p^+}\frac{1}{K^{+2}}\int d\boldsymbol{K}e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2}\frac{2K^+p^--m^2K^+/(p^+-K^+)}{\boldsymbol{K}^2-2K^+p^-+m^2K^+/(p^+-K^+)} \\
&\sim \frac{ig^2N_c\gamma^+}{8\pi^2}\sum_{0<K^+<p^+}\bigg(\frac{m^2+p^2}{p^+K^+}+\frac{m^2}{p^+(p^+-K^+)}\bigg)\ln\delta e^\gamma\frac{K^+(m^2+p^2+2K^+p^-)}{p^+-K^+} \\
&\sim \frac{ig^2N_c\gamma^+}{8\pi^2}\sum_{0<K^+<p^+}\frac{m^2}{p^+(p^+-K^+)}\ln\frac{K^{+2}m^2\delta e^\gamma}{p^+(p^+-K^+)}+(m^2+p^2)\frac{ig^2N_c\gamma^+}{8\pi^2} \\
&\quad\times\sum_{0<K^+<p^+}\frac{1}{p^+K^+}\ln\frac{K^{+2}m^2\delta e^{\gamma+1}}{p^+(p^+-K^+)}+O([m^2+p^2]^2)
\end{aligned} \tag{11}
$$

where the final form applies near mass shell $m^2+p^2\sim0$. Stripping away the γ^+ and multiplying by $2ip^+$ gives the shift in the quantity m^2+p^2, and so we find

$$
\Delta m^2=-\frac{g^2N_c}{4\pi^2}\sum_{0<K^+<p^+}\frac{m^2}{p^+-K^+}\ln\frac{K^{+2}m^2\delta e^\gamma}{p^+(p^+-K^+)},\qquad Z_2=1+\frac{g^2N_c}{4\pi^2}\sum_{0<K^+<p^+}\frac{1}{K^+}\ln\frac{K^{+2}m^2\delta e^{\gamma+1}}{p^+(p^+-K^+)}. \tag{12}
$$

The mass shift should be a numerical function of the UV cutoff δ, but the divergent sum near $K^+=p^+$ introduces an apparent infrared divergence depending on p^+. This is due to the small δ approximation used in the second line of (11), which implicitly neglected δ in comparison to the p^+ discretization unit ϵ/p^{+2}. If we go back to the on-shell limit of the first line, we see that the continuum limit of the K^+ sum is actually convergent at fixed δ.

$$
\begin{aligned}
-i\Sigma(p)|_{p^2=-m^2} &= -\frac{ig^2N_c\gamma^+}{8\pi^3}\sum_{0<K^+<p^+}\frac{m^2}{p^+(p^+-K^+)}\int d\boldsymbol{K}\frac{e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2}}{\boldsymbol{K}^2+m^2K^{+2}/p^+(p^+-K^+)} \\
&\to -\frac{ig^2N_cm^2\gamma^+}{8\pi^3p^+}\int_0^1\frac{dx}{1-x}\int d\boldsymbol{K}\frac{e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2}}{\boldsymbol{K}^2+m^2x^2/(1-x)},
\end{aligned} \tag{13}
$$

$$
\Delta m^2=\frac{g^2N_cm^2}{4\pi^2}\int_0^\infty\frac{dT}{T+\delta}\int_0^1\frac{dx}{1-x}\exp\bigg\{-\frac{m^2Tx^2}{1-x}-\frac{T\delta\boldsymbol{k}_0^2}{T+\delta}\bigg\}. \tag{14}
$$

One can easily check the large and small T behavior of the function

$$F(T) = \int_0^1 \frac{dx}{1-x} \exp\left\{-\frac{m^2Tx^2}{1-x}\right\} \sim \begin{cases} \frac{1}{2}\sqrt{\frac{\pi}{m^2T}} & \text{for } T \to \infty, \\ -\ln(m^2Te^{\gamma}) & \text{for } T \to 0, \end{cases} \tag{15}$$

which confirms that Δm^2 is finite at fixed δ. Furthermore, the small T behavior of F controls the small δ behavior of Δm^2:

$$\Delta m^2 \sim \frac{g^2 N_c m^2}{8\pi^2}(\ln^2(m^2\delta e^{\gamma}) + O(1)). \tag{16}$$

Comparing this to (11), we see that the double logarithmic UV divergence in (16) shows up as a single log UV divergence times a single log IR divergence in (11) when $\delta \to 0$ is taken before the continuum limit. This nonuniformity is because, in the latter case, the part of the UV divergence due to the zero thickness of the 1-brane is cut off by the p^+ discretization. However, for Z_2 and the more complicated diagrams considered later, it is valid to make the small δ approximation at discrete p^+, because in those cases singularities due to the zero thickness are integrable. The double log divergence in Δm^2 would not be present if the branion had not been confined to a brane. Indeed, for a p-brane with $p > 1$ the corresponding singularity would be integrable.

It is instructive to compare the transverse momentum integral using our δ regulator with that using dimensional regularization, with transverse dimension $d < 2$, which gives

$$\begin{aligned} -i\Sigma(p) &= \frac{ig^2N_c\gamma^+}{2\pi}\sum_{0<K^+<p^+}\frac{1}{K^{+2}}\int\frac{d\boldsymbol{K}}{(2\pi)^d}\frac{2K^+p^- - m^2K^+/(p^+-K^+)}{\boldsymbol{K}^2 - 2K^+p^- + m^2K^+/(p^+-K^+)} \\ &= -\frac{ig^2N_c\gamma^+}{2\pi}\sum_{0<K^+<p^+}\frac{1}{K^{+2}}\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\left[\frac{m^2K^{+2}}{p^+(p^+-K^+)} + (m^2+p^2)\frac{K^+}{p^+}\right]^{d/2} \\ &\sim -\frac{2ig^2N_c\gamma^+}{(4\pi)^{1+d/2}}\sum_{0<K^+<p^+}\frac{\Gamma(1-d/2)K^{+d-2}m^d}{p^{+d/2}(p^+-K^+)^{d/2}}\left[1 + \frac{d}{2}(m^2+p^2)\frac{p^+-K^+}{m^2K^+}\right], \end{aligned} \tag{17}$$

where in the last line we expanded about mass shell $p^2 + m^2 \sim 0$, from which we read off Δm^2:

$$\begin{aligned} \Delta m^2 &= \frac{g^2N_c}{\pi}\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\sum_{0<K^+<p^+}\frac{K^{+d-2}m^d}{p^{+d/2-1}(p^+-K^+)^{d/2}} \to \frac{g^2N_c}{\pi}\frac{m^d\Gamma(1-d/2)}{(4\pi)^{d/2}}\int_0^1 dx x^{d-2}(1-x)^{-d/2} \\ &= \frac{g^2N_c}{\pi}\frac{m^d\Gamma(1-d/2)^2}{(4\pi)^{d/2}}\frac{\Gamma(d-1)}{\Gamma(d/2)} \end{aligned} \tag{18}$$

which is finite for $d < 2$. Here the double pole at $d = 2$ reflects the double log divergence in (16). Reading off Z_2 we find

$$\begin{aligned} Z_2 &= 1 - \frac{g^2N_c}{\pi}\frac{d}{2}\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\sum_{0<K^+<p^+}\frac{K^{+d-3}m^{d-2}}{p^{+d/2-1}(p^+-K^+)^{d/2-1}} \\ &\sim 1 - \frac{g^2N_c}{4\pi^2}\sum_{0<K^+<p^+}\frac{1}{K^+}\left[\frac{\Gamma(1-d/2)}{(4\pi)^{(d-2)/2}} - \ln\frac{K^{+2}m^2e}{p^+(p^+-K^+)}\right], \end{aligned} \tag{19}$$

where in the last line we have taken $d \sim 2$. As mentioned above the singularity for $K^+ \to p^+$ is integrable at $d = 2$ here, so this is a valid procedure. We leave K^+ discrete because these expressions are divergent for $d < 2$. Comparing to the δ-regulator result for Z_2 (11), we find the correspondence

$$\frac{\Gamma(1-d/2)}{(4\pi)^{(d-2)/2}} \leftrightarrow -\ln e^{\gamma}\delta \quad \text{or} \quad \frac{2}{2-d} \leftrightarrow -\ln(4\pi\delta). \tag{20}$$

Using this correspondence we shall find that in different diagrams the two regulators are not in precise agreement as to the δ independent terms, and counterterms must be introduced to achieve equivalent results. Since dimensional regularization preserves more symmetry than the δ regulator, we shall presume that it is the latter that requires the counterterms. Then simple comparison of the two regulators in each diagram gives an efficient procedure for the identification of the required counterterms.

B. Gluon self-energy

From [20,25] the gluon self-energy diagram is given by [26]

$$\Pi^{++}(Q) \equiv Q^{+2}\Pi_1 = -\frac{g^2N_c}{4\pi^2}Q^{+2}\left(\frac{1}{6}\ln\{Q^2\delta e^\gamma\} - \frac{4}{9}\right), \tag{21}$$

$$\begin{aligned}\Pi^{\wedge\vee}(Q) &\equiv -Q^2\Pi_2 \\ &= \frac{g^2N_c}{4\pi^2}Q^2\left[\mathcal{A}(Q^2,Q^+) - \frac{11}{6}\ln\{Q^2\delta e^\gamma\} + \frac{67}{18}\right],\end{aligned} \tag{22}$$

$$\mathcal{A}(Q^2,Q^+) \equiv \sum_{q^+}\left[\frac{1}{q^+} + \frac{1}{Q^+ - q^+}\right]\ln\{x(1-x)Q^2\delta e^\gamma\}. \tag{23}$$

The corresponding gluon propagator up to one loop is

$$\begin{aligned}D^{--}(Q) &= \frac{i}{Q^{+2}}(1-\Pi_1), \\ D^{ij}(Q) &= \frac{-i\delta_{ij}}{Q^2}(1-\Pi_2).\end{aligned} \tag{24}$$

The contribution of Π to one-loop branion scattering is then

$$\begin{aligned}\mathcal{M}_{\rm SE} &= ig^2\gamma_1^+\gamma_2^+\left[-\frac{\boldsymbol{Q}^2\Pi_2}{Q^{+2}Q^2} + \frac{\Pi_1}{Q^{+2}}\right] = ig^2\gamma_1^+\gamma_2^+\left[-\frac{2Q^-\Pi_2}{Q^+Q^2} + \frac{\Pi_1 - \Pi_2}{Q^{+2}}\right] \\ &= \frac{ig^4N_c\gamma_1^+\gamma_2^+}{4\pi^2}\left[\frac{2Q^-}{Q^+Q^2}\left(\mathcal{A}(Q^2,Q^+) - \frac{11}{6}\ln\{Q^2\delta e^\gamma\} + \frac{67}{18}\right) + \frac{1}{Q^{+2}}\left(\mathcal{A}(Q^2,Q^+) - 2\ln\{Q^2\delta e^\gamma\} + \frac{25}{6}\right)\right].\end{aligned} \tag{25}$$

We note that, if the integrals are done in dimensional regularization and the correspondence (20) is assumed, then Π_2 is unchanged and the pure number in Π_1 is replaced as follows:

$$-\frac{4}{9} \to -\frac{5}{18}, \tag{26}$$

that is, $1/6$ is added, so the $25/6$ in (25) is changed to 4.

