Dynamics in nonlocal cosmological models derived from string field theory

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A general class of nonlocal cosmological models is considered. A new method for solving nonlocal Friedmann equations is proposed, and solutions of the Friedmann equations with nonlocal operator are presented. The cosmological properties of these solutions are discussed. Especially indicated is the p-adic cosmological model in which we have obtained a nonsingular bouncing solution and string field theory tachyon model in which we have obtained full solution of nonlocal Friedmann equations with $w = -1$ at large times. The possibility of obtaining a realistic value of the cosmological constant from nonlocal cosmological models is also discussed.

I. INTRODUCTION

In recent works there appears interest in nonlocal cosmological models derived from string field theory in connection with the problem of describing cosmological inflation or accelerating expansion of the Universe.

Modern cosmological data indicates that expansion of the Universe is accelerating. It may be owing to a component of the Universe with negative pressure, dark energy. Recent results of WMAP [1] together with Ia supernovae data give us the following range for the dark energy state parameter $w = -0.97^{+0.07}_{-0.09}$. In the assumption that the value of the state parameter changes in the range $0 > w > -1$ there exist different theoretical models for the dark energy (see reviews [2,3] and references therein)—quintessence models [4], K-essence [5], quintom models [6], models constructed on the DBI action (see reviews [7] and references therein, [8]) and dilatonic models [9]. The case $w = -1$ corresponds to models with cosmological constant, which are also possible. We also can find in the literature models related to $w < -1$ with modified gravity [10].

A nontrivial possibility is the case $w < -1$ that means the violation of the null energy condition. One of the possibilities is to consider the phantom Universe [11,12]. There were proposed several coupled scalar-gravity models in which the null energy condition is violated, but as an effect most such models are unstable [13]. Recently it was proposed another possibility to consider dark energy description in the context of D-brane decay in cubic superstring field theory [11]. It was shown that D-brane decay can be interpreted at least at late times as a phantom model. Note that unlike phenomenological phantom models here phantom is an effective theory. Since string field theory by itself is a consistent quantizable theory this approach does not suffer from instabilities which are inevitable for phenomenological phantom models [11]. Such nonlocal cosmological models were also considered in [14–18].

At the same time there have been a number of attempts to realize descriptions of the early Universe via nonlocal cosmological models [18–21]. One example is the p-adic inflation model [19] which is represented by nonlocal p-adic string theory coupled to gravity. For this model, a rolling inflationary solution was constructed and the interesting features were discussed and compared with cosmic microwave background (CMB) observations. The possibility of obtaining large non=Gaussian signatures in the CMB has also been considered in a general class of single field nonlocal hill-top inflation models [20]. Another example is investigation of the inflation near a maximum of the nonlocal potential when nonlocal derivative operators are included in the inflaton Lagrangian. It was found that higher-order derivative operators in the inflaton Lagrangian can support a prolonged phase of slow-roll inflation near a maximum of the potential [21].

It may happen that both early stage and contemporarily Universe can be explained with a single nonlocal model derived from string field theory. To be able to tell more about the early stage we need to consider perturbations in nonlocal models. It will be very interesting to compare obtained results, in particular, with those in [22–24]. To be able to do it the new technique for solving the nonlinear Friedmann equation is required which will hopefully be provided in the present work where for the first time construction of the solution of the full nonlocal Friedmann equations is presented. That is why in the present work we discuss numerical solution of Friedmann equations and their features in more detail.

One of the popular examples of nonlocal models is the D-brane decay in the gravitational background in the framework of string field theory which is described by the following tachyon action:

$$S = \int d^4x \sqrt{-g}\left(\frac{m_p^2}{2}R + \frac{\xi^2}{2}\phi\Box_g\phi + \frac{1}{2}\phi^2 - \frac{1}{4}\Phi^4 - T - \Lambda'\right), \tag{1}$$

where g is the metric, m_p is reduced Planck mass, ϕ is the

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tachyon field, $\Phi = e^{k\Box}\phi$, $k = \frac{1}{8}$, $\Box_g$ is the usual Beltrami-Laplace operator, T is tension of the $D3$ brane, and Λ' is the effective cosmological constant.

