Noncommutative standard model at $\mathcal{O}(\theta^2)$

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We derive the most general Seiberg-Witten maps for noncommutative gauge theories in second order of the noncommutative parameter θ. Our results reveal the existence of more ambiguities than previously known. In particular, we demonstrate that some of these ambiguities enter observables like scattering cross sections and enlarge the parameter space of the noncommutative standard model beyond $\mathcal{O}(\theta)$.

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I. INTRODUCTION

With the start of data taking at the Large Hadron Collider (LHC), particle physics will, for the first time, directly probe the Tera Scale, i.e. the scale of electroweak symmetry breaking according to the standard model (SM), around 1 TeV. While the SM has been confirmed experimentally to be a very precise effective description of the physics below the Tera Scale, there are many serious contenders for the more fundamental theory beneath the SM.

For quite some time, superstring theory has been a leading candidate for the fundamental theory unifying all known interactions. There are certain solutions of superstring theory with additional spatial dimensions, where the characteristic string scales are low enough to allow experimental tests at the LHC and the planned International e^+e^- Linear Collider (ILC). One spectacular prediction [1] of superstring theory is the emergence of a noncommutative (NC) structure of spacetime at a scale Λ_{NC} associated with nonvanishing commutators

$$[x_\mu, x_\nu] = \mathrm{i}\theta_{\mu\nu} = \mathrm{i}\frac{1}{\Lambda_{\text{NC}}^2}C_{\mu\nu} \tag{1}$$

of spacetime coordinates that correspond to oriented minimal resolvable areas of size $\mathcal{O}(\Lambda_{\text{NC}}^{-2})$. While a nonvanishing commutator like (1) had been proposed much earlier [2] as a regulator of divergencies in quantum field theory (QFT) and quantum gravity, the observation of [1] caused intense renewed interest in QFT on NC spacetimes (NCQFT).

The commutator (1) can be conveniently realized on a *commuting* spacetime by replacing all products of functions by Moyal-Weyl $*$-products

$$(f * g)(x) = f(x)e^{(\mathrm{i}/2)\overleftarrow{\partial}^\mu\theta_{\mu\nu}\overrightarrow{\partial}^\nu}g(x). \tag{2}$$

A prescription for constructing arbitrary gauge theories on an NC spacetime was presented in [1]. These so-called Seiberg-Witten Maps (SWM) realize NC gauge transformations in the NC theory as ordinary commutative gauge transformations on an effective commutative gauge theory. By going to the enveloping algebra of the Lie algebra of a given gauge group, this approach [3] circumvents obstructions like charge quantization in $U(1)$ gauge theories and the prohibition of $SU(N)$ gauge groups in the earlier attempts.

In particular, this prescription allowed the construction of the so-called Noncommutative Standard Model (NCSM) [4] as an anomaly-free [5] canonical NC extension of the SM without having to introduce additional particles.[1] In the first order of an expansion in θ, one has only three new bounded parameters that depend on the choice of the representation of the enveloping algebra of the SM Lie algebra and describe new couplings among gauge bosons [4]. These couplings vanish in the minimal NCSM, where the enveloping algebra is realized by matrices acting in the vector space of the adjoint representation. Furthermore, the bosonic sector of the minimal NCSM was shown to be renormalizable at one-loop [7], where all counter terms can be expressed through the usual field strength and coupling constant renormalizations. Also the nonminimal NCSM is renormalizable at one-loop, if a *finite* gauge invariant $\mathcal{O}(\theta)$-term is added to the action [7]. Finally, the fermionic sector can be shown to require a finite number of gauge invariant four-fermion operators as additional counter terms in one-loop order [8]. In Euclidean NC space, the renormalizability of scalar and gauge models has been shown to all orders in θ [9].

Using the effective theory in the first order of the θ-expansion, several phenomenological studies were performed for past, present, and future colliders [10–12]. In a preceding paper [12], we have studied the associated production of photons and Z-bosons at hadron colliders (Tevatron and LHC) showing that at the LHC one can reach a noncommutativity scale Λ_{NC} slightly above 1 TeV [12]. Moreover, we have found that it is necessary to go beyond the first order in θ, because of significant contributions from partonic center of mass energies ex-

[1]Other constructions of NC extensions of the SM start from a $U(3) \otimes U(2) \otimes U(1)$ gauge theory and subsequently break the extraneous symmetries, introducing additional particles that must be removed from the observable spectrum [6].

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ceeding the noncommutativity scale that can actually be probed.

While it is possible in simple cases to derive expressions for families of SWM to all orders in θ [13–15], an explicit parametrization of the most general solution has not been given. Therefore, we start in this paper by constructing the most general SWM for the NCSM in the second order of the θ-expansion. The importance of this systematic approach is stressed *a posteriori* by discovering ambiguities in the SWM that have been missed in earlier $\mathcal{O}(\theta^2)$ constructions of SWM [3,16]. While these authors expected all ambiguities to cancel in observable quantities, we find that they do *not*. In fact, using $e^+e^- \to \gamma\gamma$ as an example, we will calculate the ambiguity in the corresponding scattering amplitude explicitly.

The outline of this paper is as follows. In Sec. II we derive the general SWM up to second order in the noncommutativity θ. Particular emphasis will be given to the ambiguities resulting from the homogeneous solutions of the gauge equivalence equations. Furthermore, the Lagrangian and the Feynman rules of the neutral current sector of the NCSM are constructed in Sec. III. Section IV presents our analysis of the impact of the SWM ambiguities on physical observables. We will demonstrate by an explicit calculation of $e^+e^- \to \gamma\gamma$ that not all ambiguities cancel in the $\mathcal{O}(\theta^2)$ contribution to the cross section. In Sec. V we conclude with a brief summary. All expressions that are needed for the main results of this paper will be given in full, either in the main text or in the appendix. Complete expressions in second order in θ that are too lengthy to be included in this paper can be found in the appendix of [17].

II. SEIBERG-WITTEN MAPS

The purpose of the SWM is to realize noncommutative gauge transformations by representations of the enveloping algebra using nonlinear functions of ordinary, commutative fields that reside in representations of the given Lie algebra. This requirement is expressed by a set of so-called gauge equivalence equations for noncommutative gauge fields $\hat{A}_\mu(A, \theta)$, gauge parameters[2] $\hat{\lambda}(\alpha, A, \theta)$, and matter fields $\hat{\psi}(\psi, A, \theta)$ as functions of the commutative gauge fields A_μ, gauge parameters α, and matter fields ψ. General SWM are defined as solutions of the gauge equivalence equations:[3]

$$\hat{A}_\mu(A, \theta) \to \mathrm{e}^{\mathrm{i}\hat{\lambda}(\alpha,A,\theta)*}(\hat{A}_\mu(A, \theta) + \mathrm{i}\partial_\mu)\mathrm{e}^{-\mathrm{i}\hat{\lambda}(\alpha,A,\theta)*} \stackrel{!}{=} \hat{A}_\mu(A', \theta) \tag{3a}$$

$$\hat{\psi}(\psi, A, \theta) \to \mathrm{e}^{\mathrm{i}\hat{\lambda}(\alpha,A,\theta)*}\hat{\psi}(\psi, A, \theta) \stackrel{!}{=} \hat{\psi}(\psi', A', \theta), \tag{3b}$$

where A_μ and ψ transform as usual:

$$A_\mu \to A'_\mu = \mathrm{e}^{\mathrm{i}\alpha}(A_\mu + \mathrm{i}\partial_\mu)\mathrm{e}^{-\mathrm{i}\alpha} \tag{4a}$$

$$\psi \to \psi' = \mathrm{e}^{\mathrm{i}\alpha}\psi. \tag{4b}$$

Here, we have used the notation $A_\mu = A_\mu^a T^a$ and $\alpha = \alpha^a T^a$, T^a being the generators of the gauge group. In practice, the gauge equivalence Eqs. (3) can be solved order by order in an expansion in θ.