IV. TRIANGLE DIAGRAMS

We now calculate the triangle graphs contributing to the four-point amplitude. The Feynman diagrams for these contributions are portrayed in Fig. 3. Using the light-cone Feynman rules we immediately write the Feynman integral corresponding to the diagram on the left of Fig. 3.

$$\begin{aligned}\Gamma_{\triangle_L} &= N_c\int\frac{d^4K}{(2\pi)^4}(ig\gamma_1^+)\frac{-i}{\gamma_1^\alpha(p-K_\parallel)_\alpha + m}(ig\gamma_1^+)D^{-\mu_1}(K)D^{-\mu_2}(K+Q)D^{-\mu_3}(Q)V_{\mu_1\mu_2\mu_3}(K,-K-Q,Q)(ig\gamma_2^+) \\ &\to -\frac{4g^4N_c\gamma_1^+\gamma_2^+}{Q^+Q^2}\frac{1}{2\pi}\sum_{K^+}\int\frac{d\boldsymbol{K}}{(2\pi)^2}\frac{dK^-}{2\pi}\frac{(p^+-K^+)\times F}{((p-K_\parallel)^2+m^2)K^+K^2(K^++Q^+)(K+Q)^2},\end{aligned} \tag{27}$$

where

$$F = K^-[\boldsymbol{Q}^2(2Q^+ + K^+) + \boldsymbol{K}\cdot\boldsymbol{Q}(2K^+ + Q^+)] - Q^-[\boldsymbol{K}^2(2K^+ + Q^+) + \boldsymbol{K}\cdot\boldsymbol{Q}(2Q^+ + K^+)]. \tag{28}$$

The subscripts on the γ's distinguish between the different branes. All branion-branion-gluon vertices only include the + component of γ^μ, since the gluon propagator, $D^{\mu\nu}$, vanishes when $\mu = +$. We have replaced the K^+ integral by a sum over discretized $K^+ = \ell\epsilon$, where $\sum_{K^+}$ means $\epsilon\sum_l$. For each propagator in a Feynman diagram, the $p^+ = 0$ term in the sum is excluded. This discretization and zero mode exclusion serves two purposes: first, it systematically regulates the artificial $p^+ = 0$ divergences that crop up in light-cone gauge, and second, it provides a cutoff to regulate the physical infrared divergences due to massless gauge particles. In principle, we only take the continuum K^+ limit for properly defined physical quantities.

The diagram on the right of Fig. 3 similarly leads to the integral

$$\Gamma_{\triangle_R} = N_c \int \frac{d^4K}{(2\pi)^4}(ig\gamma_2^+)\frac{-i}{\gamma_1^\alpha(-q-K_\parallel)_\alpha + m}(ig\gamma_2^+)D^{-\mu_1}(K)D^{-\mu_2}(K+Q)D^{-\mu_3}(Q)V_{\mu_1\mu_3\mu_2}(-K,-Q,K+Q)(ig\gamma_1^+)$$
$$\to -\frac{4g^4N_c\gamma_1^+\gamma_2^+}{Q^+Q^2}\frac{1}{2\pi}\sum_{K^+}\int\frac{d\boldsymbol{K}}{(2\pi)^2}\frac{dK^-}{2\pi}\frac{(-q^+-K^+)\times F}{((q+K_\parallel)^2+m^2)K^+K^2(K^++Q^+)(K+Q)^2}, \tag{29}$$

with the same expression for F given by (28).

We shall employ dual momentum variables $\boldsymbol{l}$, $\boldsymbol{k}_0$, $\boldsymbol{k}_1$ related to the momenta via $\boldsymbol{Q} = \boldsymbol{k}_0 - \boldsymbol{k}_1$, $\boldsymbol{K} = \boldsymbol{l} - \boldsymbol{k}_0$. We shall specify our ultraviolet cutoff in these variables, by supplying a factor $e^{-\delta l^2}$. When the branions are on shell, we can use the identity

$$2(p^+-K^+)K^- = m^2 + (p-K_\parallel)^2 - \frac{m^2K^+}{p^+} \quad \text{for } \triangle_L, \tag{30}$$

$$2(-q^+-K^+)K^- = m^2 + (q+K_\parallel)^2 + \frac{m^2K^+}{q^+} \quad \text{for } \triangle_R \tag{31}$$

to rewrite the K^- term in F. Substituting (30) or (31) into (27) or (29), we see that the first two terms on the right of either identity cancel a propagator, contributing a bubble-like integral to the vertex function:

$$\Gamma^{\text{Bubble}}_{\triangle_L} = \Gamma^{\text{Bubble}}_{\triangle_R} = -\frac{2g^4N_c\gamma_1^+\gamma_2^+}{Q^+Q^2}\frac{1}{2\pi}\sum_{K^+}\int\frac{d\boldsymbol{K}}{(2\pi)^2}\frac{dK^-}{2\pi}\frac{\boldsymbol{Q}^2(2Q^++K^+)+\boldsymbol{K}\cdot\boldsymbol{Q}(2K^++Q^+)}{K^+K^2(K^++Q^+)(K+Q)^2}$$
$$= -\frac{g^4N_c\gamma_1^+\gamma_2^+}{Q^+Q^2}\frac{i\,\text{sgn}Q^+}{(2\pi)^3}\sum_{K^+}\int d\boldsymbol{K}\frac{[\boldsymbol{Q}^2(2Q^++K^+)+\boldsymbol{K}\cdot\boldsymbol{Q}(2K^++Q^+)]e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2}}{Q^+K^+(Q^++K^+)[(\boldsymbol{K}+x\boldsymbol{Q})^2+x(1-x)Q^2]}, \tag{32}$$

where we have put $x = -K^+/Q^+$, and have done $\int dK^-$ by a contour chosen to pick up the K^2 pole, closing in the upper half plane for $Q^+>0$ and in the lower half plane for $Q^+<0$. A pole is enclosed by the contour only when $0 < -K^+/Q^+ < 1$. In the rest of this paper we shall, for simplicity and definiteness, assume $Q^+>0$. We have also inserted the worldsheet-friendly cutoff $e^{-\delta l^2}$ on the dual loop momentum variable $\boldsymbol{l} = \boldsymbol{K} + \boldsymbol{k}_0$. To do the transverse momentum integral, we exponentiate the denominator with a Schwinger representation $1/D = \int_0^\infty dT e^{-DT}$ and complete the square in the exponent:

$$(\delta+T)\boldsymbol{K}^2 + 2\boldsymbol{K}\cdot(\delta\boldsymbol{k}_0 + xT\boldsymbol{Q}) + \delta\boldsymbol{k}_0^2 + x^2T\boldsymbol{Q}^2 = (\delta+T)\left(\boldsymbol{K}+\frac{\delta\boldsymbol{k}_0+xT\boldsymbol{Q}}{\delta+T}\right)^2 + \frac{\delta T(\boldsymbol{k}_0-x\boldsymbol{Q})^2}{\delta+T}. \tag{33}$$

Then doing the Gaussian integral, this term becomes

$$\frac{g^4N_c\gamma_1^+\gamma_2^+}{Q^{+3}Q^2}\frac{i}{8\pi^2}\sum_{K^+}\int\frac{dT}{T+\delta}\frac{\exp\{-Tx(1-x)Q^2-\delta T(\boldsymbol{k}_0-x\boldsymbol{Q})^2/(T+\delta)\}}{x(1-x)}\left(\boldsymbol{Q}^2(2-x)-\left[\frac{\delta\boldsymbol{k}_0+xT\boldsymbol{Q}}{\delta+T}\right]\cdot\boldsymbol{Q}(1-2x)\right). \tag{34}$$

We need the integrals

$$\int_0^\infty\frac{dT}{T+\delta}e^{-TA}\sim\Gamma'(1)-\ln(A\delta) = -\ln(A\delta e^\gamma), \qquad \int_0^\infty\frac{TdT}{(T+\delta)^2}e^{-TA}\sim-\ln(A\delta e^{\gamma+1}), \qquad \int_0^\infty\frac{dT\delta}{(T+\delta)^2}e^{-TA}\sim 1 \tag{35}$$

in the limit $\delta\to 0$, where we have introduced Euler's constant $\gamma = -\Gamma'(1)$. Then

$$\Gamma^{\text{Bubble}}_{\triangle_L} = \frac{g^4N_c\gamma_1^+\gamma_2^+}{Q^{+3}Q^2}\frac{i}{8\pi^2}\sum_{K^+}\left(\boldsymbol{Q}^2\left[\frac{2(1-x(1-x))}{x(1-x)}(-\ln(x(1-x)Q^2\delta e^\gamma))-\frac{(1-2x)^2}{2x(1-x)}\right]-(\boldsymbol{k}_0^2-\boldsymbol{k}_1^2)\frac{1-2x}{2x(1-x)}\right)$$
$$= \frac{g^4N_c\gamma_1^+\gamma_2^+}{Q^{+3}Q^2}\frac{i}{8\pi^2}\sum_{K^+}\boldsymbol{Q}^2\left[\frac{2(1-x(1-x))}{x(1-x)}(-\ln(x(1-x)Q^2\delta e^\gamma))-\frac{(1-2x)^2}{2x(1-x)}\right]$$
$$\to -\frac{ig^4N_c\gamma_1^+\gamma_2^+\boldsymbol{Q}^2}{4\pi^2Q^{+2}Q^2}\left[\mathcal{A}(Q^2,Q^+)+\frac{1}{4Q^+}\sum_{K^+}\frac{1}{x(1-x)}-\ln(Q^2\delta e^\gamma)+1\right], \tag{36}$$

where we have used $\boldsymbol{Q} = \boldsymbol{k}_0 - \boldsymbol{k}_1$. Note that the last term inside parentheses of the first equality, which depends on the $\boldsymbol{k}$'s individually, vanishes after summation on K^+ because the summand is odd under $x \to (1-x)$. We recall that the bubble contribution from the right triangle diagram is identical to this,

$$\Gamma^{\rm Bubble}_{\triangle_R} = \Gamma^{\rm Bubble}_{\triangle_L}. \tag{37}$$

The dimensional regularization evaluation of the bubble integral (32) is very simple (recall $Q^+ > 0$),

$$\begin{aligned}
\Gamma^{\rm Bubble}_{\triangle_L} &= -\frac{ig^4 N_c \gamma_1^+\gamma_2^+}{2\pi Q^+ Q^2}\sum_{K^+}\int\frac{d\boldsymbol{K}}{(2\pi)^d}\frac{1}{Q^+K^+(Q^++K^+)}\frac{\boldsymbol{Q}^2(2Q^++K^+)+\boldsymbol{K}\cdot\boldsymbol{Q}(2K^++Q^+)}{(\boldsymbol{K}+x\boldsymbol{Q})^2+x(1-x)Q^2}\\
&= \frac{ig^4N_c\gamma_1^+\gamma_2^+\boldsymbol{Q}^2}{4\pi^2Q^{+2}Q^2}\sum_{K^+}\frac{\Gamma(1-d/2)}{(4\pi)^{(d-2)/2}}\frac{(x(1-x)Q^2)^{(d-2)/2}}{Q^+x(1-x)}[1-x(1-x)] \sim \frac{ig^4N_c\gamma_1^+\gamma_2^+\boldsymbol{Q}^2}{4\pi^2Q^{+2}Q^2}\sum_{K^+}\frac{1-x(1-x)}{Q^+x(1-x)}\\
&\quad\times\left[\frac{\Gamma(1-d/2)}{(4\pi)^{(d-2)/2}} - \ln(x(1-x)Q^2)\right] \sim \frac{ig^4N_c\gamma_1^+\gamma_2^+\boldsymbol{Q}^2}{4\pi^2Q^{+2}Q^2}\sum_{K^+}\frac{1-x(1-x)}{Q^+x(1-x)}[-\ln(x(1-x)Q^2\delta e^\gamma)]\\
&= -\frac{ig^4N_c\gamma_1^+\gamma_2^+\boldsymbol{Q}^2}{4\pi^2Q^{+2}Q^2}\sum_{K^+}[\mathcal{A}(Q^2,Q^+) - \ln(Q^2\delta e^\gamma)+2].
\end{aligned} \tag{38}$$

We see that the dim-reg evaluation does not show the second term in square brackets of the δ evaluation. We shall see later that this term would spoil the cancellation of infrared divergences and should in fact be absent. So we identify it as a term to be canceled by a counterterm.

