World sheet commuting $\beta\gamma$ conformal field theory and nonrelativistic string theories

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We construct a sigma model in two dimensions with Galilean symmetry in flat target space similar to the sigma model of the critical string theory with Lorentz symmetry in 10 flat spacetime dimensions. This is motivated by the works of Gomis and Ooguri [J. Math. Phys. (N.Y.) **42,** 3127 (2001)] and Danielsson *et al.* [J. High Energy Phys. 10 (2000) 020; J. High Energy Phys. 03 (2001) 041.]. Our theory is much simpler than their theory and does not assume a compact coordinate. This nonrelativistic string theory has a bosonic matter $\beta\gamma$ conformal field theory with the conformal weight of β as 1. It is natural to identify time as a linear combination of γ and $\bar\gamma$ through an explicit realization of the Galilean boost symmetry. The angle between γ and $\bar\gamma$ parametrizes one parameter family of selection sectors. These selection sectors are responsible for having a nonrelativistic dispersion relation without a nontrivial topology in the nonrelativistic setup, which is one of the major differences from the previous works of Gomis and Ooguri and of Danielsson and co-workers. This simple theory is the nonrelativistic analogue of the critical string theory, and there are many different avenues ahead to be investigated. We mention a possible consistent generalization of this theory with different conformal weights for the $\beta\gamma$ conformal field theory. We also mention supersymmetric generalizations of these theories.

I. INTRODUCTION

String theory is a prime candidate for a quantum theory of gravity and is widely studied with the relativistic target space symmetry. A starting point at the perturbative level is the Polyakov action with flat target spacetime with Lorentz symmetry as its global symmetry. Open strings by themselves are not consistent as they can join end points together and turn into closed strings. But recently it has been realized that, in a small part of its moduli space, string theory can have only open strings without closed strings [1,2]. These theories opened up new possibilities to study the nature of string theory without complication of gravity. It has been a fascinating subject by itself revealing many novel properties including a space and time noncommutativity.

Further studies revealed that it is also possible to have a closed string theory with a different target space symmetry. That target space symmetry is Galilean symmetry, and it was named nonrelativistic closed string theory [3]. Strikingly, this nonrelativistic string theory has a world sheet description [3] as well as a target space description [4,5].[1] The world sheet action is simple, but the analysis of these papers heavily relied on the original relativistic description with an assumption of a compact coordinate in

order to preserve the effect of the Neveu-Schwarz-Neveu-Schwarz (NSNS) *B* field. Despite its strong connection to the original theory, the world sheet description attracts much attention (see, e.g., [7]). We will start to investigate this world sheet theory with some modifications of the action and of the target space topology. These modifications are considered, and are partially justified, in Sec. II.

The purpose of this paper is to propose a simpler world sheet action for the nonrelativistic string theory of Gomis and Ooguri [3] and of Danielsson *et al.* [4,5], and to study its properties. In this paper we investigate a basic bosonic sigma model with flat spatial coordinates and with a matter $\beta\gamma$ conformal field theory (CFT) replacing time and one of the spatial coordinates of the Polyakov action, similarly to the CFT of [3–5]. The main differences in our model are (i) there is no compact coordinate in our description, and (ii) there are no terms other than the $\beta\gamma$ CFT action. On the way of developing this theory, we realize that it is possible to have a one-parameter family of selection sectors which parametrize the target-space time coordinate. Each sector in this family is represented by a different linear combination of γ and $\bar\gamma$. We explicitly construct the Galilean boost transformation with this generalized time coordinate. We construct the general vertex operators following the work of Gomis and Ooguri [3]. We propose a spacetime interpretation for the $\beta\gamma$ CFT, and we also find some restrictions on the parameters in the ground state vertex operator. We explicitly quantize the theory with the "old covariant quantization" (OCQ) scheme and also with Becchi-Rouet-Stora-Tyupin (BRST) quantization. We calculate some correlation functions and demonstrate the consistency of the theory by checking the modular invariance. The $\beta\gamma$ zero modes play a crucial role in the spacetime interpretation. These are the contents of Sec. III.

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[1]The nonrelativistic nature is interesting from several aspects. Here is one: whereas the complete fundamental description of string theory is yet to be clarified, it has been put forward that a fully nonperturbative definition of noncritical *M* theory in 2 + 1 dimensions can be written in terms of a nonrelativistic Fermi liquid [6]. We will return to this point in Sec. V, where we try to justify a consideration of string theory with the nonrelativistic setup.

We then proceed to generalize this theory by allowing more general conformal weights for the $\beta\gamma$ fields (in the bosonic string theory). If we want to have a target space interpretation, there is only a finite range of the allowed conformal weights of these fields. It seems that these theories are the nonrelativistic analogues of the noncritical string theories. These are interesting because these theories can shed light on the (relativistic) noncritical string theories. We discuss these theories in Sec. IV and present some immediate observations.

We conclude in Sec. V, where we explain some implications of this bosonic nonrelativistic string theory. We also mention a supersymmetric generalization of this theory. Some justifications for considering the string theory in the nonrelativistic setup are also presented.

II. REVIEW AND SETUP THE STARTING POINT

It is interesting to look briefly at some earlier developments regarding the low energy limit of open string theory associated with a large magnetic B field (i.e., the spatial components of the NSNS 2-form field) or a critical electric B field. We therefore start with a review of noncommutative open string (NCOS), open membrane (OM), and nonrelativistic closed string (NRCS) theories. We further motivate the study of our nonrelativistic string action. The ultimate justification, though, comes when we quantize and check the one loop consistency of our nonrelativistic string theory at the end of Sec. III. If familiar with those theories, readers are encouraged to jump to Sec. III.

Before we proceed, we recall that, in contrast to NCOS, the effective description of the low energy limit associated with a large value of the magnetic B field is captured by noncommutative Yang-Mills (NCYM) theory with a space noncommutativity [8].

A low energy limit of open string theory related to a critical electric B field is not a field theory but a consistent open string theory formulated on a noncommutative spacetime (NCOS) with all the massive excitations of the open string in it [1,2]. This can be understood by thinking about the open string as a dipole whose end points carry opposite charges [1]. Then, open strings stretched along the direction of the B field are energetically favored. At the critical value of the electric B field, the energy stored in tension is almost balanced by the electric energy of the stretched string, and the effective tension of the open string goes to 0. In the low energy limit, we cannot ignore these light degrees of freedom, the open strings. On the other hand, strings on the brane cannot turn into closed strings, because it will cost a lot of energy. Effectively, this theory is the theory of open strings, and the underlying spacetime is noncommutative (NCOS) [1].

It turns out that this phenomenon is more general and can be extended to the M5 brane theory with a critical electric 3-form field in M theory. The tension of a membrane stretched along the directions of the electric 3-form field is very light and cannot be ignored in the low energy limit. This theory is OM theory [9]. And the S dual of (5 + 1) dimensional NCOS theory in type IIB is open D1 brane theory on NS5 brane in type IIB. With T duality we can get open Dp brane theories on the NS5 brane. These theories are a large class of 6 dimensional nongravitational theories with light open D-branes among their excitations, which have near critical Ramond-Ramond (RR) gauge fields of different ranks [9].

When the spatial coordinate along a critical electric B field is compactified on a circle in NCOS theory, there are finite closed string states with the positive winding number which do not decouple from the open string spectrum [10]. From these observations the authors [3,4] tried to understand the low energy limit of closed string theories with a compact coordinate in the presence of a background NSNS electric B field. They end up having a NRCS and II A/B wound and wrapped theories. For the latter case, neither a critical electric B field nor D-branes were necessary to have nonrelativistic dispersion relation in the low energy limit [4]. They also consider a low energy limit of critical RR fields and found Galilean invariant Dp brane solutions. It is useful to look at those a little further to motivate the current work.