Note that covariant string field theory as of today is constructed only in the flat background. It was built only for some special types of nonflat backgrounds, in particular, in the anti−de Sitter background [25] and in the plane wave background [26]. So the action (1) at the special choice of $V(\Phi)$ is a direct generalization for the approximated tachyon action to the case of an arbitrary metric.

The action (1) in the Friedmann-Robertson-Walker (FRW) background

$$ds^2 = -dt^2 + a^2(t)(dx_1^2 + dx_2^2 + dx_3^2)$$

leads to the following Friedmann equations:

$$3H^2 = \frac{1}{m_p^2}\mathcal{E}, \qquad (2)$$

$$3H^2 + 2\dot{H} = -\frac{1}{m_p^2}\mathcal{P}, \qquad (3)$$

which have the same form as the usual equations except for an extra term in the expression for the energy $\mathcal{E}$ and pressure $\mathcal{P}$ which appear from the nonlocal interaction and which contain the Hubble function. This term makes these Friedmann equations more complicated even for numerical consideration. More precisely,

$$\mathcal{E} = \underbrace{\mathcal{E}_k + \mathcal{E}_p + T + \Lambda'}_{\text{usual local energy expression}} + \mathcal{E}_{nl1} + \mathcal{E}_{nl2}, \qquad (4)$$

$$\mathcal{P} = \underbrace{\mathcal{E}_k - \mathcal{E}_p - T - \Lambda'}_{\text{usual local pressure expression}} - \mathcal{E}_{nl1} + \mathcal{E}_{nl2}, \qquad (5)$$

where

$$\mathcal{E}_{nl1} = k\int_0^1 d\rho (e^{k\rho\Box_g}\Phi^3)(-\Box_g e^{-k\rho\Box_g}\Phi),$$

$$\mathcal{E}_{nl2} = -k\int_0^1 d\rho (\partial e^{k\rho\Box_g}\Phi^3)(\partial e^{-k\rho\Box_g}\Phi).$$

The field equation reads

$$(\xi^2\Box_g + 1)e^{-\Box_g}\Phi = V'(\Phi),$$

here, and below we will denote by $'$ usual function derivative. The Beltrami-Laplace operator takes the form $\Box_g = -\partial_t^2 - 3H(t)\partial_t + \frac{1}{a^2}\partial_x^2$.

In the case of cubic potential the action (1) corresponds to the tachyon in bosonic string field theory [27] for the lowest level in the level-truncated scheme [28,29] with the induced twisted space-time. It is supposed that we are dealing with $D3$-brane in the $D26$ space-time and the volume of the $D22$ compactified subspace is omitted.

The case of quartic potential in the action (1) corresponds to inclusion of the metric in the tachyon action of

the fermionic cubic string [30] in the approximation of a slowly varying auxiliary field [31]. Such an action appears in the level-truncated scheme where only lowest scalar field (tachyon field) in the GSO(−) sector and correspondingly the lowest scalar field in the GSO(+) sector (this field does not have a kinetic term and is considered an auxiliary field) are considered. Integration over the auxiliary field results in the quartic potential for the tachyon field. Note that appearance of the nonextremal D-brane in the framework of fermionic string field theory requires GSO(−) sector in the spectrum [32,33].

The purpose of this work is to consider a general class of nonlocal cosmological models of the form (1) and discuss which physical properties and consequences might be obtained from such models. Among them we will be interested first in the classical solutions of the corresponding Friedmann equations which can be considered as a first approximation to the quantum solutions and might be useful for the study of ways to avoid the cosmological singularity problem [34].