The rest of each triangle diagram involves all three propagators, but in F the factor K^- is replaced by $-m^2K^+/2p^+(p^+-K^+)$ for the left triangle and by $-m^2K^+/2q^+(q^++K^+)$ for the right triangle. Thus the two numerators are replaced as follows:

$$\begin{aligned}
2(p^+-K^+)F \to &-\frac{m^2K^+}{p^+}[\boldsymbol{Q}^2(2Q^++K^+)\\
&+\boldsymbol{K}\cdot\boldsymbol{Q}(2K^++Q^+)] + (p^+-K^+)\\
&\times\frac{Q_\parallel^2}{Q^+}[\boldsymbol{K}^2(2K^++Q^+)\\
&+\boldsymbol{K}\cdot\boldsymbol{Q}(2Q^++K^+)],
\end{aligned} \tag{39}$$

$$\begin{aligned}
-2(q^++K^+)F \to &\frac{m^2K^+}{q^+}[\boldsymbol{Q}^2(2Q^++K^+)\\
&+\boldsymbol{K}\cdot\boldsymbol{Q}(2K^++Q^+)] - (q^++K^+)\\
&\times\frac{Q_\parallel^2}{Q^+}[\boldsymbol{K}^2(2K^++Q^+)\\
&+\boldsymbol{K}\cdot\boldsymbol{Q}(2Q^++K^+)].
\end{aligned} \tag{40}$$

The integration over K^-, $\boldsymbol{K}$ is evaluated in Appendixes A and B. The K^- integration restricts the range of K^+ to two distinct regions for each triangle diagram. Then the transverse integrand can have three distinct numerators, 1, $\boldsymbol{K}^2$, and $\boldsymbol{K}\cdot\boldsymbol{Q}$. In the notation of the appendixes we then have for the left triangle

$$\begin{aligned}
\Gamma^{\rm Rest}_{\triangle_L} = &-\frac{g^4N_c\gamma_1^+\gamma_2^+}{\pi Q^+Q^2}\int_0^{p^+}\frac{dK^+}{K^+(K^++Q^+)}\biggl(-\frac{m^2K^+}{p^+}[\boldsymbol{Q}^2(2Q^++K^+)I_L^1 + (2K^++Q^+)I_L^1[\boldsymbol{K}\cdot\boldsymbol{Q}]]\\
&+(p^+-K^+)\frac{Q_\parallel^2}{Q^+}[(2K^++Q^+)I_L^1[\boldsymbol{K}^2] + (2Q^++K^+)I_L^1[\boldsymbol{K}\cdot\boldsymbol{Q}]]\biggr)\\
&-\frac{g^4N_c\gamma_1^+\gamma_2^+}{\pi Q^+Q^2}\int_{-Q^+}^{0}\frac{dK^+}{K^+(K^++Q^+)}\biggl(-\frac{m^2K^+}{p^+}[\boldsymbol{Q}^2(2Q^++K^+)I_L^2 + (2K^++Q^+)I_L^2[\boldsymbol{K}\cdot\boldsymbol{Q}]]\\
&+(p^+-K^+)\frac{Q_\parallel^2}{Q^+}[(2K^++Q^+)I_L^2[\boldsymbol{K}^2] + (2Q^++K^+)I_L^2[\boldsymbol{K}\cdot\boldsymbol{Q}]]\biggr).
\end{aligned} \tag{41}$$

We have written the K^+ sums as continuous integrals, because inspection of the tables of asymptotics in Appendix B shows that the potential divergences due to the factors $1/K^+(K^++Q^+)$ are absent: The singularity at $K^+=0$ is integrable because the coefficient of $1/K^+$ is continuous through $K^+ = 0$ and the continuum limit of the sum leads to the principal value prescription The singularity at $K^+ = -Q^+$ is integrable because the coefficient of $1/(K^+ + Q^+)$ vanishes as $K^+ \to -Q^+$. However, these integrals do have some residual δ dependence. Equations (B6) and (B8) of Appendix B show that

$$I_L^1[\boldsymbol{K}^2] \equiv \hat{I}_L^1[\boldsymbol{K}^2] - \frac{i}{8\pi(p^+ - K^+)} \ln(m^2\delta e^\gamma), \tag{42}$$

$$I_L^2[\boldsymbol{K}^2] \equiv \hat{I}_L^2[\boldsymbol{K}^2] - \frac{i}{8\pi(p^+ - K^+)} \frac{K^+ + Q^+}{Q^+} \ln(m^2\delta e^\gamma), \tag{43}$$

where the notation $\hat{X}$ signifies that δe^γ in X is replaced by $1/m^2$. Then, with this same notation, we can write

$$\begin{aligned}\Gamma_{\Delta_L}^{\text{Rest}} &= \hat{\Gamma}_{\Delta_L}^{\text{Rest}} + \frac{ig^4 N_c \gamma_1^+ \gamma_2^+}{8\pi^2 Q^+ Q^2} \frac{Q_\parallel^2}{Q^+}\left(\int_0^{p^+} \frac{dK^+(2K^+ + Q^+)}{K^+(K^+ + Q^+)} + \int_{-Q^+}^0 \frac{dK^+(2K^+ + Q^+)}{K^+ Q^+}\right)\ln(m^2\delta e^\gamma) \\ &= \hat{\Gamma}_{\Delta_L}^{\text{Rest}} + \frac{ig^4 N_c \gamma_1^+ \gamma_2^+}{8\pi^2 Q^+ Q^2} \frac{Q_\parallel^2}{Q^+}\left(-\int_0^{p^+} \frac{dK^+(2K^+ + Q^+)}{Q^+(K^+ + Q^+)} + \fint_{-Q^+}^{p^+} \frac{dK^+(2K^+ + Q^+)}{K^+ Q^+}\right)\ln(m^2\delta e^\gamma) \\ &= \hat{\Gamma}_{\Delta_L}^{\text{Rest}} - \frac{ig^4 N_c \gamma_1^+ \gamma_2^+ Q^-}{4\pi^2 Q^+ Q^2}\left(\ln\frac{p^+ + Q^+}{Q^+} + \ln\frac{p^+}{Q^+} + 2\right)\ln(m^2\delta e^\gamma),\end{aligned} \tag{44}$$

where the line through the integral sign on the second line denotes a principal value prescription. Incidentally, these three lines show explicitly the infrared divergence cancellation sketched above, for the δ dependence.

Finally we quote the complete left triangle diagram:

$$\begin{aligned}\Gamma_{\Delta_L} = \hat{\Gamma}_{\Delta_L}^{\text{Rest}} - \frac{ig^4 N_c \gamma_1^+ \gamma_2^+}{4\pi^2 Q^+ Q^2}\Bigg\{&Q^-\left(\ln\frac{p^+ + Q^+}{Q^+} + \ln\frac{p^+}{Q^+} + 2\right)\ln(m^2\delta e^\gamma) \\ &+ \frac{\boldsymbol{Q}^2}{Q^+}[\mathcal{A} - \ln(Q^2\delta e^\gamma) + 2] \\ &+ \frac{\boldsymbol{Q}^2}{Q^+}\left[\frac{1}{4Q^+}\sum_{K^+}\frac{1}{x(1-x)} - 1\right]\Bigg\}.\end{aligned} \tag{45}$$

The last term in braces is absent in dim-reg. Note that the top line is finite in the infrared (continuous K^+). A similar result corresponding to the Feynman diagram on the right side of Fig. 3 may be obtained either directly or from Eq. (45) by the substitution [27], $p \to q - Q_\parallel$. Note that the only *on-shell* value of $Q^+ > 0$ is $q^+ - p^+$, so $q^+ - Q^+ = p^+$. In this case, the right triangle contribution is precisely the same as the left triangle contribution:

$$\Gamma_{\Delta_L} = \Gamma_{\Delta_L} \quad \text{on-shell.} \tag{46}$$

We shall also have use for a slight rearrangement of (45) where we use $\boldsymbol{Q}^2 = Q^2 + 2Q^+Q^-$:

$$\begin{aligned}\Gamma_{\Delta_L} = \hat{\Gamma}_{\Delta_L}^{\text{Rest}} - \frac{ig^4 N_c \gamma_1^+ \gamma_2^+}{4\pi^2 Q^+ Q^2}\Bigg\{&Q^-\Bigg[\left(\ln\frac{p^+ + Q^+}{Q^+} + \ln\frac{p^+}{Q^+}\right) \\ &\times \ln(m^2\delta e^\gamma) + 2\mathcal{A} - 2\ln\frac{Q^2}{m^2} + 4\Bigg] \\ &+ \frac{Q^2}{Q^+}\Big[\mathcal{A} - \ln(Q^2\delta e^\gamma) + 2\Big] \\ &+ \frac{\boldsymbol{Q}^2}{Q^+}\left[\frac{1}{4Q^+}\sum_{K^+}\frac{1}{x(1-x)} - 1\right]\Bigg\}.\end{aligned} \tag{47}$$

At this point we give the triangle combined with the wave function renormalization factors $\sum(Z_i - 1)/2$ associated with the three external legs:

$$\begin{aligned}
&\Gamma_{\Delta_L}+\frac{1}{2}M_{\rm SE}+\frac{1}{2}(Z_2(p)+Z_2(p+Q)-2)\frac{2ig^2\gamma_1^+\gamma_2^+Q^-}{Q^+Q^2}\\
&=\hat\Gamma^{\rm Rest}_{\Delta_L}-\frac{ig^4N_c\gamma_1^+\gamma_2^+}{4\pi^2Q^2Q^{+2}}\bigg\{-Q^+Q^-\bigg(-\bigg(\ln\frac{p^++Q^+}{Q^+}+\ln\frac{p^+}{Q^+}\bigg)\ln(m^2\delta e^\gamma)+2\ln\frac{Q^2}{m^2}-4-\frac{11}{6}\ln\{Q^2\delta e^\gamma\}+\frac{67}{18}\\
&\quad+\sum_{0<K^+<p^+}\frac{1}{K^+}\ln\frac{K^{+2}m^2\delta e^{\gamma+1}}{p^+(p^+-K^+)}+\sum_{0<K^+<p^++Q^+}\frac{1}{K^+}\ln\frac{K^{+2}m^2\delta e^{\gamma+1}}{(p^++Q^+)(p^++Q^+-K^+)}-\mathcal{A}(Q^2,Q^+)\bigg)\\
&\quad+Q^2\bigg[\frac{1}{2}\mathcal{A}(Q^2,Q^+)-\frac{25}{12}+2\bigg]+\boldsymbol{Q}^2\bigg[\frac{1}{4Q^+}\sum_{-Q^+<K^+<0}\frac{1}{x(1-x)}-1\bigg]\bigg\}\\
&\to\hat\Gamma^{\rm Rest}_{\Delta_L}-\frac{ig^4N_c\gamma_1^+\gamma_2^+}{4\pi^2Q^2Q^{+2}}\bigg\{-Q^+Q^-\bigg(2\ln\frac{Q^2}{m^2}-4+\sum_{0<K^+<p^+}\frac{1}{K^+}\ln\frac{K^{+2}e}{p^+(p^+-K^+)}\\
&\quad+\sum_{0<K^+<p^++Q^+}\frac{1}{K^+}\ln\frac{K^{+2}e}{(p^++Q^+)(p^++Q^+-K^+)}-2\sum_{0<K^+<Q^+}\frac{1}{K^+}\ln\frac{K^+(Q^+-K^+)Q^2}{m^2Q^{+2}}-\frac{11}{6}\ln\{Q^2\delta e^\gamma\}+\frac{67}{18}\bigg)\\
&\quad+Q^2\bigg[\frac{1}{2}\mathcal{A}(Q^2,Q^+)-\frac{1}{12}\bigg]+\boldsymbol{Q}^2\bigg[\frac{1}{4Q^+}\sum_{-Q^+<K^+<0}\frac{1}{x(1-x)}-1\bigg]\bigg\}\\
&\to\hat\Gamma^{\rm Rest}_{\Delta_L}-\frac{ig^4N_c\gamma_1^+\gamma_2^+}{4\pi^2Q^2Q^{+2}}\bigg\{-Q^+Q^-\bigg(-\frac{11}{6}\ln\{Q^2\delta e^\gamma\}+\frac{67}{18}+2\ln\frac{Q^2}{m^2}-4+\frac{2\pi^2}{3}+\sum_{0<K^+<Q^+}\frac{2}{K^+}\ln\frac{m^2K^+Q^+e}{Q^2p^+(p^++Q^+)}\\
&\quad+\ln\frac{p^+}{Q^+}-\ln^2\frac{p^+}{Q^+}+\ln\frac{p^++Q^+}{Q^+}-\ln^2\frac{p^++Q^+}{Q^+}\bigg)+\frac{Q^2}{2}\mathcal{A}(Q^2,Q^+)-\frac{Q^2}{12}+\boldsymbol{Q}^2\bigg[\frac{1}{4Q^+}\sum_{-Q^+<K^+<0}\frac{1}{x(1-x)}-1\bigg]\bigg\}.
\end{aligned}\tag{48}$$