Gomis and Ooguri [3] developed a world sheet description of the nonrelativistic closed string theory by taking a consistent low energy limit of the relativistic string theory. This limit is typically related to a critical value of an electric component of a background NSNS B field in order to cancel divergences which arise when one takes the low energy limit. A spatial coordinate along the electric B field should be compact to obtain nontrivial physical states. Otherwise the background B field can be gauged away without changing string spectra. Winding modes from this compact coordinate were important to obtain the nonrelativistic energy dispersion relation, and the winding number multiplied by compactified radius had a role of mass. The resulting action of the low energy limit was written as

$$S_{\text{GO}} = \int \frac{d^2z}{2\pi} \Big(\beta \bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}$$
$$+ \frac{1}{4\alpha'_{\text{eff}}} \partial\gamma \bar{\partial}\bar{\gamma} + \frac{1}{\alpha'_{\text{eff}}} \partial X^i \bar{\partial} X_i \Big), \qquad (1)$$

where $\gamma = X^0 + X^1$, $\bar{\gamma} = -X^0 + X^1$, and β, $\bar{\beta}$ are commuting auxiliary fields which were introduced as Lagrange multipliers through the process of taking the low energy limit. The index i of the fields X^i runs from 2 to 9 and α'_{eff} is an effective string "slope" related to the compactification radius.

Interestingly, at the same time Gomis and Ooguri published their paper, Danielsson *et al.* [4] provided a complementary description of this nonrelativistic string theory. It was motivated by the observation that if there is a compact coordinate "NCOS" D strings can emit wound

strings into the bulk [10]. Furthermore, through T duality along the compact coordinate, it was possible to identify NCOS theory with the discrete light-cone quantization description of the IIA string theory. The authors showed that one can define a meaningful "NCOS" limit of the IIA/B closed string theory, a theory of closed strings with positive winding modes, as long as there is a compact dimension. This theory provides a spacetime description of the string theory with the nonrelativistic energy momentum relation, which is called "wound IIA/B" theory.

From the world sheet formulation [3] a 4-point scattering amplitude was calculated, and it revealed that there exist instantaneous Newtonian gravitational interactions whose origin could not be explained. Subsequently Danielsson *et al.* further investigated this issue and provided the origin of these interactions as massless gravitons [5]. Actually massless gravitons are not dynamical degrees of freedom in the nonrelativistic string point of view. They are subleading contributions from a zero winding sector of the nonrelativistic string theory. When Gomis and Ooguri derived the low energy limit, they had a term $- \int \frac{d^2z}{2\pi} \times (\frac{2\alpha'}{1+2\pi\alpha'B}\beta\bar{\beta})$ in their action which is responsible for the subleading contributions. They then took the strict low energy limit which removed these subleading contributions. Even though this term was absent, the world sheet formulation was powerful enough to produce correct instantaneous gravitational interactions [3]. Thus we will not consider a similar term in our action.

The term $\int \frac{d^2z}{2\pi} (\frac{1}{4\alpha'_{\text{eff}}} \partial\gamma \bar{\partial}\bar{\gamma})$ in the world sheet action (1) can also be safely ignored without changing the physical spectrum of the nonrelativistic string [5]. For the nonrelativistic closed string spectrum, Danielsson *et al.* showed that this term is just a leftover after removing a divergent contribution when one takes the low energy limit. For the nonrelativistic open string spectrum, they explicitly showed that this term actually does not change the spectrum at all. Thus we will consider an action without a similar term. Gomis and Ooguri kept this term, which gives a constant contribution to the nonrelativistic energy, in order to provide a precise connection of the nonrelativistic string spectrum to the NCOS spectrum.

These explain our starting point. If we follow the steps explained above, the remaining parts of the Gomis-Ooguri action are very simple and look very familiar. It is nothing but the conventional $\beta\gamma$ CFT with the conformal weights of β and γ as 1 and 0, respectively. Now it is time to start with the simple action with $\beta\gamma$ CFT and X CFTs and to study its properties. While we proceed, we encounter many surprises.

III. "CRITICAL" NONRELATIVISTIC BOSONIC STRING THEORY

We start with a bosonic string theory action with a commuting $\beta\gamma$ CFT in addition to the spatial X^i CFT, in conformal gauge.

$$S_0 = \int \frac{d^2z}{2\pi} (\beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + \frac{1}{\alpha'}\partial X^i \bar{\partial}X_i + b_g\bar{\partial}c_g + \bar{b}_g\partial\bar{c}_g), \tag{2}$$

where i runs from 1 to 24 for X^i CFTs. The commuting matter $\beta\gamma$ CFT has conformal weights $h(\beta) = 1$ and $h(\gamma) = 0$. The central charge of the commuting $\beta\gamma$ CFT is 2. The anticommuting ghost CFT, whose central charge is -26, has weights $h(b_g) = 2$ and $h(c_g) = -1$, as usual. To be anomaly free the total central charge of the whole system should be 0, and we need 24 spatial coordinates as indicated above. We will consider the cases with general λ and corresponding d, and present a table for these theories in the next section. The case presented in this section, with $\lambda = 1$, is rather special, and we will refer to it as critical nonrelativistic string theory.

We briefly comment on the commuting $\beta\gamma$ CFT [11,12] with $\lambda = 1$. The operator product expansions (OPEs) of these fields are given by

$$\gamma(z_1)\beta(z_2) \sim \frac{1}{z_{12}}, \qquad \beta(z_1)\gamma(z_2) \sim -\frac{1}{z_{12}}. \tag{3}$$

The antiholomorphic fields satisfy similar OPEs. The mode expansions and hermiticity properties are

$$\gamma(z) = \sum_{n=-\infty}^{\infty} \frac{\gamma_n}{z^n}, \qquad \gamma_n^{\dagger} = \gamma_{-n}, \tag{4}$$

$$\beta(z) = \sum_{n=-\infty}^{\infty} \frac{\beta_n}{z^{n+1}}, \qquad \beta_n^{\dagger} = -\beta_{-n}. \tag{5}$$

The holomorphic energy momentum tensor and its mode expansion are

$$T(z) = \beta\partial\gamma = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}},$$

$$L_n = \sum_{m=-\infty}^{\infty} (n-m):\gamma_{n-m}\beta_m:. \tag{6}$$

Importantly, the normal-ordering constant turns out to be 0 for this critical case with $\lambda = 1$.

A. Galilean invariance and selection sectors

Galilean invariance of the nonrelativistic world sheet action was pointed out first in [3] and written explicitly in [5] with a particular time coordinate given by $t_{\text{GO}} = \frac{1}{\sqrt{2}}(\gamma - \bar{\gamma})$. Actually, there exists a generalized Galilean invariance in the action (2) which has generalized time $t = p\gamma(z) + q\bar{\gamma}(\bar{z}) = \cos(\phi)\gamma(z) + \sin(\phi)\bar{\gamma}(\bar{z})$, where $p = \cos(\phi)$ and $q = \sin(\phi)$. The Galilean boost transformation can be written in the following way

$$X^i \rightarrow X^i + \frac{v^i}{2}\sqrt{\alpha'}(p\gamma(z) + q\bar{\gamma}(\bar{z})),$$

$$\beta \rightarrow \beta - \frac{v^i}{\sqrt{\alpha'}}p\partial X^i - \frac{v^i v_i}{4}p\partial(p\gamma(z) + q\bar{\gamma}(\bar{z})), \quad (7)$$

$$\bar{\beta} \rightarrow \bar{\beta} - \frac{v^i}{\sqrt{\alpha'}}q\bar{\partial} X^i - \frac{v^i v_i}{4}q\bar{\partial}(p\gamma(z) + q\bar{\gamma}(\bar{z})),$$

where v^i is the Galilean boost parameter. We can easily show that the above action is invariant up to total derivatives under this generalized Galilean boost transformation. And because the fields γ and $\bar{\gamma}$ do not transform at all, there exist an infinite number of "selection sectors" parametrized by a "selection time" $t = \cos(\phi)\gamma(z) + \sin(\phi)\bar{\gamma}(\bar{z})$ for the nonrelativistic string theory. As long as one belongs to one specific sector one will never escape from that sector with combinations of Galilean transformations. Nonrelativistic closed string theory [3] and wound string theory [4,5] deal with a special selection sector with $t_{GO} = \frac{1}{\sqrt{2}}(\gamma - \bar{\gamma})$ which corresponds to the case $\phi = (2n - \frac{1}{4})\pi$ for any integer n.

In addition to this Galilean boost invariance, the action (2) is also invariant under $SO(8)$ rotations acting on the spatial coordinates X^i, and under overall spacetime translations given by

$$X^i \rightarrow X^i + a^i, \qquad \gamma \rightarrow \gamma + a^\gamma, \qquad \bar{\gamma} \rightarrow \bar{\gamma} + a^{\bar{\gamma}}, \quad (8)$$

where a^i, a^γ, and $a^{\bar{\gamma}}$ are constants.

