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Supersymmetric field theory in two-time physics

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We construct N = 1 supersymmetry in 4 + 2 dimensions compatible with the theoretical framework of 2T physics field theory and its gauge symmetries. The fields are arranged into 4 + 2-dimensional chiral and vector supermultiplets, and their interactions are uniquely fixed by supersymmetry (SUSY) and 2T-physics gauge symmetries. Many 3 + 1 spacetimes emerge from 4 + 2 by gauge fixing. Gauge degrees of freedom are eliminated as one comes down from 4 + 2 to 3 + 1 dimensions without any remnants of Kaluza-Klein modes. In a special gauge, the remaining physical degrees of freedom, and their interactions, coincide with ordinary N = 1 supersymmetric field theory in 3 + 1 dimensions. In this gauge, SUSY in 4 + 2 is interpreted as superconformal symmetry SU(2, 2|1) in 3 + 1 dimensions. Furthermore, the underlying 4 + 2 structure imposes some interesting restrictions on the emergent 3 + 1 SUSY field theory, which could be considered as part of the predictions of 2T-physics. One of these is the absence of the troublesome renormalizable *CP* violating $F \star F$ terms. This is good for curing the strong *CP* violation problem of QCD. An additional feature is that the superpotential is required to have no dimensionful parameters. To induce phase transitions, such as SUSY or electroweak symmetry breaking, a coupling to the dilaton is needed. This suggests a common origin of phase transitions that is driven by the vacuum value of the dilaton and needs to be understood in a cosmological scenario as part of a unified theory that includes the coupling of supergravity to matter. Another interesting aspect of the proposed theory is the possibility to utilize the inherent 2T gauge symmetry to explore dual versions of the N = 1 theory in 3 + 1 dimensions, such as the minimal supersymmetric standard model (MSSM) and its duals. This is expected to reveal nonperturbative aspects of ordinary 1T field theory.

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I. 2T-PHYSICS FIELD THEORY

Two-time physics (2T physics) [1–15] is a unification approach for usual one time physics (1T physics) through higher dimensions that includes one extra timelike and one extra spacelike dimension. This unification is distinctly different from Kaluza-Klein theory because in the reduction from $d + 2$ to $(d - 1) + 1$ dimensions there are no Kaluza-Klein towers of states. Instead, in the end result of the reduction one finds a variety of $(d - 1) + 1$ emerging spacetimes embedded in the same $d + 2$ spacetime, resulting in a family of 1T-physics systems in $(d - 1) + 1$ dimensions, with different dynamics from each other (i.e. different Hamiltonians), obeying duality-type relationships among themselves.

Furthermore, each 1T system in the family is a holographic image of the *same parent system* in $d + 2$ dimensions and has hidden symmetries that reflect the global symmetries of the parent theory. These hidden symmetries, and the dualities, are reflections of the hidden extra dimensions. Such properties of 2T physics are summarized with some examples in Fig. 1 of Ref. [13].

The essential ingredient underlying 2T physics is the basic gauge symmetry $Sp(2, R)$ acting on phase space X^M, P_M [1]. The role of $Sp(2, R)$ is most easily explained in the worldline description of particles. In that context it is a generalization of the 1-parameter gauge symmetry of worldline τ reparametrization to a 3-parameter non-Abelian $Sp(2, R)$ gauge symmetry acting on phase space. This gauge symmetry *requires* the particle to live in a *target* spacetime with two timelike directions, so the 2T feature is an outcome of the gauge symmetry rather than being an input by hand.

The extra 2 parameters in the gauge symmetry are able to remove 2 degrees of freedom from target spacetime in many possible ways. Through such gauge fixing one can then find many possible embeddings of *phase space* in $(d - 1) + 1$ dimensions into $d + 2$ dimensions. So, a given $d + 2$-dimensional 2T theory descends, through $Sp(2, R)$ gauge fixing, down to a family of holographic 1T images in $(d - 1) + 1$ dimensions. All images are gauge equivalent (or dual) to each other, while each one is also gauge equivalent to the same parent 2T theory in $d + 2$ dimensions. However, the various images have differing 1T-physics interpretations because of the different definitions of "time" and "Hamiltonian" inherent in the *phase space* embeddings of $(d - 1) + 1$ in $d + 2$. The rich web of dualities among the emerging 1T-physics systems is the surprising unifying power of the 2T-physics approach.

2T physics includes all cases of particles moving in all possible background fields [3]. It also describes particles with spin [2] or with target space supersymmetry, by appropriate generalizations of $Sp(2, R)$ [4,5,7]. In all such cases one finds unified families of 1T-physics systems that emerge from a unifying parent theory directly defined in $d + 2$ dimensions. So, 2T physics appears to be sufficiently general to be able to accommodate all 1T-physics systems as members of families of holographic images, with each family representing some higher system with 1 + 1 extra dimensions.

The discussion in the present paper is at the level of 2T field theory. To make the paper self-contained we have included Appendix A to give some details for how one goes from Sp$(2, R)$ gauge symmetry on the worldline to 2T field theory with its own 2T gauge symmetries.

Recently, a field theoretic description of 2T physics has been established and applied to the standard model of particles and forces [13]. In the field theoretic $4+2$ standard model (SM), the type of phenomena such as hidden symmetries, duality, holography, and emergent spacetimes are also present owing to a newly discovered 2T gauge symmetry in field theory which is actually a consequence of the gauge symmetry Sp$(2, R)$ on the worldline [12,13]. This is briefly explained in the case of scalar field in Appendix A along with brief comments on the quantization of the field theory. For the time being only one of the field theoretic images, namely, the "massless relativistic particle" gauge (see footnote 8) noted as the first item in Table I in Appendix A, has been studied in the field theory context. This $3+1$ holographic image of the $4+2$ SM coincides with the well-known $3+1$ SM and improves some of its properties as discussed in detail in [13].

In particular, some attractive features of the emergent SM include a new solution of the long-standing strong *CP* problem in QCD without an axion and novel ideas on the origins of mass as briefly reiterated below. These features emerge from the underlying $4+2$ structure which imposes some interesting restrictions on the emergent $3+1$ standard model. Such properties of field theory in $3+1$ dimensions [more generally $(d-1)+1$] could be considered as part of the predictions of 2T physics.

The goal of the present paper is to formulate the general supersymmetric version of 2T-physics field theory in $4+2$ dimensions, for fields of spins $0, \frac{1}{2}, 1$ with $N=1$ supersymmetry (SUSY). This will be a starting point for physical applications in the form of the supersymmetric version of the SM in $4+2$ dimensions, as well as for generalizations to higher $N = 2, 4, 8$, supersymmetric 2T-physics field theory, which will be presented in future papers. A summary of our results for $N=1$ SUSY has appeared as a short letter [15].

In the following, we briefly summarize the essential features of 2T field theory which will be the structure on which we will impose $N=1$ supersymmetry in the coming sections.

The field theory in $4+2$ dimensions with fields of spins $0, \frac{1}{2}, 1$ describes a set of SO(4,2) vectors $A_M^a(X)$ labeled with $M = $ SO$(4,2)$ vector, and $a =$ the adjoint representation of a Yang-Mills gauge group G [for example, $G = SU(3)\times SU(2)\times U(1)$ for the standard model, $G = SO(10)$ for grand unification]; scalars $H_i(X)$, labeled by an internal symmetry index $i = 1, 2, \cdots$ (a collection of irreducible representations of G); left- or right-handed spinors $\psi^I_{L\alpha}(X), \psi^{\tilde I}_{R\dot\alpha}(X)$ in the 4, 4^* representations of $SU(2,2) = SO(4,2)$, labeled with $\alpha = 1,2,3,4$, and $\dot\alpha = 1,2,3,4$, and internal symmetry indices $I = 1, 2, \cdots$ and $\tilde I = 1, 2, \cdots$ (again, a collection of irreducible representations of G).

The generic 2T-physics Lagrangian has the form of a Yang-Mills theory in $4+2$ dimensions (G-covariant derivatives). But it contains $4+2$ spacetime features shown explicitly in the Lagrangian below, which are needed to impose the underlying Sp$(2, R)$ gauge symmetry and the related 2T-physics gauge symmetries. For a summary of how these concepts are derived from Sp$(2, R)$ the reader can consult Appendix A. The origin of the 2T-physics gauge symmetry in field theory is more fully explained in [12,13].

We emphasize the basic important fact that the equations of motion that follow from the Lagrangian below impose the Sp$(2, R)$ gauge singlet conditions $X^2 = X\cdot P = P^2 = 0$ [or OSp$(n|2)$ gauge singlet conditions for a field with spin $n/2$], but now including also interactions [13]. The field theory Lagrangian with these properties has the general form (see Appendix A)

$$L = \Big[\delta(X^2)\{-D_M H^{i\dagger} D^M H_i\} + 2\delta'(X^2) H^{i\dagger} H_i + \delta(X^2)\Big\{\frac{i}{2}(\bar\psi^I_L X \bar D \psi_{IL} + \bar\psi^I_L \overleftarrow{D}\bar X \psi_{IL}) - \frac{i}{2}(\bar\psi^{\tilde I}_R \bar X D \psi_{\tilde I R} + \bar\psi^{\tilde I}_R \overleftarrow{\bar D} X \psi_{\tilde I R})\Big\} + \delta(X^2)\{y_I^{i\tilde I}\bar\psi^I_L X \psi_{\tilde I R} H_i + (y_I^{i\tilde I})^* H^{*i}\bar\psi^{\tilde I}_R \bar X \psi_{IL}\} + \delta(X^2)\Big\{-\frac{1}{4}F^a_{MN}F_a^{MN} - V(H, H^*, \Phi)\Big\} - \frac{1}{2}\delta(X^2)\partial_M\Phi\partial^M\Phi + \delta'(X^2)\Phi^2\Big]. \tag{1.1}$$

The left arrow on $\overleftarrow{D}_M$ means that the covariant derivative acts on the field on its left $\bar\psi_L \overleftarrow{D}_M \equiv D_M \bar\psi_L$. The distinctive spacetime features in $4+2$ dimensions include the delta function $\delta(X^2)$ and its derivative $\delta'(X^2)$ that impose $X^2 = X^M X_M = 0$ (see footnote 6), the kinetic terms of fermions that include the factors $X\bar D$, $\bar X D$, and Yukawa couplings proportional to $y_I^{i\tilde I}$, $y_I^{i\tilde I}$ that include the factors X or $\bar X$, where $X \equiv \Gamma^M X_M$, $\bar D = \bar\Gamma^M D_M$ etc., with 4×4 gamma matrices Γ^M, $\bar\Gamma^M$ in the 4, 4^* spinor bases of $SU(2,2) = SO(4,2)$. Our notation for gamma matrices for $SO(4,2) = SU(2,2)$ is given in Appendix B.

This Lagrangian is not invariant under translation of X^M but is invariant under the spacetime rotations SO(4,2). In fact, it has precisely the right spacetime, and gauge invariance, properties for the $4+2$ field theory to yield the usual $3+1$ field theory. The reduction from $4+2$ dimensions X^M to $3+1$ dimensions x^μ is obtained via gauge fixing (see footnote 8). The emergent $3+1$ field theory is invariant under translations of x^μ and Lorentz transformations SO(3,1). These Poincaré symmetries are included in SO(4,2) that takes the nonlinear form of conformal transformations in the emergent $3+1$-dimensional spacetime x^μ. The emergent $3+1$ theory contains just the right fields as functions of x^μ: all extra degrees of freedom disappear without leaving behind any Kaluza-Klein type modes or extra components of the vector and spinor fields in the extra $1+1$ dimensions. Furthermore, the emergent field theory has the usual kinetic terms and Yukawa couplings in $3+1$-dimensional Minkowski space [13].

As in the last line of the Lagrangian, one may also include an additional SO(4,2) scalar, the dilaton $\Phi(X)$, classified as a singlet under the group G. The dilaton is not optional if the action is written in $d+2$ dimensions (see [13]), as it appears in overall factors $\Phi^{2(d-4)/d-2}$, $\Phi^{d-4/d-2}$ multiplying the Yang-Mills kinetic term and Yukawa terms, respectively, in order to achieve the 2T-gauge symmetry of the action. In $4+2$ dimensions ($d=4$) these factors reduce to 1, but the dilaton can still couple to the scalars H in the potential $V(H, H^*, \Phi)$.

The 2T-physics field theory above is applied to construct the standard model in $4+2$ dimensions by choosing the gauge group $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ and including the usual matter representations for the Higgs bosons, quarks, and leptons (including right-handed neutrinos in singlets of G), but now as fields in $4+2$ dimensions. As explained in [13] this theory descends to the usual standard model in $3+1$ dimensions.

When we apply the $4+2$ approach to construct the standard model, almost all of the usual terms of the $3+1$-dimensional standard model emerge from the $4+2$ field theory above, except for two notable exceptions that play an important physical role. Namely,

(i) There is no way to generate a *renormalizable* term in the emergent $3+1$ theory that is analogous to the P and CP-violating term $\theta F_{\mu\nu} F_{\lambda\sigma} \varepsilon^{\mu\nu\lambda\sigma}$ that is possible in a purely 1T-physics approach in $3+1$ dimensions.[1] The absence of θ in the emergent standard model is due to the fact that the Levi-Civita symbol in $4+2$ dimensions has 6 indices rather than 4, and also due to the combination of 2T gauge symmetry as well as Yang-Mills gauge symmetry. The absence of this CP-violating term in 2T physics is of crucial importance in the axionless resolution of the strong CP violation problem of QCD [13].

(ii) The 2T-gauge symmetry requires the potential $V(H, H^*, \Phi)$ to be purely quartic, i.e. no mass terms are permitted. Then the emergent $3+1$ theory cannot have mass terms for the scalars and is automatically invariant under scale transformations. This makes the mass generation with the Higgs mechanism less straightforward since the tachyonic mass term is not allowed. However, by taking the Higgs potential of the form $V(\Phi, H) = \frac{\lambda}{4}(H^\dagger H - \alpha^2 \Phi^2)^2$ we obtain the breaking of the electroweak symmetry by the Higgs doublet $\langle H \rangle$ driven by the vacuum expectation value of the dilaton $\langle \Phi \rangle$, thus relating the two phase transitions to each other. In this way the $4+2$ formulation of the standard model provides an appealing deeper physical basis for mass.[2]

Having established that 2T physics field theory introduces new phenomenologically relevant constraints, it would be of great interest to find out whether the SUSY version is also constrained in phenomenologically significant ways. This is especially relevant in view of the upcoming experimental activities at the LHC starting in 2008. It would be interesting to formulate experimental signatures that could distinguish 2T-physics versions of SUSY from others, due to some extra constraints rooted in the structures of $4+2$ dimensions. The first step towards this goal is the formulation of SUSY in 2T-physics field theory which we present in this paper. We will establish the transformation rules for 2T-physics $N=1$ SUSY in $4+2$ dimensions, which are different than a straightforward higher-dimensional SUSY, and will build the general SUSY Lagrangian for fields with spins $0, \frac{1}{2}, 1$, with any Yang-Mills gauge group G, and with any representations.

The plan of the paper is as follows. Section II gives a quick outline of the paper for the reader who is interested in seeing the results without the technical details. So, in Sec. II we give a summary of our results for the general

[1] Actually there appears as if there would be a topological term of the form $\int d^6 X \varepsilon^{M_1 M_2 M_3 M_4 M_5 M_6} Tr(F_{M_1 M_2} F_{M_3 M_4} F_{M_5 M_6})$ whose density is a total divergence for any Yang-Mills gauge group G. Such a term could descend to $3+1$ dimensions $\theta F_{\mu\nu} F_{\lambda\sigma} \varepsilon^{\mu\nu\lambda\sigma}$ with an effective $\theta \sim F^{+'-'}$. However, it can be shown that this θ is 2T gauge dependent and is gauge fixed to zero in the process of descending from $4+2$ to $3+1$ dimensions. This and other possible sources of the θ term are discussed and eliminated in [13]. In this sense, the 2T gauge symmetry plays a similar role to the Peccei-Quinn symmetry in eliminating the topological term. But one must realize that the 2T gauge symmetry is introduced for other more fundamental reasons and also it is not a global symmetry. Hence, unlike the Peccei-Quinn symmetry it does not lead to an axion.

[2] As argued in [13], the dilaton-driven electroweak phase transition makes a lot more sense conceptually than the usual approach in which the electroweak phase transition is an isolated phenomenon. This is because the Higgs vacuum expectation value fills all space everywhere in the Universe. This is a hard concept to swallow without relating it to the evolution of the Universe, which then requires the participation of gravity. In the 2T-physics version of the SM, the Higgs $\langle H \rangle$ has to be driven by the dilaton $\langle \Phi \rangle$ which is a member of the gravity multiplet, so an essential part of the physics of the standard model becomes intimately related to the physics of gravity and all of its other consequences. In particular a relation is established to other phase transitions that are expected to be dilaton driven in the evolution of the Universe, such as the vacuum selection process in string theory, and perhaps even to inflation that is driven by a scalar field which could be the dilaton.

$N = 1$ supersymmetric action for a coupled system of spin $0, \frac{1}{2}, 1$ fields. We give the SUSY transformation laws that have many new features and derive the conserved SUSY current for the fully interacting system. These fields are arranged into $N = 1$ chiral and vector multiplets of SUSY in $4 + 2$ dimensions, consistent with the gauge symmetries of 2T physics, and with the gauge symmetries of a Yang-Mills group. In Sec. III we deal with the chiral multiplet by itself, discuss the SUSY symmetry in detail, and derive the conserved SUSY current. In Sec. IV we discuss the vector multiplet by itself in detail. In Sec. V we couple chiral multiplets with vector multiplets and find the unique action, supersymmetry transformation, and conserved current, justifying the outline in Sec. II. Finally, in Sec. VI we conclude with some comments and point out future directions.

In Appendix A, we briefly explain how a field theoretical formulation arises from the underlying worldline description of 2T physics and Sp(2,R) gauge symmetry on the worldline. In particualr, how the 2T gauge symmetry which is essential to restrict the form of 2T field theory action is obtained and how one does gauge fixing to obtain usual 1T field theory. In Appendix B we provide technical details on gamma matrices for SO(4,2). In Appendix C we derive some Fierz identities that are used in the proof of SUSY including interactions. In Appendix D we discuss the closure of the SUSY algebra into the supergroup $SU(2,2|1)$ when the fields are on shell, and into a larger algebra when the fields are off shell.

II. N = 1 SUSY IN 2T PHYSICS AND SUMMARY OF RESULTS

In this section we will provide a summary of our results. In the following sections, we will show how each piece in the action and the SUSY transformations arise step by step.

To some extent the well-known $3 + 1$ SUSY structures are a guide toward the $4 + 2$ SUSY structures, since after all the $3 + 1$ chiral supermultiplet and vector supermultiplet should emerge as the end result of the 2T-physics gauge fixing. Therefore, the spin $0, \frac{1}{2}, 1$ fields are members of the chiral and vector supermultiplets in the $4 + 2$ 2T-physics SUSY theory.

The chiral supermultiplet $(\varphi, \psi_L, F)_i$ in $4 + 2$ dimensions contains a set of SO(4,2) scalars $\varphi_i(X)$, left-handed spinors $\psi_{iL\alpha}(X)$ in the 4 representation of $SU(2,2) = SO(4,2)$, labeled with $\alpha = 1, 2, 3, 4$, and auxiliary complex scalar fields $F_i(X)$, all labeled by an internal symmetry index $i = 1, 2, 3, \cdots$ of a gauge symmetry group G. The internal symmetry index i is used here generically to denote any collection of *several* irreducible representations of G.

The vector supermultiplet $(A_M, \lambda_L, B)^a$ contains fields that carry SO(4,2) spacetime indices required by their spin. Thus the spin-1 $A_M^a(X)$ is the Yang-Mills gauge field, the spin-$\frac{1}{2}$ $\lambda_{\alpha L}^a(X)$ is the gaugino, and the spin-0 $B^a(X)$ is the auxiliary field.[3] They are all labeled by a which belongs to the adjoint representation of the gauge symmetry group G.

The SUSY transformations of the chiral and vector multiplets that leave the action invariant will be discussed below after we make some remarks about the significance of various terms in the action (2.3). To simplify our notation we will suppress the group G indices i, a in parts of the discussion in this paper and make it explicit when it is necessary for clarity.

A. Lagrangian

In what follows, we use mostly left-handed spinors but also find it convenient at times to use right-handed spinors as the charge conjugates of left-handed ones. The left-handed spinor $\psi_{L\alpha}(X)$, in the 4 representation of SU(2,2), is labeled with $\alpha = 1, 2, 3, 4$ while the right-handed spinor $\psi_{R\dot{\alpha}}(X)$, in the 4 representation of SU(2,2), is labeled with $\alpha = 1, 2, 3, 4$. One may also construct an 8-component spinor of SO(4,2) with a Majorana condition such that ψ_L together with ψ_R make up the 8 components of

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

and because of the Majorana condition, ψ_R and ψ_L are related to each other. One could rewrite all right-handed spinors as left-handed ones by charge conjugation which is given by

$$\psi_R \equiv C\bar{\psi}_L^T = C\eta^T(\psi_L)^*, \quad \text{or} \quad \bar{\psi}_L = -(\psi_R)^T C. \tag{2.1}$$

Using these definitions we can also write the following relations that are equivalent to Eq. (2.1):

$$\psi_L = -C\bar{\psi}_R^T, \quad \text{or} \quad \bar{\psi}_R = (\psi_L)^T C. \tag{2.2}$$

Our SO(4,2) gamma matrix notation in the Weyl basis, which includes explicit forms of the antisymmetric charge conjugation matrix $C = \tau_1 \times \sigma_2$, and the symmetric SU(2,2) metric $\eta = -i\tau_1 \times 1$ used to construct the contravariant $\bar{\psi}_L^\beta = ((\psi_L)^\dagger \eta)^\beta = (\psi_L^*)_{\dot{\alpha}} \eta^{\dot{\alpha}\beta}$, are explained in detail in Appendix B.

To satisfy the gauge symmetries of 2T physics discussed in [13], each one of the spin $0, \frac{1}{2}, 1$ fields can occur only in the form of the Lagrangian of Eq. (1.1). On this structure we now impose SUSY whose details are described in the following sections. It turns out that the general theory of the $N = 1$ chiral multiplet coupled to the $N = 1$ vector multiplet gets organized as follows:

$$L = L_{\text{chiral}} + L_{\text{vector}} + L_{\text{int}} + L_{\text{dilaton}}. \tag{2.3}$$

The vector multiplet $(A_M, \lambda_L, B)^a$ with its self-interactions

[3]The auxiliary field is usually called the D-term in $3 + 1$ SUSY, but we use here the letter B to avoid confusion with the symbol for covariant derivative D.

is described by

$$L_{\text{vector}} = \delta(X^2)\Big\{-\frac{1}{4}F^a_{MN}F^{MN}_a + \frac{i}{2}[\bar{\lambda}^a_L X\bar{D}\lambda_{aL} + \bar{\lambda}^a_L \overleftarrow{D}\bar{X}\lambda_{aL}] + \frac{1}{2}B^a B_a\Big\}. \quad (2.4)$$

The chiral multiplet $(\varphi, \psi_L, F)_i$, with its self-interactions are described by

$$\begin{aligned} L_{\text{chiral}} = \delta(X^2)\Big\{&-D_M\varphi^{i\dagger}D^M\varphi_i \\ &+ \frac{i}{2}(\bar{\psi}^i_L X\bar{D}\psi_{iL} + \bar{\psi}^i_L\overleftarrow{D}\bar{X}\psi_{iL}) + F^{\dagger i}F_i \\ &+ \Big[\frac{\partial W}{\partial\varphi_i}F_i - \frac{i}{2}\psi_{iL}(C\bar{X})\psi_{jL}\frac{\partial^2 W}{\partial\varphi_i\partial\varphi_j}\Big] + \text{H.c.}\Big\} \\ &+ 2\delta'(X^2)\varphi^{i\dagger}\varphi_i. \end{aligned} \quad (2.5)$$

Some of the interactions of the chiral multiplet with the gauge multiplet already appear through the gauge covariant derivatives $D^M\varphi_i$ and $D^M\psi_{iL}$. Additional interactions of the vector and chiral multiplets occur also through the auxiliary fields B^a and the gaugino λ^a_L as follows:

$$L_{\text{int}} = \delta(X^2)\{\alpha\varphi^{\dagger i}(t_a)^j_i\varphi_j B^a + \beta\varphi^{\dagger i}(t_a)^j_i(\psi_{jL})^T(C\bar{X})\lambda^a_L\} + \text{H.c.}, \quad (2.6)$$

where α, β will be uniquely determined by SUSY. Finally, a sketchy description of the dilaton is given by

$$\begin{aligned} L_{\text{dilaton}} = \{&-\tfrac{1}{2}\delta(X^2)\partial_M\Phi\partial^M\Phi + \delta'(X^2)\Phi^2 \\ &+ \text{superpartners of } \Phi \\ &+ \delta(X^2)\{\xi_a B^a\Phi^2 + V(\Phi, \varphi)\}. \end{aligned} \quad (2.7)$$

We note the following points on the structure of the Lagrangian:

(1) The $W(\varphi)$ in L_{chiral} is the holomorphic superpotential consisting of any combination of G-invariant *cubic* polynomials constructed from the φ_i (and excludes the $\varphi^{i\dagger}$)

$$W(\varphi) = y^{ijk}\varphi_i\varphi_j\varphi_k, \quad y^{ijk} = \text{constants compatible with } G \text{ symmetry}. \quad (2.8)$$

The purely cubic form of $W(\varphi)$ leads to a purely quartic potential energy for the scalars after the auxiliary fields F_i and B^a are eliminated through their equations of motion. A purely quartic potential is required by the 2T gauge symmetry even without SUSY.