Arrows indicate that some finite K^+ sums have been replaced by integrals and evaluated. We see that the ultraviolet divergence is that of asymptotic freedom in the first group of terms multiplying Q^-. In the last line there is still δ dependence in $\mathcal{A}$ that will be canceled by a term from the box diagram [see Eq. (54)]. Also there are uncanceled infrared divergences in the last two lines. In dimensional regularization, the last term multiplying $\boldsymbol{Q}^2$ is absent, and also the $-Q^2/12$ in the last line is absent:

$$\begin{aligned}
&\Gamma_{\Delta_L}+\frac{1}{2}M_{\rm SE}+\frac{1}{2}(Z_2(p)+Z_2(p+Q)-2)\frac{2ig^2\gamma_1^+\gamma_2^+Q^-}{Q^+Q^2}\\
&\to\hat\Gamma^{\rm Rest}_{\Delta_L}-\frac{ig^4N_c\gamma_1^+\gamma_2^+}{4\pi^2Q^2Q^{+2}}\bigg\{-Q^+Q^-\bigg(-\frac{11}{6}\ln\{Q^2\delta e^\gamma\}+\frac{67}{18}+2\ln\frac{Q^2}{m^2}-4+\frac{2\pi^2}{3}+\sum_{0<K^+<Q^+}\frac{2}{K^+}\ln\frac{m^2K^+Q^+e}{Q^2p^+(p^++Q^+)}\\
&\quad+\ln\frac{p^+}{Q^+}-\ln^2\frac{p^+}{Q^+}+\ln\frac{p^++Q^+}{Q^+}-\ln^2\frac{p^++Q^+}{Q^+}\bigg)+\frac{Q^2}{2}\mathcal{A}(Q^2,Q^+)\bigg\}\quad\text{dim-reg.}
\end{aligned}\tag{49}$$

We shall assume that dimensional regularization is correct, in which case the worldsheet-friendly δ regularization counterterms must be included which produce the contribution

$$\Gamma^{\rm C.T.}=\frac{ig^4N_c\gamma_1^+\gamma_2^+}{4\pi^2Q^2Q^{+2}}\bigg\{-\frac{Q^2}{12}+\boldsymbol{Q}^2\bigg[\frac{1}{4Q^+}\sum_{-Q^+<K^+<0}\frac{1}{x(1-x)}-1\bigg]\bigg\}\tag{50}$$

$$=\frac{ig^4N_c\gamma_1^+\gamma_2^+}{4\pi^2Q^2Q^{+2}}\bigg\{2Q^+Q^-\bigg[\frac{1}{4}\sum_{-Q^+<K^+<0}\bigg[\frac{1}{K^++Q^+}-\frac{1}{K^+}\bigg]-1\bigg]+Q^2\bigg[\frac{1}{4}\sum_{-Q^+<K^+<0}\bigg[\frac{1}{K^++Q^+}-\frac{1}{K^+}\bigg]-\frac{13}{12}\bigg]\bigg\}.\tag{51}$$

This will be a challenge for the worldsheet formalism to reproduce locally in view of the $1/K^+$ terms. We shall return to this in the concluding section.

V. BOX DIAGRAM

Finally we turn to the box diagram drawn in Fig. 4. The factors in the numerator of the box integrand can be written

$$-4(p^+ - K^+)(q^+ + K^+)K^-(K^- + Q^-) = [m^2 + (p - K_\|)^2][m^2 + (q + K_\|)^2] - [m^2 + (p - K_\|)^2]\frac{m^2(p^+ - K^+)}{p^+ + q^+} - [m^2 + (q + K_\|)^2]\frac{m^2(q^+ + K^+)}{p^+ + q^+} \quad (52)$$

which shows that the box integration can be reduced to bubblelike and trianglelike integrations.

$$\Gamma_\Box = -4g^4 N_c \gamma_1^+ \gamma_2^+ \int \frac{d^4K}{(2\pi)^4} e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2}\left[\frac{m^2(q^+ + K^+)}{(p^+ + q^+)K^+(K^+ + Q^+)K^2(K+Q)^2(m^2 + (p - K_\|)^2)} + \frac{m^2(p^+ - K^+)}{(p^+ + q^+)K^+(K^+ + Q^+)K^2(K+Q)^2(m^2 + (q + K_\|)^2)} - \frac{1}{K^+(K^+ + Q^+)K^2(K+Q)^2}\right]. \quad (53)$$

The last term in square brackets of (53) is the integrand of a bubble diagram, whose evaluation is similar to that of the corresponding term in the triangle diagram. The result is, for $Q^+ > 0$ and $\delta \sim 0$,

$$\Gamma_\Box^{\text{Bubble}} = \frac{ig^4 N_c \gamma_1^+ \gamma_2^+}{4\pi^2 Q^{+3}} \sum_{K^+} \frac{1}{x(1-x)} \ln\{x(1-x)Q^2 \delta e^\gamma\} = \frac{ig^4 N_c \gamma_1^+ \gamma_2^+}{4\pi^2 Q^{+2}} \mathcal{A}(Q^2, Q^+) \quad (54)$$

with $0 < x = -K^+/Q^+ < 1$. We see that half of this contribution precisely cancels the $\mathcal{A}$ term in the last line of (48) or (49). The other half cancels the corresponding term for a triangle vertex on the right of the four-branion graph.

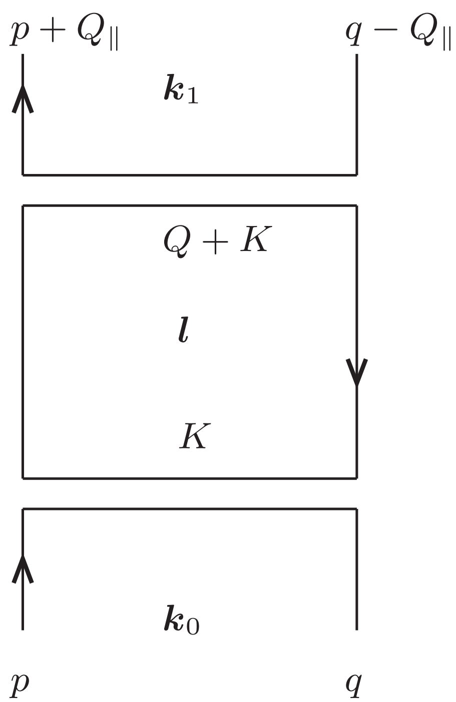


FIG. 4. Box Feynman diagram contributing to the four-point amplitude. The arrows show the direction of color flow, and p, q are incoming momenta.

The rest of the box diagram is given by two triangle integrands whose K^- and $\boldsymbol{K}$ integrations are given in Appendix A. Since these contributions are finite in the ultraviolet, we may set $\delta = 0$ in them:

$$\Gamma_\Box^{\text{rest}} = \Gamma_{\Box_L}^{\text{rest}} + \Gamma_{\Box_R}^{\text{rest}}, \quad (55)$$

$$\Gamma_{\Box_L} \equiv -\frac{4g^4 N_c \gamma_1^+ \gamma_2^+ m^2}{2\pi(p^+ + q^+)}\left\{\sum_{0<K^+<p^+} \frac{(q^+ + K^+)I_L^1}{K^+(K^+ + Q^+)} + \sum_{-Q^+<K^+<0} \frac{(q^+ + K^+)I_L^2}{K^+(K^+ + Q^+)}\right\}, \quad (56)$$

$$\Gamma_{\Box_R} \equiv -\frac{4g^4 N_c \gamma_1^+ \gamma_2^+ m^2}{2\pi(p^+ + q^+)}\left\{\sum_{-Q^+<K^+<0} \frac{(p^+ - K^+)I_R^2}{K^+(K^+ + Q^+)} + \sum_{-q^+<K^+<-Q^+} \frac{(p^+ - K^+)I_R^1}{K^+(K^+ + Q^+)}\right\}. \quad (57)$$

By consulting the asymptotics tables of Appendix B, we see that there are infrared divergences that prevent immediately converting these sums to integrals over K^+. However, we can neatly extract the divergent structure by removing the asymptotic forms from each of the I's. This can be done in many ways. Introducing a parameter ξ, we define

$$I_L^1 \equiv \check{I}_L^1(\xi) + \xi \frac{i(Q^+ + K^+)}{8\pi p^+ Q^+ Q^2} \ln \frac{Q^4 p^{+2}}{m^2 Q_\|^2 K^{+2}}, \quad (58)$$

$$I_L^2 \equiv \check{I}_L^2(\xi) + \xi\frac{i(Q^+ + K^+)}{8\pi p^+ Q^+ Q^2}\ln\frac{Q^4 p^{+2}}{m^2 Q_\parallel^2 K^{+2}} - \frac{i(Q^+ + K^+)}{8\pi Q^+ p^+ Q^2} \times \ln\frac{Q^2 p^{+2}}{m^2 Q^+(-K^+)} - \frac{iK^+}{8\pi Q^+(p^+ + Q^+)Q^2} \times \ln\frac{Q^2(p^+ + Q^+)^2}{m^2 Q^+(K^+ + Q^+)}, \tag{59}$$

$$I_R^1 \equiv \check{I}_R^1(\xi) - \xi\frac{iK^+}{8\pi(q^+ - Q^+)Q^+ Q^2} \times \ln\frac{Q^4(q^+ - Q^+)^2}{m^2 Q_\parallel^2(K^+ + Q^+)^2}, \tag{60}$$

$$I_R^2 \equiv \check{I}_R^2(\xi) - \xi\frac{iK^+}{8\pi(q^+ - Q^+)Q^+ Q^2}\ln\frac{Q^4(q^+ - Q^+)^2}{m^2 Q_\parallel^2(K^+ + Q^+)^2} + \frac{iK^+}{8\pi Q^+(q^+ - Q^+)Q^2}\ln\frac{Q^2(q^+ - Q^+)^2}{m^2 Q^+(K^+ + Q^+)} + \frac{i(Q^+ + K^+)}{8\pi Q^+(q^+ Q^2)}\ln\frac{Q^2 q^{+2}}{m^2 Q^+(-K^+)}, \tag{61}$$

where we have used $\check{X}$ to denote the part of X that leads to IR finite integrals over K^+. Notice that in the cases $I_{L,R}^2$ when both endpoints give divergences, we have multiplied the corresponding asymptotic forms by factors that are unity at the corresponding endpoint and vanish at the opposite endpoint. We included the corresponding factors in $I_{L,R}^1$ for reasons of continuity. Symmetric summation about the interior singular point shows that the $\check{I}(\xi)$'s are free of IR divergences for all ξ. The case $\xi = 1$ separates the divergent pieces of each $\check{I}$. However, we shall hereafter choose $\xi = 0$, to keep subsequent expressions as simple as possible. Then we define $\check{I} \equiv \check{I}(0)$:

$$\Gamma_{\square_L} \equiv \check{\Gamma}_{\square_L} - \frac{ig^4 N_c \gamma_1^+ \gamma_2^+ m^2}{4\pi^2(p^+ + q^+)Q^+ Q^2} \times \left\{-\sum_{-Q^+<K^+<0}\frac{(q^+ + K^+)}{K^+ p^+}\ln\frac{Q^2 p^{+2}}{m^2 Q^+(-K^+)} - \sum_{-Q^+<K^+<0}\frac{(q^+ + K^+)}{(K^+ + Q^+)(p^+ + Q^+)} \times \ln\frac{Q^2(p^+ + Q^+)^2}{m^2 Q^+(K^+ + Q^+)}\right\} \tag{62}$$

$$\rightarrow \check{\Gamma}_{\square_L} - \frac{ig^4 N_c \gamma_1^+ \gamma_2^+ m^2}{4\pi^2(p^+ + q^+)Q^+ Q^2} \times \left\{\frac{(q^+ + p^+)Q^+}{p^+(p^+ + Q^+)}\sum_{0<K^+<Q^+}\frac{1}{K^+}\ln\frac{Q^2 p^+(p^+ + Q^+)}{m^2 Q^+ K^+} + \frac{2q^+ p^+ + (q^+ - p^+)Q^+}{p^+(p^+ + Q^+)}\ln\frac{p^+}{p^+ + Q^+}\sum_{0<K^+<Q^+}\frac{1}{K^+} - \frac{Q^+}{p^+}\ln\frac{Q^2 p^{+2}}{m^2 Q^{+2}} - \frac{Q^+}{p^+ + Q^+}\ln\frac{Q^2(p^+ + Q^+)^2}{m^2 Q^{+2}}\right\}. \tag{63}$$