A. Field redefinitions vs SWM ambiguities

The physical predictions of QFT, in particular, the on-shell S-matrix elements and scattering cross sections, do not depend on the choice of interpolating fields [19–21]. In fact, any two theories which are related by nonsingular local field redefinitions

$$\Phi \leftrightarrow \Phi'(\Phi) \quad \text{with} \quad \left.\frac{\partial\Phi'}{\partial\Phi}\right|_{\Phi=0} = \mathbf{1} \tag{5a}$$

and the corresponding change in the Lagrangian

$$\mathcal{L}(\Phi) \leftrightarrow \mathcal{L}'(\Phi') = \mathcal{L}(\Phi(\Phi')) \tag{5b}$$

will predict identical scattering cross sections. This reparametrization invariance can be proven both in axiomatic QFT [19] and in perturbation theory for effective QFT [21]. It provides the basis for the application of the powerful and now ubiquitous methods of effective QFT to elementary particle physics phenomenology [20]. The fact that the reparametrization (5) corresponds to a change of integration variables in the path integral, provides a particularly intuitive proof of the invariance.

Since the SWM

$$\begin{pmatrix} \alpha \\ A_\mu \\ \psi \end{pmatrix} \to \begin{pmatrix} \hat{\lambda}(\alpha, A, \theta) \\ \hat{A}_\mu(A, \theta) \\ \hat{\psi}(\psi, A, \theta) \end{pmatrix} \tag{6}$$

appear to be just a reparametrization like (5), one might expect that its application has no effect on the calculation of observables. For a noncommutative $U(1)$ gauge theory with unit charge [22] it can be shown that this is indeed the case. However, in theories with multiple $U(1)$ charges and $SU(N)$ gauge groups, the enveloping algebra is strictly larger than the Lie algebra, and the SWM (6) must therefore be singular. The NCSM is a prominent representative of such theories, whence we must expect nontrivial effects of the SWM on observables, as can readily be seen from [10–12]. It was noted from early on, that the solutions of (3) are not unique [3,16]. Consequently, the construction of NC extensions of the SM via SWM is *a priori* not unique as

[2]The gauge parameters $\hat{\lambda}(\alpha, A, \theta)$ appear in the Lagrangian in the guise of Faddeev-Popov ghosts, if the gauge fixing is performed before application of the SWM.

[3]The gauge equivalence Eqs. (3) could be relaxed by demanding that the two sides of Eq. (3) lie in the same gauge orbit, but are not identical [18]. Since by construction the corresponding ambiguities must cancel in the gauge invariant Yang-Mills action to all orders in θ, we can ignore them in the rest of this paper.

well. Only those ambiguities that correspond to nonsingular reparametrization like (5) are guaranteed to cancel in observables. This leaves us with the crucial problem of genuine ambiguities in physical quantities.

To first order in θ it has been shown by explicit calculations that all ambiguities cancel in on-shell scattering amplitudes. In turn, one can use these cancellations as a powerful consistency check for numerical calculations of cross sections [12]. However, there is no general theorem or physical argument that implies the cancellation of all ambiguities in the SWM to all orders in the θ-expansion. Below we will identify ambiguities in the second-order SWM that affect cross sections. In an examination of the effective action of $\mathcal{O}(\theta)$-NCQED [22] showing that the SWM in first order corresponds to a field redefinition, it was already conjectured that this is not the case in second order. Our results provide a proof for this conjecture from a completely different angle. Furthermore, some of the ambiguities in second order are discussed in [16], but no claim is made of completeness. In fact, we have found additional ambiguities.

A recursive solution of the gauge equivalence Eqs. (3) to all orders in θ, including ambiguities, has been given in [13]. Unfortunately, the ambiguities are not given as an explicit function of an independent set of parameters and it is not straightforward to search for observable effects in the NCSM in this form. Thus we do not rely on [13] for a discussion of the general solution of (3) and derive the higher-order terms anew.

Still, the existence of ambiguities with observable effects does not render the NCSM proposed in [4] unphysical and thus rather useless. It merely adds more free parameters, not unlike those originating from the freedom of choosing the representation of the enveloping algebra, already discussed in [4]. The most serious aspect of such additional parameters is the question to which extent deviations from SM predictions could be hidden by particular choices of the ambiguous contributions, making the NCSM untestable by experiment.

B. Infinitesimal gauge transformations

The nonlinear gauge equivalence Eqs. (3) are most easily solved by going over to the equivalent set of equations for infinitesimal gauge transformations. In the commutative case, the latter are given by

$$\delta_\alpha A_\mu = D_\mu^{\mathrm{adj}} \alpha = \partial_\mu \alpha - \mathrm{i}[A_\mu, \alpha] \tag{7a}$$

$$\delta_\alpha \psi = \mathrm{i}\alpha\psi, \tag{7b}$$

while the noncommutative infinitesimal gauge transformations read

$$\hat{\delta}_\alpha \hat{A}_\mu(A,\theta) = \hat{D}_\mu^{\mathrm{adj}} \hat{\lambda}(\alpha, A, \theta) = \partial_\mu \hat{\lambda}(\alpha, A, \theta) - \mathrm{i}[\hat{A}_\mu(A,\theta) \overset{*}{,} \hat{\lambda}(\alpha, A, \theta)] \tag{8a}$$

$$\hat{\delta}_\alpha \hat{\psi}(\psi, A, \theta) = \mathrm{i}\hat{\lambda}(\alpha, A, \theta) * \hat{\psi}(\psi, A, \theta). \tag{8b}$$

From (7) and (8), one readily obtains the infinitesimal versions of the gauge equivalence Eqs. (3a) and (3b):

$$\hat{\delta}_\alpha \hat{A}_\mu(A,\theta) = \delta_\alpha \hat{A}_\mu(A,\theta) \tag{9a}$$

$$\hat{\delta}_\alpha \hat{\psi}(\psi, A, \theta) = \delta_\alpha \hat{\psi}(\psi, A, \theta), \tag{9b}$$

where the commutative gauge transformation δ_α acts on the arguments of the noncommutative gauge and matter fields via the chain rule.

However, the Eqs. (9) are not sufficient. The existence of a noncommutative gauge parameter $\hat{\lambda}(\alpha, A, \theta)$ in (3) requires that the commutator of two infinitesimal gauge transformations closes to another gauge transformation just as in the commutative case:

$$(\hat{\delta}_\alpha \hat{\delta}_\beta - \hat{\delta}_\beta \hat{\delta}_\alpha)\hat{\psi} = \hat{\delta}_{-\mathrm{i}[\alpha,\beta]}\hat{\psi}, \tag{10}$$

where $[\alpha, \beta]$ denotes the bracket in the commutative Lie algebra. Applying (9b) twice and taking into account that the commutative gauge transformation of the noncommutative gauge parameter $\hat{\lambda}(\alpha, A, \theta)$ does not vanish because it depends on A_μ, one has

$$\hat{\delta}_\alpha \hat{\delta}_\beta \hat{\psi} = \mathrm{i}\hat{\delta}_\alpha(\hat{\lambda}(\beta) * \hat{\psi}) = \mathrm{i}\hat{\delta}_\alpha \hat{\lambda}(\beta) * \hat{\psi} + \mathrm{i}\hat{\lambda}(\beta) * \hat{\delta}_\alpha \hat{\psi} = \mathrm{i}\hat{\delta}_\alpha \hat{\lambda}(\beta) * \hat{\psi} - \hat{\lambda}(\beta) * \hat{\lambda}(\alpha) * \hat{\psi}, \tag{11}$$

where all unnecessary arguments have been omitted. Substituting the above in (10) and factoring out $\hat{\psi}$, the additional consistency condition reads

$$\delta_\alpha \hat{\lambda}(\beta, A, \theta) - \delta_\beta \hat{\lambda}(\alpha, A, \theta) - \mathrm{i}[\hat{\lambda}(\alpha, A, \theta) \overset{*}{,} \hat{\lambda}(\beta, A, \theta)] = \hat{\lambda}(-\mathrm{i}[\alpha, \beta], A, \theta). \tag{12}$$