(2) The $\bar{X}$ in the Yukawa couplings $(\psi_{iL})^T \times (C\bar{X})\psi_{jL}\frac{\partial^2 W}{\partial\varphi_i\partial\varphi_j}$ or $\beta(\varphi^\dagger t^a\psi_L)^T(C\bar{X})\lambda_{aL}$ is consistent with the $SU(2,2) = SO(4,2)$ group theory property $(4\times 4)_{\text{antisymmetric}} = 6$, namely, two left-handed fermions must be coupled to the vector X^M to give an SO(4,2) invariant. The $\bar{X}$ insertion is also required for the 2T-gauge invariance of the Yukawa couplings, as discussed in [13].

(3) SUSY requires that the dimensionless constants α, β are all determined in terms of the gauge coupling constants g for each subgroup in G as follows[4]:

$$\alpha = g, \qquad \beta = \sqrt{2}g. \quad (2.9)$$

The only parameters that are not fixed by the symmetries are the Yang-Mills coupling constants g, and the Yukawa couplings y^{ijk} which are restricted by invariance under G-symmetry, namely,

$$\frac{\partial W}{\partial\varphi_i}(t_a\varphi)_i = 0. \quad (2.10)$$

(4) As in the nonsupersymmetric case discussed in the previous section, in the SUSY 2T-physics theory there is no way to write down a term in $4+2$ dimensions that will reduce to the CP-violating term $\theta F_{\mu\nu}F_{\lambda\sigma}\varepsilon^{\mu\nu\lambda\sigma}$ that is possible in $3+1$ dimensions in the context of purely 1T physics. The absence of this CP-violating term is of crucial importance in the axionless resolution of the strong CP violation problem of QCD suggested in [13], and which generalizes to the supersymmetric case in this paper

(5) Now we turn to the dilaton term L_{dilaton}. As mentioned above, the superpotential $W(\varphi)$ is restricted by supersymmetry to be purely cubic in φ. So for driving the spontaneous breakdown of the G symmetry the same way as in the nonsupersymmetric case (as in footnote 2), as well as for inducing soft supersymmetry breaking through the Fayet-Illiopoulos type of term $\xi_a\Phi^2B^a$, it would be desirable to couple the dilaton Φ to the chiral and vector multiplets by having interactions of the form $V(\Phi, \varphi)$ and $\xi_a \neq 0$ for U(1) gauge subgroups. However, we have not yet included the superpartners of the dilaton because this is still under development in the 2T-physics context, so we are not yet

[4]There is a separate gauge coupling g for each subgroup in G, so there are separate α, β proportional to the g for each such subgroup.

in a position to discuss the SUSY constraints on the desired couplings. So in this paper we will not be able to comment in detail on the dilaton-driven electroweak or SUSY phase transition. However, we point out that in agreement with footnote 2 this is again a consistent message from 2T physics, namely, that the physics of the standard model, in particular, the electroweak phase transition that generates mass, is not decoupled from the physics of the gravitational interactions in a complete unified theory of all the forces. The full theory may be attained by further pursuing these hints provided by the 2T-physics formulation of the standard model.

B. SUSY transformations

We now summarize the properties of the SUSY transformations for the chiral and vector multiplets that leave invariant the action $S=\int d^6xL$ based on the above Lagrangian. The supersymmetry transformation for the chiral multiplet is [in the following $\varepsilon_R\equiv C\bar{\varepsilon}_L^T$ and $\bar{\varepsilon}_R=(\varepsilon_L)^TC$, and similarly for λ_R or ψ_R, as in Eqs. (2.1) and (2.2)]

$$\delta_\varepsilon\varphi_i=\Big\{\bar{\varepsilon}_R\bar{X}\psi_{iL}+X^2\Big[-\frac{1}{2}\bar{\varepsilon}_R\bar{D}\psi_{iL}+\frac{1}{2}\frac{\partial^2W^*}{\partial\varphi^{\dagger i}\partial\varphi^{\dagger j}}\bar{\psi}_L^j\varepsilon_L-\frac{ig}{2\sqrt{2}}(\bar{\varepsilon}_L\lambda_L^a+\bar{\lambda}_L^a\varepsilon_L)(t_a\varphi)_i\Big]\Big\},\tag{2.11}$$

$$\delta_\varepsilon\psi_{iL}=i(D_M\varphi_i)(\Gamma^M\varepsilon_R)-iF_i\varepsilon_L,\tag{2.12}$$

$$\delta_\varepsilon\bar{\psi}_L^j=i\bar{\varepsilon}_R\bar{\Gamma}^M(D_M\varphi)^{\dagger i}+i\bar{\varepsilon}_LF^{\dagger i},\tag{2.13}$$

$$\delta_\varepsilon F_i=\bar{\varepsilon}_L[X\bar{D}-(X\cdot D+2)]\psi_{iL}-i\sqrt{2}g(\bar{\varepsilon}_LX\lambda_R^a)(t_a\varphi)_i.\tag{2.14}$$

The supersymmetry transformation for the vector multiplet is

$$\delta_\varepsilon A_M^a=\Big\{-\frac{1}{\sqrt{2}}\bar{\varepsilon}_L\Gamma_M\bar{X}\lambda_L^a+X^2\Big[\frac{1}{2\sqrt{2}}\bar{\varepsilon}_L\Gamma_{MN}(D^N\lambda_L^a)-\frac{ig}{4}(\bar{\varepsilon}_L\Gamma_M\psi_R^i)(t^a\varphi)_i\Big]\Big\}+\text{H.c.},\tag{2.15}$$

$$\delta_\varepsilon\lambda_L^a=i\frac{1}{2\sqrt{2}}F_{MN}^a(\Gamma^{MN}\varepsilon_L)-\frac{1}{\sqrt{2}}B^a\varepsilon_L,\tag{2.16}$$

$$\delta_\varepsilon\bar{\lambda}_L^a=i\frac{1}{2\sqrt{2}}(\bar{\varepsilon}_L\Gamma^{MN})F_{MN}^a-\frac{1}{\sqrt{2}}\bar{\varepsilon}_LB,\tag{2.17}$$

$$\delta_\varepsilon B^a=\frac{i}{\sqrt{2}}\bar{\varepsilon}_L[X\bar{D}-(X\cdot D+2)]\lambda_L^a+\text{H.c.}\tag{2.18}$$

These SUSY transformations have some parallels to naive SUSY transformations that one may attempt to write down as a direct generalization from 3 + 1 to 4 + 2 dimensions. However, there are many features that are completely different.[5] These include the insertions that involve $X=X^M\Gamma_M$ or $\bar{X}=X^M\bar{\Gamma}_M$, the terms proportional to X^2, and the terms proportional to derivative terms involving $(X\cdot D+2)$. These are *off shell* SUSY transformations that include interactions and leave invariant the off shell action. The free field limit of our transformations (i.e. $W=0$ and $g=0$) taken on shell [i.e. terms proportional to X^2 and $(X\cdot D+2)$ set to zero] agrees with previous work which was considered for on shell free fields without an action principle [16].

Despite all of the changes compared to naive SUSY, this SUSY symmetry provides a representation of the supergroup SU(2, 2|1). This is signaled by the fact that all terms are covariant under the bosonic subgroup SU(2,2), while the complex fermionic parameter ε_L and its conjugate $\bar{\varepsilon}_L$ are in the 4, 4* representations of SU(2,2), as would be expected for SU(2, 2|1).

The closure of these SUSY transformations is discussed in Appendix D in the case of the pure chiral multiplet (i.e. gauge coupling $g=0$). The commutator of two SUSY transformations closes to the bosonic part $SU(2,2)\times U(1)\subset SU(2,2|1)$ when the fields are on shell. More generally, when the fields are off shell the closure includes also a U(1) outside of SU(2, 2|1) and a 2T-physics gauge transformation, both of which are also gauge symmetries of the action.

When reduced to 3 + 1 dimensions by choosing a gauge as prescribed in footnote 8, the SU(2, 2|1) transformations give nonlinear off shell realization of superconformal symmetry in 3 + 1 dimensions.

C. Conserved supercurrent

The Lagrangian in Eq. (2.3) transforms into a total divergence under the SUSY transformations (in the absence of the dilaton). Applying Noether's theorem we

[5] Once we notice the parallels, part of the structure can be understood from SU(2,2) group theory. For example, consider the gamma matrix structures $\bar{X}$, etc. sandwiched between fermions, which are absent in 3 + 1 dimensions. $\bar{\varepsilon}_R\bar{X}\psi_{iL}$ is a SU(2,2) scalar since $\bar{\varepsilon}_R$ and ψ_{iL} are both in the 4 representation of SU(2,2), and the product $4\times4=6+10$ shows that when we couple the 6 to the $SO(4,2)=SU(2,2)$ vector X^M through the gamma matrices, we obtain a scalar.

compute the conserved SUSY current. The details are shown step by step in Secs. III, IV, and V. The result is

$$\begin{aligned} J_L^M = \delta(X^2)\Big\{ & D_K(X_N\varphi_i)(\Gamma^{KN}\Gamma^M - \eta^{MN}\Gamma^K)\psi_R^i \\ & + \frac{\partial W}{\partial \varphi_j} X_N \Gamma^{MN}\psi_{iL} \\ & + \frac{1}{2\sqrt{2}} F_{KL}^a X_N(\Gamma^{KLN}\bar{\Gamma}^M - \eta^{NM}\Gamma^{KL})\lambda_{La} \\ & + \frac{ig}{\sqrt{2}}\varphi_i(t_a\varphi^\dagger)^i X_N\Gamma^{MN}\lambda_{La}\Big\}, \end{aligned} \tag{2.19}$$

where the first line comes from L_{chiral}, the second from L_{vector}, and the third from L_{int}. The Hermitian conjugate of J_L^M can be written as the right-handed counterpart of the above $J_R^M = C(\bar{J}_L^M)^T$ (see Appendix B for Hermitian and charge conjugation properties)

$$\begin{aligned} J_R^M = \delta(X^2)\Big\{ & D_K(X_N\varphi^{\dagger i})(\bar{\Gamma}^{KN}\bar{\Gamma}^M - \eta^{MN}\bar{\Gamma}^K)\psi_{iL} \\ & + \frac{\partial W^*}{\partial \varphi^{*j}} X_N \bar{\Gamma}^{MN}\psi_R^j \\ & + \frac{1}{2\sqrt{2}} F_{KL}^a X_N(\bar{\Gamma}^{KLN}\Gamma^M - \eta^{NM}\bar{\Gamma}^{KL})\lambda_{Ra} \\ & - \frac{ig}{\sqrt{2}}\varphi^{\dagger i}(t_a\varphi)_i X_N\bar{\Gamma}^{MN}\lambda_{Ra}\Big\}. \end{aligned} \tag{2.20}$$

Using the equations of motion that follow from the action (2.3) we can verify that this SUSY current is conserved

$$\partial_M J_L^M(X) = \partial_M J_R^M(X) = 0. \tag{2.21}$$

The conservation of the current amounts also to a proof of SUSY for the theory of Eq. (2.3) that supplies the equations of motion.

In the rest of the paper we provide the details of the theory summarized above.

III. CHIRAL SUPERMULTIPLET IN 2T PHYSICS

The chiral multiplet $(\varphi, \psi_L, F)_i$ is defined in terms of left-handed spinors. As noted in Eqs. (2.1) and (2.2), right-handed spinors ψ_R are treated as the charge conjugates of left-handed spinors. Hence, for each i, there are only 4 independent complex fermionic components $(\psi_{L\alpha})_i$. If one would like to introduce right-handed independent fermions ψ_R, one may do so by introducing more $(\psi_L)_i$ with different values of i since these are equivalent to the ψ_R under charge conjugation. It is evident that the formalism in the form $(\varphi, \psi_L, F)_i$, with a range of values for i, includes all possible chiral supermultiplets (left or right) that may be needed in various applications.

A. Interacting action for chiral supermultiplets

Independent of SUSY, the free field part of the action is determined by the field theoretic formulation of 2T physics given in [13]

$$\begin{aligned} S_0 = \int d^6X\delta(X^2)\Big[& \frac{1}{2}(\varphi^\dagger\partial^2\varphi + \partial^2\varphi^\dagger\varphi) + F^\dagger F \\ & + \frac{i}{2}(\bar{\psi}_L X\bar{\partial}\psi_L + \bar{\psi}_L\overleftarrow{\partial}\bar{X}\psi_L)\Big], \end{aligned} \tag{3.1}$$

where $X \equiv X^M\Gamma_M$, $\bar{\partial} = \bar{\Gamma}^M\partial_M$, etc. By using the Hermiticity property

$$\begin{aligned} (i\bar{\psi}_{1L}\Gamma^M\bar{\Gamma}^N\cdots\bar{\Gamma}^K\psi_{2L})^\dagger = {} & i\bar{\psi}_{2L}(-\Gamma^K)\cdots(-\Gamma^N) \\ & \times(-\bar{\Gamma}^M)\psi_{1L}, \end{aligned} \tag{3.2}$$

which is explained in Eqs. (B27)–(B32), it is easily verified that this action is Hermitian.

The delta function[6] in the volume element $d^6X\delta(X^2)$ as well as the given structure of the kinetic terms are required by global SO(4, 2) = SU(2, 2) and local 2T-physics gauge symmetries [13]. The gauge symmetry is responsible for eliminating ghosts and thinning out the field degrees of freedom from 4 + 2 to 3 + 1 dimensions holographically *without* any residual Kaluza-Klein type excitations. It is also responsible for the unifying features of 2T physics as a structure above 1T physics through various definitions of time in the embeddings of 3 + 1 dimensions in 4 + 2 dimensions (see Fig. 1 in [13]).

It will be convenient to rewrite the scalar part of the free action by doing an integration by parts so that it contains only first order derivatives. The result is[7]

$$\begin{aligned} S_0(\varphi, F) = \int d^6X\Big[& \delta(X^2)(-\partial_M\varphi^\dagger\partial^M\varphi + F^\dagger F) \\ & + 2\delta'(X^2)\varphi^\dagger\varphi\Big]. \end{aligned} \tag{3.3}$$

The term that contains $2\delta'(X^2)$ with a specific coefficient is an outcome of the 2T-physics gauge symmetry. Similarly, the fermion term is invariant under a separate 2T-physics gauge symmetry [13]. It may also be integrated by parts.

[6] Some useful properties of the delta function include $\frac{\partial}{\partial X^M}\delta(X^2) = 2X_M\delta'(X^2)$, $X\cdot\frac{\partial}{\partial X}\delta(X^2) = 2X^2\delta'(X^2) = -2\delta(X^2)$, and $\partial^2\delta(X^2) = 2(d+2)\delta'(X^2) + 4X^2\delta''(X^2) = 2(d-2)\delta'(X^2)$. Here $\delta'(u)$, $\delta''(u)$ are the derivatives of the delta function with respect to its argument $u = X^2$. So we have used $u\delta'(u) = -\delta(u)$ and $u\delta''(u) = -2\delta'(u)$ as the properties of the delta function of a single variable u to arrive at the above expressions. These are to be understood in the sense of distributions under integration with smooth functions.

[7] An intermediate step in deriving Eq. (3.3) has the second term in the form $\int d^6X\delta'(X^2)X\cdot\partial(\varphi^\dagger\varphi)$. This differs from the version in Eq. (3.3) by a total derivative.

After using $\Gamma^M\bar{X} = -X\bar{\Gamma}^M + 2X^M$, and $\delta'(X^2)X\bar{X} = X^2\delta'(X^2) = -\delta(X^2)$, it takes the form

$$S_0(\psi) = i\int d^6X\delta(X^2)\bar{\psi}_L[X\bar{\partial} - (X\cdot\partial + 2)]\psi_L. \quad (3.4)$$

By using the relation $\Gamma^{MN}X_M\partial_N = X\bar{\partial} - X\cdot\partial$ this may be rewritten further in the spin-orbit coupling form

$$S_0(\psi) = -i\int d^6X\delta(X^2)\bar{\psi}_L\left[\frac{1}{2i}\Gamma^{MN}L_{MN} + 2\right]\psi_L, \quad (3.5)$$

where L_{MN} is the SO(4,2) orbital angular momentum

$$L_{MN} = -i(X_M\partial_N - X_N\partial_M). \quad (3.6)$$

The free field equations are derived by extremizing the action in Eq. (3.1)–(3.5) while treating carefully the delta function as in footnote 6. The result is [13]

$$(\partial^2\varphi)_{X^2=0} = 0, \qquad [(X\cdot\partial + 1)\varphi]_{X^2=0} = 0, \qquad (F)_{X^2=0} = 0, \quad (3.7)$$

$$(X\bar{\partial}\psi_L)_{X^2=0} = 0, \qquad [(X\cdot\partial + 2)\psi_L]_{X^2=0} = 0, \quad (3.8)$$

plus their complex conjugates. Accordingly, when the fields are on shell, they are homogeneous with a specific degree of homogeneity, namely, under rescaling they give $\varphi(tX) = t^{-1}\varphi(X)$ and $\psi_L(tX) = t^{-2}\psi_L(X)$. However, when the fields are off shell they are not restricted to be homogeneous. We emphasize that our supersymmetry transformations given below are constructed off shell without homogeneity restrictions on any of the fields $(\varphi, \psi_L, F)_i$.

With an appropriate holographic embedding of 3 + 1 dimensions in 4 + 2 dimensions as shown in the footnote,[8] the on shell equations above are equivalent to free relativistic massless fields in 3 + 1 dimensions [13].

The general interaction among chiral supermultiplets, which we will show to be supersymmetric directly in 4 + 2 dimensions is given by

$$S_{\rm int} = \int d^6X\delta(X^2)\left[\left(\frac{\partial W}{\partial\varphi_i}F_i - \frac{i}{2}(\bar{\psi}_R)_iX(\psi_L)_j\frac{\partial^2 W}{\partial\varphi_i\partial\varphi_j}\right) + \text{H.c.}\right], \quad (3.9)$$

where $W(\varphi)$ is any cubic superpotential constructed from the scalars φ_i, $i = 1, 2, 3, \cdots$, with any desired internal symmetry group G.

The structure of $S_{\rm int}$ is similar to the SUSY formalism in 3 + 1 dimensions, except for the fact that the Yukawa coupling in 4 + 2 dimensions involves the factor $\bar{X}$ in the expression $(\bar{\psi}_R)_i\bar{X}(\psi_L)_j = (\psi_{L\alpha})_i(C\bar{X})^{\alpha\beta}(\psi_{L\beta})_j$. Taking into account that $(C\bar{X})^{\alpha\beta}$ is antisymmetric in $(\alpha\leftrightarrow\beta)$, and that the fermions anticommute, this factor is symmetric under the interchange of i and j and is consistent with the symmetric $\frac{\partial^2 W}{\partial\varphi_i\partial\varphi_j}$.

As in 3 + 1 dimensions, the auxiliary fields F_i can be solved from the equations of motion

$$F^{\dagger i} = -\frac{\partial W}{\partial\varphi_i} = -3y^{ijk}\varphi_j\varphi_k, \qquad F_i = -\left(\frac{\partial W}{\partial\varphi_i}\right)^* = -3y^*_{ijk}\varphi^{\dagger j}\varphi^{\dagger k}, \quad (3.10)$$

and inserted back into the action, so that $S_{\rm total} = S_0 + S_{\rm int}$ can be expressed only in terms of the dynamical fields $(\varphi, \psi_L)_i$ in 4 + 2 dimensions. The effective scalar potential energy "F-term" in the supersymmetric 4 + 2 action is then $V_F(\varphi, \varphi^\dagger) = |\frac{\partial W}{\partial\varphi_i}|^2$, just like 3 + 1 dimensions. However, to discuss the SUSY properties of the action, it is more convenient to keep the F_i off shell in the actions S_0, $S_{\rm int}$ as written above. We will see that unlike 3 + 1 dimensions, S_0, $S_{\rm int}$ are not separately supersymmetric in 4 + 2 dimensions, but after reducing the theory to 3 + 1 dimensions, the mixing term will drop and the 4 + 2-dimensional supersymmetry will reduce to ordinary 3 + 1 *superconformal* symmetry.

Renormalizability of 2T-physics field theory in 4 + 2 dimensions is determined by the renormalizability of the equivalent 1T field theory in 3 + 1 dimensions in the 2T-physics gauge described in footnote 8. Hence renormalizability restricts W to be at the most cubic in 3 + 1 as well as 4 + 2 dimensions. Furthermore, the *2T-physics gauge symmetries* discussed in [13] require that $S_0 + S_{\rm int}$ *cannot have any dimensionful couplings or mass terms* in 4 + 2 dimensions. Hence $V_F(\varphi, \varphi^\dagger) = |\frac{\partial W}{\partial\varphi_i}|^2$ must be purely quartic, or W must be *purely cubic*, which is also what is required by supersymmetry.

[8] The "relativistic particle gauge" that provides one of the embeddings of 3 + 1 dimensions in 4 + 2 dimensions is given as follows. We choose a lightcone-type basis in 4 + 2 dimensions so that the flat metric takes the form $ds^2 = dX^MdX^N\eta_{MN} = -2dX^{+'}dX^{-'} + dX^\mu dX^\nu\eta_{\mu\nu}$, where $\eta_{\mu\nu}$, with $\mu, \nu = 0, 1, 2, 3$ is the Minkowski metric and $X^{\pm'} = \frac{1}{\sqrt{2}}(X^{0'} \pm X^{1'})$ are the lightcone coordinates for the extra space $X^{1'}$ and time $X^{0'}$ dimensions. Furthermore we choose the following parametrization $X^{+'} = \kappa$, $X^{-'} = \kappa\lambda$, $X^\mu = \kappa x^\mu$, which defines the emergent 3 + 1-dimensional spacetime x^μ as embedded in 4 + 2. The inverse relation is $\kappa = X^{+'}$, $\lambda = \frac{X^{-'}}{X^{+'}}$, $x^\mu = \frac{X^\mu}{X^{+'}}$. This provides one of the many possible embeddings of 3 + 1 dimensions in 4 + 2 dimensions. In this paper we will mainly use the 3 + 1 spacetime embedding given above. This embedding, first discussed by Dirac [17], was useful to express the usual standard model of particles and forces as a gauge fixed form of 2T physics [13]. Therefore, the same 3 + 1 embedding will connect the supersymmetric standard model in 3 + 1 dimensions to the supersymmetric formulation of 2T physics in 4 + 2 dimensions. In addition to the embedding described above there are many other embeddings of 3 + 1 in 4 + 2. Such embeddings corresponds to Sp(2, R) gauge choices in the underlying 2T-physics worldline theory and lead to a variety of 1T-physics dynamical systems as summarized in Fig. 1 of Ref. [13].

As in the case of the nonsupersymmetric standard model discussed in [13], this cubic restriction on W will require that we include a supermultiplet that includes the dilaton Φ as part of the fields in our theory so that we can generalize to $W(\varphi, \Phi)$. This would be used to drive the spontaneous breakdown of the electroweak $SU(2)\times U(1)$ gauge symmetry, as demanded by phenomenology. This point is elaborated in more detail in the comments following Eq. (2.3).