We can combine the left box, half of the box bubble, the left triangle, wave function, and half the gluon self-energy. For the last three contributions we use (49); that is, we are including the necessary counterterm (51). Remembering that on shell, $Q^- = -Q^+ m^2/[2p^+(p^+ + Q^+)]$, we find

$$\begin{aligned}
\Gamma_L &\equiv \Gamma_{\Delta_L} + \frac{1}{2}M_{\rm SE} + \frac{1}{2}(Z_2(p) + Z_2(p+Q) - 2)\frac{2ig^2\gamma_1^+\gamma_2^+ Q^-}{Q^+ Q^2} + \Gamma_{\square_L} + \frac{1}{2}\Gamma_\square^{\rm Bubble} \\
&\rightarrow \hat{\Gamma}_{\Delta_L}^{\rm Rest} + \check{\Gamma}_{\square_L} + \frac{ig^4 N_c\gamma_1^+\gamma_2^+ Q^-}{4\pi^2 Q^2 Q^+}\left(-\frac{11}{6}\ln\{Q^2\delta e^\gamma\} + \frac{67}{18} + 2\ln\frac{Q^2}{m^2} - 4 + \frac{2\pi^2}{3} + \ln\frac{p^+}{Q^+} - \ln^2\frac{p^+}{Q^+} + \ln\frac{p^+ + Q^+}{Q^+}\right. \\
&\quad \left. - \ln^2\frac{p^+ + Q^+}{Q^+}\right) - \frac{ig^4 N_c\gamma_1^+\gamma_2^+ m^2}{4\pi^2(p^+ + q^+)Q^+ Q^2}\left\{-\frac{Q^+}{p^+}\ln\frac{Q^2 p^{+2}}{m^2 Q^{+2}} - \frac{Q^+}{p^+ + Q^+}\ln\frac{Q^2(p^+ + Q^+)^2}{m^2 Q^{+2}}\right\} \\
&\quad - \frac{ig^4 N_c\gamma_1^+\gamma_2^+ m^2}{4\pi^2 Q^2 p^+(p^+ + Q^+)}\left(1 + \frac{2q^+ p^+ + (q^+ - p^+)Q^+}{Q^+(p^+ + q^+)}\ln\frac{p^+}{p^+ + Q^+}\right)\sum_{0<K^+<Q^+}\frac{1}{K^+} \\
&\rightarrow \hat{\Gamma}_{\Delta_L}^{\rm Rest} + \check{\Gamma}_{\square_L} + \frac{ig^4 N_c\gamma_1^+\gamma_2^+ Q^-}{4\pi^2 Q^2 Q^+}\left(\frac{67}{18} + 2\ln\frac{Q^2}{m^2} - 4 + \frac{2\pi^2}{3} + \ln\frac{p^+}{Q^+} - \ln^2\frac{p^+}{Q^+} + \ln\frac{p^+ + Q^+}{Q^+} - \ln^2\frac{p^+ + Q^+}{Q^+}\right) \\
&\quad - \frac{ig^4 N_c\gamma_1^+\gamma_2^+ m^2}{4\pi^2(p^+ + q^+)Q^+ Q^2}\left\{-\frac{Q^+}{p^+}\ln\frac{Q^2 p^{+2}}{m^2 Q^{+2}} - \frac{Q^+}{p^+ + Q^+}\ln\frac{Q^2(p^+ + Q^+)^2}{m^2 Q^{+2}}\right\} \\
&\quad + \Gamma^{\rm Tree}\frac{g^2 N_c}{4\pi^2}\left[\left(1 + \frac{q^{+2} + p^{+2}}{q^{+2} - p^{+2}}\ln\frac{p^+}{q^+}\right)\sum_{0<K^+<Q^+}\frac{1}{K^+} + \frac{1}{2}\left(-\frac{11}{6}\ln\{Q^2\delta e^\gamma\}\right)\right].
\end{aligned} \tag{64}$$

In the final form, we have displayed the ultraviolet and uncanceled infrared divergences in the last line as a multiple of the tree amplitude. The amplitude Γ_R can be computed directly, but it is more simply obtained from Γ_L through the substitutions $p^+ \to q^+ - Q^+$, $q^+ \to p^+ + Q^+$. But on shell we have $Q^+ = q^+ - p^+$, so in fact $\Gamma_R = \Gamma_L$. Thus

$$\Gamma^{\text{1-loop}} = \Gamma_L + \Gamma_R = 2\Gamma_L = \Gamma^{\text{Finite}} + \Gamma^{\text{Tree}} \frac{g^2 N_c}{4\pi^2} \left\{ 2\left(1 + \frac{q^{+2} + p^{+2}}{q^{+2} - p^{+2}} \ln\frac{p^+}{q^+}\right) \sum_{0<K^+<Q^+} \frac{1}{K^+} - \frac{11}{6} \ln\{Q^2 \delta e^\gamma\} \right\}. \tag{65}$$

The sign and magnitude of the ultraviolet divergent term agree exactly with asymptotic freedom. We shall see that, in its contribution to probabilities, the infrared divergences will be canceled by contributions from soft gluon bremsstrahlung. For this purpose, we need to add the tree contribution and square the result.

$$|A^{\text{Elastic}}|^2 \sim |\Gamma^{\text{Tree}} + \Gamma^{\text{Finite}}|^2 \left| 1 - 2\frac{g^2 N_c}{4\pi^2} \left(\frac{p^{+2} + q^{+2}}{q^{+2} - p^{+2}} \ln\frac{q^+}{p^+} - 1\right) \sum_{0<K^+<Q^+} \frac{1}{K^+} - \frac{11}{6} \ln\{Q^2 \delta e^\gamma\} \right|^2$$

$$\approx |\Gamma^{\text{Tree}} + \Gamma^{\text{Finite}}|^2 \left[1 - 2\frac{g^2 N_c}{4\pi^2} \left(\frac{p^{+2} + q^{+2}}{q^{+2} - p^{+2}} \ln\frac{q^{+2}}{p^{+2}} - 2\right) \sum_{0<K^+<Q^+} \frac{1}{K^+} - \frac{11}{3} \ln\{Q^2 \delta e^\gamma\} \right]. \tag{66}$$

VI. SOFT BREMSSTRAHLUNG AND PROBABILITIES

Soft gluon emission or absorption from scattered branions is dominated by the diagrams where the emitted or absorbed gluon is directly attached to external lines. In the context of large N_c we only need to sum coherently the two diagrams where the gluon is attached to neighboring lines, i.e. either emission between the two outgoing branions or absorption between the incoming branions. In the first case we simply multiply the amplitude for the core process by the factor

$$\begin{aligned} -g\frac{\boldsymbol{k}\cdot\boldsymbol{\epsilon}}{k^+}&\left[\frac{p^+ + Q^+}{(p+Q)\cdot k} - \frac{q^+ - Q^+}{(q-Q)\cdot k}\right] \\ &= 2g\boldsymbol{k}\cdot\boldsymbol{\epsilon}\left[\frac{(p^+ + Q^+)^2}{k^{+2}m^2 + \boldsymbol{k}^2(p^+ + Q^+)^2} - \frac{(q^+ - Q^+)^2}{k^{+2}m^2 + \boldsymbol{k}^2(q^+ - Q^+)^2}\right] \\ &\equiv 2g\boldsymbol{k}\cdot\boldsymbol{\epsilon}\left[\frac{1}{A + \boldsymbol{k}^2} - \frac{1}{B + \boldsymbol{k}^2}\right], \end{aligned} \tag{67}$$

the relative minus sign arising because the two branions in the final state have opposite color. Here $A = m^2 k^{+2}/(p^+ + Q^+)^2$ and $B = m^2 k^{+2}/(q^+ - Q^+)^2$. The probability for gluon emission is given by squaring the amplitude, summing over color and gluon spin, and integrating over $\boldsymbol{k}$, k^+ in a small window about zero.

$$P = |A_{\text{core}}|^2 \sum_{k^+<k_{\max}} \int_{\boldsymbol{k}^2<\Delta_T^2(k^+)} d\boldsymbol{k} \frac{4g^2 N_c}{8\pi^3 2k^+}\left[\frac{\boldsymbol{k}^2}{(\boldsymbol{k}^2 + A)^2} + \frac{\boldsymbol{k}^2}{(\boldsymbol{k}^2 + B)^2} - \frac{2\boldsymbol{k}^2}{(\boldsymbol{k}^2 + A)(\boldsymbol{k}^2 + B)}\right]. \tag{68}$$

The integrals are elementary:

$$\int_{\boldsymbol{k}^2<\Delta_T^2(k^+)} d\boldsymbol{k} \frac{\boldsymbol{k}^2}{(\boldsymbol{k}^2 + A)(\boldsymbol{k}^2 + B)} = \frac{\pi}{B - A}\left[B \ln\frac{\Delta_T^2 + B}{B} - A \ln\frac{\Delta_T^2 + A}{A}\right], \tag{69}$$

$$\int_{\boldsymbol{k}^2<\Delta_T^2(k^+)} d\boldsymbol{k} \frac{1}{(\boldsymbol{k}^2 + A)^2} = \pi\left[\ln\frac{\Delta_T^2 + A}{A} - \frac{\Delta_T^2}{\Delta_T^2 + A}\right].$$

Thus

$$P = |A_{\text{core}}|^2 \frac{g^2 N_c}{4\pi^2} \sum_{k^+<k_{\max}} \frac{1}{k^+}\left[\frac{A + B}{A - B} \ln\frac{A(\Delta_T^2 + B)}{B(\Delta_T^2 + A)} - \frac{\Delta_T^2}{\Delta_T^2 + A} - \frac{\Delta_T^2}{\Delta_T^2 + B}\right]. \tag{70}$$

Next we choose how to specify the resolutions. As discussed in [21] a nice choice is to limit the virtuality of the two "jet" momenta $p + Q + k$ and $q - Q + k$:

$$-(p + Q)\cdot k < \Delta^2, \qquad -(q - Q)\cdot k < \Delta^2 \tag{71}$$

$$\to \boldsymbol{k}^2 < \min\left(2k^+ \frac{\Delta^2 - k^+ m^2/2(p^+ + Q^+)}{(p^+ + Q^+)}, 2k^+ \frac{\Delta^2 - k^+ m^2/2(q^+ - Q^+)}{(q^+ - Q^+)}\right). \tag{72}$$

We could choose the upper limit on k^+ independently of Δ as long as it is less than the least of $2(p^+ + Q^+)\Delta^2/m^2$, $2(q^+ - Q^+)\Delta^2/m^2$. But for definiteness let us choose

$$k^+ < k_{\max} \equiv \min\left\{(q^+ - Q^+)\frac{\Delta^2}{m^2}, (p^+ + Q^+)\frac{\Delta^2}{m^2}\right\}. \tag{73}$$

With resolutions set, we now examine the small k^+ limit of the probability summand. We have required $\Delta_T^2 = O(k^+)$ and $A, B = O(k^{+2})$; we can neglect A, B in comparison to Δ_T so we find

$$\text{Summand} \sim \frac{1}{k^+}\left[\frac{A+B}{A-B}\ln\frac{A}{B}-2\right] = \frac{1}{k^+}\left[\frac{(p^+ + Q^+)^2 + (q^+ - Q^+)^2}{(p^+ + Q^+)^2 - (q^+ - Q^+)^2} \times \ln\frac{(p^+ + Q^+)^2}{(q^+ - Q^+)^2} - 2\right]. \quad (74)$$

Actually, since we are insisting that $Q^+ > 0$ the on-shell condition is $Q^+ = q^+ - p^+$ with $q^+ > p^+$. And we find the simplification

$$P_{\rm IR}^{\rm Brem} \sim 2|A_{\rm core}|^2\left[\frac{p^{+2}+q^{+2}}{q^{+2}-p^{+2}}\ln\frac{q^{+2}}{p^{+2}}-2\right]\frac{g^2N_c}{4\pi^2}\sum\frac{1}{k^+}, \quad (75)$$

where we have added the absorption probability of an extra soft gluon in the initial state, which accounts for the factor of 2. Combining this result with the square of the elastic amplitude, we see that the infrared divergence cancels.