The infinitesimal consistency Eqs. (9) and (12) still contain all orders of θ and closed expressions for their solutions are not yet known. However, we can express the SWM as formal power series in θ:

$$\hat{\lambda}(\alpha, A, \theta) = \sum_{n=0}^{\infty} \lambda^{(n)}(\alpha, A, \theta) \tag{13a}$$

$$\hat{A}_\mu(A,\theta) = \sum_{n=0}^{\infty} A_\mu^{(n)}(A,\theta) \tag{13b}$$

$$\hat{\psi}(\psi, A, \theta) = \sum_{n=0}^{\infty} \psi^{(n)}(\psi, A, \theta), \tag{13c}$$

expand the Eqs. (9) and (12) and solve them order by order in θ, starting at $n = 0$ with the commutative gauge theory:

$$\lambda^{(0)}(\alpha, A, \theta) = \alpha \tag{14a}$$

$$A_\mu^{(0)}(A,\theta) = A_\mu \tag{14b}$$

$$\psi^{(0)}(\psi, A, \theta) = \psi. \tag{14c}$$

C. First order in θ

Writing all terms involving the unknown function $\lambda^{(1)}$ on the left-hand side, the expansion of (12) results in the inhomogeneous linear equation

$$\begin{aligned}\delta_\alpha \lambda^{(1)}(\beta, A, \theta) - \delta_\beta \lambda^{(1)}(\alpha, A, \theta) - \mathrm{i}[\lambda^{(1)}(\alpha, A, \theta), \beta] \\ - \mathrm{i}[\alpha, \lambda^{(1)}(\beta, A, \theta)] - \lambda^{(1)}(-\mathrm{i}[\alpha, \beta], A, \theta) \\ = -\frac{\theta^{\mu\nu}}{2}[\partial_\mu \alpha, \partial_\nu \beta]_+. \end{aligned} \tag{15}$$

The general solution is given by

$$\lambda^{(1)}(\alpha, A, \theta) = \frac{1}{4}\theta^{\mu\nu}[\partial_\mu \alpha, A_\nu]_+ + \mathrm{i}c_\lambda^{(1)}\theta^{\mu\nu}[\partial_\mu \alpha, A_\nu] \tag{16}$$

involving one free parameter $c_\lambda^{(1)}$. Using this solution, we can proceed analogously with (9) and derive the linear equations for $A_\mu^{(1)}$,

$$\begin{aligned}\delta_\alpha A_\mu^{(1)}(A, \theta) + \mathrm{i}[A_\mu^{(1)}(A, \theta), \alpha] \\ = \partial_\mu \lambda^{(1)} - \mathrm{i}[A_\mu, \lambda^{(1)}(\alpha, A, \theta)] + \frac{\theta^{\rho\sigma}}{2}[\partial_\rho A_\mu, \partial_\sigma \alpha]_+, \end{aligned} \tag{17a}$$

and $\psi^{(1)}$,

$$\begin{aligned}\delta_\alpha \psi^{(1)}(\psi, A, \theta) - \mathrm{i}\alpha\psi^{(1)}(\psi, A, \theta) \\ = \mathrm{i}\lambda^{(1)}(\alpha, A, \theta)\psi - \frac{\theta^{\rho\sigma}}{2}\partial_\rho \alpha \partial_\sigma \psi, \end{aligned} \tag{17b}$$

where the right-hand sides depend on the free parameter $c_\lambda^{(1)}$ via $\lambda^{(1)}(\alpha, A, \theta)$. For each value of $c_\lambda^{(1)}$, we then find the general solutions

$$\begin{aligned}A_\rho^{(1)}(A, \theta) = \frac{1}{4}\theta^{\mu\nu}[F_{\mu\rho}, A_\nu]_+ - \frac{1}{4}\theta^{\mu\nu}[A_\mu, \partial_\nu A_\rho]_+ \\ + \mathrm{i}c_\lambda^{(1)}\theta^{\mu\nu}[D_\rho^{\mathrm{adj}} A_\mu, A_\nu] - 2\mathrm{i}c_A^{(1)}\theta^{\mu\nu}D_\rho^{\mathrm{adj}} F_{\mu\nu}\end{aligned} \tag{18a}$$

and

$$\begin{aligned}\psi^{(1)}(\psi, A, \theta) = -\frac{1}{2}\theta^{\mu\nu}A_\mu \partial_\nu \psi \\ + \frac{\mathrm{i}}{8}(1 - 4c_\lambda^{(1)})\theta^{\mu\nu}[A_\mu, A_\nu]\psi \\ + \frac{c_\psi^{(1)}}{2}\theta^{\mu\nu}F_{\mu\nu}\psi,\end{aligned} \tag{18b}$$

parametrized by the free parameters $c_A^{(1)}$ and $c_\psi^{(1)}$, respectively. Notice that all terms proportional to $c_\lambda^{(1)}$, $c_A^{(1)}$, or $c_\psi^{(1)}$ are Lie algebra valued. Therefore, they correspond to field reparametrizations and will cancel in observables. In fact, only the anticommutators in the expression (18a) for $A_\rho^{(1)}(A, \theta)$ require the enveloping algebra and carry the potential to affect observables.

D. Second order in θ

In second order in θ, we again start with the closure relation (12) for gauge transformations:

$$\begin{aligned}\delta_\alpha \lambda^{(2)}(\beta, A) - \delta_\beta \lambda^{(2)}(\alpha, A) - \mathrm{i}[\alpha, \lambda^{(2)}(\beta, A)] - \mathrm{i}[\lambda^{(2)}(\alpha, A), \beta] - \lambda^{(2)}(-\mathrm{i}[\alpha, \beta], A) \\ = -\frac{\mathrm{i}}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}[\partial_\mu \partial_\kappa \alpha, \partial_\nu \partial_\lambda \beta] + \mathrm{i}[\lambda^{(1)}(\alpha, A), \lambda^{(1)}(\beta, A)] - \frac{1}{2}\theta^{\mu\nu}([\partial_\mu \lambda^{(1)}(\alpha, A), \partial_\nu \beta]_+ - [\partial_\mu \lambda^{(1)}(\beta, A), \partial_\nu \alpha]_+),\end{aligned} \tag{19}$$

where we have suppressed the dependence on θ in $\lambda^{(k)}(\alpha, A)$ for brevity. Again, the right-hand side depends on the free parameter $c_\lambda^{(1)}$ via $\lambda^{(1)}(\alpha, A)$. For $c_\lambda^{(1)}$ fixed, we find that the general hermitian solution of (19) involves 15 new free parameters, $c_{\lambda,1}^{(2)}, \ldots, c_{\lambda,15}^{(2)}$. For the present discussion it is sufficient to describe the general characteristics of the solutions and to give explicit expressions only for a specific choice of the free parameters. In the Appendix we spell out the specific solution corresponding to $c_{\lambda,i}^{(2)} = 0$ in the case $c_\lambda^{(1)} = 0$. The lengthy complete expression can be found in the appendix of [17]. It can be shown that the solutions given in [3,16] are both contained in our 16-parameter family of solutions.