B. Supersymmetry transformations

SUSY transformations in 2T physics in $4+2$ dimensions with N supersymmetries were formulated for the 2T superparticle on the worldline [4,5]. These form the supergroup $SU(2,2|N)$ as the global symmetry of the superparticle, and therefore this is the supersymmetry in the field theory version in $4+2$ dimensions. Here we concentrate on $N=1$ supersymmetry with the supergroup $SU(2,2|1)$. The fermionic parameter ε_L is a left-handed spinor (just like ψ_L) in the 4 representation of SU(2,2). We also note the right-handed charge conjugate $\varepsilon_R = C\bar{\varepsilon}_L^T$ which is not independent of ε_L (just like ψ_R), and classified as the $\bar{4}$ of SU(2,2)

We introduce the SUSY transformations of the chiral multiplet $(\varphi, \psi_L, F)_i$ off shell

$$\delta_\varepsilon\varphi_i = \bar{\varepsilon}_R\bar{X}\psi_{iL} - \tfrac{1}{2}X^2\bar{\varepsilon}_R(\bar{\partial}\psi_{iL} + U_{ij}^\dagger\psi_R^j) + \delta_\varepsilon^1\varphi_i, \tag{3.11}$$

$$\delta_\varepsilon\psi_{iL} = i(\partial\varphi_i)\varepsilon_R - iF_i\varepsilon_L, \tag{3.12}$$

$$\delta_\varepsilon F_i = \bar{\varepsilon}_L[X\bar{\partial} - (X\cdot\partial + 2)]\psi_{iL} + \delta_\varepsilon^1 F_i. \tag{3.13}$$

The additional pieces $\delta_\varepsilon^1\varphi_i$, $\delta_\varepsilon^1 F_i$ [given in Eq. (2.11) and (2.14) and explained in Sec. V] are proportional to the gauge coupling constants g and are needed when vector supermultiplets are coupled to chiral supermultiplets. In this section we assume the chiral supermultiplets on their own, so we will take $\delta_\varepsilon^1\varphi_i|_{g=0} = \delta_\varepsilon^1 F_i|_{g=0} = 0$. In the absence of interactions among the chiral multiplets the coefficient $U_{ij}^\dagger$ is also absent, but with interactions we will see that $U_{ij}^\dagger$ must satisfy $U_{ij}^\dagger = \frac{\partial^2 W^*}{\partial\varphi^{\dagger i}\partial\varphi^{\dagger j}} = 3!y_{ijk}^*\varphi^{\dagger k}$ where $W(\varphi)$ will turn out to be the superpotential of Eqs. (2.8) and (2.10). The last line may be rewritten as $\delta_\varepsilon F_i = -\bar{\varepsilon}_L(\frac{1}{2i}\Gamma^{MN}L_{MN} + 2)\psi_{iL} + \delta_\varepsilon^1 F_i$ as in Eqs. (3.4) and (3.5).

There are some parallels and some differences between these $4+2$ SUSY transformations and the familiar ones in $3+1$ dimensions. In particular, the terms proportional to X^2 and $(X\cdot\partial + 2)$ in $\delta_\varepsilon\varphi_i$ and $\delta_\varepsilon F_i$, respectively, have no parallels in $3+1$ dimensions, as noted following Eq. (2.18).

The transformation of the Hermitian conjugate fields $(\varphi^*, \bar{\psi}_L, F^*)^i$ is derived from above by using the Hermiticity properties of gamma matrices given in Eqs. (B27)–(B32)

$$\delta_\varepsilon\varphi^{\dagger i} = \bar{\psi}_L^i X\varepsilon_R - \tfrac{1}{2}X^2(\bar{\psi}_L^i\overleftarrow{\bar{\partial}} - U^{ij}\bar{\psi}_{Ri})\varepsilon_R + \delta_\varepsilon^1\varphi^{\dagger i}, \tag{3.14}$$

$$\delta_\varepsilon\bar{\psi}_L^i = i\bar{\varepsilon}_R(\bar{\partial}\varphi^{\dagger i}) + i\bar{\varepsilon}_L F^{\dagger i}, \tag{3.15}$$

$$\delta_\varepsilon F^{\dagger i} = -\bar{\psi}_L^i[\overleftarrow{\partial}\bar{X} - (\partial\cdot X + 2)]\varepsilon_L + \delta_\varepsilon^1 F^{\dagger i}. \tag{3.16}$$

The transformation of the charge conjugate fields $(\varphi^*, \psi_R, F^*)^i$, in terms of ψ_R^i instead of $\bar{\psi}_L^i$, are obtained by using the properties given in Eqs. (B42)–(B47)

$$\delta_\varepsilon\varphi^{\dagger i} = \bar{\varepsilon}_L X\psi_R^i - \tfrac{1}{2}X^2\bar{\varepsilon}_L(\partial\psi_R^i + U^{ij}\psi_{iL}) + \delta_\varepsilon^1\varphi^{\dagger i}, \tag{3.17}$$

$$\delta_\varepsilon\psi_R^i = -i(\bar{\partial}\varphi^{\dagger i})\varepsilon_L + iF^{\dagger i}\varepsilon_R, \tag{3.18}$$

$$\delta_\varepsilon F^{\dagger i} = \bar{\varepsilon}_R(\bar{X}\partial - (X\cdot\partial + 2))\psi_R^i + \delta_\varepsilon^1 F^{\dagger i}. \tag{3.19}$$

The last line may be rewritten as $\delta_\varepsilon F^{\dagger i} = -\bar{\varepsilon}_R(\frac{1}{2i}\bar{\Gamma}^{MN}L_{MN} + 2)\psi_R^i + \delta_\varepsilon^1 F^{\dagger i}$.

We will first show that the free action S_0 is invariant off shell in the absence of interactions provided the matrix U, $U^\dagger$ is dropped in the transformation rules (3.11)–(3.19). When $S_{\rm int}$ is included we will show that, unlike SUSY in $3+1$ dimensions, S_0, $S_{\rm int}$ cannot be made separately invariant. However, by including the $U, U^\dagger$ terms in the transformation rules (3.11)–(3.19), the total action $S_{\rm tot}^{\rm chiral} = S_0 + S_{\rm int}$ will be invariant off shell in $4+2$ dimensions.

We begin with the free action in the form of Eqs. (3.3)–(3.5). Its variation is

$$\begin{aligned}\delta_\varepsilon S_0 = \int d^6X[&-\delta(X^2)\partial_M\varphi^\dagger\partial^M(\delta_\varepsilon\varphi) + 2\delta'(X^2)\varphi^\dagger\delta_\varepsilon\varphi \\ &+ \delta(X^2)\{i(\delta_\varepsilon\bar{\psi}_L)[X\bar{\partial} - (X\cdot\partial + 2)]\psi_L \\ &+ F^\dagger(\delta_\varepsilon F)\}] + \text{H.c.}\end{aligned} \tag{3.20}$$

Inserting the SUSY transformation given above we get

$$\delta_\varepsilon S_0 = \int d^6X\Big[-\delta(X^2)\partial_M\varphi^\dagger\partial^M\Big[\bar\varepsilon_R\bar X\psi_L - \frac{1}{2}X^2\bar\varepsilon_R(\bar\partial\psi_{iL} + U^*\psi_R)\Big] + 2\delta'(X^2)\varphi^\dagger\Big[\bar\varepsilon_R\bar X\psi_L - \frac{1}{2}X^2\bar\varepsilon_R(\bar\partial\psi_L + U^*\psi_R)\Big]$$
$$+ i\delta(X^2)(i\bar\varepsilon_R(\bar\partial\varphi^\dagger) + i\bar\varepsilon_L F^\dagger)[X\bar\partial - (X\cdot\partial + 2)]\psi_L + \delta(X^2)F^\dagger\bar\varepsilon_L[X\bar\partial - (X\cdot\partial+2)]\psi_L\Big] + \text{H.c.} \tag{3.21}$$

In the first line, after using the properties $X^2\delta(X^2) = 0$ and $X^2\delta'(X^2) = -\delta(X^2)$, the terms containing X^2 simplify to $\delta(X^2)(X\cdot\partial + 1)\varphi^\dagger\bar\varepsilon_R(\bar\partial\psi_{iL} + U^*_{ij}\psi^j_R)$. In the last two lines, the terms proportional to $F^\dagger\bar\varepsilon_L$ cancel each other. The surviving terms from all lines take the form of a total divergence plus a term proportional to U, $U^\dagger$ as follows:

$$\delta_\varepsilon S_0 = \int d^6X\{\partial_M[\delta(X^2)V_0^M] + \delta(X^2)[(X\cdot\partial + 1)\varphi^{\dagger i}]U^*_{ij}\bar\varepsilon_R\psi^j_R)\} + \text{H.c.} = 0. \tag{3.22}$$

Hence, in the free theory, by dropping the U, $U^\dagger$ terms in the transformation laws, and dropping the total divergence with proper boundary conditions, we have demonstrated that we have a supersymmetric action $\delta_\varepsilon S_0 = 0$. Here V_0^M is given by[9]

$$V_0^M = \bar\varepsilon_R\{-(\partial^M\varphi^\dagger)\bar X + \bar\Gamma^M(X\cdot\partial + 1)\varphi^\dagger - \bar\partial\varphi^\dagger X\bar\Gamma^M + X^M(\bar\partial\varphi^\dagger)\}\psi_L \tag{3.23}$$

$$= \bar\varepsilon_R\{\bar\Gamma^M\varphi^{\dagger i} + \bar\Gamma^{MNK}X_N\partial_K\varphi^{\dagger i}\}\psi_{iL}. \tag{3.24}$$

[9]The various terms in Eq. (3.27) contribute to the various terms in Eq. (3.21) as follows. The subscripts in $\{\cdots\}_n$ denote terms that should be combined together for the same n

$$\partial_M[-\delta(X^2)(\partial^M\varphi^\dagger)\bar\varepsilon_R\bar{\not X}\psi_L] = \{-\delta(X^2)\partial^M\varphi^\dagger\partial_M(\bar\varepsilon_R\bar{\not X}\psi_L)\}_1 + \{-2\delta'(X^2)(X\cdot\partial\varphi^\dagger)\bar\varepsilon_R\bar{\not X}\psi_L\}_2 + \{-\delta(X^2)(\partial^2\varphi^\dagger)\bar\varepsilon_R\bar{\not X}\psi_L\}_6,$$

$$\partial_M[\delta(X^2)\bar\varepsilon_R\bar\Gamma^M((X\cdot\partial+1)\varphi^\dagger)\psi_L] = \{\delta'(X^2)2(X\cdot\partial+1)\varphi^\dagger\bar\varepsilon_R\bar{\not X}\psi_L\}_2 + \{\delta(X^2)[(X\cdot\partial+1)\varphi^\dagger]\bar\varepsilon_R\bar{\not\partial}\psi_L\}_4 + \{\delta(X^2)\bar\varepsilon_R((X\cdot\partial+2)(\bar{\not\partial}\varphi^\dagger))\psi_L\}_7,$$

$$\partial_M[-\delta(X^2)\bar\varepsilon_R(\bar{\not\partial}\varphi^\dagger\not X\bar\Gamma^M)\psi_L] = \{-\delta(X^2)[(\bar\varepsilon_R(\bar{\not\partial}\varphi^\dagger)\not X\bar{\not\partial})\psi_L]\}_3 + \{\delta(X^2)(\bar\varepsilon_R(\partial^2\varphi^\dagger)\bar{\not X})\psi_L\}_6 + \{-2\delta(X^2)(\bar\varepsilon_R((X\cdot\partial+2)\bar{\not\partial}\varphi^\dagger))\psi_L\}_7,$$

$$\partial_M[\delta(X^2)X^M\bar\varepsilon_R(\bar{\not\partial}\varphi^\dagger)\psi_L] = \{\delta(X^2)[\bar\varepsilon_R(\bar{\not\partial}\varphi^\dagger)((X\cdot\partial+2)\psi_L)]\}_5 + \{\delta(X^2)\bar\varepsilon_R[(X\cdot\partial+2)\bar{\not\partial}\varphi^\dagger]\psi_L\}_7.$$

The sum of these terms give the $\delta_\varepsilon S_0$ in Eq. (3.21) after cancelling the $F^\dagger\bar\varepsilon_L$ terms as follows:

$$\delta_\varepsilon S_0 = \int d^6X[\{-\delta(X^2)\partial_M\varphi^\dagger\partial^M(\bar\varepsilon_R\bar{\not X}\psi_L)\}_1 + \{2\delta'(X^2)\varphi^\dagger\bar\varepsilon_R\bar{\not X}\psi_L\}_2 + \{\delta(X^2)(X\cdot\partial+1)\varphi^\dagger\bar\varepsilon_R\bar{\not\partial}\psi_L\}_4 - \delta(X^2)\bar\varepsilon_R\bar{\not\partial}\varphi^\dagger[\{\not X\bar{\not\partial}\}_3 - \{(X\cdot\partial+2)\}_5]\psi_L] + \text{H.c.}$$

Now using the generalized Noether's theorem we obtain the part of the conserved current $(J_R^M)_0$ coming from the free action S_0

$$\bar\varepsilon_R(J_R^M)_0 + \text{H.c.} = \Big(\frac{\partial L}{\partial(\partial_M\varphi)}\delta_\varepsilon\varphi + \delta\bar\psi_L\frac{\partial L}{\partial(\partial_M\bar\psi_L)} + \frac{\partial L}{\partial(\partial_M F)}\delta_\varepsilon F - \delta(X^2)V_0^M\Big) + \text{H.c.}, \tag{3.25}$$

where the last term is obtained from the total divergence in Eq. (3.22). For consistency of this computation we must use again the form of the action in Eqs. (3.3)–(3.5). Noting that $\frac{\partial L}{\partial(\partial_M\bar\psi_L)} = \frac{\partial L}{\partial(\partial_M F)} = 0$, only the first and last terms contribute. This gives the current which we write in several equivalent forms

$$\bar\varepsilon_R(J_R^M)_0 = \delta(X^2)\bar\varepsilon_R[\bar\partial\varphi^\dagger X\bar\Gamma^M - X^M(\bar\partial\varphi^\dagger) - \bar\Gamma^M(X\cdot\partial+1)\varphi^\dagger]\psi_L \tag{3.26}$$

$$= \delta(X^2)\bar\varepsilon_R[-\bar\Gamma^{MPQ}X_P\partial_Q\varphi^\dagger - \partial^M(\bar X\varphi^\dagger)]\psi_L \tag{3.27}$$

$$= \delta(X^2)\bar\varepsilon_R(\bar\Gamma^{QP}\bar\Gamma^M - \eta^{MP}\bar\Gamma^Q)\psi_L\partial_Q(X_P\varphi^\dagger) \tag{3.28}$$

$$= \delta(X^2)\bar\varepsilon_R\Big[\frac{1}{2i}(\bar\Gamma^{MPQ}L_{PQ}\varphi^\dagger)\bar\Gamma^M - \partial^M(\bar X\varphi^\dagger)\Big]\psi_L. \tag{3.29}$$

One can check explicitly that this current is conserved when we use the equations of motion for the free action.

Now we turn to the interaction term S_{int} in Eq. (3.9) and investigate its transformation properties under SUSY for any W. Inserting the transformation rules above we get after some simplifications

$$\delta_\varepsilon S_{\rm int} = \int d^6X\delta(X^2)\bigg[\bigg(\frac{\partial W}{\partial\varphi_i}\bar\varepsilon_L[X\bar\partial - (X\cdot\partial + 2)]\psi_{iL}. - \frac{\partial}{\partial\varphi_i}\bigg(\frac{\partial W}{\partial\varphi_j}\bigg)\partial_M\varphi_i\bar\varepsilon_L\Gamma^M\bar X(\psi_L)_j - \bigg(\frac{i}{2}(\psi_{Li})^TC\bar X\psi_{Lj})((\varepsilon_L)^TC\bar X\psi_{Lk}\bigg)\frac{\partial^3W}{\partial\varphi_i\partial\varphi_j\partial\varphi_k}\bigg) + \text{H.c.}\bigg]. \quad (3.30)$$

One of the crucial observations here is the gamma matrix identities for $SU(2,2) = SO(4,2)$

$$\tfrac{1}{8}(\Gamma^{MN})_\alpha{}^\beta(\Gamma_{MN})_\gamma{}^\delta = \tfrac{1}{4}\delta_\alpha{}^\beta\delta_\gamma{}^\delta - \delta_\alpha{}^\delta\delta_\gamma{}^\beta. \quad (3.31)$$

Using this identity, and the fact that $C\bar X$ is an antisymmetric matrix, we see that the $\frac{\partial^3W}{\partial\varphi_i\partial\varphi_j\partial\varphi_k}$ term drops out due to a Fierz identity proven in Appendix C. Then $\delta_\varepsilon S_{\rm int}$ takes the form of a total divergence plus a term proportional to $\frac{\partial W}{\partial\varphi_j}$, $\frac{\partial W^*}{\partial\varphi^{*j}}$, as follows:

$$\delta_\varepsilon S_{\rm int} = \int d^6X\bigg\{\partial_M[\delta(X^2)V_1^M] + \delta(X^2)\frac{\partial W^*}{\partial\varphi^{*j}}\bar\varepsilon_R(X\cdot\partial + 2)\psi_R^j\bigg\} + \text{H.c.}, \quad (3.32)$$

where

$$V_1^M = -\frac{\partial W^*}{\partial\varphi^{*j}}\bar\varepsilon_R(\bar\Gamma^MX)\psi_R^j. \quad (3.33)$$

So, in the presence of the U, $U^\dagger$ terms in the transformation laws, neither S_0 nor $S_{\rm int}$ on their own are invariant. However, the terms proportional to U, $U^\dagger$, in Eq. (3.22) combine to a total divergence with the terms proportional to $\frac{\partial W}{\partial\varphi_j}$, $\frac{\partial W^*}{\partial\varphi^{*j}}$ in Eq. (3.32), in the form $\partial_M[\delta(X^2)V_2^M + \text{H.c.}]$, with

$$V_2^M = X^M\frac{\partial W^*}{\partial\varphi^{*j}}\bar\varepsilon_R\psi_R^j, \quad (3.34)$$

provided

$$U^{ij} = \frac{\partial^2W}{\partial\varphi_i\partial\varphi_j}, \qquad U_{ij}^\dagger = \frac{\partial^2W^*}{\partial\varphi^{*i}\partial\varphi^{*j}}, \quad (3.35)$$

and

$$U^{ij}\varphi_j = 2\frac{\partial W}{\partial\varphi_i}, \qquad U_{ij}^\dagger\varphi^{*j} = 2\frac{\partial W^*}{\partial\varphi^{*i}}. \quad (3.36)$$

These conditions require $W(\varphi)$ to be a purely cubic, but otherwise arbitrary, function in the scalar fields φ_i. Thus only in the case of a cubic superpotential

$$W(\varphi) = y^{ijk}\varphi_i\varphi_j\varphi_k, \quad (3.37)$$

with arbitrary dimensionless constants y^{ijk} which should be made compatible with other desired symmetries, we get a total divergence for the SUSY variation of the total action by putting together Eqs. (3.24), (3.33), and (3.34),

$$\delta_\varepsilon S_{\rm tot}^{\rm chiral} = \int d^6X\partial_M[\delta(X^2)(V_0^M + V_1^M + V_2^M)] + \text{H.c.}, \quad (3.38)$$

which implies that the total action is supersymmetric $\delta_\varepsilon(S_0 + S_{\rm int}) = 0$ off shell.

The fact that the superpotential $W(\varphi)$ is purely cubic, and therefore the F-term of the potential $V_F = |\partial W/\partial\varphi_i|^2$ is purely quartic, is in agreement with what we should have expected on the basis of the 2T-gauge symmetry, even without supersymmetry, as discussed in [12,13]. However, it is interesting that by demanding supersymmetry we also arrive independently at the same conclusion that only purely quartic interactions are admitted in the field theoretic formulation of 2T-physic in $4 + 2$ dimensions. This automatically implies a renormalizable field theory as easily seen from the perspective of $3 + 1$ dimensions.

Since the total divergence is not trivial, the conserved current gets contributions from both S_0 and $S_{\rm int}$ and is given by

$$\bar\varepsilon_R(J_R^M)^{\rm chiral} = \bar\varepsilon_R[(J_R^M)_0 - \delta(X^2)(V_1^M + V_2^M)], \quad (3.39)$$

where $\bar\varepsilon_R(J_R^M)_0$ is given in Eq. (3.26). Hence the supercurrent is

$$(J_R^M)^{\rm chiral} = \delta(X^2)\bigg\{(\bar\Gamma^{QP}\bar\Gamma^M - \eta^{MP}\bar\Gamma^Q)\psi_{iL}\partial_Q(X_P\varphi^{\dagger i}) + \frac{\partial W^*}{\partial\varphi^{*j}}X_N\bar\Gamma^{MN}\psi_R^j\bigg\}. \quad (3.40)$$

By using the equations of motion for the self interacting chiral multiplets that follow from $(S_0 + S_{\rm int})$, one can verify that the full SUSY current constructed above is conserved

$$\partial_M(J_R^M)_{\rm total} = 0. \quad (3.41)$$

IV. VECTOR SUPERMULTIPLET IN 2T PHYSICS

We now turn to the vector supermultiplet $(A_M, \lambda_L, B)^a$ in the adjoint representation of the Yang-Mills gauge group G and at first examine it by itself without coupling it to the chiral supermultiplet.

We begin with an action of the following form suggested by 2T-physics field theory for any Yang-Mills type gauge theory in $4 + 2$ dimensions [13]

$$L_{\rm vector} = \delta(X^2)\bigg\{-\frac{1}{4}F_{MN}^aF_a^{MN} + \frac{1}{2}B^aB_a + \frac{i}{2}[\bar\lambda_L^aX\bar D\lambda_{aL} + \bar\lambda_L^aD\bar X\lambda_{aL}]\bigg\}. \quad (4.1)$$

This action is invariant under 2T-physics gauge symmetries and has just the right structure to get reduced to a gauge theory in $3 + 1$ dimensions when gauge fixed as described in footnote 8, without any Kaluza-Klein leftovers. In this form B^a could be integrated out and set equal to zero through its equations of motion. But after coupling to the chiral multiplet, integrating out the auxiliary field B^a in the interacting theory will give rise to the so-called D-term which is a φ^4 interaction for the scalar fields in the chiral multiplet.

We now propose the following supersymmetry transformations in $4+2$ dimensions:

$$\delta_\varepsilon A_M^a = \{-2b\bar{\varepsilon}_L\Gamma_M\bar{X}\lambda_L^a + bX^2\bar{\varepsilon}_L\Gamma_{MN}D^N\lambda_L^a + \delta^1 A_M^a\} + \text{H.c.}, \tag{4.2}$$

$$\delta_\varepsilon\lambda_L^a = ib^* F_{MN}^a(\Gamma^{MN}\varepsilon_L) - ia^* B^a\varepsilon_L, \tag{4.3}$$

$$\delta_\varepsilon\bar{\lambda}_L^a = ib(\bar{\varepsilon}_L\Gamma^{MN})F_{MN}^a + ia\bar{\varepsilon}_L B, \tag{4.4}$$

$$\delta_\varepsilon B^a = a\bar{\varepsilon}_L[X\bar{D} - (X\cdot D + 2)]\lambda_L^a + \text{H.c.} \tag{4.5}$$

Here a and b are complex numbers whose values remain arbitrary until they are fixed later when we include interaction with chiral multiplets [see (5.16)]. Also, $\delta^1 A_M^a$ as given in Eq. (2.15) is an additional piece that arises only when chiral multiplets are coupled to vector multiplets and is determined later in Eq. (5.23) for the coupled theory. When vector supermultiplets are considered in isolation, as in this section, the extra term vanishes $(\delta^1 A_M^a)\to 0$; however, we will include it in part of the computation for later use.

The transformation of the gaugino λ_L^a is similar to ordinary supersymmetry transformation in $3+1$ dimension while the transformation of the auxiliary field B^a is similar to the transformation law of F in the chiral multiplet except that here B^a is Hermitian. The first term in the transformation of A_M^a also resembles the one in $3+1$ dimensions except that there is a $\bar{X}$ inserted between Γ_M and λ_L^a which breaks translation symmetry in $4+2$ dimensions. This insertion is required by the 2T-physics structures and has just the correct form such that the SUSY transformations in $4+2$ dimensions reduce to ordinary superconformal transformations in $3+1$ dimensions when one fixes the 2T gauge symmetry as described in footnote 8.