VII. CONCLUDING REMARKS

In this article we have calculated physical on-shell branion-branion scattering through one loop for the case that the branions are Dirac fermions living on parallel 1-branes. This work refines and completes a calculation initiated in [17] by carrying out a careful treatment of the on-shell limit including a proper definition of scattering probabilities allowing for the emission and absorption of extra soft gluons. The ambiguity of the on-shell limit found in [17] came from attempting the continuous K^+ limit for an unphysical off-shell quantity. This is therefore another example of the novel aspects of light-cone gauge. In a normal covariant gauge no infrared cutoff is needed when computing off-shell quantities.

We worked on shell from the beginning in this paper, so the entire calculation was actually quite different from that carried out in [17]. Besides this, we also used the worldsheet-friendly ultraviolet cutoff of [14,15,19–21] rather than the one employed in [17]. By comparing our results to those given by dimensional regularization we were able to identify all the one-loop counterterms that will be required for the construction of the light-cone worldsheet description of this system. In this concluding section we shall briefly indicate how the worldsheet formalism can handle these counterterms. But since there remain some unresolved issues in the worldsheet construction with 1-brane sources, we stress that it is only illustrative, and the final "best" solution may be quite different.

First of all, the counterterms for the self-energy diagrams are no different than those we required in [20,21]. There is of course the branion mass shift (16), which is nothing but mass renormalization. There is some novelty in the fact that the zero thickness of the 1-brane promotes a single log divergence to a double log one, but that does not change the fact that the shift is a Lorentz invariant constant, and mass renormalization proceeds as usual. But there is also a contribution to the self-energy calculation that is "tadpolelike" coming from the instantaneous longitudinal gluon and that does not involve a propagating intermediate state. This is just the term we associated with the added semicircular contours:

$$\begin{aligned} -i\Sigma^{\rm Instant} &= g^2N_c\gamma^+\int\frac{d\boldsymbol{K}}{(2\pi)^4}e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2} \times\left[-2i\pi\sum_{0<K^+<p^+}\frac{1}{K^{+2}}\right] \\ &= -\frac{ig^2N_c\gamma^+}{8\pi^2\delta}\sum_{0<K^+<p^+}\frac{1}{K^{+2}} \\ &= -\frac{ig^2N_c\gamma^+}{8\pi^2\delta\epsilon}\sum_{n=0}^{M-1}\frac{1}{n^2} \\ &= -\frac{ig^2N_c\gamma^+}{8\pi^2\delta\epsilon}\left[\frac{\pi^2}{6}-\frac{1}{M}+O\left(\frac{1}{M^2}\right)\right] \\ &\sim -\frac{ig^2N_c\gamma^+}{48\delta\epsilon}+\frac{ig^2N_c\gamma^+}{8\pi^2\delta p^+}. \end{aligned} \quad (76)$$

The second term has the right behavior to be absorbed in mass renormalization. The first term is a divergent p^+ *independent* shift in p^-, the light-cone "energy" of the branion. On the light-cone worldsheet it therefore has the interpretation as a boundary energy or boundary "cosmological constant." Again, such a term has already been encountered in the gluon self-energy as discussed in [20,21], and introduces no new problems for the light-cone worldsheet.

The branion-gluon vertex counterterm (51) looks more problematic because of the nonpolynomial p^+ dependence. It is helpful to rearrange it a little,

$$\Gamma^{\rm C.T.} = \frac{ig^4N_c\gamma_1^+\gamma_2^+}{4\pi^2Q^{+2}}\left\{-\frac{1}{12}+\left(1+\frac{2Q^+Q^-}{Q^2}\right) \times\left[\frac{1}{4Q^+}\sum_{-Q^+<K^+<0}\frac{1}{x(1-x)}-1\right]\right\}. \quad (77)$$

Since this expression will be multiplied by $e^{i\boldsymbol{Q}\cdot\boldsymbol{L}}$ and integrated over $\boldsymbol{Q}$, the $\boldsymbol{Q}$ independent terms will be proportional to $\delta(\boldsymbol{L}) = 0$ for the process we are analyzing since $\boldsymbol{L} \neq 0$. Thus we are left with the problem of representing

$$\frac{ig^4N_c\gamma_1^+\gamma_2^+Q^-}{4\pi^2Q^+Q^2}\left[\sum_{0<k^+<Q^+}\frac{1}{k^+}-2\right] \quad (78)$$

locally on the worldsheet. Although awkward looking, there is a way to do it. First of all, a factor of $1/k^+$ can be produced by the insertion of a local worldsheet field, call it $\phi(\sigma,\tau)$, at a point a distance k^+ from the boundary of the strip representing the gluon propagator (see [11] in connection with the representation of $1/p^+$ factors in vertex functions). Then integrating this point across the gluon strip reproduces the desired nonpolynomial terms. A truly local prescription, however, should integrate the field insertion point over the whole worldsheet, not just a single time slice on a single propagator. So we need to arrange things so that the integral over the whole worldsheet contributes only at one time and only on the gluon propagator. Again, there is precedent for this sort of effect in the way the worldsheet can produce quartic vertices. Briefly, the way this works is that one can introduce freely any number of extra worldsheet fields $\boldsymbol{\chi}_i$ which satisfy $\boldsymbol{\chi}_i = 0$ on all boundaries together with ghost fields β_i, γ_i such that the path integral over them all gives unity. Denoting $\partial/\partial\sigma$ by $'$, then $\langle \boldsymbol{\chi}_i' \rangle = 0$ but $\langle \boldsymbol{\chi}_i'(\sigma, \tau)\boldsymbol{\chi}_i'(\sigma', \tau')\rangle \propto \delta(\tau - \tau')$. By attaching one of these extra fields to the branion-gluon interaction point and another to the local field ϕ, the contribution can be restricted in the desired way. We content ourselves here with this feasibility argument and leave a definitive solution for future work, in which we hope to resolve the other difficulties posed by the introduction of 1-brane sources into the light-cone worldsheet formalism.

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APPENDIX A: TRIANGLE INTEGRALS WITHOUT NUMERATOR FACTORS

In light-cone evaluations we always reserve the K^+ integrations till last. Starting with the left triangle integrand, we do the K^- integral first:

$$\int \frac{dK^-}{2\pi} \frac{1}{K^2(K+Q)^2(m^2 + (p - K_{\|})^2} = \frac{i\theta(p^+ - K^+)\theta(K^+)}{2(p^+ - K^+)[\boldsymbol{K}^2 + B][(\boldsymbol{K}+\boldsymbol{Q})^2 + C]} + \frac{i\theta(Q^+ + K^+)\theta(-K^+)(K^+ + Q^+)}{2Q^+(p^+ - K^+)[(\boldsymbol{K}' - \boldsymbol{Q}')^2 + D][(\boldsymbol{K}')^2 + C]}, \tag{A1}$$

$$B = \frac{m^2 K^{+2}}{p^+(p^+ - K^+)}, \qquad C = \frac{m^2(K^+ + Q^+)^2}{(p^+ + Q^+)(p^+ - K^+)},$$
$$D = -K^+(K^+ + Q^+)\left[\frac{\boldsymbol{Q}^2}{Q^{+2}} + \frac{m^2}{p^+(p^+ + Q^+)}\right] = -\frac{K^+(K^+ + Q^+)}{Q^{+2}} Q^2, \qquad \boldsymbol{K}' = \boldsymbol{K} + \boldsymbol{Q}, \qquad \boldsymbol{Q}' = \frac{Q^+ + K^+}{Q^+}\boldsymbol{Q}. \tag{A2}$$

Next the transverse momentum integral can be done after combining denominators with the Feynman trick:

$$\int \frac{d^2\boldsymbol{K}}{4\pi^2} \int_0^1 dx \frac{1}{[(\boldsymbol{K} + x\boldsymbol{Q})^2 + x(1-x)\boldsymbol{Q}^2 + B(1-x) + Cx]^2} = \frac{1}{4\pi} \int_0^1 dx \frac{1}{x(1-x)\boldsymbol{Q}^2 + B(1-x) + Cx},$$

$$\int \frac{d^2\boldsymbol{K}}{4\pi^2} \int_0^1 dx \frac{1}{[(\boldsymbol{K}' - x\boldsymbol{Q}')^2 + x(1-x)\boldsymbol{Q}'^2 + Dx + C(1-x)]^2} = \frac{1}{4\pi} \int_0^1 dx \frac{1}{x(1-x)\boldsymbol{Q}'^2 + Dx + C(1-x)}.$$

The x integral can be done by factoring the denominator:

$$\int_0^1 dx \frac{1}{ax(1-x) + bx + c(1-x)} = \frac{1}{a(r_+ - r_-)} \ln\frac{r_+(1-r_-)}{-r_-(r_+ - 1)}, \tag{A3}$$

$$r_\pm = \frac{1}{2} + \frac{b-c}{2a} \pm \frac{1}{2a}\sqrt{a^2 + b^2 + c^2 + 2a(b+c) - 2bc}. \tag{A4}$$

Let us denote the roots for $a = \boldsymbol{Q}^2$ and $b = B$, $c = C$ by $r_\pm$ without primes, and the roots with $a = \boldsymbol{Q}'^2$ and $b = C$, $c = D$ by $r'_\pm$. Then

$$I_L \equiv \int \frac{d^2\boldsymbol{K}dK^-}{8\pi^3} \frac{1}{K^2(K+Q)^2(m^2 + (p - K_{\|})^2} = \frac{1}{8\pi(p^+ - K^+)}\left[\frac{i\theta(p^+ - K^+)\theta(K^+)}{\boldsymbol{Q}^2(r_+ - r_-)} \ln\frac{r_+(1-r_-)}{-r_-(r_+ - 1)} + \frac{i\theta(Q^+ + K^+)\theta(-K^+)(K^+ + Q^+)}{Q^+\boldsymbol{Q}'^2(r'_+ - r'_-)} \ln\frac{r'_+(1-r'_-)}{-r'_-(r'_+ - 1)}\right] \tag{A5}$$

$$\equiv \theta(p^+ - K^+)\theta(K^+)I_L^1 + \theta(Q^+ + K^+)\theta(-K^+)I_L^2, \tag{A6}$$

$$I_L^1 = \frac{i}{8\pi(p^+ - K^+)\boldsymbol{Q}^2(r_+ - r_-)} \ln\frac{r_+(1-r_-)}{-r_-(r_+ - 1)}, \tag{A7}$$

$$I_L^2 = \frac{iQ^+}{8\pi(p^+ - K^+)(K^+ + Q^+)\boldsymbol{Q}^2(r'_+ - r'_-)} \times \ln\frac{r'_+(1-r'_-)}{-r'_-(r'_+ - 1)}. \tag{A8}$$

The K^+ integration of these results is infrared divergent for K^+ near 0 and $-Q^+$. The singular behavior for K^+ near p^+ is integrable. To extract the infrared structure we examine the behavior of I_L near each of these dangerous points.