B. Vertex operators

Following Gomis and Ooguri [3] we will start with a ground state vertex operator V_0 acting on the $SL(2, C)$ invariant vacuum.

$$V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z}) = g{:}e^{ik^\gamma\gamma + ik^{\bar{\gamma}}\bar{\gamma} - ip'\int^z \beta - iq'\int^{\bar{z}} \bar{\beta} + ik^i \cdot X_i}{:}, \quad (9)$$

where k^γ, $k^{\bar{\gamma}}$, and k^i represent overall continuous momenta along coordinates γ, $\bar{\gamma}$, and X^i, respectively.

The OPEs of the vertex operator V_0 with the fields β and γ are

$$\beta(z_1)V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z_2, \bar{z}_2) \sim -\frac{ik^\gamma}{z_{12}}V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z_2, \bar{z}_2), \quad (10)$$

$$\gamma(z_1)V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z_2, \bar{z}_2) \sim ip'\ln(z_{12})V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z_2, \bar{z}_2). \quad (11)$$

From the first equation we can read off the fact that the state corresponding to the vertex operator $V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z})$ is an eigenstate of the zero mode β_0 of the field $\beta(z)$, and the eigenvalue is ik^γ. Similarly, the vertex operator has eigenvalue $ik^{\bar{\gamma}}$ with respect to the zero mode of the field $\bar{\beta}(\bar{z})$.

We can calculate the conformal weight of the vertex operator $V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z})$ using OPE with the energy momentum tensor

$$T^{\text{matter}}(z_1)V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z_2, \bar{z}_2)$$

$$\sim \frac{(\frac{\alpha'}{4}k^i k_i - k^\gamma p')}{z_{12}^2}V_0(k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z}) + \cdots, \quad (12)$$

where $T^{\text{matter}}(z_1) = T_X(z_1) + T_{\beta\gamma}(z_1)$. Thus the ground state vertex operator has the following conformal weights for left and right moving sectors:

$$(h_0, \tilde{h}_0) = \left(\frac{\alpha'}{4}k^i k_i - k^\gamma p', \frac{\alpha'}{4}k^i k_i - k^{\bar{\gamma}}q'\right). \quad (13)$$

To be a physical vertex operator we need to have $(h_0, \tilde{h}_0) = (1, 1)$. Thus there is a constraint to be imposed on all the physical operators,

$$k^\gamma p' = k^{\bar{\gamma}}q'. \quad (14)$$

Even though the commuting $\beta\gamma$ CFT is described by the first order formulation, it actually gives us zero-mode contributions through an integral form of the field β in the exponent of the vertex operator. These are related to the overall motion of these nonrelativistic string states after we change fields γ and $\bar{\gamma}$ into time and space in target spacetime.

As we already constructed the ground state vertex operator, it is relatively easy to construct first excited vertex operators, which can be written[2]

$$V_e(k^\gamma, k^{\bar{\gamma}}, k^i; z, \bar{z}) = g{:}e_{MN}\partial X^M \bar{\partial} X^N$$

$$\times e^{ik^\gamma\gamma + ik^{\bar{\gamma}}\bar{\gamma} - ip'\int^z \beta - iq'\int^{\bar{z}} \bar{b} + ik^i \cdot X_i}{:}, \quad (15)$$

with

$$\partial X^M = (\partial\gamma, \beta, (2/\alpha')^{1/2}\partial X^i), \quad (16)$$

$$\bar{\partial} X^N = (\bar{\partial}\bar{\gamma}, \bar{\beta}, (2/\alpha')^{1/2}\bar{\partial} X^j). \quad (17)$$

Note that ∂X^M and $\bar{\partial} X^N$ have conformal weights $(1, 0)$ and $(0, 1)$, respectively. So the overall conformal weights of the first excited vertex operators are

$$(h_e, \tilde{h}_e) = \left(\frac{\alpha'}{4}k^i k_i - k^\gamma p' + 1, \frac{\alpha'}{4}k^i k_i - k^{\bar{\gamma}}q' + 1\right). \quad (18)$$

For this vertex to be physical we need to impose conditions $(h_e, \tilde{h}_e) = (1, 1)$. This gives us a familiar nonrelativistic dispersion relation as we will see later. Higher excited vertex operators can also be constructed in a similar way.

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[2]Covariant notation here is only for convenience and compactness.

C. OCQ

In the old covariant quantization scheme we can just forget about the ghosts, and can restrict to a smaller physical Hilbert space so that the missing equations of motion of the energy momentum tensor hold for matrix elements. Resulting requirements for the physical states $|\psi\rangle$ can be written as [12]

$$(L_n^{\text{matter}} + a\delta_{n,0})|\psi\rangle = 0 \quad \text{for } n \geq 0, \qquad (19)$$

$$\langle\psi|L_n^{\text{matter}}|\psi'\rangle = \langle L_{-n}^{\text{matter}}\psi|\psi'\rangle = 0 \quad \text{for } n < 0. \qquad (20)$$

Here $a = -1$. This normal-ordering constant comes from the ghost contribution only because bosonic matter $\beta\gamma$ CFT with $\lambda = 1$ has the vanishing normal-ordering constant.

1. Ground state

The ground state is denoted by $|V_0\rangle$,

$$|V_0\rangle = V_0(k^\gamma, k^{\bar\gamma}, k^i; z = 0, \bar z = 0)|0; 0\rangle_{X^i} \otimes |0\rangle_{\beta\gamma\bar\beta\bar\gamma}$$
$$= |0; k^i\rangle_{X^i} \otimes |0; k^\gamma, k^{\bar\gamma}\rangle_{\beta\gamma\bar\beta\bar\gamma}. \qquad (21)$$

Norm of this ground state is given by

$$\langle V_0(k_2)|V_0(k_1)\rangle = (2\pi)^{26}\delta^{24}(k_2^i - k_1^i)\delta(k_2^\gamma - k_1^\gamma)$$
$$\times \delta(k_2^{\bar\gamma} - k_1^{\bar\gamma}). \qquad (22)$$

The first physical state condition can be evaluated to give us the result

$$(L_0^X + L_0^{\beta\gamma} - 1)|V_0\rangle = \left(\frac{\alpha'}{4}k^ik_i - k^\gamma p' - 1\right)|V_0\rangle = 0. \qquad (23)$$

When we evaluate the $\beta\gamma$ CFT part, we cannot just ignore the term involved with the zero modes in $L_0^{\beta\gamma}$ because a divergent contribution from the vertex operator renders it finite.[3] For this ground state to be physical, the condition $\frac{\alpha'}{4}k^ik_i - k^\gamma p' = 1$ should be imposed. And there is a similar condition from antiholomorphic sector as $\frac{\alpha'}{4}k^ik_i - k^{\bar\gamma}q' = 1$. This is consistent with the previous result (12) that the conformal weight of the vertex operator should be $(h, \tilde h) = (1, 1)$ to become a physical vertex operator.

From these physical conditions we have nontrivial relations between the parameters in the vertex operator. The arbitrary looking parameters, p' and q', in the vertex operator are uniquely fixed by the overall momenta k^γ, $k^{\bar\gamma}$, and k^i as follows:

$$p' = \frac{1}{k^\gamma}\left(\frac{\alpha'}{4}k^ik_i - 1\right), \qquad q' = \frac{1}{k^{\bar\gamma}}\left(\frac{\alpha'}{4}k^ik_i - 1\right). \quad (24)$$

Or we can view these equations as equations of k^γ and $k^{\bar\gamma}$ in terms of p', q' and transverse momenta. This viewpoint will give us a more familiar dispersion relation of the nonrelativistic string.