Transformation of the action is

$$\delta_\varepsilon L = \delta(X^2)B^a(\delta_\varepsilon B_a) - \delta(X^2)F_a^{MN}D_M(\delta_\varepsilon A_N^a) \tag{4.6}$$

$$+\frac{i}{2}\delta(X^2)[(\delta_\varepsilon\bar{\lambda}_L^a)(X\bar{D} + \overleftarrow{D}\bar{X})\lambda_{aL}] + \text{H.c.} \tag{4.7}$$

$$+ i\delta(X^2)f_{abc}(\delta_\varepsilon A_M^a)\bar{\lambda}_L^b\Gamma^{MN}\lambda_L^c X_N. \tag{4.8}$$

The last term $i\delta(X^2)f_{abc}(\delta_\varepsilon A_M^a)\bar{\lambda}_L^b\Gamma^{MN}\lambda_L^c X_N$ vanishes by itself, partly due to $X^2\delta(X^2)=0$ and partly because of the nontrivial Fierz rearrangement identity proven in Appendix C that follows the form of Eq. (3.31). The first term in line (4.6) proportional to $(\delta_\varepsilon B_a)$ which we define as $(\delta_\varepsilon L)_1$ gives

$$(\delta_\varepsilon L)_1 = a\delta(X^2)B\bar{\varepsilon}_L[X\bar{D} - (X\cdot D + 2)]\lambda_L + \text{H.c.} \tag{4.9}$$

In the second term in line (4.6) proportional to $\delta_\varepsilon A_N^a$, which we define as $(\delta_\varepsilon L)_2$, we first collect a total derivative (suppressing the adjoint index a for less clutter)

$$\begin{aligned}(\delta_\varepsilon L)_2 = \{&-\partial_M\{\delta(X^2)F^{MN}\delta_\varepsilon A_N\} \\ &+ \delta(X^2)(D_M F^{MN})(\delta_\varepsilon A_N) \\ &+ \delta'(X^2)2X_M F^{MN}(\delta_\varepsilon A_N)\}\end{aligned} \tag{4.10}$$

and then insert $\delta_\varepsilon A_N^a$ in the remainder. After using $X^2\delta(X^2)=0$ and $X^2\delta'(X^2) = -\delta(X^2)$, we obtain

$$\begin{aligned}(\delta_\varepsilon L)_2 = \{&\partial_M\{-\delta(X^2)F^{MN}\delta_\varepsilon A_N\} - 2b\delta(X^2)(D_M F^{MN}) \\ &\times[\bar{\varepsilon}_L\Gamma_N\bar{X}\lambda_L^a + \text{H.c.}] - 2b(X_M F^{MN}) \\ &\times[\delta(X^2)(\bar{\varepsilon}_L\Gamma_{NP}D^P\lambda_L) + 2\delta'(X^2)\bar{\varepsilon}_L\Gamma_{NP}\lambda_L^a X^P] \\ &+ \delta'(X^2)2X_N F^{NM}(\delta^1 A_M^a)\}.\end{aligned} \tag{4.11}$$

Note that since $(\delta^1 A_M^a)$ is proportional to X^2 it survives only when multiplied by $\delta'(X^2)$. This term will be dropped in this section since it is present only when there is coupling to chiral multiplets; it will be taken into account later in Eq. (5.22).

The term in line (4.7) proportional to $\delta_\varepsilon\bar{\lambda}_L^a$ which we define as $(\delta_\varepsilon L)_3$ takes the form

$$\begin{aligned}(\delta_\varepsilon L)_3 = &-\tfrac{1}{2}\delta(X^2)[b\bar{\varepsilon}_L(\Gamma_{MN}F^{MN}) + a\bar{\varepsilon}_L B] \\ &\times(X\bar{D}\lambda_L + \overleftarrow{D}\bar{X}\lambda_L) + \text{H.c.}\end{aligned} \tag{4.12}$$

We do an integration by parts to change the covariant derivative hitting on λ_L to covariant derivative hitting on F^{MN} and change the covariant derivative hitting on B to covariant derivative hitting on λ_L, and in this process collect a total divergence

$$\begin{aligned}(\delta_\varepsilon L)_3 = \{&\partial_M(-\tfrac{1}{2}a\delta(X^2)B\bar{\varepsilon}_L\Gamma^M\bar{X}\lambda_L - \tfrac{1}{2}b\delta(X^2\bar{\varepsilon}_L) \\ &\times(\Gamma_{NP}F^{NP})X\bar{\Gamma}^M\lambda_L) \\ &- \delta(X^2)aB\bar{\varepsilon}_L[X\bar{D} - (X\cdot D + 2)]\lambda_L \\ &- \delta(X^2)b\bar{\varepsilon}_L(\Gamma_{MN}F^{MN})\Gamma_{PQ}\overleftarrow{D}^P X^Q\lambda_L \\ &+ 2b\delta(X^2)\bar{\varepsilon}_L(\Gamma_{MN}F^{MN})\lambda_L\} + \text{H.c.}\end{aligned} \tag{4.13}$$

This is further developed with some gamma matrix algebra

$$\begin{aligned}\Gamma_{MN}\Gamma_{PQ} = \Gamma_{MNPQ} &+ \{\eta_{NP}\Gamma_{MQ} - \eta_{MP}\Gamma_{NQ} + \eta_{MQ}\Gamma_{NP} \\ &- \eta_{NQ}\Gamma_{MP}\} + \{\eta_{NP}\eta_{MQ} - \eta_{MP}\eta_{NQ}\},\end{aligned} \tag{4.14}$$

and simplifications due to the Bianchi identity $\Gamma_{MNPQ}(D^P F^{MN}) = 0$.

Combining all the terms we find a total divergence after cancellations

$$(\delta_\varepsilon L)_1 + (\delta_\varepsilon L)_2 + (\delta_\varepsilon L)_3 = \partial_M[\delta(X^2)(\bar{\varepsilon}_L V_L^M + \bar{\varepsilon}_R V_R^M)^{\text{vector}}], \quad (4.15)$$

where $\bar{\varepsilon}_L V_L^M$ is

$$(\bar{\varepsilon}_L V^M)^{\text{vector}} = \{-F^{MN}(\delta_{\bar{\varepsilon}_L} A_N) - \tfrac{1}{2} aB\bar{\varepsilon}_L \Gamma^M \bar{X}\lambda_L - \tfrac{1}{2} bF^{PQ}\bar{\varepsilon}_L \Gamma_{PQ} X\bar{\Gamma}^M \lambda_L + 2b(X^Q F_{QP})\bar{\varepsilon}_L \Gamma^{MP}\lambda_L\}, \quad (4.16)$$

and similarly for $\bar{\varepsilon}_R V_R^M$ (obtained by replacing left $\leftrightarrow$ right) which is the Hermitian conjugate of $\bar{\varepsilon}_L V_L^M$ (verified via the formulas in Appendix B).

Hence we have shown that the Lagrangian (4.1) is symmetric under the given SUSY transformations for any complex numbers a and b. This means that in the transformation rules (4.2), (4.3), (4.4), and (4.5) we can replace $a\varepsilon_L$ by an independent SUSY parameter $b\varepsilon_L$, so that the SUSY symmetry of the Lagrangian (4.1) is actually twice as large. However, we will see that we will have to fix a and b relative to each other when there is interaction with chiral multiplets [see Eq. (5.16)].

Noether's theorem for the theory with only vector supermultiplets in Eq. (4.1) gives the following supercurrent:

$$(\bar{\varepsilon}_L J_L^M)^{\text{vector}} = \frac{\partial L}{\partial(\partial_M A_N)}(\delta_{\bar{\varepsilon}_L} A_N) + (\delta_\varepsilon \bar{\lambda}_L)\frac{\partial L}{\partial(\partial_M \bar{\lambda}_L)} - \delta(X^2)(\bar{\varepsilon}_L V_L^M)^{\text{vector}} \quad (4.17)$$

and similarly for $(\bar{\varepsilon}_R J_R^M)^{\text{vector}}$. The part $\frac{\partial L}{\partial(\partial_M A_N)}(\delta_{\bar{\varepsilon}_L} A_N) = -F^{MN}(\delta_{\bar{\varepsilon}_L} A_N)$ cancels against an equal term in $\bar{\varepsilon}_L V^M_{\text{vector}}$ of Eq. (4.16). The part $(\delta_\varepsilon \bar{\lambda}_L)\frac{\partial L}{\partial(\partial_M \bar{\lambda}_L)} = \delta(X^2)(\delta_\varepsilon \bar{\lambda}_L) \times (\frac{i}{2}\Gamma^M X\lambda_L)$ gives

$$(\delta_\varepsilon \bar{\lambda}_L)\frac{\partial L}{\partial(\partial_M \bar{\lambda}_L)} = \delta(X^2)\left\{-\frac{1}{2} bF_{PQ}\bar{\varepsilon}_L \Gamma^{PQ}\Gamma^M \bar{X}\lambda_L - \frac{a}{2} B\bar{\varepsilon}_L \Gamma^M \bar{X}\lambda_L\right\}. \quad (4.18)$$

The piece proportional to a cancels against an equal term in $\bar{\varepsilon}_L V^M_{\text{vector}}$. After these simplifications we are left with the following terms proportional only to $b\bar{\varepsilon}_L$:

$$(\bar{\varepsilon}_L J_L^M)^{\text{vector}} = \delta(X^2)\{-\tfrac{1}{2} bF_{PQ}\bar{\varepsilon}_L \Gamma^{PQ}(\Gamma^M \bar{X} - X\bar{\Gamma}^M)\lambda_L - 2b(X^Q F_{QP})\bar{\varepsilon}_L \Gamma^{MP}\lambda_L\}. \quad (4.19)$$

By using gamma matrix identities (4.14) and (B2)–(B6) it is convenient to bring this to the following alternative forms:

$$(\bar{\varepsilon}_L J_L^M)^{\text{vector}} = b\delta(X^2)F_{PQ}X_N\bar{\varepsilon}_L(\Gamma^{PQNM} + 2\eta^{MP}\Gamma^Q\bar{\Gamma}^N)\lambda_L \quad (4.20)$$

$$= b\delta(X^2)F_{PQ}X_N\bar{\varepsilon}_L(\Gamma^{PQN}\bar{\Gamma}^M - \eta^{NM}\Gamma^{PQ} + 2\eta^{MP}\eta^{NQ})\lambda_L. \quad (4.21)$$

The form in the last line makes it easy to check that this current is conserved $\partial_M(\bar{\varepsilon}_L J_L^M)^{\text{vector}} = 0$ as follows. After using some of the equations of motion, in particular $X^N F_{MN} = 0$ and[10] $(X \cdot D + 2)\lambda_L = 0$ plus the Bianchi identity $D_{[M}F_{PQ]} = 0$, the divergence of the current becomes proportional to the remaining equations of motion $X\bar{D}\lambda_L^a$ and $D^M F_{MQ}^a$ as follows:

$$\partial_M(\bar{\varepsilon}_L J_L^M)^{\text{vector}} = \delta(X^2)\{F_{PQ}\bar{\varepsilon}_L\Gamma^{PQ}X\bar{D}\lambda_L + 2\bar{\varepsilon}_L\Gamma^{QN}\lambda_L X_N(D^M F_{MQ})\} = \text{sources.} \quad (4.22)$$

In the absence of coupling between the vector and chiral multiplets the sources in the equations of motion are

$$X\bar{D}\lambda_L^a = 0, \qquad D^M F_{MQ}^a = gf^{abc}(\bar{\lambda}_L^b \Gamma_{QP}\lambda_L^c)X^P. \quad (4.23)$$

Therefore, we obtain

$$\partial_M(\bar{\varepsilon}_L J_L^M)^{\text{vector}} = \delta(X^2)2f^{abc}X_N X^P \bar{\varepsilon}_L \Gamma^{QN}\lambda_L^a \bar{\lambda}_L^b \Gamma_{QP}\lambda_L^c = 0, \quad (4.24)$$

where in the last step we have used the Fierz identity (C2) in Appendix C which is valid only in special dimensions [in particular valid for SO(4,2)]. Hence, the pure vector-multiplet current is conserved by itself, $\partial_M(J_L^M)^{\text{vector}} = 0$. The conservation of the current amounts also to a proof of SUSY for the theory of Eq. (4.1) that supplies the equations of motion.

The Hermitian conjugate of this conserved current can be written as the right-handed current (see Appendix B for Hermitian and charge conjugation properties of gamma matrices)

$$(\bar{\varepsilon}_R J_R^M)^{\text{vector}} = b^*\delta(X^2)X_N F_{PQ}^a \bar{\varepsilon}_R(\bar{\Gamma}^{PQN}\Gamma^M - \eta^{NM}\bar{\Gamma}^{PQ})\lambda_R^a. \quad (4.25)$$

Here we have dropped the term proportional to $X_N F^{MN}$

[10]From $X^N F_{MN}^a = 0$ it follows that $(X \cdot D + 2)F_{MN}^a = 0$. This is needed, along with the other equations, to prove Eq. (4.22).

since the current can be modified by terms proportional to the equations of motion, and one of them happens to be $X_N F^{MN} = 0$. The corresponding term should then be dropped also from $\bar{\varepsilon}_L J_L^M$ in Eq. (4.21).

Although we do not discuss it in detail, it is worth mentioning that the currents $(J_L^M, J_R^M)^{\text{vector}}$ are invariant under the 2T-gauge transformations [13] for the fields $(A_M^a, \lambda_R^a, B^a)$, and therefore they are gauge invariant physical observables from the point of view of all the gauge symmetries.

V. SUPERSYMMETRIC 2T PHYSICS WITH FIELDS OF SPINS 0, $\frac{1}{2}$, 1

The next step is to couple chiral supermultiplets $(\varphi, \psi_L, F)_i$ minimally to vector supermultiplets $(\lambda_L, A_M, B)^a$ to describe gauge interactions. This requires more than promoting the ordinary derivatives to covariant derivatives, namely, more interaction terms also need to be added (the α, β terms) as in the full action in Eqs. (2.3) and (2.6). Once SUSY is achieved we will calculate the full conserved supercurrent for the coupled theory.

The full SUSY transformation rules for the chiral multiplet are the gauge-covariantized versions of those given in Eqs. (3.11)–(3.19), but including also the nonzero extra terms $(\delta_\varepsilon^1 F, \delta_\varepsilon^1 \varphi)$ which will be determined below in Eqs. (5.6) and (5.24). Similarly, the SUSY transformation rules for the vector multiplet are those given in Eqs. (2.15)–(4.5) but with the extra nonzero term $\delta_\varepsilon^1 A$ which is determined below in Eq. (5.23).

We will see that the parameters α, β, a, b will be fully fixed [see Eq. (5.16)]. Eventually when we construct supergravity in the 2T formalism, the dilaton and its partners will also contribute to the transformation rules and restrict the possible parameters such as ξ and $V(\Phi, \varphi)$ which appear in L_{dilaton}. In this section the dilaton and its partners will be neglected, so we will assume the case with L_{dilaton} set to zero. With all these points taken into account, the full SUSY transformation rules will be shown to be those given in Eqs. (2.11)–(2.18).

Thus we consider the full Lagrangian of Eq. (2.3), without a dilaton Φ written in three pieces $L = L_{\text{vector}} + L_{\text{chiral}} + L_{\text{int}}$. Here L_{vector} is identical to Eq. (4.1), L_{chiral} is the gauge-covariantized version of $S_0 + S_{\text{int}}$ of Eqs. (3.3), (3.4), and (3.9), and L_{int} is the expression given in Eq. (2.6). The full SUSY variation of the parts $L_{\text{vector}} + L_{\text{chiral}}$ is almost identical to the variations discussed in the previous sections for the uncoupled multiplets, except for replacing all derivatives by covariant derivatives in L_{chiral}, and taking into account extra terms that appear as follows:

(1) Varying A_M that occurs in the covariant derivatives $\delta_\varepsilon L_{\text{chiral}} \rightarrow \frac{\partial L_{\text{chiral}}}{\partial A_M^a}(\delta_\varepsilon A_M^a)$.

(2) The effect of the extra $\delta_\varepsilon^1 F$ term $\delta_\varepsilon L_{\text{chiral}} \rightarrow (\frac{\partial L_{\text{chiral}}}{\partial F_i}\delta_\varepsilon^1 F_i + \text{H.c.})$.

(3) New terms that arise in $\delta_\varepsilon L_{\text{chiral}}$ due to changing of orders of noncommuting covariant derivatives.[11]

(4) The effect of the extra $\delta_\varepsilon^1 \varphi$, $\delta_\varepsilon^1 A$ terms in the variation of $\delta_\varepsilon(L_{\text{vector}} + L_{\text{chiral}})$. We leave these for last because they are both proportional to X^2 so they drop out in most terms due to the overall $\delta(X^2)$ in the Lagrangian. They can contribute only through the variation of the kinetic terms and through the terms in the action proportional to $\delta'(X^2)$.

Hence, for $\delta_\varepsilon(L_{\text{vector}} + L_{\text{chiral}})$ we can use the results of the previous sections plus the extra modifications listed above, and then add the full variation of the coupling term $\delta_\varepsilon L_{\text{int}}$. So, the computation is organized as follows:

$$\delta_\varepsilon L = \partial_M((\bar{\varepsilon}_L V_L^M + \bar{\varepsilon}_R V_R^M)^{\text{chiral}} + (\bar{\varepsilon}_L V_L^M + \bar{\varepsilon}_R V_R^M)^{\text{vector}}) \tag{5.1}$$

$$+ \delta_\varepsilon L_{1+2+3}^{\text{extra}} + \delta_\varepsilon L_{\text{int}} + \delta_\varepsilon L_4^{\text{extra}}. \tag{5.2}$$

In the second line the subscripts indicate the variations that correspond to the items listed above. The first line is the total divergence results of Eqs. (3.38) and (4.15), where $(V_{L,R}^M)^{\text{vector}}$ are identical to Eq. (4.16), while $(V_{L,R}^M)^{\text{chiral}}$ is given in Eq. (3.41) except for replacing all derivatives by covariant derivatives.

The three items in $\delta_\varepsilon L_{1+2+3}^{\text{extra}}$ give the following contributions:

$$\delta_\varepsilon L_{1+2+3}^{\text{extra}} = \delta(X^2) g(\delta_\varepsilon A_M^a)[-i\varphi^\dagger t_a \overleftrightarrow{D}_M \varphi + X^N \bar{\psi}_L \Gamma_{NM} t_a \psi_L] \tag{5.3}$$

$$+ \delta(X^2)(\delta_\varepsilon^1 F^{\dagger i})\left(F_i + \frac{\partial W^*}{\partial \varphi^{*i}}\right) + \text{H.c.} \tag{5.4}$$

$$+ i\frac{g}{2}\delta(X^2)(F_{MN}\varphi)_i \bar{\varepsilon}_L[-\Gamma^{MN} X + 2\Gamma^M X^N]\psi_R^i + \text{H.c.} \tag{5.5}$$

It is evident that $(\delta_\varepsilon^1 F^{\dagger i})$ must be chosen to cancel terms proportional to F_i coming from varying the coupling term $\delta_\varepsilon L_{\text{int}}$. The only new contribution proportional to F_i is the variation of ψ in the coupling term $\delta_\varepsilon L_{\text{int}}$ given explicitly below. As will be verified below, this fixes uniquely the extra piece in the SUSY transformation of F_i, $F^{\dagger i}$ as

$$(\delta_\varepsilon^1 F^{\dagger i}) = i\beta(\varepsilon_L^T (C\bar{X})\lambda_{aL})(\varphi^\dagger t^a)^i. \tag{5.6}$$

[11]The terms proportional to $[D_M, D_N] \sim F_{MN}$ can be obtained by going over the computations in footnote 9 and replacing covariant derivatives in all appropriate places. The terms that arise from commuting covariant derivatives has the form $\delta(X^2)([D_M, D_N]\varphi^\dagger)^i \bar{\varepsilon}_R[\frac{1}{2}\bar{\Gamma}^{MN}\bar{X} - \bar{\Gamma}^M X^N]\psi_{Li}$ + H.c., where we replace $[D_M, D_N]\varphi^\dagger = ig(\varphi^\dagger F_{MN})^i$. To obtain the expression in Eq. (5.5) we prefer to use the Hermitian conjugate version of this expression $-ig\delta(X^2)(F_{MN}\varphi)_i \bar{\psi}_L^i[-\frac{1}{2}X\Gamma^{MN} - \Gamma^M X^N]\varepsilon_R$ + H.c., where we have used Eqs. (B28), and then use the Majorana properties of Eqs. (B38)–(B40), to rewrite it in the form $-ig\delta(X^2)(F_{MN}\varphi)_i \bar{\varepsilon}_L[\frac{1}{2}\Gamma^{MN} X - \Gamma^M X^N]\psi_R^i$ + H.c.

Inserting this $(\delta^1_\varepsilon F^{\dagger i})$ and $(\delta_\varepsilon A^a_M)$ from Eq. (4.2) into $\delta_\varepsilon L^{\rm extra}_{1+2+3}$ we obtain

$$\delta_\varepsilon L^{\rm extra}_{1+2+3} = -2bg\delta(X^2)\{-i[\bar\varepsilon_L\Gamma_M\bar X\lambda^a_L][\varphi^\dagger t_a D_M\varphi] + (\bar\varepsilon_L\Gamma^{MN}\lambda^a_L)(\bar\psi_L\Gamma_{PM}t_a\psi_L)X_NX^P\} + \text{H.c.} \tag{5.7}$$

$$+\delta(X^2)i\beta(\varepsilon^T_L(C\bar X)\lambda_{aL})(\varphi^\dagger t^a)^i\left(F_i+\frac{\partial W^*}{\partial\varphi^{*i}}\right) + \text{H.c.} \tag{5.8}$$

$$+ i\frac{g}{2}\delta(X^2)\bar\varepsilon_L(-\Gamma^{MN}X+2\Gamma^MX^N)\psi^i_R(F_{MN}\varphi)_i + \text{H.c.} \tag{5.9}$$

Next we compute $\delta_\varepsilon L_{\rm int}$ by varying Eq. (2.6)

$$\delta_\varepsilon L_{\rm int} = \alpha\delta(X^2)\{[(\varphi^\dagger t_a\varphi)+\xi_a\Phi^2](\delta_\varepsilon B^a) + (\delta_\varepsilon\varphi^\dagger t_a\varphi)B^a + \text{H.c.}\} \tag{5.10}$$

$$+\beta\delta(X^2)\{(\delta_\varepsilon\varphi^\dagger)t^a((\psi_L)^T(C\bar X)\lambda_{aL}) + \varphi^\dagger t^a((\delta_\varepsilon\psi_L)^T(C\bar X)\lambda_{aL}) + (\delta_\varepsilon\bar\lambda_{aL})X\psi_R t_a\varphi\} + \text{H.c.} \tag{5.11}$$

We insert the $(\delta_\varepsilon\bar\lambda_{aL})$ in Eq. (4.4) and $\delta_\varepsilon\varphi^{\dagger i}$, $\delta_\varepsilon\psi_{iL}$ in Eqs. (3.11)–(3.19) with gauge covariant derivatives replacing ordinary derivatives. After dropping the terms $X^2\delta(X^2)=0$ we obtain

$$\delta_\varepsilon L_{\rm int} = \alpha\delta(X^2)\{a(\varphi^\dagger t_a\varphi)[\bar\varepsilon_L(\Gamma^{MN}X_MD_N-2)\lambda^a_L] + B^a(\bar\varepsilon_LX\psi^i_R)(t_a\varphi)_i\} + \text{H.c.} \tag{5.12}$$

$$\begin{aligned}&+\beta\delta(X^2)\{(\bar\varepsilon_LX\psi_Rt^a)^i(\psi^T_{iL}(C\bar X)\lambda_{aL})\\&-i(\varphi^\dagger t^aD_M\varphi)\bar\varepsilon_L\Gamma^M\bar X\lambda_{aL}\\&-i(\varphi^\dagger t^a)^iF_i(\varepsilon^T_L(C\bar X)\lambda_{aL})ibF^a_{MN}(\bar\varepsilon_L\Gamma^{MN}X\psi^i_R)(t_a\varphi)_i\\&+iaB^a(\bar\varepsilon_LX\psi^i_R)(t_a\varphi)_i\} + \text{H.c.}\end{aligned} \tag{5.13}$$

Note that the terms proportional to F_i, that appear in Eqs. (5.8) and the third line of Eq. (5.13), cancel by the choice of $(\delta^1_\varepsilon F^{\dagger i})$ of Eq. (5.6) as anticipated. To cancel some of the other terms in the sum $\delta_\varepsilon L^{\rm extra}_{\rm chiral}+\delta_\varepsilon L_{\rm int}$ we fix the unknown coefficients α, β, a, b as follows:

(i) $\alpha=-i\beta a$, cancels terms proportional to B^a that appear in the last lines of Eqs. (5.12) and (5.13).

(ii) $\beta b=\frac{g}{2}$, cancels partially terms proportional to F^a_{MN} that appear in Eqs. (5.9) and (5.13). The leftover is

$$ig\delta(X^2)(\bar\varepsilon_L\Gamma^M\psi^i_R)(F_{MN}\varphi)_iX^N + \text{H.c.} \tag{5.14}$$

(iii) $\beta=4bg$, cancels partially terms proportional to $(\varphi^\dagger t^aD_M\varphi)$ that appear in Eqs. (5.7) and (5.13). The leftover is $-2ibg\delta(X^2)\times(\bar\varepsilon_L\Gamma_M\bar X\lambda^a_L)D_M(\varphi^\dagger t_a\varphi)$ which can be rewritten in the following form by using the gamma matrix identities $\Gamma_M\bar X=\Gamma_{MN}X^N+X_M$

$$\begin{aligned}&-2ibg\delta(X^2)(D_M(\varphi^\dagger t_a\varphi)[\bar\varepsilon_L\Gamma_{MN}\lambda^a_L]X^N\\&+(\bar\varepsilon_L\lambda^a_L)[(X\cdot D+2)(\varphi^\dagger t_a\varphi)]\\&-2(\bar\varepsilon_L\lambda^a_L)(\varphi^\dagger t_a\varphi)).\end{aligned} \tag{5.15}$$

Furthermore, using the same $4bg=\beta$, the charge conjugation property $\psi_R=C\bar\psi^T_L$, and a Fierz identity, we cancel the terms of the form $\bar\varepsilon\lambda\bar\psi\psi XX$ that appear in (5.7) and (5.13).