Proceeding with the expansion of the gauge equivalence Eqs. (9) in powers of θ, one obtains the second-order equations for gauge and matter fields:

$$\begin{aligned}\delta_\alpha A_\rho^{(2)}(A) - \mathrm{i}[\alpha, A_\rho^{(2)}(A)] = \partial_\rho \lambda^2(\alpha, A) - \mathrm{i}[A_\rho, \lambda^{(2)}(\alpha, A)] - \mathrm{i}[A_\rho^{(1)}(A), \lambda^{(1)}(\alpha, A)] + \frac{1}{2}\theta^{\mu\nu}([\partial_\mu A_\rho^{(1)}(A), \partial_\nu \alpha]_+ \\ + [\partial_\mu A_\rho, \partial_\nu \lambda^{(1)}(\alpha, A)]_+) + \frac{\mathrm{i}}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu \partial_\kappa A_\rho \partial_\nu \partial_\lambda \alpha\end{aligned} \tag{20a}$$

and

$$\delta_\alpha \psi^{(2)}(A) - \mathrm{i}\alpha\psi^{(2)}(A) = \mathrm{i}\lambda^{(1)}(\alpha, A)\psi^{(1)}(A) + \mathrm{i}\lambda^2(\alpha, A)\psi - \frac{1}{2}\theta^{\mu\nu}(\partial_\mu\lambda^{(1)}(\alpha, A)\partial_\nu\psi + \partial_\mu\alpha\partial_\nu\psi^{(1)}(A)) - \frac{\mathrm{i}}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu\partial_\kappa\alpha\partial_\nu\partial_\lambda\psi, \tag{20b}$$

where the right-hand sides now depend on $c_\lambda^{(1)}$, $c_A^{(1)}$, $c_\psi^{(1)}$, $c_{\lambda,1}^{(2)}$, ..., and $c_{\lambda,15}^{(2)}$.

1. Gauge fields

Substituting (16) and (18a) for $\lambda^{(1)}(\alpha, A)$ and $A_\mu^{(1)}(A)$, respectively, and the general solution $\lambda^{(2)}(\alpha, A)$ from [17] in (20a), one gets the general hermitian solution for $A_\mu^{(2)}(A)$ which depends on 6 additional parameters $c_{A,1,\pm}^{(2)}$. The latter define a convenient basis for the solutions of the homogeneous equation:

$$A_{\rho,1,\pm}^{(2)} = c_{A,1,\pm}^{(2)}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}[D_\kappa^{\mathrm{adj}}F_{\mu\nu}, F_{\lambda\rho}]_\pm \tag{21a}$$

$$A_{\rho,2,\pm}^{(2)} = c_{A,2,\pm}^{(2)}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}[F_{\mu\kappa}, D_\rho^{\mathrm{adj}}F_{\nu\lambda}]_\pm \tag{21b}$$

$$A_{\rho,3,\pm}^{(2)} = c_{A,3,\pm}^{(2)}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}[F_{\kappa\lambda}, D_\rho^{\mathrm{adj}}F_{\mu\nu}]_\pm. \tag{21c}$$

The three commutator terms $A_{\rho,i,-}^{(2)}$ are contained in the Lie algebra and correspond to field redefinitions that must cancel in observables. In contrast, the three anticommutator terms $A_{\rho,i,+}^{(2)}$ need not be part of the Lie algebra and can, in principle, give nonvanishing contributions to scattering amplitudes as we will demonstrate in Sec. IV B. An explicit expression for a specific solution of $A_\mu^{(2)}$ is given in the Appendix.

The solution presented in [3] can be shown to be contained in our set of general solutions [17]. In contrast, the solution presented in [16] is not. In fact, one can verify that the solution as written in [16] does not satisfy the gauge equivalence Eq. (20a) and must therefore be considered incorrect. In addition, the solutions $A_{\rho,1,\pm}^{(2)}$ of the homogeneous equation are missing. As we will see later, $A_{\rho,1,+}^{(2)}$ plays a particularly important role. While the size of the expressions makes it hard to pinpoint the actual error in [16], we want to mention that our results have been obtained and verified by long but straightforward algebraic manipulations using FORM [23].

2. Matter fields

Plugging (16) for $\lambda^{(1)}(\alpha, A)$, (18b) for $\psi^1(\psi, A)$, and the general solution for $\lambda^{(2)}(\alpha, A)$ given in [17] into (20b), we obtain a three-parameter family of matter field SWM. A suitable basis for the solutions to the homogeneous equations is given by

$$\psi_1^{(2)} = \mathrm{i}c_{\psi,1}^{(2)}\theta^{\mu\nu}\theta^{\kappa\lambda}(D_\mu^{\mathrm{adj}}F_{\nu\kappa})D_\lambda\psi \tag{22a}$$

$$\psi_2^{(2)} = -\frac{c_{\psi,2}^{(2)}}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}F_{\mu\nu}F_{\kappa\lambda}\psi \tag{22b}$$

$$\psi_3^{(2)} = \frac{c_{\psi,3}^{(2)}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}F_{\mu\kappa}F_{\nu\lambda}\psi. \tag{22c}$$

This result confirms the corresponding result in [16]. In the Appendix we present a specific solution for $\psi^{(2)}$. For the general solution we again refer to [17].

III. NCSM LAGRANGIAN AND FEYNMAN RULES

Following the prescription of [1] for constructing a NCQFT, we replace all field products in the Yang-Mills Lagrangian

$$L_{\mathrm{YM}} = -\frac{1}{2}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(\mathrm{i}\not{D} - m)\psi \tag{23}$$

by $*$-products, and obtain an action for fields in the enveloping algebra that is invariant under NC gauge transformations and follows from the Lagrangian

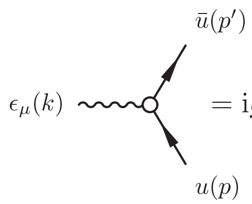
$$L_{\mathrm{YM},*} = -\frac{1}{2}\operatorname{tr}(F_{\mu\nu,*} * F_*^{\mu\nu}) + \bar{\psi} * (\mathrm{i}\not{D} - m) * \psi \tag{24}$$

with $F_{\mu\nu} = \mathrm{i}[D_\mu, D_\nu]$ and $F_{\mu\nu,*} = \mathrm{i}[D_\mu \stackrel{*}{,} D_\nu]$, respectively. In a second step, we apply the SWM (6) to obtain an action for fields residing in the Lie algebra that is invariant under commutative gauge transformations and results from

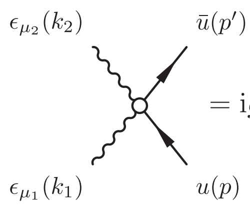
$$L_{\mathrm{NCYM}} = -\frac{1}{2}\operatorname{tr}(\hat{F}_{\mu\nu,*}(A) * \hat{F}_*^{\mu\nu}(A)) + \bar{\hat{\psi}}(\psi, A) * (\mathrm{i}\hat{\not{D}}(A) - m) * \hat{\psi}(\psi, A). \tag{25}$$

From (25), one can derive the Feynman rules for the NC extension of the original commutative gauge theory. Since the terms in (23) and (25) that are quadratic in the fields are identical, we can perform gauge-fixing and introduce Faddeev-Popov ghosts directly to (25) in terms of the commuting gauge fields. Hence, the ghost interactions are not modified by NC contributions.

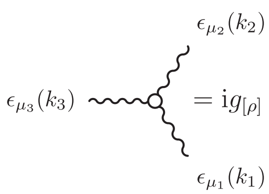
For the purposes of the present paper, we are mainly interested in the cubic and quartic couplings in the neutral current sector of the NCSM that contribute to boson pair production in fermion-antifermion annihilation, $f\bar{f} \to VV$, at tree level up to second order in θ. With *all* momenta incoming, including outgoing fermions, we define the vertex factors as follows:



$$= \mathrm{i}g \cdot V_\mu(p', k, p) \tag{26a}$$



$$= \mathrm{i}g^2 \cdot V_{\mu_2\mu_1}(p', k_2, k_1, p) \tag{26b}$$



$$= \mathrm{i}g_{[\rho]} \cdot V_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3)\,. \tag{26c}$$

where $g_{[\rho]}$ denotes the three-gauge boson coupling that depends on the choice for the representation ρ of the enveloping algebra. In a $U(1)$ gauge theory we can choose an arbitrary hermitian matrix $\rho(T)$ as a generator, normalized to $\mathrm{tr}(\rho(T)\rho(T)) = 1$. Consequently the squares of the eigenvalues of $\rho(T)$ are bounded, $0 \leq \lambda_i^2 \leq 1$, and we find $-g \leq g_{[\rho]} \leq g$, with $g_{[\rho]} = 0$ for $\rho(T) = \sigma_3/\sqrt{2}$ and $g_{[\rho]} = g$ for $\rho(T) = 1$. Only in the latter case, the anti-commutator remains in the Lie algebra representation.