We start with the description of some general class of nonlocal models. In Sec. III we will describe a new numerical algorithm for solving nonlocal Friedmann equations. In Sec. IV we will present the numerical solutions of Friedmann equations for different parameters and will provide an analytical explanation for some of their interesting features. In Sec. V we will consider which cosmological solutions can be provided by nonlocal models.

II. GENERAL CLASS OF NONLOCAL COSMOLOGICAL MODELS

In this paper we will consider the following nonlocal scalar field on the D-brane coupled to the gravity

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R + \frac{M_s^4}{g_4}\left(\frac{\xi^2}{2}\phi(\Box_g/M_s^2)\phi + \frac{1}{2}\phi^2 - V(e^{k\Box_g/M_s^2}\phi) - \Lambda' - T\right)\right], \qquad (6)$$

where g is the metric, $\Box_g = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$, M_p is a Planck mass, M_s is a characteristic string scale related with the string tension α' as $M_s = 1/\sqrt{\alpha'}$, ϕ is a scalar field, g_4 is a dimensionless four-dimensional effective coupling constant related with the ten-dimensional string coupling constant g_0 and the compactification scale, $\Lambda = \frac{M_s^4}{g_4}\Lambda'$ is an effective four-dimensional cosmological constant, T is the brane tension, and k is a nonlocal coupling constant [31,33].

For further investigation, let us rewrite our action in dimensionless space-time variables

$$S = \int d^4x \sqrt{-g} \left(\frac{m_p^2}{2} R + \frac{\xi^2}{2} \phi \Box_g \phi + \frac{1}{2} \phi^2 \right.$$
$$\left. - V(\Phi) - \Lambda' - T \right), \tag{7}$$

where ϕ is a dimensionless scalar field, $\Phi = e^{k\Box_g}\phi$ and $m_p^2 = g_4 \frac{M_p^2}{M_s^2}$.

As a particular metric we take FRW one

$$ds^2 = -dt^2 + a^2(t)(dx_1^2 + dx_2^2 + dx_3^2). \tag{8}$$

We will consider spatially homogeneous configurations for which the Beltrami-Laplace operator takes the form $\Box_g = -\partial_t^2 - 3H(t)\partial_t$. For the convenience of numerical calculations let us introduce the following notation $\mathcal{D}_H^2 \equiv \partial_t^2 + 3H(t)\partial_t$, which we will understand as a generalization of the second order time derivative, which contains the extra term $3H(t)\partial_t$ associated with gravitational background.

Equation of motion for the space homogeneous configurations for the scalar field Φ takes the form

$$(-\xi^2 \mathcal{D}_H^2 + 1)e^{2k\mathcal{D}_H^2}\Phi = V'(\Phi). \tag{9}$$

The Friedmann equations have the following form:

$$3H^2 = \frac{1}{m_p^2}\mathcal{E}, \qquad 3H^2 + 2\dot{H} = -\frac{1}{m_p^2}\mathcal{P}. \tag{10}$$

For the case of nonlocal potentials the energy and the pressure have additional nonlocal terms $\mathcal{E}_{nl1}$ and $\mathcal{E}_{nl2}$

$$\mathcal{E} = \underbrace{\mathcal{E}_k + \mathcal{E}_p + T + \Lambda'}_{\text{usual local energy expression}} + \mathcal{E}_{nl1} + \mathcal{E}_{nl2}, \tag{11}$$

$$\mathcal{P} = \underbrace{\mathcal{E}_k - \mathcal{E}_p - T - \Lambda'}_{\text{usual local pressure expression}} - \mathcal{E}_{nl1} + \mathcal{E}_{nl2}, \tag{12}$$

where

$$\mathcal{E}_k = \frac{\xi^2}{2}(\partial\phi)^2, \qquad \mathcal{E}_p = -\frac{1}{2}\phi^2 + V(\Phi), \tag{13a}$$

$$\mathcal{E}_{nl1} = k \int_0^1 d\rho (e^{-k\rho \mathcal{D}_H^2} V'(\Phi))(\mathcal{D}_H^2 e^{k\rho \mathcal{D}_H^2}\Phi), \tag{13b}$$

$$\mathcal{E}_{nl2} = -k \int_0^1 d\rho (\partial e^{-k\rho \mathcal{D}_H^2} V'(\Phi))(\partial e^{k\rho \mathcal{D}_H^2}\Phi). \tag{13c}$$

As we can see, the structure of Friedmann equations and of equations for scalar fields in this model is rather complicated and their study is already a very interesting mathematical problem and can be considered as a separate mathematical investigation.