(iv) $2\alpha a=4ibg$, cancels the terms of the form $(\varphi^\dagger t_a\varphi)(\bar\varepsilon_L\lambda^a_L)$ that appear in the first line of Eq. (5.12) and the last line of Eq. (5.15)

From these conditions the unknown coefficients are completely fixed as $a=\pm i\sqrt{1/2}$, $b=\pm'\sqrt{1/8}$, $\alpha=\pm\pm' g$, $\beta=\pm'\sqrt{2}g$. The $\pm$, $\pm'$ signs can be absorbed by a redefinition of the signs of λ, B wherever they appear in the Lagrangian and transformation rules. Therefore it is sufficiently general to choose one set of signs for these coefficients, thus we settle with the upper signs

$$a=i\frac{1}{\sqrt2},\qquad b=\frac{1}{2\sqrt2},\qquad \alpha=g,\qquad \beta=\sqrt2 g \tag{5.16}$$

to agree with conventions in the case of 3 + 1 dimensions.

The remaining terms that have not canceled so far come from the first line of Eq. (5.12), the first and second lines of Eq. (5.15), the remainder in Eq. (5.14), and the term proportional to $\frac{\partial W^*}{\partial\varphi^{*i}}$ in Eq. (5.8). These are collected below after inserting the constants above

$$\delta_\varepsilon L^{\rm extra}_{1+2+3}+\delta_\varepsilon L_{\rm int} \tag{5.17}$$

$$=\frac{ig}{\sqrt2}\delta(X^2)((\varphi^\dagger t_a\varphi)[\bar\varepsilon_L\Gamma^{MN}X_MD_N\lambda^a_L]) + \text{H.c.} \tag{5.18}$$

$$-\frac{ig}{\sqrt2}\delta(X^2)(D_M(\varphi^\dagger t_a\varphi)[\bar\varepsilon_L\Gamma_{MN}\lambda^a_L]X^N + (\bar\varepsilon_L\lambda^a_L)[(X\cdot D+2)(\varphi^\dagger t_a\varphi)]) + \text{H.c.} \tag{5.19}$$

$$+ig\delta(X^2)(\bar\varepsilon_L\Gamma^M\psi^i_R)(F_{MN}\varphi)_iX^N + \text{H.c.} \tag{5.20}$$

$$-i\sqrt2 g\delta(X^2)(\varphi^\dagger t^a)^i\frac{\partial W^*}{\partial\varphi^{*i}}(\varepsilon^T_L(C\bar X)\lambda_{aL}) + \text{H.c.} \tag{5.21}$$

The last term vanishes because W^* is gauge invariant, which requires $(\varphi^\dagger t^a)^i\frac{\partial W^*}{\partial\varphi^{*i}}=0$ as in Eq. (2.10).

The remaining terms assemble into a total divergence plus terms that are proportional to the subset of equations of motion that imply homogeneity conditions on the fields $[(X\cdot D+2)(\varphi^\dagger t_a\varphi)]$ and $F_{MN}X^N$. These would vanish by the (homogeneity) subset of equations of motion on mass shell. However, off-mass shell they can be canceled by additional pieces $\delta^1_\varepsilon\varphi_i$, $\delta^1_\varepsilon A^a_M$ in the SUSY transformation of φ, A_M^a, by taking $\delta_\varepsilon^1\varphi_i$, $\delta_\varepsilon^1 A_M^a$ to be proportional to X^2. These extra pieces generally drop out in most terms in the SUSY variation of the Lagrangian due to $X^2\delta(X^2)=0$ but survive in some of the kinetic terms and the terms that contain $\delta'(X^2)$. The $\delta_\varepsilon^1\varphi_i$, $\delta_\varepsilon^1 A_M^a$ variation of the Lagrangian $\delta_\varepsilon(L_{\rm vector}+L_{\rm chiral})$ are also proportional to the subset of equations of motion $[(X\cdot D+2)(\varphi^\dagger t_a\varphi)]$ or $F_{MN}X^N$, as follows:

$$\delta_\varepsilon L_4^{\rm extra} = 2\delta'(X^2)(X_N F_a^{NM})\delta_\varepsilon^1 A_M^a + 2\delta'(X^2)[(X\cdot D+1)\varphi^{\dagger i}]\delta_\varepsilon^1\varphi_i + {\rm H.c.} \quad (5.22)$$

Therefore, we can add the extra pieces $\delta_\varepsilon^1\varphi_i$, $\delta_\varepsilon^1 A_M^a$ to the variation of φ_i, A_M^a to cancel the terms noted above. So we choose

$$\delta_\varepsilon^1 A_M^a = -i\frac{g}{4}X^2(\bar\varepsilon_L\Gamma_M\psi_R^i)(t_a\varphi)_i + {\rm H.c.}, \quad (5.23)$$

$$\delta_\varepsilon^1\varphi_i = -i\frac{g}{2\sqrt{2}}X^2(\bar\varepsilon_L\lambda_L^a + \bar\lambda_L\varepsilon_L)(t_a\varphi)_i. \quad (5.24)$$

Then we obtain the following expression

$$\delta_\varepsilon L_{1+2+3}^{\rm extra} + \delta_\varepsilon L_{\rm int} + \delta_\varepsilon L_4^{\rm extra} \quad (5.25)$$

$$= \frac{ig}{\sqrt{2}}\delta(X^2)((\varphi^\dagger t_a\varphi)[\bar\varepsilon_L\Gamma^{MN}X_M D_N\lambda_L^a] - D_M(\varphi^\dagger t_a\varphi)[\bar\varepsilon_L\Gamma_{MN}\lambda_L^a]X^N) + {\rm H.c.} \quad (5.26)$$

$$= \partial_M(\delta(X^2)(\bar\varepsilon_L V_L^M + \bar\varepsilon_L V_R^M)^{\rm int}). \quad (5.27)$$

In the final form we see that we have obtained a total divergence, with $\bar\varepsilon_L(V_L^M)^{\rm int}$ and its Hermitian conjugate $\bar\varepsilon_R(V_R^M)^{\rm int}$ given by

$$\bar\varepsilon_L(V_L^M)^{\rm int} = -\frac{ig}{\sqrt{2}}\delta(X^2)(\varphi^\dagger t_a\varphi)[\bar\varepsilon_L\Gamma^{MN}X_N\lambda_L^a], \quad (5.28)$$

$$\bar\varepsilon_R(V_R^M)^{\rm int} = \frac{ig}{\sqrt{2}}\delta(X^2)(\varphi^\dagger t_a\varphi)[\bar\varepsilon_R\bar\Gamma^{MN}X_N\lambda_R^a]. \quad (5.29)$$

We have shown that under the SUSY transformations the total Lagrangian (in the absence of the dilaton) transforms into a total divergence. Using the form of the divergence given in (5.27), and the previous pieces in the total divergence noted in Eq. (5.1), we compute the conserved SUSY current by applying Noether's theorem. The result is given by Eqs. (2.19) and (2.20).

a. Conservation of the supercurrent

In this section, we prove the conservation of the supercurrent obtained above. For clarity, we separate the supercurrent into pieces and calculate one by one

$$(J_R^M)^{\rm total} = \{(J_R^M)^{\rm chiral} + (J_R^M)^{\rm vector} + (J_R^M)^{\rm int}\}, \quad (5.30)$$

$$(J_R^M)^{\rm chiral} = \delta(X^2)\Big\{D_K(X_N\varphi^{\dagger i})(\bar\Gamma^{KN}\bar\Gamma^M - \eta^{MN}\bar\Gamma^K)\psi_{iL} + \frac{\partial W^*}{\partial\varphi^{*j}}X_N\bar\Gamma^{MN}\psi_R^j\Big\}, \quad (5.31)$$

$$(J_R^M)^{\rm vector} = \delta(X^2)\frac{1}{2\sqrt{2}}\{F_{KL}^a X_N(\bar\Gamma^{KLN}\Gamma^M - \eta^{NM}\bar\Gamma^{KL})\lambda_{Ra}\}, \quad (5.32)$$

$$(J_R^M)^{\rm int} = \delta(X^2)\Big\{-\frac{ig}{\sqrt{2}}\varphi^{\dagger i}(t_a\varphi)_i\bar\Gamma^{MN}\lambda_{Ra}X_N\Big\}. \quad (5.33)$$

By using the equations of motion that follow from (2.3) one can check explicitly that this current is conserved as follows. We first drop terms that vanish because of the homogeneity conditions for on shell fields. Then we get

$$\partial_M(J_R^M)^{\rm chiral} = \Big\{\bar D\varphi^{\dagger i}X\bar D\psi_{iL} - (D^2\varphi^{\dagger i})\bar X\psi_{iL} + \frac{-g}{2}(F_{MN}\varphi^\dagger)^i\Gamma^{MN}\psi_{iL} - \frac{\partial W^\dagger}{\partial\varphi^{\dagger i}}\bar X D\psi_R^i + \frac{\partial^2 W^\dagger}{\partial\varphi^{\dagger i}\partial\varphi^{\dagger j}}\bar D\varphi^{\dagger j}X\psi_R^i\Big\}, \quad (5.34)$$

$$\partial_M(J_R^M)^{\rm vector} = \Big\{+\frac{1}{\sqrt{2}}D^M(F_{MP}^a)X_N\bar\Gamma^{PN}\lambda_{Ra} + \frac{1}{2\sqrt{2}}F^{PQa}\bar\Gamma_{PQ}\bar X D\lambda_{Ra}\Big\}, \quad (5.35)$$

$$\partial_M(J_R^M)^{\rm int} = \Big\{+\frac{ig}{\sqrt{2}}\partial_N[\varphi^{\dagger i}(t_a\varphi)_i]\bar\Gamma^{MN}\lambda_{Ra}X_M + \frac{ig}{\sqrt{2}}\varphi^{\dagger i}(t_a\varphi)_i\bar X D\lambda_{Ra} + 2\frac{ig}{\sqrt{2}}\varphi^{\dagger i}(t_a\varphi)_i\lambda_{Ra}\Big\}. \quad (5.36)$$

Next we use the equations of motion to verify the conservation of the current. All of the following equations, and their Hermitian conjugates, should be multiplied by $\delta(X^2)$, so they are required to be satisfied only at $X^2=0$

$$(X\cdot D+1)\varphi_i = (X\cdot D+2)F_i = (X\cdot D+2)B^a = X^N F_{NM}^a = 0, \quad (5.37)$$

$$(X\cdot D+2)\psi_R^i = (X\cdot D+2)\lambda_R^a = 0, \quad (5.38)$$

$$D^2\varphi^{\dagger i} + \frac{\partial^2 W}{\partial\varphi_i\partial\varphi_j}F_j - \frac{i}{2}\bar{\psi}_{Rj}C\bar{X}\psi_{Lk}\frac{\partial^3 W}{\partial\varphi_i\partial\varphi_j\partial\varphi_k} + g(\varphi^\dagger B)^i + \sqrt{2}g(\bar{\psi}_L t^a)^i X\lambda_R^a = 0, \quad (5.39)$$

$$(D_M F^{MN})^a - if^{abc}\bar{\lambda}_{Lb}\Gamma^{MN}\lambda_{Lc}X_M - ig\varphi^\dagger t^a \overleftrightarrow{D}_N\varphi + gX_M\bar{\psi}_L\Gamma^{MN}t^a\psi_L = 0, \quad (5.40)$$

$$i\bar{X}D\psi_R^i + i\bar{X}\psi_{Lj}\frac{\partial^2 W}{\partial\varphi_i\partial\varphi_j} - \sqrt{2}g(\varphi^\dagger t_a\bar{X}\lambda_L^a)^i = 0, \quad (5.41)$$

$$B^a + g\varphi^{\dagger i}(t_a\varphi)_i = 0, \qquad F_i + \frac{\partial W^\dagger}{\partial\varphi^{\dagger i}} = 0, \quad (5.42)$$

$$i\bar{X}D\lambda_R^a + \sqrt{2}g\varphi^{\dagger i}(t_a\bar{X}\psi_L)_i = 0. \quad (5.43)$$

The first two equations impose homogeneity conditions on the fields, while the others control the dynamics. In the absence of the interaction given by $L_{\rm int}$, we had already proven that $(J_R^M)^{\rm chiral}$ and $(J_R^M)^{\rm vector}$ were conserved. In the presence of the interaction $L_{\rm int}$ the surviving terms come from the sources in the equations of motion provided by $L_{\rm int}$. Indeed we find

$$\partial_M(J_R^M)^{\rm chiral} = \text{source (interaction with vector multiplet from } L_{\rm int}) \quad (5.44)$$

$$= \delta(X^2)\Big\{i\sqrt{2}g\bar{D}\varphi^{\dagger i}X(\lambda_R\varphi)_i + g(\varphi^\dagger B)^i\bar{X}\psi_{iL} + \sqrt{2}g(\bar{\psi}_L X\lambda_R)^i\bar{X}\psi_{iL} + \frac{-g}{2}(F_{MN}\varphi^\dagger)^i\Gamma^{MN}\bar{X}\psi_{iL}\Big\} \quad (5.45)$$

$$= \delta(X^2)\Big\{-i\sqrt{2}gX_M D_N\varphi^{\dagger i}\Gamma^{MN}(\lambda_R\varphi)_i + i\sqrt{2}g[X\cdot D\varphi^{\dagger i}](\lambda_R\varphi)_i + g(\varphi^\dagger t^a)^i[-g\varphi^{\dagger i}(t_a\varphi)_i]\bar{X}\psi_{iL} + \sqrt{2}g(\bar{\psi}_L X\lambda_R)^i\bar{X}\psi_{iL} + \frac{-ig}{2}(F_{MN}\varphi^\dagger)^i\Gamma^{MN}\bar{X}\psi_{iL}\Big\}, \quad (5.46)$$

where we have used the Fierz identity of Eq. (C1), the gauge invariance of $W(\varphi)$ as given in Eq. (2.10), and the equations of motion to substitute for B^a.

For the supercurrent arising from vector multiplet a similar argument gives

$$\partial_M(J_R^M)^{\rm vector} = \text{source (interaction with chiral multiplet from } L_{\rm int}) \quad (5.47)$$

$$= \delta(X^2)\Big\{\frac{1}{\sqrt{2}}(ig\varphi^\dagger t^a\overleftrightarrow{D}_P\varphi - gX_M\bar{\psi}_L\Gamma^M{}_P t^a\psi_L)X_N\bar{\Gamma}^{PN}\lambda_{Ra} + \frac{ig}{2}F^{PQa}\bar{\Gamma}_{PQ}\varphi^{\dagger i}(t_a\bar{X}\psi_L)_i\Big\}, \quad (5.48)$$

where we have used the Fierz identity of Eq. (C2). Inserting these results in Eqs. (5.34), (5.35), and (5.36) to construct $\partial_M(J_R^M)^{\rm total}$ as the sum of these, and cancelling terms due to the relation

$$\delta(X^2)\Big[\sqrt{2}g(\bar{\psi}_L X\lambda_R)^i\bar{X}\psi_{iL} - \frac{1}{\sqrt{2}}gX_M\bar{\psi}_L\Gamma^M{}_P t^a\psi_L X_N\bar{\Gamma}^{PN}\lambda_{Ra}\Big] = 0, \quad (5.49)$$

which follows from the Fierz identity in Eq. (C1), we get

$$\partial_M(J_R^M)^{\rm total} = \delta(X^2)\Big\{-i\sqrt{2}g(D_N\varphi^{\dagger i})(t_a\varphi)_i(X_M\Gamma^{MN}\lambda_R^a) - \frac{ig}{\sqrt{2}}((\varphi^\dagger t_a)^i\overleftrightarrow{D}_N\varphi_i)(X_M\bar{\Gamma}^{MN}\lambda_R^a) + \frac{ig}{\sqrt{2}}\partial_N[\varphi^{\dagger i}(t_a\varphi)_i](X_M\bar{\Gamma}^{MN}\lambda_R^a)\Big\} = 0, \quad (5.50)$$

which is seen to sum up to zero. This proves the conservation of the total supercurrent.

VI. PHYSICS CONSEQUENCES AND FUTURE DIRECTIONS

In this paper we have explicitly constructed $N=1$ supersymmetric field theory with fields of spin $0, \frac{1}{2}, 1$ in $4+2$ dimensions, which is compatible with the theoretical framework of 2T field theory and its gauge symmetries.

This represents another significant step in demonstrating that 2T physics is sufficiently general to encompass all possible physical phenomena in 1T physics. The importance of this is in the fact that 2T physics unifies many 1T-physics systems with different dynamics in different spacetimes (so different meanings of time and Hamiltonian). By further pursuing this concept in the context of supersymmetry we expect to obtain dually related $3 + 1$-dimensional supersymmetric field theories. This could be used both as a tool to perform possibly nonperturbative computations in supersymmetric $3 + 1$ field theory, as well as a new avenue to investigate what is meant by ''spacetime'' and ''unification.''

The $4 + 2$ supersymmetry transformation is given here off shell and is shown to leave invariant the action with all consistent interactions included. The SUSY transformations are different from higher-dimensional $N = 1$ supersymmetry transformations one would write down in $4 + 2$ dimensions naively. If we specialize to the on shell and noninteracting version of our equations, we find agreement with previous work [16] which was done at the level of equations of motion without an action and only for free fields. Despite the differences, the SUSY algebra, combined with the $SU(2,2) = SO(4,2)$ global symmetry of any 2T field theory, close to form the Lie superalgebra of $SU(2,2|1)$ for on shell fields, including interactions. We checked this explicitly for the chiral multiplet as shown in Appendix D, but we believe it to be true for the full interacting theory. However, for off shell fields the closure involves a tower of additional 2T gauge transformations.

The coupling of chiral and vector multiplets is studied and is uniquely fixed by the supersymmetry algebra. But unlike ordinary 1T supersymmetry, the supersymmetry for 2T field theory requires the superpotential in the theory to be purely cubic, which is consistent with what is required by 2T gauge symmetry. In the framework of 2T field theory dimensionful parameters are not permitted by the 2T-gauge symmetry. Therefore, to induce soft supersymmetry breaking it is desirable to couple the dilaton whose vacuum expectation value plays the role of the desired dimensionful parameter. To maintain SUSY, the superpartners of the dilaton should also be included.

After fixing the 2T gauge symmetry in a particular gauge as mentioned in footnote 8, the $4 + 2$ supersymmetry transformation $SU(2,2|1)$ reduces to the nonlinear superconformal transformation of the corresponding massless fields in $3 + 1$ dimensions.

The emergent $3 + 1$ SUSY field theory *in this gauge* is in most respects similar to standard SUSY field theory. However, there are some interesting additional constraints from the $4 + 2$ structure which would not be present in the general $3 + 1$ SUSY theory. One of these is the banishing of the troublesome *renormalizable CP* violating terms [18,19] of the type $\theta\varepsilon_{\mu\nu\lambda\sigma}Tr(F^{\mu\nu}F^{\lambda\sigma})$. This is good for solving the strong *CP* violation problem in QCD without an axion. This property of the emergent $3 + 1$ theory already occurs in the nonsupersymmetric 2T field theory as described in [13], and continues to be true also in the supersymmetric case.

Recalling also that the superpotential can only be purely cubic, we see that phase transitions like supersymmetry breaking and electroweak breaking need to be driven by the dilaton vacuum expectation value, and hence according to 2T physics such phase transitions must be intimately related to the physics of the supergravity multiplet. The fact that these phenomena are not allowed to be independent of each other makes the 2T-physics approach physically more appealing as described in footnote 2.

It appears that to investigate phenomenological consequences of SUSY in the context of 2T physics, we will need to construct the 2T formulation of supergravity which includes the dilaton and its couplings to matter along with the graviton. This is one of our immediate projects. We will then be in a position to describe a 2T version of the minimal supersymmetric standard model (MSSM), or its extensions, including the dilaton. The new restrictions imposed on it by 2T physics, and the corresponding phenomenological consequences, could be of great interest for phenomenological predictions at the LHC.

Generalization to extended supersymmetry with $N = 2$, 4, 8 is another research direction which is straightforward and will be discussed in a following paper. This would proceed by constructing the higher N theories from $N = 1$ blocks discussed generally in this paper. In this way the 2T gauge symmetry is maintained while the higher N structure puts more severe symmetry restrictions on the theory. The higher N theories in $4 + 2$ dimensions will then become laboratories for investigating nonperturbative phenomena both from the point of view of the new 2T vistas as well as from the point of view of earlier nonperturbative studies. The latter would include studies such as the $N = 2$ Seiberg-Witten solution [20] or the $N = 4$ AdS-CFT phenomena [21], but now directly in $4 + 2$ dimensions.

There are two ways to quantize the theory. First, solving the kinematical equations at the classical level as described in Appendix A, the theory can be quantized in the usual way in 1T-physics field theory. This would yield a variety of dual supersymmetric *quantum field theories*, which is related to one another under duality transformations of the type described in [22]. This should be valid at least at the level of tree diagrams and most likely also at higher loops. An alternative and maybe safer way to quantize the field theory is to quantize directly in $4 + 2$ dimensions and then solve the kinematic equations as Ward-type identities for the covariantly quantized theory, thus arriving at the various dual versions directly in the quantum theory. The quantization of 2T-physics field theory directly in $4 + 2$ dimensions is still under development. This involves the path integral approach, taking into account the 2T-physics gauge symmetries discussed in Appendix A, and the corresponding Faddeev-Popov ghosts. More on this effort will be reported in future publications.

Ultimately, the main impact of the 2T point of view is likely to be along the ideas described in the first paragraph of this section, so we emphasize this again: In coming down to $3 + 1$ dimensions there are a variety of spacetimes that can be obtained through the gauge fixing of the 2T gauge symmetry, and this is expected to generate a web of dual supersymmetric field theories one of which is the well-known 3 + 1-dimensional chiral multiplets coupled to the vector multiplets. We expect that nonperturbative information can be obtained from such dualities. The methods for performing this research will be discussed in another paper [22].

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APPENDIX A: FROM SP(2, R) WORLDLINE GAUGE SYMMETRY TO 2T FIELD THEORY

In this appendix we briefly outline the SP$(2, R)$ gauge symmetry in the worldline particle formalism and explain its relation to the field theoretic action principle of Eq. (1.1) and the 2T gauge symmetry at the field theory level. In particular we want to clarify the origin of the delta function $\delta(X^2)$ that appears as part of the volume element in the action principle. These issues have been explained in detail in other papers [12,13], and the brief outline in this appendix is an attempt to make the present paper self contained.

The idea of a fundamental SP$(2, R)$ gauge symmetry begins by noting that position and momentum appear at an equal footing in the commutation rules of quantum mechanics, or the Poisson brackets of classical mechanics, as well as in initial boundary conditions, before a particular system (Hamiltonian) is specified. The SP$(2, R)$ symmetry is part of canonical transformations that leave the symplectic form $dX^M P_M$ invariant. 2T physics arises by requiring that this position-momentum Sp$(2, R)$ global symmetry is promoted to a *gauge symmetry* on the worldline that holds at any instant, and for all motions.

It is useful to first consider the worldline description of a 1T massless scalar particle in Minkowski space described by a worldline action of the form

$$S = \int d\tau\left(\dot{x}^\mu p_\mu - \frac{e}{2} p_\mu p_\nu \eta^{\mu\nu}\right),$$

where $\eta^{\mu\nu}$ is the Minkowski metric. This action has a τ-reparametrization symmetry on the worldline which is sufficient to remove the negative norm states in the theory. 2T physics starts from generalizing this parametrization symmetry to Sp$(2, R)$ gauge symmetry on the worldline. The Sp$(2, R)$ gauge symmetry is just sufficient and necessary for the existence of two timelike dimensions and removing all ghost states. Further generalizations of this approach to higher groups and/or higher timelike dimensions does not seem to work for a variety of reasons to be explained elsewhere, so 2T physics seems to be special.