To have a nonrelativistic dispersion relation, we need to take into account the selection time, $t = p\gamma + q\bar\gamma = \cos(\phi)\gamma + \sin(\phi)\bar\gamma$. And we need to introduce another coordinate x which can be written as $x = -q\gamma + p\bar\gamma = -\sin(\phi)\gamma + \cos(\phi)\bar\gamma$. Then we have

$$\gamma = \cos(\phi)t - \sin(\phi)x, \qquad \bar\gamma = \sin(\phi)t + \cos(\phi)x. \qquad (25)$$

From the action Eq. (2) it is clear that β and $\bar\beta$ are conjugate variables of γ and $\bar\gamma$. So we can identify the energy and momentum as follows:

$$\frac{\beta}{2\pi} = \cos(\phi)P_t - \sin(\phi)P_x,$$
$$\frac{\bar\beta}{2\pi} = \sin(\phi)P_t + \cos(\phi)P_x. \qquad (26)$$

If one picks up eigenvalues for both sides one will have the equations $i\beta_0 = \cos(\phi)p_t - \sin(\phi)p_x$ and $i\bar\beta_0 = \sin(\phi)p_t + \cos(\phi)p_x$. With the eigenvalues $\beta_0 = -ik^\gamma$ and $\bar\beta_0 = -ik^{\bar\gamma}$ given in Eq. (10) we can get

$$p_t = \cos(\phi)k^\gamma + \sin(\phi)k^{\bar\gamma}, \qquad (27)$$

$$p_x = -\sin(\phi)k^\gamma + \cos(\phi)k^{\bar\gamma}. \qquad (28)$$

So the energy can be decided by the momenta k^γ, $k^{\bar\gamma}$ and the selection parameter ϕ. At first glance, these expressions look a little bit strange but we can reexpress these results in terms of other parameters using the constraint given by equation $pp' = qq'$.[4]

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[3]This divergent contribution of the ground state vertex operator presents in the critical string theory and is nothing new. In our case it is important to keep this divergent contribution in mind because of the first order form of the energy momentum tensor. We can check explicitly that this is consistent with the observations given in the previous subsection.

[4]This part is a little subtle. After changing fields from $\beta\gamma$ to t and x, we need to examine the ground state vertex operator in terms of these new variables. Concentrating on zero modes we have

$$\exp(ip_t(t + i[pp'\log z + qq'\log\bar z]) + ip_x(x - i[qp'\log z - pq'\log\bar z])). \qquad (29)$$

To be consistent we need to impose one of the following conditions. (a) $pp' = qq'$. Then $p_t = \frac{1}{pp'}(\frac{\alpha'}{4}k^ik_i - 1)$ and $p_x = 0$. (b) $qp' = -pq'$. Then $p_t = 0$ and $p_x = \frac{p'}{p}(\frac{\alpha'}{4}k^ik_i - 1)$. Effectively the $\beta\gamma$ zero modes are mapped into one bosonic coordinate and its momentum. What about the other coordinate? I think it is hidden somewhere because if there is only t coordinate, the central charges do not match after changing variables. It will be interesting to investigate further along this line. One can consider the coordinate x as a compact coordinate by introducing winding modes for γ and $\bar\gamma$ fields. It remains to be seen whether this compact coordinate can make the hidden coordinate visible or not. In the main body we will follow with the first condition.

Then the results are

$$p_t = \frac{1}{pp'}\left(\frac{\alpha'}{4}k^i k_i - 1\right), \qquad p_x = 0. \qquad (30)$$

Surely we are familiar with the first expression. The mass of the nonrelativistic particle corresponding to this vertex operator is given by the selection sector parameter and the parameter p' in the vertex operator. This energy spectrum has the same structure as that of Gomis and Ooguri [3] if one identifies the mass of the particle with $\omega R/2$

$$\epsilon_{GO} = \frac{\omega R}{2\alpha'} + \frac{2}{\omega R}\left(\frac{\alpha'}{4}k^i k_i - 1\right), \qquad (31)$$

where ω and R are the winding number and the radius of the compact coordinate. Note that the first term in Eq. (31), which is constant, comes from the tension of the winding string and is related to the term proportional to $\partial \beta \bar{\partial} \bar{\beta}$, which we ignored. They need a compact coordinate to have nonzero result on the energy spectrum and energy is crucially related to the positive winding number of the compact coordinate.

2. First excited states

The first excited states can be constructed with the corresponding vertex operator

$$|V_e\rangle = V_e(k^\gamma, k^{\bar{\gamma}}, k^i; z = 0, \bar{z} = 0)(|0; 0\rangle_{X^i} \otimes |0\rangle_{\beta\gamma\bar{\beta}\bar{\gamma}})$$
$$(32)$$

$$= (e_{M\bar{N}}\alpha^M_{-1}\bar{\alpha}^N_{-1})(|0; k^i\rangle_{X^i} \otimes |0; k^\gamma, k^{\bar{\gamma}}\rangle_{\beta\gamma\bar{b}\,\bar{\gamma}}). \quad (33)$$

Where the index M runs γ, β, and i, and the index $\bar{N}$ runs $\bar{\gamma}$, $\bar{\beta}$, and $\bar{i}$. Thus $e_M\alpha^M_{-1} = \gamma_{-1}e_\gamma + \beta_{-1}e_\beta + \alpha^i_{-1}e_i$. Again the notation is for convenience.

These states should satisfy the physical state conditions $(L^m_0 - 1)|V_e\rangle = 0$ and $(\bar{L}^m_0 - 1)|V_e\rangle = 0$ and we have

$$\frac{\alpha'}{4}k^i k_i - k^\gamma p' = 0, \qquad \frac{\alpha'}{4}k^i k_i - k^{\bar{\gamma}}q' = 0. \qquad (34)$$

And there are other nontrivial conditions we need to impose for these states, $L^m_1|V_e\rangle = 0$ for the holomorphic part and similarly $\bar{L}^m_1|V_e\rangle = 0$ for the antiholomorphic part. Concentrating on the holomorphic part we can get

$$L^m_1|V_e\rangle = (k^M e_{M\bar{N}}\bar{\alpha}^N_{-1})|V_e\rangle = 0, \qquad (35)$$

where

$$k^M = (p', k^\gamma, (\alpha'/2)^{1/2}k^i), \qquad e_M = (e_\gamma, e_\beta, e_i). \quad (36)$$

Thus we have conditions $k^M e_M = p'e_\gamma + k^\gamma e_\beta + (\alpha'/2)^{1/2}k^i e_i = 0$ for general $\bar{N}$

There is a spurious state at this level,

$$L^m_{-1}|V_0\rangle = l_M\alpha^M_{-1}|V_0\rangle$$
$$= (-k^\gamma\gamma_{-1} - p'\beta_{-1} + (\alpha'/2)^{1/2}k_i\alpha^i_{-1})|V_0\rangle,$$
$$(37)$$

where

$$l_M = (-k^\gamma, -p', (\alpha'/2)^{1/2}k^i),$$
$$\alpha^M_{-1} = (\gamma_{-1}, \beta_{-1}, \alpha^i_{-1}).$$
$$(38)$$

From the observation $k^M(e_{M\bar{N}} + l_M\bar{\beta}_N + A_M\bar{l}_N) = 0$, and with the conditions $k^M A_M = \bar{\beta}_N\bar{k}^N = 0$, we can check that this spurious state $L^m_{-1}|V_0\rangle$ is actually physical and null. To derive this result we use the fact $k^M l_M = 0$, which is a direct consequence of the first physical state condition (34). These nontrivial equations reveal the equivalent relation

$$e_{M\bar{N}} \approx e_{M\bar{N}} + l_M\bar{\beta}_N + A_M\bar{l}_N, \quad \text{with}$$
$$k^M A_M = \bar{\beta}_N\bar{k}^N = 0.$$
$$(39)$$

These equations for the equivalence relation are similar to the relativistic string theory. From this observation we can conclude that this nonrelativistic string theory has the same number of degrees of freedom with the corresponding relativistic string theory.

We can also analyze the energy dispersion relation for the first excited state. The derivation is almost the same as the previous subsection and the result is

$$E_e = \frac{k^i k_i}{2M}, \quad \text{where } M = \frac{2}{\alpha'}pp' = \frac{2}{\alpha'}\cos(\phi)p'. \quad (40)$$

This energy dispersion relation is exactly the same as that of the known nonrelativistic particles.

D. BRST quantization

We already quantize the nonrelativistic bosonic string theory with the OCQ method. But it is still interesting to see the equivalence between the OCQ result and the BRST quantization result.