Consider first $K^+ \sim 0$ from the positive side. Then $B \sim 0$ and $r_+ \to 1$, $r_+ - 1 \sim B/(\boldsymbol{Q}^2 + C)$, $r_- \to -C/\boldsymbol{Q}^2$, $1 - r_- \to (\boldsymbol{Q}^2 + C)/\boldsymbol{Q}^2$, $\boldsymbol{Q}^2 + C \to \boldsymbol{Q}^2 + m^2 Q^{+2}/p^+(p^+ + Q^+) = \boldsymbol{Q}^2 + Q_\parallel^2 = Q^2$, and

$$I_L \sim \frac{i}{8\pi p^+ Q^2} \ln\frac{Q^4 p^{+2}}{m^2 Q_\parallel^2 K^{+2}} \quad \text{for } K^+ \to 0_+. \tag{A9}$$

It is simple to check that $I_L = O(1)$ as $K^+ \to p^+$. Next we consider $K^+ \sim 0$ from below. Then $D \sim 0$, $r'_- \to 0$, $r'_+ \to (A + C)/A$, and

$$I_L \sim \frac{i}{8\pi p^+ Q^2} \ln\frac{Q^2 Q^+}{Q_\parallel^2(-K^+)} \quad \text{for } K^+ \to 0_-. \tag{A10}$$

Finally, we consider $K^+ \sim -Q^+$. In this case, $D \sim Q^2(K^+ + Q^+)/Q^+$, $C \sim m^2(K^+ + Q^+)^2/(p^+ + Q^+)^2$, so $r'_+ - 1 \sim C/D$, $r'_- \sim -D/\boldsymbol{Q}'^2$, so

$$I_L \sim \frac{i}{8\pi(p^+ + Q^+)Q^2} \ln\frac{Q^2(p^+ + Q^+)^2}{m^2 Q^+(K^+ + Q^+)} \quad \text{for } K^+ \to -Q^+. \tag{A11}$$

Turning now to the right triangle integrand, we do the K^- integral first:

$$\int \frac{dK^-}{2\pi} \frac{1}{K^2(K+Q)^2(m^2 + (q + K_\parallel)^2} = \frac{i\theta(q^+ + K^+)\theta(-Q^+ - K^+)}{2(q^+ + K^+)[\boldsymbol{K}^2 + \bar{B}][(\boldsymbol{K} + \boldsymbol{Q})^2 + \bar{C}]} + \frac{i\theta(Q^+ + K^+)\theta(-K^+)(-K^+)}{2Q^+(q^+ + K^+)[(\boldsymbol{K} + \boldsymbol{Q}')^2 + \bar{D}][\boldsymbol{K}^2 + \bar{B}]}, \tag{A12}$$

$$\begin{aligned} \bar{B} &= \frac{m^2 K^{+2}}{q^+(q^+ + K^+)}, \\ \bar{C} &= \frac{m^2(K^+ + Q^+)^2}{(q^+ - Q^+)(q^+ + K^+)}, \\ \bar{D} &= -K^+(K^+ + Q^+)\left[\frac{\boldsymbol{Q}^2}{Q^{+2}} + \frac{m^2}{q^+(q^+ - Q^+)}\right] \\ &= -\frac{K^+(K^+ + Q^+)}{Q^{+2}} Q^2 = D, \\ \boldsymbol{Q}' &= -\frac{K^+}{Q^+}\boldsymbol{Q}. \end{aligned} \tag{A13}$$

Next the transverse momentum integrals are given by (A3) with appropriate substitutions, and the x integral by (A4). Let us denote the roots for $a = \boldsymbol{Q}^2$ and $b = \bar{B}$, $c = \bar{C}$ by $\bar{r}_\pm$ without primes, and the roots with $a = \boldsymbol{Q}'^2$ and $b = \bar{C}$, $c = \bar{D}$ by $\bar{r}'_\pm$. Then

$$\begin{aligned} I_R &\equiv \int \frac{d^2\boldsymbol{K} dK^-}{8\pi^3} \frac{1}{K^2(K+Q)^2(m^2 + (q + K_\parallel)^2)} \\ &= \frac{1}{8\pi(q^+ + K^+)}\left[\frac{i\theta(q^+ + K^+)\theta(-K^+ - Q^+)}{\boldsymbol{Q}^2(\bar{r}_+ - \bar{r}_-)} \times \ln\frac{\bar{r}_+(1-\bar{r}_-)}{-\bar{r}_-(\bar{r}_+ - 1)} + \frac{i\theta(Q^+ + K^+)\theta(-K^+)(-K^+)}{Q^+\boldsymbol{Q}'^2(\bar{r}'_+ - \bar{r}'_-)} \times \ln\frac{\bar{r}'_+(1-\bar{r}'_-)}{-\bar{r}'_-(\bar{r}'_+ - 1)}\right] \end{aligned} \tag{A14}$$

$$\equiv \theta(q^+ + K^+)\theta(-K^+ - Q^+)I_R^1 + \theta(Q^+ + K^+)\theta(-K^+)I_R^2, \tag{A15}$$

$$I_R^1 = \frac{i}{8\pi(q^+ + K^+)\boldsymbol{Q}^2(\bar{r}_+ - \bar{r}_-)} \ln\frac{\bar{r}_+(1-\bar{r}_-)}{-\bar{r}_-(\bar{r}_+ - 1)}, \quad -q^+ < K^+ < -Q^+, \tag{A16}$$

$$I_R^2 = \frac{iQ^+}{8\pi(q^+ + K^+)(-K^+)\boldsymbol{Q}^2(\bar{r}'_+ - \bar{r}'_-)} \ln\frac{\bar{r}'_+(1-\bar{r}'_-)}{-\bar{r}'_-(\bar{r}'_+ - 1)}, \quad -Q^+ < K^+ < 0. \tag{A17}$$

Again, the K^+ integration is infrared divergent for K^+ near 0 and $-Q^+$. The singular behavior for K^+ near $-q^+$ is integrable. To extract the infrared structure we examine the behavior of I_R near each of these dangerous points.

Consider first $K^+ \sim -Q^+$ from the negative side. Then $\bar{C} \sim 0$ and $\bar{r}_+ \to (\boldsymbol{Q}^2 + \bar{B})/\boldsymbol{Q}^2$, $\bar{r}_+ - 1 \sim \bar{B}/\boldsymbol{Q}^2$, $\bar{r}_- \to -\bar{C}/(\boldsymbol{Q}^2 + \bar{B})$, $1 - r_- \to 1$, $\boldsymbol{Q}^2 + \bar{B} \to Q^2$, and

$$I_R \sim \frac{i}{8\pi(q^+ - Q^+)Q^2} \ln\frac{Q^4(q^+ - Q^+)^2}{m^2 Q_\parallel^2(K^+ + Q^+)^2} \quad \text{for } K^+ + Q^+ \to 0_-. \tag{A18}$$

It is simple to check that $I_R = O(1)$ as $K^+ \to -q^+$. Next we consider $K^+ \sim -Q^+$ from above. Then $\bar{D} \sim 0$, $r'_- \to 0$, $r'_+ \to (\boldsymbol{Q}^2 + \bar{B})/\boldsymbol{Q}^2$, and

$$I_R \sim \frac{i}{8\pi(q^+ - Q^+)Q^2} \ln \frac{Q^2 Q^+}{Q_\parallel^2 (K^+ + Q^+)} \quad \text{for } K^+ + Q^+ \to 0_+. \tag{A19}$$

Finally, we consider $K^+ \sim 0-$. In this case, $\bar{D} \sim -Q^2 K^+/Q^+$, $\bar{B} \sim m^2 K^{+2}/q^{+2}$, so $\bar{r}'_+ - 1 \sim \bar{B}/\bar{D}$, $r'_- \sim -\bar{D}/\boldsymbol{Q}'^2$, so

$$I_R \sim \frac{i}{8\pi q^+ Q^2} \ln \frac{Q^2 q^{+2}}{m^2 Q^+(-K^+)} \quad \text{for } K^+ \to 0_-. \tag{A20}$$

We have remarked in the text that right triangle integrals can be obtained from left triangle integrals through the substitutions $p \to q - Q$, $q \to p + Q$. In the context of the integrals in this section which have left K^+ integration unperformed, we see by direct inspection that $I_R^{1,2}(q^+, K^+) = I_L^{1,2}(q^+ - Q^+, -K^+ - Q^+)$. Note in this context that the range $-q^+ < K^+ < 0$ can be expressed as $-Q^+ < -K^+ - Q^+ < q^+ - Q^+$, analogous to the range $-Q^+ < K^+ < p^+$.

APPENDIX B: TRIANGLE INTEGRALS WITH NUMERATOR FACTORS

Some of the triangle integrals we need contain numerator factors involving K^-, $\boldsymbol{K}^2$, or $\boldsymbol{K}\cdot\boldsymbol{Q}$. In the text we have shown how to replace K^- factors with polynomials in K^+ together with canceled propagator terms, which have the structure of bubble diagrams, explicitly evaluated in the text. Once the numerators are free of K^- factors, the K^- integration is done by contours, leaving denominators which are quadratic in $\boldsymbol{K}$. We can then replace numerator factors of $\boldsymbol{K}^2$ or $\boldsymbol{K}\cdot\boldsymbol{Q}$ with functions of K^+ times the integrals of the previous section plus integrals with only one denominator. We evaluate these one denominator integrals in this section. Since they are log divergent in the UV we give both delta-regulator and dimensional-regulator forms of the answers.

After integration over K^- there are four distinct transverse integrals to do: the left and right triangle integrals and for each of these, two distinct regions of K^+. For each of these four transverse integrals there can be three numerators: 1, $\boldsymbol{K}^2$, $\boldsymbol{K}\cdot\boldsymbol{Q}$. We adopt the notation $I_{L,R}^{1,2}[X]$ with X symbolizing the numerator. In the previous section we evaluated all of the $I_{L,R}^{1,2}[1] \equiv I_{L,R}^{1,2}$.