The free spinless massless particle in 1T physics above, as well as a host of other dynamical systems involving the spinless particle in 1T physics, are all described and unified by the following action in the 2T-physics formulation:

$$S = \int d\tau\left(\dot{X} \cdot P - \frac{1}{2} A^{ij}(\tau) Q_{ij}(X, P)\right). \tag{A1}$$

Here $A^{ij}(\tau) = A^{ji}(\tau)$ with $i = 1, 2$ is a symmetric matrix that describes the three gauge fields for Sp$(2, R)$ The three charges $Q_{ij}(X, P)$ are required to satisfy the algebra of Sp$(2, R)$ under Poisson brackets. Then this action is invariant under the following local Sp$(2, R)$ transformations:

$$\delta_\omega X^M = \frac{1}{2}\omega^{ij}(\tau)\{X^M, Q_{ij}(X, P)\} = \frac{1}{2}\omega^{ij}(\tau)\frac{\partial Q_{ij}}{\partial P_M}, \tag{A2}$$

$$\delta_\omega P_M = \frac{1}{2}\omega^{ij}(\tau)\{P_M, Q_{ij}(X, P)\} = -\frac{1}{2}\omega^{ij}(\tau)\frac{\partial Q_{ij}}{\partial X^M}, \tag{A3}$$

$$\delta_\omega A^{ij} = D_\tau \omega^{ij} = \partial_\tau \omega^{ij} + A^i{}_k \omega^{kj} - \omega^i{}_k A^{kj}, \tag{A4}$$

where indices are raised/lowered with the completely antisymmetric Levi-Civita symbol ε_{ij} which is the Sp$(2, R)$ metric.

There are an infinite set of $Q_{ij}(X, P)$'s that satisfy the required Sp$(2, R)$ Lie algebra under Poisson brackets as shown in [3]. The general case for $Q_{ij}(X, P)$ allows for all possible background fields in which the spinless particle can propagate.

The simplest form of $Q_{ij}(X, P)$ occurs for the flat background metric η_{MN} used to construct dot products

$$Q_{11} = \tfrac{1}{2}X \cdot X, \qquad Q_{12} = \tfrac{1}{4}(X \cdot P + P \cdot X), \qquad Q_{22} = \tfrac{1}{2}P \cdot P. \tag{A5}$$

At the outset η_{MN} is allowed to have any signature, but the signature is later fixed by the gauge invariance of the physical sector and by unitarity as described in the next paragraph. This example of $Q_{ij}(X, P)$'s is just the tip of an "iceberg" in the realm of 2T physics. Most (but not all) of the 2T physics discussion to date is based on this simplest and manageable case, which is rather rich in content. The construction of the 2T-physics version of the standard model [13] as a field theory in 4 + 2 dimensions, as well as the supersymmetric generalizations of 2T-field theory that we discuss in this paper and in [15] are also done in the context of this simplest $Q_{ij}(X, P)$ in the flat background.

The equations of motion for the gauge field A^{ij} require that all Sp$(2, R)$ charges $Q_{ij}(X, P) = 0$ must vanish in the physical sector. This simply means that the physical sector is that part of phase space (X^M, P_M) which is gauge invariant under Sp$(2, R)$. It is easy to see that a flat metric η_{MN} with signature for 0 or 1 timelike dimensions gives only trivial solutions to $X^2 = P^2 = X \cdot P = 0$ in the classical case. So, the highly symmetric $\mathrm{Sp}(2,R)$ invariant motions cannot be realized in a spacetime with zero or one timelike dimensions. Nontrivial $\mathrm{Sp}(2,R)$ invariant classical solutions are possible only if spacetime has two timelike dimensions, or more. However, with more than two timelike dimensions there is not enough gauge symmetry to remove ghosts. Hence there can be no more and no less than two timelike dimensions for correct description of physics with the action in Eq. (A1). The same result holds true for the general $Q_{ij}(X,P)$. So the two timelike dimensions is an output of the $\mathrm{Sp}(2,R)$ gauge symmetry; it is not an arbitrary input.

Examining further the possible gauge choices, one finds that the highly symmetric motions that satisfy $Q_{ij}(X,P) = 0$ in $d+2$ dimensions are effectively motions in one fewer time as well as one fewer space dimensions, resulting in an effective spacetime in $(d-1)+1$ dimensions with a single timelike dimension. Hence the physical sector is in agreement with the description of phenomena as formulated in 1T physics. Furthermore, the theory is unitary and satisfies causality.

The big payoff in 2T physics is the surprising fact that a given set of $Q_{ij}(X,P)$, such as those in Eq. (A5), produce many 1T-physics systems upon gauge fixing. An example is provided by the particle systems listed in Table I which emerge just from the special example with $Q_{ij}(X,P) = (X^2, P^2, X\cdot P)$. These systems are in different $(d-1)+1$ spacetimes with one time, with the same total dimension d. They describe spinless particles with or without mass, freely moving or subject to various forces, in curved or flat spacetimes. The "etc." is included because there are a few more known cases of gauge choices, and also because the complete set of emergent systems has not yet been classified. For some recent details on these solutions see Tables 1,2,3 in [22].

TABLE I. Particle systems that emerge in $(d-1)+1$ dimensions from gauge fixing and solving two constraints $X^2 = X\cdot P = 0$ in $d+2$ dimensions.

Massless relativistic particle in Minkowski spacetime
Massive relativistic particle in Minkowski spacetime
Massive particle in nonrelativistic spacetime
Nonrelativistic particle in the α/r potential (H-atom, celestial mechanics)
Harmonic oscillator in $d-2$ space dims, with mass = another dimension
Particle in the $\mathrm{AdS}_{d-n}\times \mathrm{S}^n$ spacetimes, $n=0,1,\cdots,(d-2)$
Particle in the $\mathrm{R}\times \mathrm{S}^{d-1}$ spacetime
Particle in any maximally symmetric spacetime
Particle in the Robertson-Walker cosmological spacetimes
Particle in the cosmological constant spacetime
Particle in any conformally flat spacetime
Particle in a spacetime with a general function $\alpha(x)$, including singularities etc.

Such results in 2T physics describe relationships between these systems akin to dualities, with parameters such as mass, coupling strength, curvature, etc. that emerge as moduli in the embedding of phase space in $(d-1)+1$ dimensions into phase space in $d+2$ dimensions. Furthermore the formalism reveals a hidden symmetry $\mathrm{SO}(d,2)$ in all of these systems, which is just the global symmetry $\mathrm{SO}(d,2)$ of the flat background metric η_{MN}.

Although the results above relate to a simple example of $Q_{ij}(X,P)$, they are sufficient to make the point that 1T physics is incomplete since it is not equipped to predict such new phenomena that emerge as natural outcomes from 2T physics. These predicted phenomena can be verified both by computation within 1T physics and by experimentation, showing the predictive power of 2T physics and its status as a unifying structure above 1T physics.

One way to promote the worldline theory to field theory is to do covariant first quantization by imposing the constraints $Q_{ij}(X,P)$ on physical states

$$X^2|\Phi\rangle = 0, \qquad P^2|\Phi\rangle = 0, \qquad (X\cdot P + P\cdot X)|\Phi\rangle = 0, \tag{A6}$$

and interpret the wave function in position space $\hat{\Phi}(X) = \langle X|\Phi\rangle$ as a field that satisfies the covariant quantization conditions [11,13]. Since momentum is represented as a derivative in position space $\langle X|P_M = -i\partial_M\langle X|$, the free field (before interactions) must satisfy the following equations:

$$X^2\hat{\Phi}(X) = 0, \qquad \partial_M\partial^M\hat{\Phi}(X) = 0, \\ X^M\partial_M\hat{\Phi}(X) + \partial_M(X^M\hat{\Phi}(X)) = 0. \tag{A7}$$

One can also find a unique self-interaction consistent with extending the $\mathrm{Sp}(2,R)$ gauge transformations in the presence of interactions. This was done most clearly by implementing covariant quantization in the BRST formalism in a way analogous to string field theory including interactions [12], leading to the interacting action given below, as described in detail in [12,13]. It turns out that the first and last equations in (A7) are not modified, but the Klein-Gordon type equation in the middle gets modified with a source term on the right-hand side involving interactions whose details are uniquely fixed by the BRST approach [12] or by the 2T gauge symmetry [13] discussed below.

The field $\hat{\Phi}(X)$ that satisfies the equations in (A7) is a general superposition of the physical states that are gauge invariant under $\mathrm{Sp}(2,R)$ including interactions. The general solution of the first equation is

$$\hat{\Phi}(X) = \delta(X^2)\Phi(X), \tag{A8}$$

where $\Phi(X)$ (without the hat) is any function of X^M which is not singular at $X^2 = 0$. This delta function, coming from one of the $\mathrm{Sp}(2,R)$ conditions, is the origin of the delta function that appears in the action in Eq. (1.1). The remaining equations become the following conditions on $\Phi(X)$

[rather than $\hat{\Phi}(X)$]

$$\delta(X^2)\left(X\cdot\partial+\frac{d-2}{2}\right)\Phi=0,$$
$$\delta(X^2)\left(\partial_M\partial^M\Phi+\frac{\partial V(\Phi)}{\partial\Phi}\right)+4\delta'(X^2)\left(X\cdot\partial+\frac{d-2}{2}\right)\Phi=0, \tag{A9}$$

where the self-interaction is determined uniquely up to the overall coupling constant λ as mentioned above

$$V(\Phi)=\lambda\frac{d-2}{2d}\Phi^{(2d/d-2)}. \tag{A10}$$

Equations of motion that are equivalent to these are derived from the following action

$$S(\Phi)=\int d^{d+2}X\delta(X^2)\left[\frac{1}{2}\Phi\partial^2\Phi-V(\Phi)\right]. \tag{A11}$$

After an integration by parts, this action may also be written as

$$S(\Phi)=\int d^{d+2}X\left\{\delta(X^2)\left[-\frac{1}{2}\partial_M\Phi\partial^M\Phi-V(\Phi)\right]+\delta'(X^2)\Phi^2\right\}. \tag{A12}$$

As it turns out this action has local symmetries which are evident already from the general solution $\hat{\Phi}(X)=\delta(X^2)\Phi(X)$. Namely, physics should not change under the transformation

$$\Phi(X)\rightarrow\Phi(X)+X^2\Lambda(X) \tag{A13}$$

for arbitrary $\Lambda(X)$, since the physical state $\hat{\Phi}(X)$ remains unchanged due to the property of the delta function $X^2\delta(X^2)=0$. Indeed it is argued in [13] that this symmetry is valid for the theory in Eq. (A11) and is the underlying reason for the uniqueness of the interaction given in Eq. (A10).

The action as well the gauge symmetry Eq. (A13) was extended to all fields with spin 0, 1/2 and 1 in [13]. In the case of spinning fields, in particular, for fermions, the gauge transformations Eq. (A13) were generalized to include a kappa-type fermionic symmetry. Such generalized symmetries were called the 2T-gauge symmetry in 2T-physics field theory. This gauge symmetry can be used to eliminate parts of all these fields proportional to X^2, as well as parts of the field components (e.g. fermions in higher dimensions) as gauge degrees of freedom, such that the remaining degrees of freedom are in agreement with physics in $3+1$ dimensions.

It was shown in [13] that the standard model in $3+1$ dimensions emerges from such a field theory in $4+2$ dimensions after insuring that the $SU(3)\times SU(2)\times U(1)$ gauge bosons and quarks and lepton fields in $4+2$ dimensions have been made part of the $4+2$ theory.

The reduction from $4+2$ dimensions to $3+1$ dimensions involves two steps. The first is the elimination of gauge degrees of freedom in the parts proportional to X^2 by using the 2T-gauge symmetries in field theory described in Eq. (A13) and its generalization for every field. Generally this step is SO(4,2) covariant. The second step is solving two of the three equations in Eqs. (A8) and (A9) leaving behind degrees of freedom in $3+1$ dimensions. The two equations that are solved explicitly, namely $X^2=0$ and the homogeneity condition $(X\cdot\partial+\frac{d-2}{2})\Phi=0$ do not involve interactions. We call these two equations the kinematic equations, while the remaining one is called the dynamical equation.

Solving the two kinematical equations can be done in a way parallel to the gauge fixing and solving the $X^2=X\cdot P=0$ constraints for the particle on the worldline as indicated in Table I. This is the step that produces the surprising variety of the $3+1$ systems as listed in Table I. These systems are therefore unified by dualities which amount to $Sp(2,R)$ gauge transformations. The same dualities emerge in field theory including interactions as discussed in detail in [22].

The standard model in $3+1$ dimensions correspond to the case of solving the kinematic equations along the lines of the massless particle gauge, which is the first item in Table I. The same $4+2$ theory has also all the other dual versions in $3+1$ dimensions listed in Table I, leading to a set of field theories that are dual to the standard model. The methods for investigating these dualities has been initiated in [22] and is expected to lead to some nonperturbative insights in field theory.

Much of the discussion above is at the level of classical field theory. After solving the kinematical equations at the classical level, the theory can be quantized in the usual way in 1T-physics field theory. This would yield a variety of dual *quantum field theories* as in Table I, which may be compared to one another under duality transformations of the type described in [22]. This should be valid at least at the level of tree diagrams and most likely also at higher loops. However, there is some danger that the procedure of first solving the kinematic equations at the classical level may miss some subtleties. It is safer to first quantize the field theory directly in $d+2$ dimensions and then solve the kinematic equations as Ward-type identities for the covariantly quantized theory, thus arriving at the various dual versions directly in the quantum theory. The quantization of 2T-physics field theory directly in $d+2$ dimensions is still under development. This involves the path integral approach, taking into account the 2T-physics gauge symmetries discussed above, and the corresponding Faddeev-Popov ghosts. More on this effort will be reported in future publications.

This 2T-physics field theory formalism is the starting point of our paper, as in Eq. (1.1), for generalizations to the supersymmetric version of field theory in $4+2$ dimensions.

APPENDIX B: GAMMA MATRICES FOR SO(d, 2) AND SO(4, 2) = SU(2, 2)

We consider at first even dimensions $d+2$ in general, for a spacetime X^M labeled by M, which forms the vector basis of SO(d, 2). There are two Weyl spinors labeled by α, $\dot{\alpha}$,

$$\psi^L_\alpha, \qquad \psi^R_{\dot{\alpha}}, \tag{B1}$$

so there are two representations of gamma matrices $(\Gamma^M)_\alpha^{\dot{\beta}}$ and $(\bar{\Gamma}^M)_{\dot{\alpha}}^{\beta}$ in the left/right Weyl bases. The gamma matrices must satisfy the anticommutation rules

$$\Gamma^M\bar{\Gamma}^N + \Gamma^N\bar{\Gamma}^M = 2\eta^{MN}, \qquad \bar{\Gamma}^M\Gamma^N + \bar{\Gamma}^N\Gamma^M = 2\eta^{MN}, \tag{B2}$$

where η^{MN} is the SO(d, 2) metric with signature $\eta^{MN} = \mathrm{diag}(-,-,+,+,\cdots,+)$. Then the correctly normalized SO(d, 2) generator $J^{MN} = L^{MN} + S^{MN}$ is represented on the two spinors by the spin $S^{MN} = \frac{1}{2i}\Gamma^{MN}$ or $\frac{1}{2i}\bar{\Gamma}^{MN}$ where

$$\Gamma^{MN} = \tfrac{1}{2}(\Gamma^M\bar{\Gamma}^N - \Gamma^N\bar{\Gamma}^M), \qquad \bar{\Gamma}^{MN} = \tfrac{1}{2}(\bar{\Gamma}^M\Gamma^N - \bar{\Gamma}^N\Gamma^M). \tag{B3}$$

Thus, when the M, N indices are different one gets $\Gamma^{12} = \Gamma^1\bar{\Gamma}^2$, etc. Similarly, antisymmetrized products of gamma matrices applied on the two spinors are given by

$$\Gamma^{MNK} = \tfrac{1}{3}(\Gamma^{MN}\bar{\Gamma}^K + \Gamma^{KM}\bar{\Gamma}^N + \Gamma^{NK}\bar{\Gamma}^M), \tag{B4}$$

$$\bar{\Gamma}^{MNK} = \tfrac{1}{3}(\bar{\Gamma}^{MN}\Gamma^K + \bar{\Gamma}^{KM}\Gamma^N + \bar{\Gamma}^{NK}\Gamma^M), \tag{B5}$$

$$\Gamma_{MNKL} = \tfrac{1}{4}(\Gamma_{MNK}\Gamma_L - \Gamma_{NKL}\Gamma_M + \Gamma_{KLM}\Gamma_N - \Gamma_{LMN}\Gamma_K), \text{ etc.} \tag{B6}$$

Thus, when the M, N, K indices are different one gets $\Gamma^{123} = \Gamma^1\bar{\Gamma}^2\Gamma^3$ and $\bar{\Gamma}^{123} = \bar{\Gamma}^1\Gamma^2\bar{\Gamma}^3$, etc.

An explicit form of SO(d, 2) gamma matrices Γ^M in even dimensions, labeled by $M = 0'$, $1'$, μ and $\mu = 0, i$ is given by

$$\Gamma^{0'} = -i\tau_1 \times 1, \qquad \Gamma^{1'} = \tau_2 \times 1, \qquad \Gamma^0 = 1 \times 1, \qquad \Gamma^i = \tau_3 \times \gamma^i, \tag{B7}$$

where γ^i are the SO($d-1$) gamma matrices. The $\bar{\Gamma}^M$ are the same as the Γ^M for $M = 0'$, $1'$, i, but for $M = 0 = \mu$ we have

$$\bar{\Gamma}^0 = -\Gamma^0 = -1 \times 1. \tag{B8}$$

It is useful to define a lightcone-type basis $X^{\pm'} = \frac{1}{\sqrt{2}} \times (X^{0'} \pm X^{1'})$ and the corresponding gamma matrices

$$\Gamma^{\pm'} = \frac{1}{\sqrt{2}}(\Gamma^{0'} \pm \Gamma^{1'}) = -i\sqrt{2}\tau^{\pm} \times 1. \tag{B9}$$

In this basis the metric takes the form $ds^2 = dX^M dX^N \eta_{MN} = -2dX^{+'}dX^{-'} + dX^\mu dX^\nu \eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric for SO(d, 1). Explicitly we write

$$\Gamma^{+'} = \begin{pmatrix} 0 & -i\sqrt{2} \\ 0 & 0 \end{pmatrix}, \qquad \Gamma^{-'} = \begin{pmatrix} 0 & 0 \\ -i\sqrt{2} & 0 \end{pmatrix}, \qquad \Gamma^\mu = \begin{pmatrix} \bar{\gamma}^\mu & 0 \\ 0 & \bar{\gamma}^\mu \end{pmatrix}, \tag{B10}$$

$$\bar{\Gamma}^{+'} = \begin{pmatrix} 0 & -i\sqrt{2} \\ 0 & 0 \end{pmatrix}, \qquad \bar{\Gamma}^{-'} = \begin{pmatrix} 0 & 0 \\ -i\sqrt{2} & 0 \end{pmatrix}, \qquad \bar{\Gamma}^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \bar{\gamma}^\mu \end{pmatrix}, \tag{B11}$$

where

$$\gamma_\mu = (1, \gamma_i), \qquad \bar{\gamma}_\mu = (-1, \gamma_i), \quad \text{or} \quad \gamma^\mu = (-1, \gamma^i), \qquad \bar{\gamma}^\mu = (1, \gamma^i), \tag{B12}$$

paying attention to the lower or upper μ indices since the SO(d, 1) metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \cdots, 1)$. It should be emphasized that the γ^μ, $\bar{\gamma}^\mu$ in $\bar{\Gamma}^\mu$ are switched relative to Γ^μ. We can further write

$$\gamma^1 = \sigma^1 \times 1, \qquad \gamma^2 = \sigma^2 \times 1, \qquad \gamma^r = \sigma^3 \times \rho^r, \tag{B13}$$

where the ρ^r are the gamma matrices for SO($d-3$). With the explicit form of the gamma matrices above we have

$$\Gamma^{+'-'} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma^{+'\mu} = i\sqrt{2}\begin{pmatrix} 0 & \bar{\gamma}^\mu \\ 0 & 0 \end{pmatrix}, \tag{B14}$$

$$\Gamma^{\mu\nu} = \begin{pmatrix} \bar{\gamma}^{\mu\nu} & 0 \\ 0 & \gamma^{\mu\nu} \end{pmatrix}, \qquad \Gamma^{-'\mu} = i\sqrt{2}\begin{pmatrix} 0 & 0 \\ \gamma^\mu & 0 \end{pmatrix}, \tag{B15}$$

similarly

$$\bar{\Gamma}^{+'-'} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \bar{\Gamma}^{+'\mu} = i\sqrt{2}\begin{pmatrix} 0 & \gamma^\mu \\ 0 & 0 \end{pmatrix}, \tag{B16}$$

$$\bar{\Gamma}^{\mu\nu} = \begin{pmatrix} \gamma^{\mu\nu} & 0 \\ 0 & \bar{\gamma}^{\mu\nu} \end{pmatrix}, \qquad \bar{\Gamma}^{-'\mu} = i\sqrt{2}\begin{pmatrix} 0 & 0 \\ \bar{\gamma}^\mu & 0 \end{pmatrix}. \tag{B17}$$

Then $X = X^M\Gamma_M$, $\bar{X} = X^M\bar{\Gamma}_M$, $\frac{1}{2}\Gamma_{MN}J^{MN}$, $\frac{1}{2}\bar{\Gamma}_{MN}J^{MN}$ etc. take explicit matrix forms, such as

$$X^M\Gamma_M = -X^{+'}\Gamma^{-'} - X^{-'}\Gamma^{+'} + X^\mu\Gamma_\mu = \begin{pmatrix} X^\mu\bar{\gamma}_\mu & i\sqrt{2}X^{-'} \\ i\sqrt{2}X^{-'} & -X^\mu\gamma_\mu \end{pmatrix} \tag{B18}$$

and

$$\frac{1}{2}\Gamma_{MN}J^{MN} = -\Gamma^{+'-'}J^{+'-'} + \frac{1}{2}J_{\mu\nu}\Gamma^{\mu\nu} - \Gamma^{+'}_{\mu}J^{-'\mu} - \Gamma^{-'}_{\mu}J^{+'\mu} \tag{B19}$$

$$= \begin{pmatrix} \frac{1}{2}J_{\mu\nu}\bar{\gamma}^{\mu\nu} + J^{+'-'} & -i\sqrt{2}\bar{\gamma}_{\mu}J^{-'\mu} \\ i\sqrt{2}\gamma_{\mu}J^{+'\mu} & \frac{1}{2}J_{\mu\nu}\gamma^{\mu\nu} - J^{+'-'} \end{pmatrix}. \tag{B20}$$

If we specialize to $SO(4,2) = SU(2,2)$ with $d+2=6$. Then the ρ^r are replaced just by the number 1 and then the γ_μ, $\bar{\gamma}_\mu$ are given in terms of the 2×2 Pauli matrices

$$\gamma_\mu = (1, \vec{\sigma}), \qquad \bar{\gamma}_\mu = (-1, \vec{\sigma}), \quad \text{or} \quad \gamma^\mu = (-1, \vec{\sigma}), \quad \bar{\gamma}^\mu = (1, \vec{\sigma}). \tag{B21}$$

1. Metric, Hermitian conjugation

To be specific we now specialize to $SO(4,2) = SU(2,2)$ with $d+2=6$. The gamma matrices we have defined are consistent with the metric $\eta^{\dot{\alpha}\beta}$ or $\eta^{\alpha\dot{\beta}}$ in spinor space given as follows:

$$\eta^{\dot{\alpha}\beta} = -i\tau_1 \times 1 = \Gamma^{0'} = \bar{\Gamma}^{0'}, \qquad \eta^{\alpha\dot{\beta}} = \Gamma^{0'} = \bar{\Gamma}^{0'}, \tag{B22}$$

$$\bar{\psi}^{\ \beta}_{L} = (\psi_L^\dagger)_{\dot{\alpha}}\eta^{\dot{\alpha}\beta} = (\psi_L^\dagger\bar{\Gamma}^{0'})^\beta, \tag{B23}$$

$$\bar{\psi}^{\ \dot{\beta}}_{L} = (\psi_R^\dagger)_{\alpha}\eta^{\alpha\dot{\beta}} = (\psi_R^\dagger\Gamma^{0'})^{\dot{\beta}}. \tag{B24}$$

The metric η has the following properties:

$$\eta = \Gamma^{0'}, \qquad \eta^2 = -1, \qquad \eta^{-1} = -\eta, \qquad \eta^T = \eta, \qquad \eta^\dagger = -\eta. \tag{B25}$$

We then note that

$$\Gamma^{0'}\Gamma^{M}\Gamma^{0'} = \begin{pmatrix} 0' & 1' & 0 & i \\ \tau_1\times 1, & \tau_2 \times 1, & -1\times 1, & \tau_3\times\sigma^i \end{pmatrix} = (\bar{\Gamma}^M)^\dagger. \tag{B26}$$

From this we obtain the following properties:

$$\eta\Gamma^M\eta^{-1} = -(\bar{\Gamma}^M)^\dagger \quad \text{and} \quad \eta\bar{\Gamma}^M\eta^{-1} = -(\bar{\Gamma}^M)^\dagger, \tag{B27}$$

$$\eta\Gamma^{MN}(\eta)^{-1} = -(\Gamma^{MN})^\dagger \quad \text{and} \quad \eta\bar{\Gamma}^{MN}\eta^{-1} = -(\bar{\Gamma}^{MN})^\dagger, \tag{B28}$$

$$\eta\Gamma^{MN}\Gamma^K\eta^{-1} = (\bar{\Gamma}^K\Gamma^{MN})^\dagger \quad \text{and} \quad \eta\Gamma^K\bar{\Gamma}^{MN}\eta^{-1} = (\Gamma^{MN}\bar{\Gamma}^K)^\dagger, \tag{B29}$$

$$\eta\Gamma^{MNK}\eta^{-1} = (\bar{\Gamma}^{MNK})^\dagger \quad \text{and} \quad \eta\bar{\Gamma}^{MNK}\eta^{-1} = (\Gamma^{MNK})^\dagger. \tag{B30}$$

The second line is derived from the first line: $\eta(\Gamma^M\bar{\Gamma}^N)\eta^{-1} = (\bar{\Gamma}^M)^\dagger(\Gamma^N)^\dagger = (\Gamma^N\bar{\Gamma}^M)^\dagger$ which leads to $(\eta\Gamma^M\bar{\Gamma}^N) = -(\eta\Gamma^N\bar{\Gamma}^M)^\dagger$. Similarly the third line is derived from the first and second lines $\eta\Gamma^{MN}\Gamma^K\eta^{-1} = [-(\Gamma^{MN})^\dagger][-(\bar{\Gamma}^K)^\dagger] = (\bar{\Gamma}^K\Gamma^{MN})^\dagger$ etc., while the fourth line follows from the third. Note that the patterns of Γ, $\bar{\Gamma}$ on the left or right are not the same in each line. From these we obtain the following properties of the matrices η, $\eta\Gamma^M$, $\eta\Gamma^{MN}$, $\eta\Gamma^{MNK}$, etc. under Hermitian conjugation

$$\eta = -\eta^\dagger, \qquad \eta\Gamma^M = (\eta\bar{\Gamma}^M)^\dagger, \qquad \eta\Gamma^{MN} = (\eta\Gamma^{MN})^\dagger, \qquad \eta\Gamma^{MNK} = -(\eta\bar{\Gamma}^{MNK})^\dagger \tag{B31}$$

and similarly for $\eta\bar{\Gamma}^M$, $\eta\bar{\Gamma}^{MN}$, $\eta\bar{\Gamma}^{MNK}$, etc.