We are given with the gauge-fixed action (2) and we will closely follow [12]. The only difference comes from the $\beta\gamma$ matter sector and we have the BRST transformations concentrating on the holomorphic part

$$\delta_B X^i = i\epsilon c_g \partial X^i, \qquad \delta_B\beta = i\epsilon c_g \partial\beta,$$
$$\delta_B\gamma = i\epsilon c_g \partial\gamma, \qquad \delta_B b_g = i\epsilon(T^m + T^g), \qquad (41)$$
$$\delta_B c_g = i\epsilon c_g \partial c_g.$$

BRST transformations for the $\beta\gamma$ matter CFT are nothing but the conformal transformations. $T^{\text{matter}} = T_X + T_{\beta\gamma}$, where $T_{\beta\gamma}$ is given above. And the energy momentum tensor for the ghost part has a usual form $T^g = (\partial b_g)c_g - \lambda_g\partial(b_g c_g)$. The BRST current and charge have the follow-

ing forms:

$$j_B = c_g T^m + \frac{1}{2} :c_g T^g: + \frac{3}{2} \partial^2 c_g, \qquad (42)$$

$$Q_B = \frac{1}{2\pi i} \oint dz j_B(z) = \sum_{n=-\infty}^{\infty} (c_{gn} L^m_{-n} + c_{gn} L^g_{-n}) - c_{g0}. \qquad (43)$$

The OPE between the BRST current and the energy momentum tensor are

$$j_B(z_1) j_B(z_2) \sim -\frac{c^m - 18}{2z_{12}^3} c_g \partial_2 c_g(z_2) - \frac{c^m - 18}{4z_{12}^2} c_g \partial_2^2 c_g(z_2) \\ - \frac{c^m - 26}{12z_{12}} c_g \partial_2^3 c_g(z_2), \qquad (44)$$

$$T(z_1) j_B(z_2) \sim \frac{c^m - 26}{2z_{12}^4} c_g(z_2) + \frac{1}{z_{12}^2} j_B(z_2) + \frac{1}{z_{12}} \partial_2 j_B(z_2). \qquad (45)$$

From these equations we can read off the facts that the BRST charge is nilpotent and the BRST current is a conformal tensor only if the central charge of the matter sector $c^m = 26$.

The physical states of this nonrelativistic string theory can be systematically constructed with the BRST cohomology using the nilpotent BRST operator $Q_B^2 = 0$. The inner product of the states can be defined with the identifications

$$\alpha_m^{i\dagger} = \alpha_{-m}^i, \qquad \beta_m^\dagger = -\beta_{-m}, \qquad \gamma_m^\dagger = \gamma_{-m}, \\ b_{gm}^\dagger = b_{g-m}, \qquad c_{gm}^\dagger = c_{g-m}. \qquad (46)$$

The zero modes of the ghost fields force the inner product of the ground state to have the form

$$\langle V_0'(k_2)|\bar{c}_{g0} c_{g0}|V_0'(k_1)\rangle = i(2\pi)^{26} \delta^{24}(k_2^i - k_1^i) \\ \times \delta(k_2^\gamma - k_1^\gamma) \delta(k_2^{\bar\gamma} - k_1^{\bar\gamma}), \quad (47)$$

where $|V_0'\rangle = |V_0\rangle \otimes |0\rangle_g$ denotes product of the matter ground state with the ghost ground state. The c_{g0} and $\bar{c}_{g0}$ insertions are necessary for nonzero results.

To get the physical ground state we need two conditions

$$b_{g0}|V_0'\rangle = 0, \qquad (48)$$

$$Q_B|V_0'\rangle = 0. \qquad (49)$$

These imply $L_0|V_0'\rangle = \{Q_B, b_{g0}\}|V_0'\rangle = 0$. This condition with the antiholomorphic part actually gives us the same mass shell conditions as the OCQ result given in Eq. (24). There is no exact state at this level and the state $|V_0'\rangle$ is a BRST cohomology class which is physical.

At the first excited level there are $(26 + 2)^2$ states.

$$|V_e'\rangle = (e_{\mu\bar\nu} \alpha_{-1}^\mu \bar\alpha_{-1}^\nu)|V_0'\rangle, \qquad (50)$$

where the index μ includes i, γ, β, c_g, and b_g, and the index $\bar\nu$ includes i, $\bar\gamma$, $\bar\beta$, $\bar{c}_g$, and $\bar{b}_g$. The norm of this excited state for the holomorphic part is given by

$$\langle V_e'|\bar{c}_{g0} c_{g0}|V_e'\rangle = (e_M^* e_M + e_{bg}^* e_{bg} + e_{cg}^* e_{bg}) \\ \times \langle V_0'|\bar{c}_{g0} c_{g0}|V_0'\rangle, \qquad (51)$$

where $e_M^* e_M = e_i^* e_i + e_\beta^* e_\gamma + e_\gamma^* e_\beta$, and the last expression $\langle V_0'|\bar{c}_{g0} c_{g0}|V_0'\rangle$ is already evaluated in Eq. (47). The antiholomorphic counter part also has a similar form.

The BRST invariant condition for this first excited state can be analyzed independently for holomorphic and antiholomorphic sectors. Concentrating on the holomorphic sector, we have

$$0 = Q_B|V_e'\rangle \\ = (c_{g-1}\{\alpha_0^i e_i + k^\gamma e_\beta + p' e_\gamma\} \\ + e_{bg}\{\alpha_0 \alpha_{-1} - p'\beta_{-1} - k^\gamma \gamma_{-1}\})|V_g'\rangle \\ = (c_{g-1} k^M e_M + e_{gb} l_M \alpha_{-1}^M)|V_g'\rangle, \qquad (52)$$

where the detail calculation can be done with the same procedure given in previous subsection. k^M and l_M are given by the equations (36) and (38), respectively. Because c_{g-1} and α_{-1}^M are creation operators, the BRST closed condition forces us to have

$$k^M e_M = 0, \qquad e_{bg} = 0. \qquad (53)$$

This is very similar to the relativistic case.

There is an additional zero norm state created by c_{g-1} and $l_M \alpha_{-1}^M$. A general state is of the same form as the first excited state with different coefficients

$$|\psi\rangle = (e^{M'} \alpha_{M-1} + e_{bg}' b_{g-1} + e_{cg}' c_{g-1})|V_g'\rangle. \qquad (54)$$

The BRST exact state at this level is

$$Q_B|\psi\rangle = (c_{g-1} k^M e_M' + e_{gb}' l_M \alpha_{-1}^M)|V_g'\rangle. \qquad (55)$$

Thus the ghost state $c_{g-1}|V_0'\rangle$ is BRST exact, while the "polarization" has the following equivalence relation

$$e^M \sim e^M + e_{bg}' l^M. \qquad (56)$$

This is the same result as the OCQ given in the previous section. So there are total $(24)^2$ states in the first excited level which is exactly the same as the physical spectra of the relativistic closed string.

E. Scattering amplitudes with vertex operators

In this section we calculate various scattering amplitudes with the ground state vertex operators following the paper [3]. And we also show that the amplitudes factorize properly into nonrelativistic string poles. The scattering amplitude can be written

$$\left\langle \prod_{s=1}^{n} V_{0s}(z_s) \right\rangle = \int DX^i D\gamma D\bar{\gamma} D\beta D\bar{\beta} e^{-S}$$

$$\times \prod_{s=1}^{n} V_0(k_s^\gamma, k_s^{\bar{\gamma}}, k_s^i; z_s, \bar{z}_s), \quad (57)$$

where the action is given in Eq. (2) and the vertex operator is given in Eq. (9).

These scattering amplitudes can be calculated with the functional integral. The calculation for the quadratic X^i part can be done with the Gaussian integral and the result is

$$\left\langle \prod_{s=1}^{n} V_{gs}(z_s) \right\rangle = g^n \prod_{s \neq t} (z_s - z_t)^{-k_s^\gamma p_t'} (\bar{z}_s - \bar{z}_t)^{-k_s^{\bar{\gamma}} q_t'} |z_s - z_t|^{(\alpha'/2)k_s^i k_{ti}}. \quad (59)$$

From this calculation it is straightforward to evaluate scattering amplitudes for any number of the ground state vertex operator V_0.

To evaluate the string poles we need to go a little bit further. The second exponent can be expressed

$$k_s^{\bar{\gamma}} q_t' = \frac{\tan(\phi_s)}{\tan(\phi_t)} k_s^\gamma p_t' = k_s^\gamma p_t', \quad (60)$$

where we use the relations $k^\gamma p' = k^{\bar{\gamma}} q'$ given in (14), constraint $pp' = qq'$ and $\phi_s = \phi_t$, which is guaranteed by the fact that any two particles in different selection sectors cannot interact with each other with the Galilean transformations. Then the n-point vertex scattering amplitude (59) can be simplified

$$\left\langle \prod_{s=1}^{n} V_{0s}(z_s) \right\rangle = g^n \prod_{s<t} |z_s - z_t|^{-2k_s^\gamma p_t' - 2k_t^\gamma p_s' + \alpha' k_s^i k_{ti}} = g^n \prod_{s<t} |z_s - z_t|^{-2\cos(\phi)(p_s' + p_t')(E_s + E_t) + (\alpha'/2)(k_s^i + k_t^i)^2 - 4}, \quad (61)$$

where we used the energy relation $E_s = \frac{1}{p_s' \cos(\phi)} (\frac{\alpha'}{4} k_s^i k_{si} - 1)$. So we have the following closed string poles

$$E_s + E_t = \frac{\frac{\alpha'}{4}(k_s^i + k_t^i)^2 + m - 1}{\cos(\phi)(p_s' + p_t')}. \quad (62)$$

This is the closed string spectrum of nonrelativistic string theory, and can be identified with the general formula given in [3] with appropriate modifications.