Since the chiral structure of the fermionic currents remains unaffected by the SWM, we have written the following vertex factors for pure vector currents. The necessary substitutions $\gamma_\mu \to g_V\gamma_\mu - g_A\gamma_\mu\gamma_5$ depending on the fermion flavor and the type of vector boson coupled can be copied directly from the SM Lagrangian. Using the notations $p\theta q = p_\mu \theta^{\mu\nu} q_\nu$ and $p\theta^\nu = p_\mu\theta^{\mu\nu}$, the vertex factors (26) up to second order in θ are given by

$$V^{(1)}_\mu(p', k, p) = \frac{\mathrm{i}}{2}[k\theta_\mu \not{p}(1 - 4c^{(1)}_\psi) + 2k\theta_\mu \not{k}(c^{(1)}_A - c^{(1)}_\psi) - p\theta_\mu \not{k} - (k\theta p)\gamma_\mu] \tag{27a}$$

$$V^{(2)}_\mu(p', k, p) = \frac{1}{8}(k\theta p)[k\theta_\mu \not{p}(1 - 16c^{(2)}_\psi) + 4k\theta_\mu \not{k}(c^{(1)}_A - 2c^{(2)}_\psi) - p\theta_\mu \not{k} - (k\theta p)\gamma_\mu] \tag{27b}$$

$$V^{(1)}_{\mu_2\mu_1}(p', k_2, k_1, p) = \frac{\mathrm{i}}{2}[k_2\theta_{\mu_1}\gamma_{\mu_2} - k_1\theta_{\mu_1}\gamma_{\mu_2}(1 - 4c^{(1)}_\psi) - \theta_{\mu_1\mu_2}\not{k}_1 + (\mu_1 \leftrightarrow \mu_2, k_1 \leftrightarrow k_2)] \tag{27c}$$

$$\begin{aligned} V^{(2)}_{\mu_2\mu_1}(p', k_2, k_1, p) = {} & +\frac{1}{8}[k_1\theta k_2 k_1\theta_{\mu_1}\gamma_{\mu_2}(8c^{(2)}_{A,1,+} - 4c^{(1)}_\psi + 8c^{(2)}_\psi - 1) + k_1\theta p k_1\theta_{\mu_1}\gamma_{\mu_2}(16c^{(2)}_\psi - 1) \\ & + 2k_2\theta p k_1\theta_{\mu_1}\gamma_{\mu_2}(4c^{(1)}_\psi - 1) - k_1\theta k_2 k_2\theta_{\mu_1}\gamma_{\mu_2} + 3k_1\theta p k_2\theta_{\mu_1}\gamma_{\mu_2} + 2k_2\theta p k_2\theta_{\mu_1}\gamma_{\mu_2} \\ & - 3k_1\theta k_2 p\theta_{\mu_1}\gamma_{\mu_2} + 4k_1\theta_{\mu_1}k_1\theta_{\mu_2}\not{k}_1(2c^{(2)}_{A,1,+} - c^{(1)}_A - c^{(1)}_\psi) + 2k_1\theta_{\mu_1}p\theta_{\mu_2}\not{k}_1(1 - 4c^{(1)}_\psi) \\ & + 2k_2\theta_{\mu_1}p\theta_{\mu_2}\not{k}_1 - 4\theta_{\mu_1\mu_2}k_1\theta p\not{k}_1 + (\mu_1 \leftrightarrow \mu_2, k_1 \leftrightarrow k_2)] \\ & + \text{terms vanishing by equations of motion} \end{aligned} \tag{27d}$$

$$\begin{aligned} V^{(1)}_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = {} & \theta_{\mu_1\mu_2}[(k_1k_3)k_{2,\mu_3} - (k_2k_3)k_{1,\mu_3}] + (k_1\theta k_2)[k_{3,\mu_1}g_{\mu_2\mu_3} - g_{\mu_1\mu_3}k_{3,\mu_2}] \\ & + [(k_1\theta)_{\mu_1}[k_{2,\mu_3}k_{3,\mu_2} - (k_2k_3)g_{\mu_2\mu_3}] - (\mu_1 \leftrightarrow \mu_2) - (\mu_1 \leftrightarrow \mu_3)] \\ & + \text{cyclical permutations of } \{(\mu_1, k_1), (\mu_2, k_2), (\mu_3, k_3)\} \end{aligned} \tag{27e}$$

$$\begin{aligned} V^{(2)}_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = {} & \mathrm{i}[k_1\theta k_2 k_1\theta_{\mu_1}((c^{(2)}_{A,1,+} - c^{(1)}_A)(k_{1,\mu_3}k_{3,\mu_2} - g_{\mu_2\mu_3}(k_1k_3)) + c^{(2)}_{A,1,+}(k_{2,\mu_3}k_{3,\mu_2} - g_{\mu_2\mu_3}(k_2k_3))) \\ & + k_1\theta_{\mu_1}k_1\theta_{\mu_2}(c^{(2)}_{A,1,+} - c^{(1)}_A)(k_{2,\mu_3}(k_1k_3) - k_{1,\mu_3}(k_2k_3)) + ((\mu_2, k_2) \leftrightarrow (\mu_3, k_3))] \\ & + \text{cyclical permutations of } \{(\mu_1, k_1), (\mu_2, k_2), (\mu_3, k_3)\}. \end{aligned} \tag{27f}$$

Note that the triple-gauge boson vertex (27f) at $\mathcal{O}(\theta^2)$ is generated by the SWM alone. There are no contributions from the Moyal-Weyl $*$-product. In this paper, we will apply the NCSM Feynman rules to the process $f\bar{f} \to VV$ at tree level. In the above, we have therefore given only the on-shell expression for the $\bar{f}VVf$ contact term (27d), dropping terms which vanish by equation of motion. The complete expressions can be found in [17].

IV. INFLUENCE OF SWM AMBIGUITIES ON OBSERVABLES

As noted in Sec. II C, all SWM ambiguities to first order in θ, i.e. all terms in the SWM proportional to $c^{(1)}_\lambda$, $c^{(1)}_A$, and $c^{(1)}_\psi$, correspond to Lie algebra valued field redefinitions and must cancel in observables such as on-shell scattering amplitudes to this order. This is explicitly checked for

$f\bar{f} \rightarrow VV$ in (32). Note, however, that the above $\mathcal{O}(\theta)$ parameters reappear in the SWM in higher order.

A. Ambiguities to second order in θ

We have already pointed out that beyond $\mathcal{O}(\theta)$ there are no such general arguments for or against the cancellation of the SWM ambiguities that do not correspond to nonsingular field redefinitions in observables. Therefore, we can approach this question presently only by studying specific examples. For that we choose the NCQED process $e^+e^- \rightarrow \gamma\gamma$ as a prototype process in the neutral current sector of the NCSM. The additional Z-boson couplings, their chiral structure, and the Z-mass in the NCSM will not add to our conclusions about the SWM ambiguities. The Feynman diagrams contributing to $e^+e^- \rightarrow \gamma\gamma$ in NCQED are depicted in Figs. 1–3. The Feynman rules to $\mathcal{O}(\theta)$ and $\mathcal{O}(\theta^2)$ are given in Sec. III.