Using these properties of the metric we obtain the following Hermiticity properties for pairs of fermions:

$$\begin{aligned} (\bar{\psi}_{1L}\psi_{2L})^\dagger &= -\bar{\psi}_{2L}\psi_{1L}, \\ (\bar{\psi}_{1L}\Gamma^M\psi_{2R})^\dagger &= \bar{\psi}_{2R}\bar{\Gamma}^M\psi_{1L}, \\ (\bar{\psi}_{1L}\Gamma^M\bar{\Gamma}^N\psi_{2L})^\dagger &= -\bar{\psi}_{2L}\Gamma^N\bar{\Gamma}^M\psi_{1L}, \\ (\bar{\psi}_{1L}\Gamma^{MN}\psi_{2L})^\dagger &= -\bar{\psi}_{2L}\Gamma^{MN}\psi_{1L} \\ (\bar{\psi}_{1L}\Gamma^{MN}\Gamma^K\psi_{2R})^\dagger &= \bar{\psi}_{2R}\bar{\Gamma}^K\Gamma^{MN}\psi_{1L}, \\ (\bar{\psi}_{1L}\Gamma^{MNK}\psi_{2R})^\dagger &= -\bar{\psi}_{2R}\Gamma^{MNK}\psi_{1L}. \end{aligned} \tag{B32}$$

These are used to verify the Hermiticity of the action, transformation properties, and consistency of $SU(2,2|1)$ group theoretical structures that appear in the text.

2. Charge conjugation, transposition, Majorana spinors

Next we define the charge conjugation matrix C for $SU(2,2)$ by

$$C = \tau_1 \times \sigma_2 = -\tilde{C}\eta, \quad \text{where } \tilde{C} \equiv C\Gamma^{0'} = -1\times i\sigma_2. \tag{B33}$$

It has the following properties:

$$C = \tau_1\times\sigma_2, \qquad C^2 = 1, \qquad C^{-1} = C, \qquad C^T = -C, \qquad C^\dagger = C. \tag{B34}$$

Then we see explicitly that

$$C\Gamma^M = \begin{pmatrix} 0' & 1' & 0 & i \\ -1\times i\sigma_2, & \tau_3\times i\sigma_2, & \tau_1\times\sigma_2, & -\tau_2\times i\sigma_2\sigma^i \end{pmatrix},$$

which shows that $C\Gamma^M$ are all antisymmetric matrices. Similarly, $C\bar{\Gamma}^M$ are also antisymmetric. Therefore we derive the following properties:

$$C\Gamma^M C^{-1} = (\Gamma^M)^T, \qquad C\bar{\Gamma}^M C^{-1} = (\bar{\Gamma}^M)^T,$$
$$C\Gamma^{MN} C^{-1} = -(\bar{\Gamma}^{MN})^T, \qquad C\bar{\Gamma}^{MN} C^{-1} = -(\Gamma^{MN})^T,$$
$$C\Gamma^{MNK} C^{-1} = (\Gamma^{MNK})^T, \qquad C\bar{\Gamma}^{MNK} C^{-1} = (\bar{\Gamma}^{MNK})^T. \tag{B35}$$

The second and third lines are derived from the first line: $C(\Gamma^M\bar{\Gamma}^N)C^{-1} = (\Gamma^M)^T(\bar{\Gamma}^N)^T = (\bar{\Gamma}^N\Gamma^M)^T$ which leads to $(C\Gamma^M\bar{\Gamma}^N) = -(C\bar{\Gamma}^N\Gamma^M)^T$, etc. Note that the patterns of Γ, $\bar{\Gamma}$ on the left or right of each equation are not the same in each line. From these we obtain the following properties of $C\Gamma^M$, $C\Gamma^{MN}$, $C\Gamma^{MNK}$ etc. under transposition:

$$(C\Gamma^M)^T = -(C\Gamma^M), \qquad (C\Gamma^{MN})^T = (C\bar{\Gamma}^{MN}), \qquad (C\Gamma^{MNK})^T = (C\Gamma^{MNK}) \tag{B36}$$

and similarly for $C\bar{\Gamma}^M$, $C\bar{\Gamma}^{MN}$, $C\bar{\Gamma}^{MNK}$.

The charge conjugate spinor of a left-handed spinor $(\psi_L)_\alpha$ [a 4 of SU(2,2)] is a right-handed spinor which we denote as $(\psi_R^c)_{\dot{\alpha}}$ [a 4 of SU(2,2)] and define it by

$$(\psi_R^c)_{\dot{\alpha}} = (C\bar{\psi}_L^T)_{\dot{\alpha}} = C_{\dot{\alpha}\beta}(\bar{\psi}_L^T)^\beta = (C(\Gamma^{0'})^T\psi_L^*)_{\dot{\alpha}} = (\tilde{C}\psi_L^*)_{\dot{\alpha}}, \tag{B37}$$

with $\tilde{C} \equiv -1 \times i\sigma_2$. From this we extract $(\psi_R^c)^* = (\tilde{C}^*\psi_L) = (\tilde{C}\psi_L)$ which gives the following form after multiplying both sides with $\tilde{C}$:

$$\psi_L = -\tilde{C}(\psi_R^c)^* = -C(\Gamma^{0'})^T(\psi_R^c)^* = -C(\bar{\psi}_R^c)^T. \tag{B38}$$

So, for consistency with these equations, the charge conjugate spinors need to be defined with the following patterns of chiralities and signs:

$$(\psi_R^c) = (C\bar{\psi}_L^T), \qquad \psi_L = -C(\bar{\psi}_R^c)^T, \tag{B39}$$

$$(\psi_L^c) = -(C\bar{\psi}_R^T), \qquad \psi_R = C(\bar{\psi}_L^c)^T. \tag{B40}$$

We now define a Majorana fermion for SO(4,2) as one that satisfies the following condition:

$$\psi_{L,R}^c \overset{\text{Majorana}}{=} = \psi_{L,R}. \tag{B41}$$

Then from the above definition of $\psi_{L,R}^c$ we derive consistently that a Majorana fermion has the following properties:

$$\psi_R \overset{\text{Majorana}}{=} C\bar{\psi}_L^T, \qquad (\psi_L) \overset{\text{Majorana}}{=} C\bar{\psi}_R^T, \tag{B42}$$

$$\bar{\psi}_R \overset{\text{Majorana}}{=} (\psi_L)^T C, \qquad \bar{\psi}_L \overset{\text{Majorana}}{=} -(\psi_R)^T C. \tag{B43}$$

Using this and Eq. (B36) we now see the following permutation properties of Majorana spinors when ψ_1, ψ_2 are interchanged (treated as anticommuting Grassmann numbers). Thus we find $\bar{\psi}_{1L}\Gamma^M\psi_{2R}$ and $\bar{\psi}_{1R}\bar{\Gamma}^M\psi_{2L}$ are symmetric under the interchange of $1 \leftrightarrow 2$

$$\bar{\psi}_{1L}\Gamma^M\psi_{2R} \overset{\text{Majorana}}{=} -(\psi_{1R})^T C\Gamma^M\psi_{2R} \tag{B44}$$

$$= -(\psi_{2R})^T C\Gamma^M\psi_{1R} = \bar{\psi}_{2L}\Gamma^M\psi_{1R}. \tag{B45}$$

With similar manipulations we establish the following properties under the interchange of $1 \leftrightarrow 2$ for Majorana spinors. These follow from Eqs. (B35) and (B36)

$$\begin{array}{ll}
\text{for Majorana fermions only} & \bar{\psi}_{1L}\psi_{2L} = -\bar{\psi}_{2R}\psi_{1R} \\
\bar{\psi}_{1L}\Gamma^M\psi_{2R} = \bar{\psi}_{2L}\Gamma^M\psi_{1R}, & \bar{\psi}_{1R}\bar{\Gamma}^M\psi_{2L} = \bar{\psi}_{2R}\bar{\Gamma}^M\psi_{1L}, \\
\bar{\psi}_{1L}\Gamma^M\bar{\Gamma}^N\psi_{2L} = -\bar{\psi}_{2R}\bar{\Gamma}^N\Gamma^M\psi_{1R}, & \bar{\psi}_{1L}\Gamma^{MN}\psi_{2L} = \bar{\psi}_{2R}\bar{\Gamma}^{MN}\psi_{1R} \\
\bar{\psi}_{1L}\Gamma^{MN}\Gamma^K\psi_{2R} = -\bar{\psi}_{2L}\Gamma^K\bar{\Gamma}^{MN}\psi_{1R}, & \bar{\psi}_{1R}\bar{\Gamma}^{MN}\bar{\Gamma}^K\psi_{2L} = -\bar{\psi}_{2R}\bar{\Gamma}^K\Gamma^{MN}\psi_{1L}, \\
\bar{\psi}_{1L}\Gamma^{MNK}\psi_{2R} = -\bar{\psi}_{2L}\Gamma^{MNK}\psi_{1R}, & \bar{\psi}_{1R}\bar{\Gamma}^{MNK}\psi_{2L} = -\bar{\psi}_{2R}\bar{\Gamma}^{MNK}\psi_{1L}.
\end{array} \tag{B46}$$

Note that for the gamma matrices Γ^M, Γ^{MNK} the interchange $1 \leftrightarrow 2$ is symmetric or antisymmetric, but for the gamma matrices 1, Γ^{MN} the interchange $1 \leftrightarrow 2$ is neither symmetric nor antisymmetric since left-handed fermions are replaced by right-handed ones, and vice versa. These properties are used to manipulate various terms in proving the SUSY properties of the action and to check the consistency of $SU(2,2|1)$ group theoretical structures that appear in the text. In particular, from the third line above we deduce the following properties of the fermion kinetic term

$$i(\bar{\psi}_L X\bar{D}\psi_L + \bar{\psi}_L\overleftarrow{D}\bar{X}\psi_L) \overset{\text{Majorana}}{=} -i(\bar{\psi}_R\bar{X}D\psi_R + \bar{\psi}_R\overleftarrow{\bar{D}}X\psi_R). \tag{B47}$$

This agrees with the correct overall signs of the kinetic terms for fermions of left/right chiralities.

3. 8-component SO(4,2) Majorana spinor

Although we prefer to use the 4-component left or right $SU(2,2) = SO(4,2)$ spinor notation in this paper, for completeness and for future reference we also discuss the 8-

component spinor in this appendix. The left or right $SU(2,2) = SO(4,2)$ spinor can be rewritten as the 8-component Majorana spinor of SO(4,2). To see this we now introduce an 8-dimensional Majorana spinor ψ and its conjugate $\bar{\psi}$ through the following definitions in which ψ_L, ψ_R are related to each other as shown in Eqs. (B42) and (B43)

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \qquad \bar{\psi} = (\psi_L^\dagger, \psi_R^\dagger)\begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} = (\bar{\psi}_L, \bar{\psi}_R), \tag{B48}$$

$$\psi^c = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}\begin{pmatrix} \bar{\psi}_L^T \\ \bar{\psi}_R^T \end{pmatrix} \overset{\text{Majorana}}{=} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \psi. \tag{B49}$$

Note that ψ satisfies the Majorana condition $\psi^c = \psi$. We can also write [see Eq. (B43)]

$$\bar{\psi} = \psi^T c, \quad \text{with} \quad c = -\begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}, \qquad c^T = c. \tag{B50}$$

Then $\bar{\psi}_1 \gamma^{M_1 \cdots M_n} \psi_2$ take the form

$$\bar{\psi}_1 \gamma^M \psi_2 = (\bar{\psi}_{1L}, \bar{\psi}_{1R})\begin{pmatrix} 0 & \Gamma^M \\ \bar{\Gamma}^M & 0 \end{pmatrix}\begin{pmatrix} \psi_{2L} \\ \psi_{2R} \end{pmatrix}, \tag{B51}$$

$$\bar{\psi}_1 \gamma^{MN} \psi_2 = (\bar{\psi}_{1L}, \bar{\psi}_{1R})\begin{pmatrix} \Gamma^{MN} & 0 \\ 0 & \bar{\Gamma}^{MN} \end{pmatrix}\begin{pmatrix} \psi_{2L} \\ \psi_{2R} \end{pmatrix}, \tag{B52}$$

etc. For 8-component Majorana spinors we obtain the following permutation properties when ψ_1, ψ_2 are interchanged

$$\begin{aligned} \bar{\psi}_1 \psi_2 &= \bar{\psi}_{1L} \psi_{2L} + \bar{\psi}_{1R} \psi_{2R} \\ &: \overset{\text{Majorana}}{=} -\bar{\psi}_{2R} \psi_{1R} - \bar{\psi}_{2L} \psi_{1L} = -\bar{\psi}_2 \psi_1, \end{aligned} \tag{B53}$$

$$\begin{aligned} \bar{\psi}_1 \gamma^M \psi_2 &= \bar{\psi}_{1L} \Gamma^M \psi_{2R} + \bar{\psi}_{1R} \bar{\Gamma}^M \psi_{2L} \\ &: \overset{\text{Majorana}}{=} -\bar{\psi}_{2L} \Gamma^M \psi_{1R} + \bar{\psi}_{2R} \bar{\Gamma}^M \psi_{1L} = \bar{\psi}_2 \gamma^M \psi_1, \end{aligned} \tag{B54}$$

$$\begin{aligned} \bar{\psi}_1 \gamma^{MN} \psi_2 &= \bar{\psi}_{1L} \Gamma^{MN} \psi_{2L} + \bar{\psi}_{1R} \bar{\Gamma}^{MN} \psi_{2R} \\ &: \overset{\text{Majorana}}{=} -\bar{\psi}_{2R} \bar{\Gamma}^{MN} \psi_{1R} + \bar{\psi}_{2L} \Gamma^{MN} \psi_{1L} \\ &= \bar{\psi}_2 \gamma^{MN} \psi_1, \end{aligned} \tag{B55}$$

$$\begin{aligned} \bar{\psi}_1 \gamma^{MNK} \psi_2 &= \bar{\psi}_{1L} \Gamma^{MNK} \psi_{2R} + \bar{\psi}_{1R} \bar{\Gamma}^{MNK} \psi_{2L} \\ &: \overset{\text{Majorana}}{=} -\bar{\psi}_{2L} \Gamma^{MNK} \psi_{1R} - \bar{\psi}_{2R} \bar{\Gamma}^{MNK} \psi_{1L} \\ &= -\bar{\psi}_2 \gamma^{MNK} \psi_1. \end{aligned} \tag{B56}$$

In summary, $\bar{\psi}_i \gamma^{M_1 \cdots M_n} \psi_j$ have the following symmetry or antisymmetry properties under the interchange of i, j

$$\begin{aligned} &\text{symmetric}: \bar{\psi}_i(\gamma^M)\psi_j, \bar{\psi}_i(\gamma^{MN})\psi_j, \\ &\text{antisymmetric}: \bar{\psi}_i(1)\psi_j, \bar{\psi}_i(\gamma^{MNK})\psi_j. \end{aligned} \tag{B57}$$

We can also introduce an additional 8×8 gamma matrix

$$\gamma^7 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which anticommutes with the other six gamma matrices $\{\gamma_7, \gamma_M\} = 0$, and construct $\gamma^7\gamma^M$, $\gamma^7\gamma^{MN}$ and γ^7 as the additional traceless 8×8 gamma matrices that complete the set of all 8×8 matrices. For these we have the following permutation properties

$$\begin{aligned} \bar{\psi}_1 \gamma^7 \psi_2 &= \bar{\psi}_{1L} \psi_{2L} - \bar{\psi}_{1R} \psi_{2R} \\ &: \overset{\text{Majorana}}{=} -\bar{\psi}_{2R} \psi_{1R} + \bar{\psi}_{2L} \psi_{1L} = \bar{\psi}_2 \gamma^7 \psi_1, \end{aligned} \tag{B58}$$

$$\begin{aligned} \bar{\psi}_1 \gamma^7 \gamma^M \psi_2 &= \bar{\psi}_{1L} \Gamma^M \psi_{2R} - \bar{\psi}_{1R} \bar{\Gamma}^M \psi_{2L} \\ &: \overset{\text{Majorana}}{=} -\bar{\psi}_{2L} \Gamma^M \psi_{1R} - \bar{\psi}_{2R} \bar{\Gamma}^M \psi_{1L} \\ &= \bar{\psi}_2 \gamma^7 \gamma^M \psi_1, \end{aligned} \tag{B59}$$

$$\begin{aligned} \bar{\psi}_1 \gamma^7 \gamma^{MN} \psi_2 &= \bar{\psi}_{1L} \Gamma^{MN} \psi_{2L} - \bar{\psi}_{1R} \bar{\Gamma}^{MN} \psi_{2R} \\ &: \overset{\text{Majorana}}{=} -\bar{\psi}_{2R} \bar{\Gamma}^{MN} \psi_{1R} - \bar{\psi}_{2L} \Gamma^{MN} \psi_{1L} \\ &= -\bar{\psi}_2 \gamma^7 \gamma^{MN} \psi_1. \end{aligned} \tag{B60}$$

In summary, $\bar{\psi}_i \gamma^7 \gamma^{M_1 \cdots M_n} \psi_j$ have the following symmetry or antisymmetry properties under the interchange of ψ_i, ψ_j

$$\begin{aligned} &\text{symmetric}: \bar{\psi}_i(\gamma^7)\psi_j, \bar{\psi}_i(\gamma^7\gamma^M)\psi_j, \\ &\text{antisymmetric}: \bar{\psi}_i(\gamma^7\gamma^{MN})\psi_j. \end{aligned} \tag{B61}$$

The symmetry or antisymmetry properties given above can be related to the properties of SO(5,2) gamma matrices given by $\gamma^m = (\gamma^M, \gamma^7)$. Specifically we note the 8×8 SO(5,2) gamma matrices c and $c\gamma^{mnk}$ are symmetric while $c\gamma^m$, $c\gamma^{mn}$ are antisymmetric, where c is given in Eq. (B44). This is easily understood by simple counting of dimensions in spinor and vector spaces of SO(5,2). That is, the symmetric products in spinor space gives $(8 \times 8)_s = \frac{8 \cdot 9}{1 \cdot 2} = 36$, while for gamma matrices $c \oplus c\gamma^{mnk}$ we count the same dimension, namely, $1 + \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 36$. Similarly for the antisymmetric product in spinor space we have $(8 \times 8)_a = \frac{8 \cdot 7}{1 \cdot 2} = 28$, while for $c\gamma^m \oplus c\gamma^{mn}$ we count the same dimension, namely, $7 + \frac{7 \cdot 6}{1 \cdot 2} = 28$. From this we immediately conclude that the SO(5,2) gamma matrices have definite symmetry properties, namely, $c \oplus c\gamma^{mnk}$ are symmetric and $c\gamma^m \oplus c\gamma^{mn}$ are antisymmetric. Taking into account that $\bar{\psi} = \psi^T c$ [see Eq. (B44)], and an extra minus sign due to the interchange of Grassmann numbers, we obtain the permutation properties of fermion bilinears $\bar{\psi}_i(\gamma^{m_1 \cdots m_n})\psi_j = \psi_i^T(c\gamma^{m_1 \cdots m_n})\psi_j$ under the interchange of ψ_i, ψ_j as follows:

$$\begin{aligned} &\text{symmetric}: \bar{\psi}_j(\gamma^m)\psi_i,\ \bar{\psi}_i(\gamma^{mn})\psi_j,\\ &\text{antisymmetric}: \bar{\psi}_i(1)\psi_j,\ \bar{\psi}_i(\gamma^{mnk})\psi_j. \end{aligned} \tag{B62}$$

These SO(5, 2) properties reduce to the SO(4,2) properties for Majorana fermions as given in Eqs. (B57) and (B61) by specializing the indices $m = (M, 7)$.

The charge conjugation or Majorana properties described in this appendix are used to verify the SU(2, 2|1) group theoretical consistency of the fermion bilinears that appear in the transformation laws and other structures given in the text.

APPENDIX C: FIERZ IDENTITIES

In this appendix we prove the two Fierz identities

$$\delta(X^2)\frac{\partial^3 W}{\partial\varphi_i\partial\varphi_j\partial\varphi_k}(\bar{\psi}_{Ri}\bar{X}\psi_{Lk})(\bar{\varepsilon}_R\bar{X}\psi_{Lj}) = 0 \tag{C1}$$

and

$$\delta(X^2)f_{abc}(\bar{\varepsilon}_L[\Gamma_M, \bar{X}]\lambda_L^a)(\bar{\lambda}_L^b[\Gamma^M, \bar{X}]\lambda_L^c) = 0. \tag{C2}$$

We start with the gamma matrix identity of Eq. (3.31), which allows us to write

$$\begin{aligned} &[(\bar{\psi}_{Ri}\bar{X})^\alpha\delta_\alpha{}^\delta(\psi_{Lj})_\delta][(\bar{\varepsilon}_R\bar{X})^\gamma\delta_\gamma{}^\beta(\psi_{Lk})_\beta]\\ &\quad = -\tfrac{1}{4}(\bar{\psi}_{Ri}\bar{X}\psi_{Lk})(\bar{\varepsilon}_R\bar{X}\psi_{Lj}) + \tfrac{1}{8}(\bar{\psi}_{Ri}\bar{X}\Gamma_{MN}\psi_{Lk})\\ &\quad\quad\times(\bar{\varepsilon}_R\bar{X}\Gamma^{MN}\psi_{Lj}). \end{aligned} \tag{C3}$$

This equation is rearranged by moving the first term on the right side to the left side. After multiplying both sides with the totally symmetric $\frac{\partial^3 W}{\partial\varphi_i\partial\varphi_j\partial\varphi_k}$ and summing over i, j, k we derive

$$\begin{aligned} &\frac{5}{4}\frac{\partial^3 W}{\partial\varphi_i\partial\varphi_j\partial\varphi_k}(\bar{\psi}_{Ri}\bar{X}\psi_{Lk})(\bar{\varepsilon}_R\bar{X}\psi_{Lj})\\ &\quad = \frac{1}{8}\frac{\partial^3 W}{\partial\varphi_i\partial\varphi_j\partial\varphi_k}(\bar{\psi}_{Ri}\bar{X}\Gamma_{MN}\psi_{Lk})(\bar{\varepsilon}_R\bar{X}\Gamma^{MN}\psi_{Lj}). \end{aligned} \tag{C4}$$

We focus on the term $\bar{\psi}_{Ri}\bar{X}\Gamma_{MN}\psi_{Lk}$ on the right-hand side which can be rewritten by using gamma matrix identities as $\bar{\psi}_{Ri}\bar{X}\Gamma_{MN}\psi_{Lk} = X^P\bar{\psi}_{Ri}(\Gamma_{PMN} + \eta_{PM}\Gamma_N - \eta_{PN}\Gamma_M)\psi_{Lk}$. The Γ_{PMN} term can be rewritten as $X^P(\psi_{Li})^T(C\Gamma_{PMN})\psi_{Lk}$ by using the charge conjugation property $\bar{\psi}_{Ri} = (\psi_{Li})^T C$. We argue that the Γ_{PMN} term can be dropped due to the symmetric property of $\frac{\partial^3 W}{\partial\varphi_i\partial\varphi_j\partial\varphi_k}$ under the interchange of i and k, the symmetric property of $(C\Gamma_{PMN})^{\alpha\beta}$ under the interchange of α and β, and the antisymmetry under the interchange of two fermions. The remaining terms on the right-hand side take the form $\frac{1}{8}\frac{\partial^3 W}{\partial\varphi_i\partial\varphi_j\partial\varphi_k}(\bar{\psi}_{Ri}X_{[M}\Gamma_{N]}\psi_{Lk})\times(\bar{\varepsilon}_R X^{[M}\Gamma^{N]}\psi_{Lj})$. We drop the term proportional to X^2 since there is an overall $\delta(X^2)$. Then we obtain the relation

$$\begin{aligned} &\frac{5}{4}\delta(X^2)\frac{\partial^3 W}{\partial\varphi_i\partial\varphi_j\partial\varphi_k}(\bar{\psi}_{Ri}\bar{X}\psi_{Lk})(\bar{\varepsilon}_R\bar{X}\psi_{Lj})\\ &\quad = -\frac{1}{2}\delta(X^2)\frac{\partial^3 W}{\partial\varphi_i\partial\varphi_j\partial\varphi_k}(\bar{\psi}_{Ri}\bar{X}\psi_{Lk})(\bar{\varepsilon}_R\bar{X}\psi_{Lj}). \end{aligned} \tag{C5}$$

Pulling all terms to the same side of the equation and rearranging, we obtain the desired Fierz identity of Eq. (C1).