These scattering amplitudes also can be calculated with the operator formulation, and these two results are equivalent. Scattering amplitude for the excited states can also be calculated without difficulty following the procedure given here. We present a four vertex scattering amplitude for the excited states in the Appendix. Because this nonrelativistic theory is less symmetric and lacks covariant notation, the expression for the scattering amplitude is quite complicated.

F. One loop partition function

It is important to check the modular invariance of this nonrelativistic string theory because a breakdown of the modular invariance may be thought of as a global anomaly of the reparametrization invariance in string theories. The one loop partition function with a modulus $\tau = \tau_1 + i\tau_2$ on torus can be given in the operator language as

$$Z(\tau) = \text{Tr}(\exp(2\pi i \tau_1 P - 2\pi \tau_2 H)), \quad (63)$$

where $P = L_0 - \bar{L}_0$ and $H = L_0 + \bar{L}_0 - \frac{1}{24}(c + \bar{c})$.

the same as the one in [3,12]. The first order functional integral can be evaluated with the Lagrangian (2) and the extremum is given by

$$\beta(z) = \sum_s -\frac{ik_s^\gamma}{z - z_s}, \qquad \gamma(z) = \sum_s ip_s' \ln(z - z_s). \quad (58)$$

Thus all the functional integrals can be trivially evaluated and there are also the contributions from the various OPEs between the vertex operators. The result can be given

There are three independent parts to be evaluated, the X^i CFTs, the ghost bc CFT, and the matter $\beta\gamma$ CFT. Contributions from the X^i CFTs and the ghost bc CFT are well known [12]

$$Z_X^{\text{tot}} = V_{24} Z_X^{24} = V_{24}((4\pi^2 \alpha' \tau_2)^{-1/2} |\eta(\tau)|^{-2})^{24},$$

$$Z_{bc} = |\eta(\tau)|^4, \quad (64)$$

where Dedekind $\eta(\tau)$ function is given by $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$ with $q = e^{2\pi i\tau}$.

A new contribution from the matter $\beta\gamma$ CFT can be calculated similarly[5]

$$Z_{\beta\gamma} = V_{\beta\gamma}(q\bar{q})^{-2/24} \int \frac{dk^\gamma dk^{\bar{\gamma}}}{(2\pi)^2} \delta(k^\gamma p' - k^{\bar{\gamma}} q')$$

$$\times q^{-k^\gamma p'} \bar{q}^{-k^{\bar{\gamma}} q'} \prod_{m=-\infty}^{\infty} \sum_{N_m, \bar{N}_m = 0}^{\infty} q^{mN_m} \bar{q}^{m\bar{N}_m}$$

$$= \frac{V_{\beta\gamma}}{2p'q'}((4\pi^2 \alpha' \tau_2)^{-1/2} |\eta(\tau)|^{-2})^2. \quad (65)$$

The matter part of the partition function $Z_X \cdot Z_{\beta\gamma}$ is the same as that of the relativistic string theory up to the volume factor. And the total partition function is

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[5]The delta function is inserted because any physical state needs this condition. When we evaluate the integral we only integrated over the range $0 \leq k^\gamma \leq \infty$ without losing a generality which can be achieved by adjusting p'.

$$Z_{\text{tot}} = \frac{V_{24}V_{\beta\gamma}}{2p'q'} \int \frac{d^2\tau}{16\pi^2\alpha'\tau_2^2} (Z_X^{24}). \qquad (66)$$

This shows that the nonrelativistic string theory is modular invariant because the integrand $\frac{d^2\tau}{\tau_2^2}$ is modular invariant and $Z_X = (4\pi^2\alpha'\tau_2)^{-1/2}|\eta(\tau)|^{-2}$ is itself modular invariant. The contributions from the excitations of the $\beta\gamma$ CFT in the partition function are cancelled by those of the bc CFT. The contributions from the zero modes of the $\beta\gamma$ CFT actually contribute to the factor $\frac{1}{\tau_2}$ in order to ensure the modular invariance.

G. Open string

For the open string, we can impose the boundary condition $T_{zz} = T_{\bar{z}\bar{z}}$ at $\text{Im}z = 0$. The fields need to have the conditions,

$$\partial X^i = \bar{\partial}X^i, \qquad \beta = \bar{\beta}, \qquad \gamma = \bar{\gamma} \quad \text{at} \quad z = 0. \quad (67)$$

Then the usual doubling trick to write the holomorphic and antiholomorphic fields in the upper half-plane in terms of holomorphic fields in the whole plane,

$$\beta(z) \equiv \bar{\beta}(\bar{z}'), \qquad \gamma(z) \equiv \bar{\gamma}(\bar{z}'), \qquad \text{Im}(z) \leq 0, \qquad z' = \bar{z} \qquad (68)$$

The holomorphic energy momentum tensor should match the antiholomorphic energy momentum tensor after Galilean boost transformation at $z = 0$ for the open string case. This imposes a condition $p = q$ for the selection sector, in which the nonrelativistic open string theory is meaningful. Consequently there is a constraint $p' = q'$ for the open string vertex operator. We expect the rest of the quantization procedure is similar to the closed string case and is straightforward after the work of the closed string theory.

IV. BOSONIC THEORY WITH GENERAL COMMUTING $\beta\gamma$ CFT

After quantizing the critical case with $\lambda = 1$ for the nonrelativistic string, it is natural to ask if there are also meaningful theories for the $\beta\gamma$ CFT with different conformal weights. Thus we start with the bosonic string theory action with a general commuting $\beta\gamma$ CFT in addition to the spatial X^i CFT in conformal gauge.

$$S = \int \frac{d^2z}{2\pi} \left(\beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + \frac{1}{\alpha'}\partial X^i\bar{\partial}X_i \right. $$
$$\left. + b_g\bar{\partial}\gamma_g + \bar{b}_g\partial\bar{c}_g \right), \qquad (69)$$

where i runs from 1 to d for X^i CFTs. The commuting matter $\beta\gamma$ CFT has the conformal weights $h(\beta) = \lambda$ and $h(\gamma) = 1 - \lambda$. The anticommuting ghost bc CFT has the weight $h(b_g) = 2$ and $h(c_g) = -1$.

For the general commuting $\beta\gamma$ CFT, the OPEs are given by [11,12]

$$\gamma(z_1)\beta(z_2) \sim \frac{1}{z_{12}}, \qquad \beta(z_1)\gamma(z_2) \sim -\frac{1}{z_{12}}. \qquad (70)$$

And the antiholomorphic fields satisfy the similar OPE. The mode expansion and hermiticity property are

$$\gamma(z) = \sum_{n=-\infty}^{\infty} \frac{\gamma_n}{z^{n+(1-\lambda)}}, \qquad \gamma_n^\dagger = \gamma_{-n}, \qquad (71)$$

$$\beta(z) = \sum_{n=-\infty}^{\infty} \frac{\beta_n}{z^{n+\lambda}}, \qquad \beta_n^\dagger = -\beta_{-n}. \qquad (72)$$

The holomorphic energy momentum tensor and its mode expansion are

$$T = (\partial\beta)\gamma - \lambda\partial(\beta\gamma) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}},$$
$$L_n = \sum_{m=-\infty}^{\infty} (n\lambda - m):\gamma_{n-m}\beta_m: + a\delta_{n,0}, \qquad (73)$$

where $a = -\frac{\lambda(\lambda-1)}{2}$ for the commuting bosons with periodic boundary condition. For the interesting case $\lambda = 1$, this ordering constant vanishes as we saw in the previous section.

The central charge of the commuting $\beta\gamma$ CFT is $2(6\lambda^2 - 6\lambda + 1)$. To have a consistent theory, central charge from the matter CFTs (the X^i CFTs and the $\beta\gamma$ CFT) should cancel the central charge -26 from the reparametrization ghost bc CFT. Thus we can have the following condition:

$$d = 26 - 2(6\lambda^2 - 6\lambda + 1). \qquad (74)$$

In Table I we present different λ and d for the possible consistent theories.