Actually, it is possible to demonstrate the dependence of the scattering amplitude on the free $\mathcal{O}(\theta^2)$ parameter $c^{(2)}_{A,1,+}$ and the interplay of reparametrization invariance and the enveloping algebra even without performing a complete calculation. The term in (21a) relevant for the tree-level diagrams of Fig. 3 contains two gauge fields. Suppressing terms with more than two gauge fields, one has

$$A^{(2)}_{\rho,1,+} = \mathrm{i}c^{(2)}_{A,1,+}\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu\partial_\kappa A_\nu(\partial_\lambda A_\rho - \partial_\rho A_\lambda) + \ldots. \tag{28}$$

This term contributes both to the contact and the three-gauge boson vertex. Explicitly, from the Feynman rule (27d) for the contact term one finds the following contribution to the scattering amplitude:

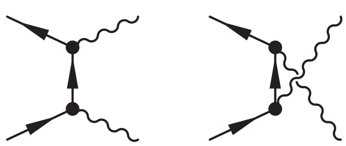


FIG. 1. Feynman diagrams contributing to $e^+e^- \rightarrow \gamma\gamma$ in QED.

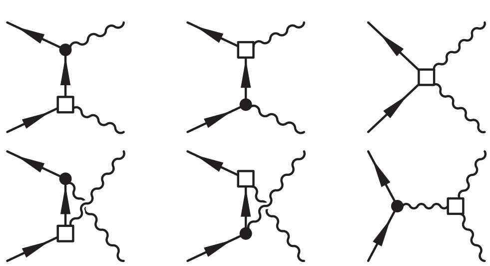


FIG. 2. Feynman diagrams contributing to $e^+e^- \rightarrow \gamma\gamma$ in $\mathcal{O}(\theta)$ in NCQED. The open squares denote $\mathcal{O}(\theta)$-vertices.

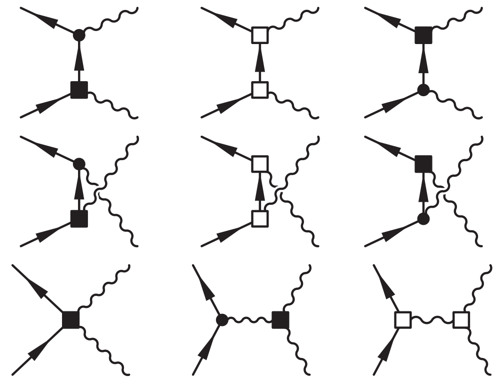


FIG. 3. Feynman diagrams contributing to $e^+e^- \rightarrow \gamma\gamma$ in $\mathcal{O}(\theta^2)$ in NCQED. The filled squares denote $\mathcal{O}(\theta^2)$-vertices.

$$A^{(2)}_{\text{contact}} = \mathrm{i}g^2c^{(2)}_{A,1,+}[k_1\theta\varepsilon_1(k_1\theta k_2\not{\varepsilon}_2 + k_1\theta\varepsilon_2\not{k}_1) + k_2\theta\varepsilon_2(k_2\theta k_1\not{\varepsilon}_1 + k_2\theta\varepsilon_1\not{k}_2)] + \ldots. \tag{29}$$

Since $c^{(2)}_{A,1,+}$ is absent from the on-shell $\bar{f}Vf$-vertex, this contribution cannot be cancelled by any term coming from the t- or u-channel diagrams. Therefore, a cancellation must involve s-channel diagrams, which, however, are proportional to the representation dependent coupling $g_{[\rho]}$ as can be seen from (26c). Consequently, the cancellation can at most take place for a particular value of $g_{[\rho]}$, namely, for $g_{[\rho]} = g$, when the noncommutative gauge fields do not leave the Lie algebra and the SWM are just field reparametrizations.

As a side remark, the nonvanishing, *a priori* undetermined contribution (29) to the tree-level amplitude of $f\bar{f} \rightarrow VV'$ results from one of the SWM ambiguities missed in [16].

B. $e^+e^- \rightarrow \gamma\gamma$ scattering amplitude

We will now corroborate these preliminary remarks by a complete tree-level calculation of $e^+e^- \rightarrow \gamma\gamma$ up to second order in θ. It is useful to split the full scattering amplitude in the following pieces:

$$A(e^+e^- \rightarrow \gamma\gamma) = g^2A^{\text{SM}} + g^2A^{(1)} + gg_{[\rho]}A^{(1)}_s + g^2A^{(2)} + gg_{[\rho]}A^{(2)}_s, \tag{30}$$

which are self-explaining. It should be noted that the s-channel contributions $A^{(i)}_s$ must be separately gauge invariant, because their normalization depends on the choice of the representation of the enveloping algebra.

For completeness, we restate the familiar QED amplitude from the diagrams in Fig. 1:

$$A^{\text{SM}} = -\frac{\mathrm{i}}{q_u^2}\bar{v}(p_2)\not{\varepsilon}_1\not{q}_u\not{\varepsilon}_2u(p_1) - \frac{\mathrm{i}}{q_t^2}\bar{v}(p_2)\not{\varepsilon}_2\not{q}_t\not{\varepsilon}_1u(p_1), \tag{31}$$

with $q_t = p_1 - k_1$ and $q_u = p_1 - k_2$, as well as the

$\mathcal{O}(\theta)$-amplitude in NCQED [11]:

$$A^{(1)} = -\frac{\mathrm{i}}{q_u^2}\left(\frac{p_1\theta p_2 + k_1\theta k_2}{2\mathrm{i}}\right)\bar{v}(p_2)\not\varepsilon_1\not q_u\not\varepsilon_2 u(p_1) - \frac{\mathrm{i}}{q_t^2} \times \left(\frac{p_1\theta p_2 - k_1\theta k_2}{2\mathrm{i}}\right)\bar{v}(p_2)\not\varepsilon_2\not q_t\not\varepsilon_1 u(p_1) + A^{(1)}_{\text{no pole}}, \tag{32}$$

with

$$A^{(1)}_{\text{no pole}} = \bar{v}(p_2)\left[\frac{1}{2}\varepsilon_1\theta\varepsilon_2(\not k_1 - \not k_2) - k_1\theta\varepsilon_2\not\varepsilon_1 - k_2\theta\varepsilon_1\not\varepsilon_2\right]u(p_1), \tag{33}$$

which collects all contributions from the t- and u-channel diagrams as well as the contact diagram in Fig. 2 containing no pole. As expected, all contributions from ambiguous terms in the SWM cancel in $A^{(1)}$ after application of the equations of motion. Note that the $1/t$- and $1/u$-pole terms in (32) result solely from the expansion of the Moyal phases to order θ. This can be easily seen by combining the phase factors which multiply the vertices in a given diagram. For momenta p_1, p_2, and p_3 flowing into a vertex momentum conservation implies the identities

$$\mathrm{e}^{-\mathrm{i}p_1\theta p_2} = \mathrm{e}^{-\mathrm{i}p_2\theta p_3} = \mathrm{e}^{-\mathrm{i}p_3\theta p_1}. \tag{34}$$

In the t-channel one thus obtains the total phase factor

$$\mathrm{e}^{-\mathrm{i}(-k_2)\theta(p_1-k_1)}\mathrm{e}^{-\mathrm{i}p_1\theta(p_2-k_2)} = \mathrm{e}^{-\mathrm{i}(p_1\theta p_2 - k_1\theta k_2)}, \tag{35}$$

from which the corresponding phase factor in the u-channel follows by exchanging $k_1 \leftrightarrow k_2$:

$$\mathrm{e}^{-\mathrm{i}(p_1\theta p_2 + k_1\theta k_2)}. \tag{36}$$

Turning to the s-channel amplitude $A_s^{(1)}$, one finds that the $1/s$-pole contributions cancel in all contributions from the SWM yielding

$$A_s^{(1)} = A_{s,*}^{(1)} - A^{(1)}_{\text{no pole}}, \tag{37}$$

with the $1/s$-pole coming from the Moyal-Weyl $*$-product alone:

$$A_{s,*}^{(1)} = \frac{1}{s}(k_1\theta k_2)\bar{v}(p_2)\left[\frac{1}{2}(\varepsilon_1\varepsilon_2)(\not k_2 - \not k_1) + (k_1\varepsilon_2)\not\varepsilon_1 - (k_2\varepsilon_1)\not\varepsilon_2\right]u(p_1), \tag{38}$$

and $A^{(1)}_{\text{no pole}}$ given in (33). This ensures that all effects from the SWM cancel for $g_{[\rho]} = g$ as they should, since in this case the SWM are nonsingular field reparametrizations.