Next we prove the Fierz identity of Eq. (C2) where $[\Gamma_M, \bar{X}]$ is *defined* as

$$[\Gamma_M, \bar{X}] \equiv \Gamma_M\bar{X} - X\bar{\Gamma}_M = 2\Gamma_{MN}X^N. \tag{C6}$$

We start again with the gamma matrix identity of Eq. (3.31) to write

$$\begin{aligned} &f_{abc}(\bar{\varepsilon}_L[\Gamma_M, \bar{X}]\lambda_L^a)(\bar{\lambda}_L^b[\Gamma^M, \bar{X}]\lambda_L^c)\\ &\quad = f_{abc}[\tfrac{1}{8}(\bar{\lambda}_{bL}[\Gamma^M, \bar{X}]\Gamma^{RQ}\lambda_{aL})(\bar{\varepsilon}_L[\Gamma_M, \bar{X}]\Gamma_{RQ}\lambda_{cL})\\ &\quad\quad - \tfrac{1}{4}(\bar{\lambda}_{bL}[\Gamma^M, \bar{X}]\lambda_{aL})(\bar{\varepsilon}_L[\Gamma_M, \bar{X}]\lambda_{cL})]. \end{aligned} \tag{C7}$$

The last term on the right side has the same form as the left side, so the equation is rearranged as

$$\begin{aligned} &\tfrac{3}{4}f_{abc}\bar{\varepsilon}_L[\Gamma_M, \bar{X}]\lambda_L^a\bar{\lambda}_L^b[\Gamma^M, \bar{X}]\lambda_L^c\\ &\quad = \tfrac{1}{8}f_{abc}(\bar{\lambda}_{bL}[\Gamma^M, \bar{X}]\Gamma^{RQ}\lambda_{aL})(\bar{\varepsilon}_L[\Gamma_M, \bar{X}]\Gamma_{RQ}\lambda_{cL}). \end{aligned} \tag{C8}$$

By using gamma matrix identities (4.14), and setting X^2 terms to zero since there is an overall $\delta(X^2)$, the right-hand side can be rewritten as

$$\begin{aligned} &\tfrac{4}{8}f^{abc}(\bar{\lambda}_{bL}\Gamma^{MNRQ}\lambda_{aL})(\bar{\varepsilon}_L\Gamma_{MPRQ}\lambda_{cL})X_N X^P\\ &\quad + \tfrac{4}{8}f^{abc}(\bar{\lambda}_{bL}[\Gamma^M, \bar{X}]\lambda_{aL})(\bar{\varepsilon}_L[\Gamma_M, \bar{X}]\lambda_{cL}). \end{aligned} \tag{C9}$$

The last term in Eq. (C9) is similar to the left side of Eq. (C7), so they combine, and we are left with

$$\begin{aligned} &\tfrac{1}{4}\delta(X^2)f_{abc}\bar{\varepsilon}_L[\Gamma_M, \bar{X}]\lambda_L^a\bar{\lambda}_L^b[\Gamma^M, \bar{X}]\lambda_L^c\\ &\quad = \tfrac{4}{8}\delta(X^2)f^{abc}X_N X^P(\bar{\lambda}_{bL}\Gamma^{MNRQ}\lambda_{aL})(\bar{\varepsilon}_L\Gamma_{MPRQ}\lambda_{cL}). \end{aligned} \tag{C10}$$

Next we use the fact that $\Gamma^{M_1M_2M_3M_4} = \frac{1}{2}\varepsilon^{M_1M_2M_3M_4M_5M_6}\Gamma_{M_5M_6}$ and we perform the sum

$$\left(\frac{1}{2}\right)^2 X_N X^P\varepsilon^{MNRQM_5M_6}\varepsilon_{MPRQN_5N_6} = \frac{3!}{4}\delta^N_{[P}\delta^{M_5}_{N_5}\delta^{M_6}_{N_6]}X_N X^P. \tag{C11}$$

Here we drop the terms proportional to X^2 since there is an overall delta function $\delta(X^2)$. Inserting this in Eq. (C10), we find the right-hand side of (C10) looks the same as the left side of (C10), but with the numerical coefficient $-3/4$ on the right versus $1/4$ on the left. Pulling all terms to the same side of Eq. (C10) we obtain the desired Fierz identity of Eq. (C2).

APPENDIX D: OFF SHELL CLOSURE FOR CHIRAL SUPERMULTIPLET

We now consider the closure of the SUSY transformations $(\delta_{\varepsilon_1}\delta_{\varepsilon_2}-\delta_{\varepsilon_2}\delta_{\varepsilon_1})$ applied on each field in the chiral multiplet $(\varphi,\psi_L,F)_i$, in the absence of interactions with the vector multiplet (i.e. $g=0$), with each δ_ε given in Eqs. (3.11)–(3.13).

1. Closure for scalars

$$\delta_{[\varepsilon_1}\delta_{\varepsilon_2]}\varphi=\bar{\varepsilon}_{R[2}\bar{X}(\delta_{\varepsilon_1]}\psi_L)-\tfrac{1}{2}X^2(\bar{\varepsilon}_{R[2}\bar{\partial}(\delta_{\varepsilon_1]}\psi_L) + \bar{\varepsilon}_{R[2}U^\dagger(\delta_{\varepsilon_1]}\psi_R)) \tag{D1}$$

$$\begin{aligned}=\{&i\bar{\varepsilon}_{R[2}\bar{\Gamma}^{MN}\varepsilon_{R1]}X_M\partial_N\varphi+i\bar{\varepsilon}_{R[2}\varepsilon_{R1]}[X\cdot\partial\varphi-\tfrac{1}{2}X^2\partial^2\varphi]\\ &-i\bar{\varepsilon}_{R[2}\bar{X}\varepsilon_{L1]}F+\tfrac{1}{2}X^2i\bar{\varepsilon}_{R[2}\bar{\Gamma}^M\varepsilon_{L1]}\partial_MF\\ &-\tfrac{1}{2}X^2i\bar{\varepsilon}_{R[2}\varepsilon_{R1]}U^\dagger F^\dagger+\tfrac{1}{2}X^2i\bar{\varepsilon}_{R[2}\bar{\Gamma}^M\varepsilon_{L1]}U^\dagger\partial_M\varphi\}\end{aligned} \tag{D2}$$

$$\begin{aligned}=\{&-\tfrac{1}{2}\bar{\varepsilon}_{R[2}\bar{\Gamma}^{MN}\varepsilon_{R1]}L_{MN}\varphi-i\bar{\varepsilon}_{R[2}\varepsilon_{R1]}\varphi\\ &+i\bar{\varepsilon}_{R[2}\varepsilon_{R1]}(X\cdot\partial+1)\varphi-\tfrac{1}{2}X^2i\bar{\varepsilon}_{R[2}\varepsilon_{R1]}(\partial^2\varphi+U^\dagger F^\dagger)\},\end{aligned} \tag{D3}$$

where we have used

$$\bar{\varepsilon}_{R[2}\bar{\Gamma}^M\varepsilon_{L1]}=0 \tag{D4}$$

from the second line to the third line.

From Appendix B, one can conclude that $\bar{\varepsilon}_{R[2}\Gamma^{MN}\varepsilon_{R1]}$ and $i\bar{\varepsilon}_{R[2}\varepsilon_{R1]}$ are imaginary numbers. These effective parameters in the first line of (D3) are interpreted as the closure to the global bosonic subgroup $SU(2,2)\times U(1)\subseteq SU(2,2|1)$, where the $U(1)$ is the so-called R-symmetry. The second line of (D3) is proportional to the 2T-gauge symmetry generators connected to the phase space constraints $X\cdot P$ and X^2. So these terms in the closure are 2T-gauge transformations of the scalar field [13]. If the field is partially on shell by setting $X^2=0$ and $(X\cdot\partial+1)\varphi=0$, to satisfy these constraints [derived as equations of motion in Eq. (5.37)], then the closure for such fields is purely into the bosonic subgroup of $SU(2,2|1)$.

This makes it clear that for fields that satisfy the $Sp(2,R)$ gauge invariance conditions (i.e. partially on shell), the closure is into $SU(2,2)\times U(1)\subseteq SU(2,2|1)$. However, for general off shell fields the closure of two SUSY transformations is into the global $SU(2,2)\times U(1)\subseteq SU(2,2|1)$, plus 2T-gauge transformations connected to the underlying $Sp(2,R)$ [13]. The same pattern is observed for the other components of the chiral multiplet as follows.

2. Closure for auxiliary fields

The closure of the auxiliary field F works as usual,

$$\delta_{[\varepsilon_1}\delta_{\varepsilon_2]}F=-\bar{\varepsilon}_{R[2}\left(\frac{1}{2i}\Gamma^{MN}L_{MN}+2\right)\delta_{\varepsilon_1]}\psi_L \tag{D5}$$

$$\begin{aligned}=\Big\{&\frac{1}{2}\bar{\varepsilon}_{R[2}\Gamma^{MN}\varepsilon_{L1]}L_{MN}F+2i\bar{\varepsilon}_{R[2}\varepsilon_{L1]}F\\ &-2i\bar{\varepsilon}_{R[2}\bar{\Gamma}^M\varepsilon_{R1]}\partial_M\varphi-\frac{1}{2}\bar{\varepsilon}_{R[2}\Gamma^{MNP}\varepsilon_{L1]}L_{MN}\partial_p\varphi\\ &-\frac{1}{2}\bar{\varepsilon}_{R[2}\bar{\Gamma}^{[M}\varepsilon_{R1]}\eta^{N]P}L_{MN}\partial_p\varphi\Big\}\end{aligned} \tag{D6}$$

$$=-\frac{1}{2}\bar{\varepsilon}_{R[2}\bar{\Gamma}^{MN}\varepsilon_{R1]}L_{MN}F+2i\bar{\varepsilon}_{R[2}\varepsilon_{R1]}F, \tag{D7}$$

where we have used (see Appendix B)

$$\bar{\varepsilon}_{R[2}\varepsilon_{R1]}=\bar{\varepsilon}_{R[2}\varepsilon_{L1]}, \tag{D8}$$

$$\bar{\varepsilon}_{R[2}\bar{\Gamma}^{MN}\varepsilon_{R1]}=-\bar{\varepsilon}_{R[2}\Gamma^{MN}\varepsilon_{L1]}. \tag{D9}$$

The closure on F consists again of the global bosonic subgroup $SU(2,2)\times U(1)\subseteq SU(2,2|1)$.

3. Closure for spinors

To calculate the closure on the spinor, we use the following Fierz identities which will be derived later in this subsection:

$$\bar{\varepsilon}_{R[1}\bar{\Gamma}^M\psi_L\Gamma_M\varepsilon_{R2]}=-\tfrac{3}{2}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}\psi_L+\tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}\Gamma_{MN}\psi_L, \tag{D10}$$

$$\begin{aligned}\bar{\varepsilon}_{R[1}\bar{\Gamma}^M(X_M\partial_N\psi_L)\Gamma^N\varepsilon_{R2]}=[&-\tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}X_M\partial_N\psi_L-\tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}X\cdot\partial\psi_L+\tfrac{1}{8}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{PRMN}\varepsilon_{R2]}\Gamma_{PR}X_M\partial_N\psi_L\\ &+\tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MR}\varepsilon_{R2]}\Gamma_RX_M\partial\psi_L-\tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}X_M\partial_N\psi_L+\tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{PN}\varepsilon_{R2]}X\Gamma_P\partial_N\psi_L\\ &-\tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}X_M\partial_N\psi_L+\tfrac{1}{8}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}\Gamma_{MN}X\cdot\partial\psi_L+\tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}X\partial\psi_L-\tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}X\cdot\partial\psi_L],\end{aligned} \tag{D11}$$

$$\bar{\varepsilon}_{L[1}\psi_L\varepsilon_{L2]}=-\tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}\psi_L-\tfrac{1}{8}\Gamma_{PR}\psi_L\bar{\varepsilon}_{R[1}\bar{\Gamma}^{PR}\varepsilon_{R2]}, \tag{D12}$$

and

$$\bar{\varepsilon}_{L[1}\Gamma^{MN}(X_M\partial_N\psi_L)\varepsilon_{L2]} = \{\tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}(X_M\partial_N\psi_L) + \tfrac{1}{8}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MNPR}\varepsilon_{R2]}\Gamma_{PR}(X_M\partial_N\psi_L) - \tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MR}\varepsilon_{R2]}\Gamma^P{}_R X_M\partial_P\psi_L + \tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MR}\varepsilon_{R2]}\Gamma^P{}_R X_P\partial_M\psi_L - \tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}\Gamma^{MN}X_M\partial_N\psi_L\}. \tag{D13}$$

Then we compute

$$\delta_{[\varepsilon_1}\delta_{\varepsilon_2]}\psi_L = i\partial(\delta_{[\varepsilon_1}\varphi)\varepsilon_{R2]} - i\delta_{[\varepsilon_1}F\varepsilon_{L2]} \tag{D14}$$

$$= \{i\bar{\varepsilon}_{R[1}\partial[\bar{X}\psi_L - \tfrac{1}{2}X^2(\bar{\partial}\psi_L + U^\dagger\psi_R)]\varepsilon_{R2]} - i\bar{\varepsilon}_{L[1}(\Gamma^{MN}X_M\partial_N - 2)\psi_L\varepsilon_{R2]}\}. \tag{D15}$$

This becomes

$$= \{[i\bar{\varepsilon}_{R[1}\bar{\Gamma}^M\psi_L\Gamma_M\varepsilon_{R2]}] + [i\bar{\varepsilon}_{R[1}\bar{\Gamma}^M(X_M\partial_N\psi_L)\Gamma^N\varepsilon_{R2]}] + [-i\bar{\varepsilon}_{L[1}\Gamma^{MN}X_M\partial_N\psi_L\varepsilon_{R2]}] + [2i\bar{\varepsilon}_{L[1}\psi_L\varepsilon_{R2]}] + (X\zeta + X^2\varrho)\}. \tag{D16}$$

Here we note that everything of the form $X\zeta + X^2\varrho$ in the transformation of ψ_L is a 2T-gauge transformation of the spinor [13].

This can further be put into the form

$$\delta_{[\varepsilon_1}\delta_{\varepsilon_2]}\psi_L = \left\{-\frac{1}{2}\bar{\varepsilon}_{R[2}\bar{\Gamma}^{MN}\varepsilon_{R1]}\left(L_{MN} + \frac{1}{2i}\Gamma_{MN}\right)\psi_L + \frac{i}{2}\bar{\varepsilon}_{R[2}\varepsilon_{R1]}\psi_L\right\} \tag{D17}$$

$$+ \left\{-\frac{i}{8}\bar{\varepsilon}_{R[2}\bar{\Gamma}^{MN}\varepsilon_{R1]}\Gamma_{MN}(X\cdot\partial + 2)\psi_L + \frac{3}{4}i\bar{\varepsilon}_{R[2}\varepsilon_{R1]}(X\cdot\partial + 2)\psi_L + (X\zeta + X^2\varrho)\right\} \tag{D18}$$

In this form, we see that the first bracket represents the closure into the bosonic subgroup $SU(2,2)\times U(1) \subseteq SU(2,2|1)$, with the correct SU(2,2) generator $(L_{MN} + \frac{1}{2i}\Gamma_{MN})$ for the spin 1/2 fermion. The second bracket is again a 2T-gauge transformation since $(X\cdot\partial + 2)\psi_L$ is the action of the $Sp(2,R)$ generator $X\cdot P$ on the fermion [13]. For a partially on shell homogeneous field $(X\cdot\partial + 2)\psi_L = 0$ that is $Sp(2,R)$ gauge invariant [which is an equation of motion at $g = 0$ as in Eq. (5.38)], the second bracket drops out. Hence for $Sp(2,R)$ gauge invariant fields the closure is purely into the bosonic subgroup of $SU(2,2|1)$.

If we gauge fix the 2T gauge symmetry as in footnote 8, the transformations become the familiar hidden superconformal symmetry of $N = 1$ chiral multiplet.

4. Proof of the identities (D10)–(D13)

The first two identities are proved as follows. Using the Fierz identity in Eq. (3.31), we can write

$$\bar{\varepsilon}_{R[1}\bar{\Gamma}^M\psi_L\Gamma^N\varepsilon_{R2]} = -\tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^M\Gamma^N\varepsilon_{R2]}\psi_L + \tfrac{1}{8}\bar{\varepsilon}_{R[1}\bar{\Gamma}^M\Gamma^{PR}\Gamma^N\varepsilon_{R2]}\Gamma_{PR}\psi_L. \tag{D19}$$

Using the commutation relation $[\bar{\Gamma}^M, \Gamma^{PR}] = 2\eta^{MP}\bar{\Gamma}^R - 2\eta^{MR}\bar{\Gamma}^P$ we change the order of $\bar{\Gamma}^M$ and Γ^{PR} for the second term on the right-hand side. After that, use $\bar{\Gamma}^M\Gamma^N = \bar{\Gamma}^{MN} - \eta^{MN}$ and (4.14), to get

$$\bar{\varepsilon}_{R[1}\bar{\Gamma}^M\psi_L\Gamma^N\varepsilon_{R2]} = \{-\tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}\psi_L - \tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}\eta^{MN}\psi_L + \tfrac{1}{8}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{PRMN}\varepsilon_{R2]}\Gamma_{PR}\psi_L - \tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{P\{M}\varepsilon_{R2]}\Gamma_P{}^{N\}}\psi_L + \tfrac{1}{8}\eta^{MN}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{PR}\varepsilon_{R2]}\Gamma_{PR}\psi_L + \tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}\Gamma^{MN}\psi_L\}. \tag{D20}$$

Then we can use (D20) to derive the first two identities

$$\bar{\varepsilon}_{R[1}\bar{\Gamma}^M\psi_L\Gamma_M\varepsilon_{R2]} = -\tfrac{3}{2}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}\psi_L + \tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}\Gamma_{MN}\psi_L,$$

and

$$\bar{\varepsilon}_{R[1}\bar{\Gamma}^M(X_M\partial_N\psi_L)\Gamma^N\varepsilon_{R2]} = [-\tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}X_M\partial_N\psi_L - \tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}X\cdot\partial\psi_L + \tfrac{1}{8}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{PRMN}\varepsilon_{R2]}\Gamma_{PR}X_M\partial_N\psi_L - \tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{PM}\varepsilon_{R2]}\Gamma_{PN}X_M\partial^N\psi_L + \tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{PN}\varepsilon_{R2]}\Gamma_{MP}X^M\partial_N\psi_L + \tfrac{1}{8}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}\Gamma_{MN}X\cdot\partial\psi_L + \tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}\Gamma^{MN}X_M\partial_N\psi_L] \tag{D21}$$

$$= [-\tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}X_M\partial_N\psi_L - \tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}X\cdot\partial\psi_L + \tfrac{1}{8}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{PRMN}\varepsilon_{R2]}\Gamma_{PR}X_M\partial_N\psi_L + \tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MR}\varepsilon_{R2]}\Gamma_R X_M\partial\psi_L - \tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}X_M\partial_N\psi_L + \tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{PN}\varepsilon_{R2]}X\Gamma_P\partial_N\psi_L - \tfrac{1}{4}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}X_M\partial_N\psi_L + \tfrac{1}{8}\bar{\varepsilon}_{R[1}\bar{\Gamma}^{MN}\varepsilon_{R2]}\Gamma_{MN}X\cdot\partial\psi_L + \tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}X\partial\psi_L - \tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}X\cdot\partial\psi_L]. \quad \text{(D22)}$$

On the other hand, using the Fierz identities (D8) and (D9) we can easily derive

$$\bar{\varepsilon}_{L[1}\psi_L\varepsilon_{L2]} = -\tfrac{1}{4}\bar{\varepsilon}_{L[1}\varepsilon_{L2]}\psi_L + \tfrac{1}{8}\Gamma_{PR}\psi_L\bar{\varepsilon}_{L[1}\Gamma^{PR}\varepsilon_{L2]} \quad \text{(D23)}$$

$$= -\tfrac{1}{4}\bar{\varepsilon}_{R[1}\varepsilon_{R2]}\psi_L - \tfrac{1}{8}\Gamma_{PR}\psi_L\bar{\varepsilon}_{R[1}\bar{\Gamma}^{PR}\varepsilon_{R2]}. \quad \text{(D24)}$$

Now let us tackle the last identity. First, we use the Fierz identity,

$$\bar{\varepsilon}_{L[1}\Gamma^{MN}(X_M\partial_N\psi_L)\varepsilon_{L2]} \quad \text{(D25)}$$

$$= -\tfrac{1}{4}\bar{\varepsilon}_{L[1}\Gamma^{MN}\varepsilon_{L2]}(X_M\partial_N\psi_L) + \tfrac{1}{8}\bar{\varepsilon}_{L[1}\Gamma^{MN}\Gamma^{PR}\varepsilon_{L2]}\Gamma_{PR}(X_M\partial_N\psi_L). \quad \text{(D26)}$$

Then we use (4.14) to get

$$\bar{\varepsilon}_{L[1}\Gamma^{MN}(X_M\partial_N\psi_L)\varepsilon_{L2]} = \{\tfrac{1}{4}\bar{\varepsilon}_{R[1}\Gamma^{MN}\varepsilon_{R2]}(X_M\partial_N\psi_L) + \tfrac{1}{8}\bar{\varepsilon}_{L[1}\Gamma^{MNPR}\varepsilon_{L2]}\Gamma_{PR}(X_M\partial_N\psi_L) + \tfrac{1}{4}\bar{\varepsilon}_{L[1}\Gamma^{MR}\varepsilon_{L2]}\Gamma^P{}_R X_M\partial_P\psi_L - \tfrac{1}{4}\bar{\varepsilon}_{L[1}\Gamma^{MR}\varepsilon_{L2]}\Gamma^P{}_R X_P\partial_M\psi_L - \tfrac{1}{4}\bar{\varepsilon}_{L[1}\varepsilon_{L2]}\Gamma^{MN}X_M\partial_N\psi_L\}. \quad \text{(D27)}$$

Finally, we use (D8) and (D9) to change left-handed spinors to the charge-conjugated right-handed spinors, and similarly

$$\bar{\varepsilon}_{L[1}\Gamma^{MNPR}\varepsilon_{L2]} = -i\epsilon^{MNPRQS}\bar{\varepsilon}_{L[1}\Gamma_{QS}\varepsilon_{L2]} \quad \text{(D28)}$$

$$= i\epsilon^{MNPRQS}\bar{\varepsilon}_{R[1}\bar{\Gamma}_{QS}\varepsilon_{R2]} \quad \text{(D29)}$$

$$= \bar{\varepsilon}_{R[1}\bar{\Gamma}^{MNPR}\varepsilon_{R2]} \quad \text{(D30)}$$

to obtain the last identity.

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