We present the immediate observations on these possible consistent nonrelativistic string theories, which have Galilean symmetry mentioned in the previous section with the corresponding rotational invariance, $SO(d)$. First, there exist only a finite range of conformal weight $-1 \leq \lambda \leq 2$ to have a space and time interpretation for the theories with the general $\beta\gamma$ CFT. The maximum number of the spatial coordinates excluding the $\beta\gamma$ CFT is 27 for

TABLE I. Table for the possible consistent bosonic string theories with the $\beta\gamma$ CFT with integer and half integer conformal weights. The conformal weight λ, the central charge, and the number of the spatial dimensions of the target space are presented. "$\cdots$" represents the case with the "negative number" spatial dimensions.

λ	$\cdots$	2	3/2	1	1/2	0	$-1/2$	-1	$\cdots$
$c_{\beta\gamma}$	$c_{\beta\gamma} > 26$	26	11	2	-1	2	11	26	$c_{\beta\gamma} > 26$
d	$\cdots$	0	15	24	27	24	15	0	$\cdots$

the case of $\lambda = 1/2$. This is a rather special case, and is worthy of further investigation.

Second, as we increase λ, the central charge of $\beta\gamma$ CFT decreases until $c_{\beta\gamma} = -1$ for $\lambda = 1/2$ and then it increases. Usually two different conformal weights correspond to the same central charge and the same number of spatial coordinates. We already considered the critical case of $\lambda = 1$. The $\lambda = 0$ case seems to be exactly the same as the critical case if we change the role of β and γ, except possibly different space and time interpretations.

Third, there are two possible theories with only the $\beta\gamma$ CFT and the $b_g c_g$ CFT without spatial coordinates. We would like to call these "topological theories." These topological theories have only the zero modes corresponding to spacetime coordinates and momenta without any excitations. There are some curious facts in these theories as we mention at the end of this subsection.

Fourth, it will be interesting to understand the theories with half integer conformal weight fields. At first glance, it seems that there is no possible interpretation of time in target space because the fields with half integer conformal weight do not give us zero modes. But as is well known, it is required to have two different sectors, NS and R sectors, for the half integer conformal weight fields. These fields are very similar to the superconformal ghost fields. Of course there we need be careful with the continuous zero modes in order to have spacetime interpretations.

These nonrelativistic string theories are very similar to the critical case except the zero modes of the $\beta\gamma$ CFT. Establishing zero modes will be a challenge for these theories, but we think these problem could be solved with spectral flow. Generally noncritical relativistic string theories are hard to understand. In some sense, these theories are "noncritical." If we understand these theories better, we may have more insights for the relativistic counterpart.

Unification of the first order CFTs

There is a curiosity about a unification of the first order CFTs. The bosonic $\beta\gamma$ CFT and the ghost $b_g c_g$ CFT can be unified in bigger multiplets, "grand multiplets," $\mathbf{v}$ and $\mathbf{w}$ with new field Θ_{gh} which carries the conformal weight, the U(1) ghost charge, and the U(1) matter charge

$$\mathbf{v} = \beta + \Theta_{gh} b_g, \qquad \mathbf{w} = c_g + \Theta_{gh} \gamma.$$

If one investigates these grand multiplets a little further, one can read off that Θ_{gh} is anticommuting field with the conformal weight $\lambda - \lambda_g$, the matter U(1) charge -1, and the ghost number 1. $\mathbf{v}$ is a commuting multiplet with the conformal weight $\lambda - 1/2$, the U(1) matter charge -1, and the ghost number 0, whereas $\mathbf{w}$ is an anticommuting multiplet with the conformal weight $1 - \lambda$, the U(1) matter charge 0, and the ghost number 1.

With these observations we can rewrite the bosonic string action in a very simple form for the holomorphic part

$$S_{\mathbf{vw}} = \int \frac{d^2z}{2\pi} d\Theta_{gh}(\mathbf{v}\bar{\partial}\mathbf{w}) = \int \frac{d^2z}{2\pi} (\beta\bar{\partial}\gamma + b_g\bar{\partial}c_g).$$

Here we did not gauge the field Θ_{gh} as indicated in the usual derivative ∂.

For the cases of $\lambda = 2$ and of $\lambda = -1$ mentioned as "topological" theories, only the $\beta\gamma$ CFT and the bc CFT are present. It will be interesting to investigate these theories further. There are only the zero modes without any oscillator excitations.

V. CONCLUSION

In this paper we investigate a new possibility of string theory with Galilean symmetry as a global symmetry. The action has the matter $\beta\gamma$ CFT in addition to the usual X CFTs and the ghost $b_g c_g$ CFT in the conformal gauge. This Galilean symmetry is realized by the combinations of simplest theories, $\beta\gamma$ CFT and X CFTs. We quantize this theory in an elementary fashion. We would like to say that this theory has the full-fledged form of a string theory and is at the same level as the perturbative relativistic string theory described by the Polyakov action. We think there are many different avenues ahead to be investigated.

Why do we consider the nonrelativistic string theories? First, the complete fundamental M-theory and string theory formulations are not available yet. Recently a nonperturbative formulation of noncritical M theory using a nonrelativistic setup has been put forward [6]. It seems to us that it is important to investigate other possibilities[6] and to construct explicit examples, which will provide insights into the issues to be solved in the relativistic string theory. In this spirit, this nonrelativistic string theory provides an example of a full bosonic string theory similar to the relativistic bosonic string theory in many aspects. In this nonrelativistic setup, it is possible to ask questions related to the nature of string theory even without the complication of gravity. Second, it is possible that diffeomorphism invariance and Lorentz invariance can emerge at a special point of the moduli space of less symmetric theories such as the nonrelativistic theories (e.g., [14,15]). Third, even though there are many string theories with broken Lorentz symmetries such as lightlike linear dilaton theory, there were not so many attempts to understand these theories with emphasis on their manifest symmetries from the starting point.[7]

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[6]Lorentz violating effects in string theory is very interesting and there are a lot of efforts to investigate them. For example, see, e.g., [13].

[7]We thank Professor Petr Hořava for pointing out this to us. This is actually one of the strong motivations for the current work.

Here is an another motivation. There is evidence which suggests that time is rather different from space in string theory. "Emergent spaces" in string theory is not hard to find in the literature, while "emergent time" poses many challenges for current understanding of quantum theory (e.g., [16]). As a specific example without the complications of gravity, we can contrast "Noncommutative open string theory" (NCOS), which is a string theory with all the massive excitations of the open string in it, and "noncommutative Yang-Mills theory" (NCYM), which is a field theory. These two theories are related to consistent low energy limits of the relativistic string theory with D-branes in the presence of NSNS B field with an electric and a magnetic component, respectively. This nonrelativistic string theory is different from the relativistic string theory by having the $\beta\gamma$ fields instead of the X^0 and X^1 fields, and it provides an example which treats time and space in a different footing.

We will conclude with a few comments and future directions. This theory seems to be very similar to the relativistic string theory in many aspects. The total degrees of freedom of this nonrelativistic string theory are the same as those of the relativistic string theory, because all the excited degrees of freedom are the same. The consistency conditions of the two dimensional conformal field theory put strong constraints on the spectrum of the nonrelativistic theory, which suppress the excitations of the $\beta\gamma$ CFT in the physical spectrum. Differences between the relativistic string and the nonrelativistic string are related to the zero modes and their interpretations. The zero modes of the $\beta\gamma$ CFT are very important for the modular invariance. By the way, the conventional $\beta\gamma$ CFT has a U(1) symmetry. On the other hand, we want to have $\beta\gamma$ zero modes in the ground state vertex operator in order to have a space and time interpretation. Thus this U(1) symmetry is broken. Related facts for the zero modes of the superconformal $\beta_g \gamma_g$ CFT were already considered in [17] long ago.

Even though we started with the $\beta\gamma$ CFT and X CFTs, we are able to identify the time of the target space as a linear combination of γ and $\bar{\gamma}$ through the explicit Galilean boost transformation. There is a parameter of selection sectors which is responsible for the nontrivial dispersion relation. While we change the variables from $\beta\gamma$ to time t and space x in target space, we encounter a peculiar fact that space x is actually hidden and only time t is visible, as explained in the main text. It will be interesting to check whether it is possible to make space x be visible by including winding modes in this theory by compactifying the coordinate x similarly to the case of Gomis and Ooguri.