To second order in θ, we obtain from the diagrams in Fig. 3

$$A^{(2)} = -\frac{\mathrm{i}}{q_u^2}\frac{1}{2}\left(\frac{p_1\theta p_2 + k_1\theta k_2}{2i}\right)^2\bar{v}(p2)\not\varepsilon_1\not q_u\not\varepsilon_2 u(p_1) - \frac{\mathrm{i}}{q_t^2}\frac{1}{2}\left(\frac{p_1\theta p_2 - k_1\theta k_2}{2i}\right)^2\bar{v}(p_2)\not\varepsilon_2\not q_t\not\varepsilon_1 u(p_1) + A^{(2)}_{\text{no pole}} \tag{39}$$

with

$$A^{(2)}_{\text{no pole}} = -\frac{\mathrm{i}}{2}p_1\theta p_2 A^{(1)}_{\text{no pole}} + \mathrm{i}(c^{(2)}_{A,1,+} - c^{(1)}_A)\bar{v}(p_2) \times [k_1\theta\varepsilon_1 k_1\theta\varepsilon_2\not k_1 + k_2\theta\varepsilon_1 k_2\theta\varepsilon_2\not k_2 + k_1\theta k_2(k_1\theta\varepsilon_1\not\varepsilon_2 - k_2\theta\varepsilon_2\not\varepsilon_1)]u(p_1). \tag{40}$$

Note that the $1/t$- and $1/u$-pole terms in (39) follow again from the expansion of the combined Moyal phases and contain no contribution from the SWM. However, in contrast to $A^{(1)}_{\text{no pole}}$, the second-order amplitude $A^{(2)}_{\text{no pole}}$ does depend on ambiguous terms in the SWM. In the case at hand, this is signalled by the appearance of the free parameters $c^{(1)}_A$ and $c^{(2)}_{A,1,+}$. The exact cancellation for the choice $c^{(2)}_{A,1,+} = c^{(1)}_A$ appears to be accidental.

As in first order in θ, we find that all $1/s$-poles cancel in the s-channel terms coming from the SWM. Since the Moyal-Weyl $*$-product only contributes to the three-photon couplings in odd orders of θ there is no $1/s$-pole term at all in $\mathcal{O}(\theta^2)$. The result is exactly the negative of 40:

$$A_s^{(2)} = -A^{(2)}_{\text{no pole}}, \tag{41}$$

leading again to the cancellation of all contributions from the SWM in the case $g_{[\rho]} = g$, including the ambiguous terms involving the free parameter $c^{(2)}_{A,1,+} - c^{(1)}_A$. However, in general, the cross section for $e^+e^- \longrightarrow \gamma\gamma$ is affected by this SWM ambiguity.

V. CONCLUSIONS

We have investigated the noncommutative extension of the standard model up to second order in θ. As our main result, we find that the general solution for the corresponding Seiberg-Witten maps contains more ambiguous terms than those reported previously, and that the SWM ambiguities do not necessarily cancel in observables. Furthermore, studying the scattering amplitude for $e^+e^- \longrightarrow \gamma\gamma$ as an explicit example we have shown that the ambiguities remaining in the scattering amplitude can be traced to the necessary extension of the Lie algebra of the gauge group to its enveloping algebra, which elevates the SWM from a unobservable field reparametrization to a source of new effective interactions.

Our results imply that the parameter space of the NCSM [4] in $\mathcal{O}(\theta^2)$ is larger than assumed so far. There is every reason to expect that higher orders in θ will introduce even more ambiguities and evidence for this has been found in NCQED [15].

As an outlook, phenomenological consequences of the NCSM at $\mathcal{O}(\theta^2)$ with the focus on collider searches will be presented in an upcoming paper [24].

ACKNOWLEDGMENTS

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APPENDIX: SEIBERG-WITTEN MAPS IN $\mathcal{O}(\theta^2)$

1. Gauge parameter

For each value of $c_\lambda^{(1)}$, we find a family of hermitian solutions to (19) depending on the free parameters $c_{\lambda,1}^{(2)}, \ldots, c_{\lambda,15}^{(2)}$. The specific solution corresponding to $c_{\lambda,i}^{(2)} = 0$ for the case $c_\lambda^{(1)} = 0$ is given by

$$\begin{aligned}
\lambda^{(2)}(\alpha, A) = {}& \frac{i}{32}\theta^{\kappa\lambda}\theta^{\mu\nu}(-3A_\kappa A_\lambda \partial_\nu \alpha A_\mu - 4A_\kappa A_\nu \partial_\lambda \alpha A_\mu - 3A_\kappa \partial_\lambda \alpha A_\mu A_\nu - 2A_\lambda \partial_\nu \alpha A_\mu A_\kappa - 2A_\mu A_\kappa A_\lambda \partial_\nu \alpha \\
& - A_\mu A_\nu A_\kappa \partial_\lambda \alpha - 2A_\nu A_\kappa \partial_\lambda \alpha A_\mu - 4A_\nu \partial_\lambda \alpha A_\mu A_\kappa - 2\partial_\mu A_\kappa \partial_\lambda \partial_\nu \alpha - 2\partial_\lambda \alpha A_\mu A_\nu A_\kappa - \partial_\nu \alpha A_\mu A_\kappa A_\lambda \\
& + 2\partial_\lambda \partial_\nu \alpha \partial_\mu A_\kappa) + \frac{1}{16}\theta^{\kappa\lambda}\theta^{\mu\nu}(4A_\kappa \partial_\lambda \partial_\nu \alpha A_\mu + A_\lambda \partial_\nu \alpha \partial_\mu A_\kappa + 2A_\mu A_\kappa \partial_\lambda \partial_\nu \alpha - 2A_\mu \partial_\kappa A_\nu \partial_\lambda \alpha \\
& - \partial_\kappa A_\nu \partial_\lambda \alpha A_\mu + \partial_\mu A_\kappa A_\lambda \partial_\nu \alpha - \partial_\lambda \alpha A_\mu \partial_\kappa A_\nu + 2\partial_\nu \alpha \partial_\mu A_\kappa A_\lambda + 2\partial_\lambda \partial_\nu \alpha A_\mu A_\kappa).
\end{aligned} \tag{A1}$$

For the general expression we refer to the appendix of [17].

2. Gauge fields

A representative of the six-parameter family of second-order SWM $A_\rho^{(2)}(A)$ corresponding to the parameter choice $c_\lambda^{(1)} = c_{\lambda,i}^{(2)} = c_A^{(1)} = 0$ is given by