The one denominator integrals have the general form

$$\int \frac{d^2K}{(2\pi)^2} \frac{1}{(\boldsymbol{K}+\boldsymbol{L})^2 + Z} \to \int \frac{d^2K}{(2\pi)^2} \frac{e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2}}{(\boldsymbol{K}+\boldsymbol{L})^2 + Z} \qquad \delta\text{-reg} \tag{B1}$$

$$\to \int \frac{d^dK}{(2\pi)^d} \frac{1}{(\boldsymbol{K}+\boldsymbol{L})^2 + Z} \qquad \text{dim-reg} \tag{B2}$$

with δ and dimensional regularization, respectively. In the first case we have

$$\int \frac{d^2K}{(2\pi)^2} \frac{e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2}}{(\boldsymbol{K}+\boldsymbol{L})^2 + Z} = \int_0^\infty dT \int \frac{d^2K}{(2\pi)^2} e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2 - T[(\boldsymbol{K}+\boldsymbol{L})^2 + Z]} = \frac{1}{4\pi}\int_0^\infty \frac{dT}{T+\delta} \exp\left\{-TZ - \frac{T\delta}{T+\delta}(\boldsymbol{L}-\boldsymbol{k}_0)^2\right\}. \tag{B3}$$

The second term in the exponent is $O(\delta)$ for all T and the divergence as $\delta \to 0$ is only logarithmic, so this term is negligible for $\delta \sim 0$. In the limit these integrals are therefore independent of $\boldsymbol{L}$ and $\boldsymbol{k}_0$.

$$\int \frac{d^2K}{(2\pi)^2} \frac{e^{-\delta(\boldsymbol{K}+\boldsymbol{k}_0)^2}}{(\boldsymbol{K}+\boldsymbol{L})^2 + Z} \sim \frac{1}{4\pi}\int_0^\infty \frac{dT}{T+\delta} e^{-TZ} \sim -\frac{1}{4\pi}\ln(Z\delta e^\gamma). \tag{B4}$$

In dim-reg, we have simply

$$\int \frac{d^dK}{(2\pi)^d} \frac{1}{(\boldsymbol{K}+\boldsymbol{L})^2 + Z} = \frac{\Gamma(1-d/2)}{(4\pi)^{d/2} Z^{(d-2)/2}} \sim \frac{1}{4\pi}\left[\frac{\Gamma(1-d/2)}{(4\pi)^{(d-2)/2}} - \ln Z\right] \to -\frac{1}{4\pi}\ln(Z\delta e^\gamma) \tag{B5}$$

with the correspondence (20). We see that the two regularizations exactly agree for these integrals.

It remains to obtain the eight distinct integrals with nontrivial numerators.

$$I_L^1[\boldsymbol{K}^2] = -BI_L^1 + I_L^1[\boldsymbol{K}^2 + B] = -BI_L^1 - \frac{i}{8\pi(p^+ - K^+)}\ln(C\delta e^\gamma), \tag{B6}$$

$$I_L^1[\boldsymbol{K}\cdot\boldsymbol{Q}] = \frac{1}{2}I_L^1[(\boldsymbol{K}+\boldsymbol{Q})^2 + C - \boldsymbol{K}^2 - B] + \frac{1}{2}(B - C - \boldsymbol{Q}^2)I_L^1 = \frac{1}{2}(B - C - \boldsymbol{Q}^2)I_L^1 - \frac{i}{16\pi(p^+ - K^+)}\ln(B/C), \tag{B7}$$

$$I_L^2[\boldsymbol{K}^2] = I_L^2[\boldsymbol{K}'^2 - 2\boldsymbol{K}' \cdot \boldsymbol{Q}] + \boldsymbol{Q}^2 I_L^2$$
$$= -\left(\frac{K^+ C}{K^+ + Q^+} + \frac{K^+}{Q^+}\boldsymbol{Q}^2 + \frac{Q^+ D}{K^+ + Q^+}\right) I_L^2 - \frac{i}{8\pi(p^+ - K^+)}\left(\frac{K^+ + Q^+}{Q^+}\ln(D\delta e^\gamma) + \ln\frac{C}{D}\right), \quad \text{(B8)}$$

$$I_L^2[\boldsymbol{K} \cdot \boldsymbol{Q}] = \left(\boldsymbol{Q}^2 \frac{K^+ - Q^+}{2Q^+} - \frac{Q^+(C - D)}{2(K^+ + Q^+)}\right) I_L^2 + \frac{i}{16\pi(p^+ - K^+)}\ln\frac{C}{D}, \quad \text{(B9)}$$

$$I_R^1[\boldsymbol{K}^2] = -\bar{B} I_R^1 - \frac{i}{8\pi(q^+ + K^+)}\ln(\bar{C}\delta e^\gamma), \quad \text{(B10)}$$

$$I_R^1[\boldsymbol{K} \cdot \boldsymbol{Q}] = \frac{1}{2}(\bar{B} - \bar{C} - \boldsymbol{Q}^2) I_R^1 - \frac{i}{16\pi(q^+ + K^+)}\ln(\bar{B}/\bar{C}), \quad \text{(B11)}$$

$$I_R^2[\boldsymbol{K}^2] = -\bar{B} I_R^2 + \frac{iK^+}{8\pi Q^+(q^+ + K^+)}\ln(D\delta e^\gamma), \quad \text{(B12)}$$

$$I_R^2[\boldsymbol{K} \cdot \boldsymbol{Q}] = -\frac{Q^+}{2K^+}\left(\bar{B} - D - \frac{K^{+2}}{Q^{+2}}\boldsymbol{Q}^2\right) I_R^2 - \frac{i}{16\pi(q^+ + K^+)}\ln\frac{\bar{B}}{D}. \quad \text{(B13)}$$

These results are used in the evaluation of not only the triangle diagrams themselves, but also trianglelike integrals that contribute to the box diagrams. They must still be integrated over K^+, for which there are several potential divergences when $K^+ \to p^+, 0, -Q^+, -q^+$. The behavior at these points is tabulated in Figs. 5 and 6. The points p^+ and $-q^+$ cause no difficulty. This is because $\boldsymbol{K}^2$ is always multiplied by $p^+ - K^+$ or $q^+ + K^+$ whenever it occurs in the left or right triangle diagram, respectively. The points 0 and $-Q^+$ can cause infrared divergences. However, the integrals contributing to the actual triangle diagrams turn out to be convergent. As we discuss in the text, there are some residual infrared divergences in the trianglelike integrals contributing to the box diagram. Inspection of these tables easily allows their extraction.

Finally, we have remarked several times in the text that right triangle integrals can be obtained from left triangle integrals through the substitutions $p \to q - Q$, $q \to p + Q$. In the context of the integrals in these appen-

Left Triangle Asymptotics, $-Q^+ < K^+ < p^+$			
	$K^+ \to p^+$	$K^+ \to 0$	$K^+ \to -Q^+$
I_L^1	$\frac{i}{8\pi m^2 Q^+}\ln\frac{p^+ + Q^+}{p^+}$	$\frac{i}{8\pi p^+ Q^2}\ln\frac{Q^4 p^{+2}}{m^2 Q_\parallel^2 K^{+2}}$	NA
$I_L^1[\boldsymbol{K}^2]$	$-\frac{i}{8\pi(p^+ - K^+)}\left[\frac{p^+}{Q^+}\ln\frac{p^+ + Q^+}{p^+} + \ln\frac{(p^+ + Q^+)m^2\delta e^\gamma}{p^+ - K^+}\right]$	$-\frac{i}{8\pi p^+}\ln(Q_\parallel^2\delta e^\gamma)$	NA
$I_L^1[\boldsymbol{K}\cdot\boldsymbol{Q}]$	$-\frac{i\boldsymbol{Q}^2}{16\pi Q^+}\ln\frac{p^+ + Q^+}{p^+}$	$-\frac{i}{16\pi p^+}\ln\frac{Q^4}{Q_\parallel^4}$	NA
I_L^2	NA	$\frac{i}{8\pi p^+ Q^2}\ln\frac{Q^2 Q^+}{Q_\parallel^2(-K^+)}$	$\frac{i}{8\pi(p^+ + Q^+)Q^2}\ln\frac{Q^2(p^+ + Q^+)^2}{m^2 Q^+(K^+ + Q^+)}$
$I_L^2[\boldsymbol{K}^2]$	NA	$-\frac{i}{8\pi p^+}\ln(Q_\parallel^2\delta e^\gamma)$	$\frac{i\boldsymbol{Q}^2}{8\pi(p^+ + Q^+)Q^2}\ln\frac{Q^2(p^+ + Q^+)^2}{m^2 Q^+(K^+ + Q^+)}$
$I_L^2[\boldsymbol{K}\cdot\boldsymbol{Q}]$	NA	$-\frac{i}{16\pi p^+}\ln\frac{Q^4}{Q_\parallel^4}$	$\frac{-i\boldsymbol{Q}^2}{16\pi(p^+ + Q^+)Q^2}\ln\frac{Q^2(p^+ + Q^+)^2}{m^2 Q^+(K^+ + Q^+)}$

FIG. 5. Asymptotic behavior of the left triangle K^+ integrands near singular points.

Right Triangle Asymptotics, $-q^+ < K^+ < 0$			
	$K^+ \to -q^+$	$K^+ \to -Q^+$	$K^+ \to 0$
I_R^1	$\frac{i}{8\pi m^2 Q^+} \ln \frac{q^+}{q^+ - Q^+}$	$\frac{i}{8\pi(q^+-Q^+)Q^2} \ln \frac{Q^4(q^+-Q^+)^2}{m^2 Q_\parallel^2 K^{+2}}$	NA
$I_R^1[\boldsymbol{K}^2]$	$-\frac{i}{8\pi(q^++K^+)}\left[\frac{q^+}{Q^+}\ln\frac{q^+}{q^+-Q^+} + \ln\frac{(q^+-Q^+)m^2\delta e^\gamma}{q^++K^+}\right]$	$\boldsymbol{Q}^2 I_R^1 - \frac{i}{8\pi(q^+-Q^+)} \ln \frac{Q^4 \delta e^\gamma}{Q_\parallel^2}$	NA
$I_R^1[\boldsymbol{K}\cdot\boldsymbol{Q}]$	$-\frac{i\boldsymbol{Q}^2}{16\pi Q^+} \ln \frac{q^+}{q^+-Q^+}$	$-\boldsymbol{Q}^2 I_R^1 + \frac{i}{16\pi(q^+-Q^+)} \ln \frac{Q^4}{Q_\parallel^4}$	NA
I_R^2	NA	$\frac{i}{8\pi(q^+-Q^+)Q^2} \ln \frac{Q^2 Q^+}{Q_\parallel^2 (K^++Q^+)}$	$\frac{i}{8\pi q^+ Q^2} \ln \frac{Q^2 q^{+2}}{m^2 Q^+(-K^+)}$
$I_R^2[\boldsymbol{K}^2]$	NA	$\boldsymbol{Q}^2 I_R^2 - \frac{i}{8\pi(q^+-Q^+)} \ln \frac{Q^4\delta e^\gamma}{Q_\parallel^2}$	0
$I_R^2[\boldsymbol{K}\cdot\boldsymbol{Q}]$	NA	$-\boldsymbol{Q}^2 I_R^2 + \frac{i}{16\pi(q^+-Q^+)} \ln \frac{Q^4}{Q_\parallel^4}$	0

FIG. 6. Asymptotic behavior of the right triangle K^+ integrands near singular points.

dixes, which have left K^+ integration unperformed, we see by direct inspection that $I_R^{1,2}(q^+, K^+) = I_L^{1,2}(q^+ - Q^+, -K^+ - Q^+)$. Note in this context that the range $-q^+ < K^+ < 0$ can be expressed as $-Q^+ < -K^+ - Q^+ < q^+ - Q^+$, analogous to the range $-Q^+ < K^+ < p^+$. There are similar relations between the other integrals. All together we have

$$I_R^{1,2}(q^+, K^+) = I_L^{1,2}(q^+ - Q^+, -K^+ - Q^+), \qquad \text{(B14)}$$

$$I_R^{1,2}[(\boldsymbol{K}+\boldsymbol{Q})^2](q^+, K^+) = I_L^{1,2}[\boldsymbol{K}^2](q^+ - Q^+, -K^+ - Q^+), \qquad \text{(B15)}$$

$$I_R^{1,2}[-(\boldsymbol{K}+\boldsymbol{Q})\cdot\boldsymbol{Q}](q^+, K^+) = I_L^{1,2}[\boldsymbol{K}\cdot\boldsymbol{Q}](q^+ - Q^+, -K^+ - Q^+). \qquad \text{(B16)}$$

After the K^+ integrals have been performed, these relations just produce the general substitution rule quoted above.

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[26] We correct an error in Eqs. (2.2) and (2.12) of [25] which gave -2 instead of $-4/9$ for the constant in Π^{++}.
[27] Strictly speaking, we should also substitute $\boldsymbol{k}_{0,1} \rightarrow -\boldsymbol{k}_{1,0}$. But this is not necessary since Eq. (45) only depends on the dual momenta through $\boldsymbol{k}_1 - \boldsymbol{k}_0$ which is invariant under this last substitution.