There are other possibly consistent string theories with $\beta\gamma$ of different conformal weights. We have a viewpoint that these theories are the nonrelativistic analogues of noncritical string theories. And the analysis seems rather involved because it is not clear how to put $\beta\gamma$ fields explicitly in the vertex operator for these zero modes to

give the spacetime interpretations. We think that spectral flow can have a role for this analysis.

We are currently investigating the supersymmetric generalizations of this nonrelativistic string theory. For the critical case with 8 spatial coordinates, we have anticommuting bc CFT in addition to the usual $\psi^i, i = 2, \ldots, 9$ CFTs. They all have identical conformal weights $1/2$. Naively we can change fields from bc to ψ^0 and ψ^1 and we could get an $SO(9, 1)$ symmetry in the fermionic sector. But this is too naive and there is no transformation which maps from bc to the other ψ^i's. Quantization seems to be straightforward at least for the critical case. And there also exist similar noncritical supersymmetric generalizations for the noncritical nonrelativistic string theories. It turns out that there is an infinite range of possible consistent string theories for this nonrelativistic setup. We hope to report this progress in the near future. It will also be interesting to investigate the connection between these theories and supercritical string theories [18,19].

APPENDIX: SCATTERING AMPLITUDE FOR EXCITED VERTEX OPERATORS

The scattering amplitude for the 4 excited state vertex operators can be expressed as

$$\left\langle \prod_{s=1}^{n} V_{es}(z_s) \right\rangle = \int DX^i D\gamma D\bar{\gamma} D\beta D\bar{\beta} e^{-S_b}$$
$$\times \prod_{s=1}^{n} V_e(k_s^\gamma, k_s^{\bar{\gamma}}, k_s^i; z_s, \bar{z}_s), \qquad \text{(A1)}$$

where the action is again given in (2) and the vertex operator is given in (15).

It will be very convenient to evaluate this functional integration using the technique given in [12], which evaluates the exponential factors first at the minimum of each field. The minimum of each field is again given by Eq. (58). With these minimum values we can change variables from ∂x^M to q^M which will be given explicitly below. And we

write down everything in terms of these new variables. After that we have the following scattering amplitude

$$g^n \prod_{s \neq t} (z_s - z_t)^{-k_s^\gamma p_t'} (\bar{z}_s - \bar{z}_t)^{-k_s^{\bar{\gamma}} q_t'} |z_s$$

$$- z_t|^{(\alpha'/2) k_s^i k_{ti}} \left\langle \prod_{s=1}^n e_{M\bar{N}}^s [v_s^M + q_s^M][\bar{v}_s^N + \bar{q}_s^N] \right\rangle, \quad \text{(A2)}$$

where

$$q_s^M = \partial X_s^M - v_s^M, \qquad \bar{q}_s^M = \bar{\partial} X_s^M - \bar{v}_s^M, \qquad \text{(A3)}$$

$$v_s^M = \sum_{t \neq s} \frac{p_s^M}{z_s - z_t} \quad \text{with}$$

$$p_s^M = (ik_s^\gamma, -ip_s', -i(\alpha'/2)^{1/2} k_s^i), \qquad \text{(A4)}$$

$$\bar{v}_s^N = \sum_{t \neq s} \frac{\bar{p}_s^N}{\bar{z}_s - \bar{z}_t} \quad \text{with}$$

$$\bar{p}_s^M = (ik_s^{\bar{\gamma}}, -iq_s', -i(\alpha'/2)^{1/2} k_s^j). \qquad \text{(A5)}$$

We still need to evaluate the expectation value of q^M and $\bar{q}^M$. These are given by the sum over all contractions. The only nonvanishing two contractions are given by

$$\langle q_s^\gamma q_t^\beta \rangle = \langle q_s^\beta q_t^\gamma \rangle = \frac{1 - \delta_{st}}{(z_s - z_t)^2},$$

$$\langle \bar{q}_s^\gamma \bar{q}_t^\beta \rangle = \langle \bar{q}_s^\beta \bar{q}_t^\gamma \rangle = \frac{1 - \delta_{st}}{(\bar{z}_s - \bar{z}_t)^2}, \qquad \text{(A6)}$$

$$\langle q_s^i q_t^j \rangle = \langle q_s^j q_t^i \rangle = \frac{(1 - \delta_{st}) \delta^{ij}}{(z_s - z_t)^2},$$

$$\langle \bar{q}_s^i \bar{q}_t^j \rangle = \langle \bar{q}_s^j \bar{q}_t^i \rangle = \frac{(1 - \delta_{st}) \delta^{ij}}{(\bar{z}_s - \bar{z}_t)^2}. \qquad \text{(A7)}$$

Rather than evaluating this general scattering amplitude we can evaluate one specific example with polarization $e_{\beta\bar{\gamma}}^1 e_{\gamma\bar{\beta}}^2 e_{\partial X^i \bar{\partial} X^j}^3 e_{\partial X^k \bar{\partial} X^l}^4$ for the 4 excited state vertex operators as an illustration. The expectation value can be evaluated straightforwardly and we have the following expression without polarization factor which will be introduced later:

$$\frac{4}{\alpha'^2} \langle [v_1^\beta + q_1^\beta][v_2^\gamma + q_2^\gamma][v_3^{\partial X^i} + q_3^{\partial X^i}][v_4^{\partial X^k} + q_4^{\partial X^k}][\bar{v}_1^{\bar{\gamma}} + \bar{q}_1^{\bar{\gamma}}][\bar{v}_2^{\bar{\beta}} + \bar{q}_2^{\bar{\beta}}][\bar{v}_3^{\bar{\partial} X^j} + \bar{q}_3^{\bar{\partial} X^j}][\bar{v}_4^{\bar{\partial} X^l} + \bar{q}_4^{\bar{\partial} X^l}] \rangle$$

$$= \left(\left(\sum_{s \neq 1} \frac{-ip_s'}{z_1 - z_s} \right) \left(\sum_{t \neq 2} \frac{ik_t^\gamma}{z_2 - z_t} \right) + \frac{1}{(z_1 - z_2)^2} \right) \left(\left(\sum_{u \neq 3} \frac{k_u^i}{z_3 - z_u} \right) \left(\sum_{v \neq 4} \frac{k_v^k}{z_4 - z_v} \right) + \frac{\delta^{ik} \cdot 2/\alpha'}{(z_3 - z_4)^2} \right) \left(\left(\sum_{s \neq 1} \frac{ik_s^{\bar{\gamma}}}{\bar{z}_1 - \bar{z}_s} \right) \left(\sum_{t \neq 2} \frac{-iq_t'}{\bar{z}_2 - \bar{z}_t} \right) \right.$$

$$+ \frac{1}{(\bar{z}_1 - \bar{z}_2)^2} \right) \left(\left(\sum_{u \neq 3} \frac{k_u^i}{\bar{z}_3 - \bar{z}_u} \right) \left(\sum_{v \neq 4} \frac{k_v^k}{\bar{z}_4 - \bar{z}_v} \right) + \frac{\delta^{ik} \cdot 2/\alpha'}{(\bar{z}_3 - \bar{z}_4)^2} \right)$$

$$\equiv \frac{4}{\alpha'^2} A_1^{\beta\bar{\gamma}\gamma\bar{\beta}i\bar{j}k\bar{l}}.$$

Thus the scattering amplitude can be summarized as using the result given in (61)

$$\langle V_{e_{\beta\bar{\gamma}}}(z_1) V_{e_{\gamma\bar{\beta}}}(z_2) V_{e_{ij}}(z_3) V_{e_{kl}}(z_4) \rangle = g^4 (e_{\beta\bar{\gamma}}^1 e_{\gamma\bar{\beta}}^2 e_{\partial X^i \bar{\partial} X^j}^3 e_{\partial X^k \bar{\partial} X^l}^4 \cdot A_1^{\beta\bar{\gamma}\gamma\bar{\beta}i\bar{j}k\bar{l}}) \prod_{s < t} |z_s - z_t|^{-2k_s^\gamma p_t' - 2k_t^\gamma p_s' + \alpha' k_s^i k_{ti}}. \quad \text{(A8)}$$

The scattering amplitudes with both ground state vertex operators and first excited vertex operators can be evaluated in a straightforward manner with the procedure given in this Appendix.

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