$$\begin{aligned}
A_\rho^{(2)}(A) = {}& \frac{i}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}(2[\partial_\nu \partial_\lambda A_\rho, \partial_\mu A_\kappa] + [\partial_\mu A_\kappa, \partial_\rho \partial_\lambda A_\nu]) + \frac{1}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}(+2A_\mu A_\kappa \partial_\nu \partial_\lambda A_\rho - A_\mu A_\kappa \partial_\nu \partial_\rho A_\lambda \\
& + A_\mu A_\kappa \partial_\rho \partial_\lambda A_\nu + 4A_\mu \partial_\nu A_\kappa \partial_\lambda A_\rho - 4A_\mu \partial_\nu A_\kappa \partial_\rho A_\lambda - 2A_\mu \partial_\kappa A_\nu \partial_\lambda A_\rho - 2A_\nu \partial_\lambda A_\rho \partial_\mu A_\kappa + 3A_\nu \partial_\rho A_\lambda \partial_\mu A_\kappa \\
& + 4A_\kappa \partial_\nu \partial_\lambda A_\rho A_\mu + 2A_\lambda \partial_\nu A_\rho \partial_\mu A_\kappa - A_\lambda \partial_\rho A_\nu \partial_\mu A_\kappa - A_\rho \partial_\mu A_\kappa \partial_\lambda A_\nu - 4\partial_\mu A_\kappa A_\nu \partial_\lambda A_\rho + \partial_\mu A_\kappa A_\nu \partial_\rho A_\lambda \\
& + \partial_\mu A_\kappa A_\lambda \partial_\rho A_\nu - \partial_\mu A_\kappa \partial_\lambda A_\nu A_\rho + 2\partial_\nu A_\kappa \partial_\lambda A_\rho A_\mu - 3\partial_\nu A_\kappa \partial_\rho A_\lambda A_\mu + 2\partial_\nu A_\rho \partial_\mu A_\kappa A_\lambda - 2\partial_\kappa A_\nu \partial_\lambda A_\rho A_\mu \\
& + \partial_\kappa A_\nu \partial_\rho A_\lambda A_\mu + 2\partial_\lambda A_\nu A_\rho \partial_\mu A_\kappa + 2\partial_\lambda A_\rho A_\mu \partial_\nu A_\kappa - 4\partial_\lambda A_\rho \partial_\mu A_\kappa A_\nu - \partial_\rho A_\lambda A_\mu \partial_\nu A_\kappa - \partial_\rho A_\lambda A_\mu \partial_\kappa A_\nu \\
& + 4\partial_\rho A_\lambda \partial_\mu A_\kappa A_\nu + 2\partial_\nu \partial_\lambda A_\rho A_\mu A_\kappa + \partial_\nu \partial_\rho A_\lambda A_\mu A_\kappa - \partial_\rho \partial_\lambda A_\nu A_\mu A_\kappa). + \frac{i}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}(-4A_\mu A_\nu A_\kappa \partial_\lambda A_\rho \\
& + 3A_\mu A_\nu A_\kappa \partial_\rho A_\lambda - 2\partial_\rho A_\lambda A_\mu A_\kappa A_\nu + 4A_\mu A_\kappa A_\nu \partial_\lambda A_\rho - 2A_\mu A_\kappa A_\nu \partial_\rho A_\lambda - 4A_\mu A_\kappa A_\lambda \partial_\nu A_\rho - 2A_\mu A_\kappa \partial_\nu A_\lambda A_\rho \\
& + 2A_\mu A_\kappa \partial_\lambda A_\nu A_\rho - 8A_\mu \partial_\nu A_\kappa A_\lambda A_\rho - 4A_\nu A_\kappa \partial_\lambda A_\rho A_\mu + 4A_\nu A_\kappa \partial_\rho A_\lambda A_\mu + 8A_\nu A_\lambda A_\rho \partial_\mu A_\kappa - 4A_\nu \partial_\lambda A_\rho A_\mu A_\kappa \\
& - 2A_\nu \partial_\rho A_\lambda A_\mu A_\kappa - 4A_\kappa A_\nu \partial_\lambda A_\rho A_\mu - 2A_\kappa A_\nu \partial_\rho A_\lambda A_\mu - 4A_\kappa A_\lambda \partial_\nu A_\rho A_\mu + A_\kappa A_\lambda \partial_\rho A_\nu A_\mu - 8A_\kappa \partial_\lambda A_\nu A_\rho A_\mu \\
& - 4A_\kappa \partial_\lambda A_\rho A_\mu A_\nu + A_\kappa \partial_\rho A_\lambda A_\mu A_\nu - 4A_\lambda A_\nu A_\rho \partial_\mu A_\kappa - 4A_\lambda A_\rho A_\mu \partial_\nu A_\kappa + 4A_\lambda A_\rho A_\mu \partial_\kappa A_\nu + 8A_\lambda A_\rho \partial_\mu A_\kappa A_\nu \\
& - 4A_\lambda \partial_\nu A_\rho A_\mu A_\kappa + 4A_\lambda \partial_\rho A_\nu A_\mu A_\kappa - 2A_\rho A_\mu A_\kappa \partial_\nu A_\lambda - 2A_\rho A_\mu A_\kappa \partial_\lambda A_\nu + 8A_\rho A_\mu \partial_\kappa A_\nu A_\lambda - 2A_\rho \partial_\mu A_\kappa A_\nu A_\lambda \\
& + 2A_\rho \partial_\mu A_\kappa A_\lambda A_\nu + 2\partial_\mu A_\kappa A_\nu A_\lambda A_\rho + 2\partial_\mu A_\kappa A_\lambda A_\nu A_\rho - 4\partial_\nu A_\kappa A_\lambda A_\rho A_\mu + 4\partial_\nu A_\lambda A_\rho A_\mu A_\kappa - 4\partial_\nu A_\rho A_\mu A_\kappa A_\lambda \\
& + 4\partial_\kappa A_\nu A_\lambda A_\rho A_\mu - 8\partial_\lambda A_\nu A_\rho A_\mu A_\kappa - 4\partial_\lambda A_\rho A_\mu A_\nu A_\kappa + 4\partial_\lambda A_\rho A_\mu A_\kappa A_\nu + 3\partial_\rho A_\nu A_\mu A_\kappa A_\lambda) \\
& + \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}(-3A_\mu A_\nu A_\kappa A_\lambda A_\rho + 2A_\mu A_\kappa A_\nu A_\lambda A_\rho - 4A_\nu A_\kappa A_\lambda A_\rho A_\mu + 4A_\nu A_\lambda A_\rho A_\mu A_\kappa - 4A_\nu A_\rho A_\mu A_\kappa A_\lambda \\
& + 4A_\kappa A_\nu A_\lambda A_\rho A_\mu - 4A_\kappa A_\lambda A_\nu A_\rho A_\mu - 2A_\kappa A_\lambda A_\rho A_\mu A_\nu - 8A_\lambda A_\nu A_\rho A_\mu A_\kappa - 4A_\lambda A_\rho A_\mu A_\nu A_\kappa + 4A_\lambda A_\rho A_\mu A_\kappa A_\nu \\
& - 3A_\rho A_\mu A_\nu A_\kappa A_\lambda + 2A_\rho A_\mu A_\kappa A_\nu A_\lambda).
\end{aligned} \tag{A2}$$

Here, we have arranged the terms according to the power of A_μ, for the benefit of deriving Feynman rules. The general solution can be found in [17]. As pointed out in Sec. II D 1, this solution is related to the one presented in [3], but incompatible with the one in [16].

3. Matter fields

Choosing $c_\lambda^{(1)} = c_{\lambda,i}^{(2)} = c_\psi^{(1)} = 0$, we obtain the following representative of the three-parameter family of matter field SWM:

$$\begin{aligned}\psi^{(2)}(\psi, A) = {}& \frac{\mathrm{i}}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}(-\partial_\mu A_\kappa \partial_\nu \partial_\lambda \psi) + \frac{1}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}(-2A_\mu \partial_\kappa A_\nu \partial_\lambda \psi - 2\partial_\mu A_\kappa A_\nu \partial_\lambda \psi + 2A_\mu A_\kappa \partial_\nu \partial_\lambda \psi + 4A_\mu \partial_\nu A_\kappa \partial_\lambda \psi \\ & - \partial_\mu A_\kappa \partial_\nu A_\lambda \psi) + \frac{\mathrm{i}}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}(-2A_\mu \partial_\nu A_\kappa A_\lambda \psi + \partial_\mu A_\kappa A_\nu A_\lambda \psi - A_\mu A_\kappa A_\lambda \partial_\nu \psi + A_\mu A_\kappa A_\nu \partial_\lambda \psi \\ & - A_\mu A_\nu A_\kappa \partial_\lambda \psi) + \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}(-3A_\mu A_\nu A_\kappa A_\lambda \psi + 4A_\mu A_\kappa A_\nu A_\lambda \psi - 2A_\mu A_\kappa A_\lambda A_\nu \psi). \end{aligned} \tag{A4}$$

Again, the terms are ordered according to the power in the gauge field. For the general solution one may consult [17].

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