To avoid calculation of the $e^{-k\rho \mathcal{D}_H^2}$ term which is much harder to compute than $e^{k\rho \mathcal{D}_H^2}$ $(k > 0)$ as computation of the former results in an ill-posed problem [35], we will use the following representation for nonlocal energy terms on

the equation of motion for the scalar field

$$\mathcal{E}_{nl1} = k \int_0^1 d\rho ((-\xi^2 \mathcal{D}_H^2 + 1)e^{(2-\rho)k\mathcal{D}_H^2}\Phi)(\mathcal{D}_H^2 e^{k\rho \mathcal{D}_H^2}\Phi), \tag{14a}$$

$$\mathcal{E}_{nl2} = -k \int_0^1 d\rho (\partial(-\xi^2 \mathcal{D}_H^2 + 1)e^{(2-\rho)k\mathcal{D}_H^2}\Phi)(\partial e^{k\rho \mathcal{D}_H^2}\Phi). \tag{14b}$$

III. METHOD FOR SOLUTION OF NONLOCAL FRIEDMANN EQUATIONS

A. Iterative procedure for solution construction

For numerical calculations let us rewrite our system in the following form:

$$3H^2 = \frac{1}{m_p^2}\mathcal{E}, \tag{15a}$$

$$\dot{H} = -\frac{1}{2m_p^2}(\mathcal{P} + \mathcal{E}), \tag{15b}$$

$$(-\xi^2 \mathcal{D}_H^2 + 1)e^{2k\mathcal{D}_H^2}\Phi = V'(\Phi). \tag{15c}$$

In this paper we would like to point out which physical results we can extract from solutions of this system. First, we will find the Hubble function and the scalar field from Eqs. (15b) and (15c). It happens that Eq. (15a) plays a role of energy conservation from which the effective cosmological constant is extracted and is unique for every field configuration.

Integrating Eq. (15b) we get the following system:

$$(-\xi^2 \mathcal{D}_H^2 + 1)e^{2k\mathcal{D}_H^2}\Phi = V'(\Phi), \tag{16a}$$

$$H = -\frac{1}{m_p^2} \int_0^t d\tau \left[\frac{\xi^2}{2}(\partial\phi)^2 - k \int_0^1 d\rho (\partial(-\xi^2 \mathcal{D}_H^2 + 1) \right.$$
$$\left. \times e^{(2-\rho)k\mathcal{D}_H^2}\Phi)(\partial e^{k\rho \mathcal{D}_H^2}\Phi) \right]. \tag{16b}$$

To find solutions of the system (16) we construct the following iterative process

$$\Phi = \lim_{n\to\infty} \Phi_n, \qquad H = \lim_{n\to\infty} H_n, \tag{17a}$$

where iterations Φ_n and H_n are in turn obtained as limits of subiterations

$$\Phi_{n+1} = \lim_{m\to\infty} \Phi_{n,m}, \qquad H_{n+1} = \lim_{m\to\infty} H_{n,m}, \tag{17b}$$

which are recursively defined as $(m \geq 0)$

$$\Phi_{n,m+1} = V'^{-1}((-\xi^2 \mathcal{D}_H^2 + 1)e^{2k\mathcal{D}_H^2}\Phi_{n,m}), \tag{17c}$$

$$H_{n,m+1} = -\frac{1}{m_p^2} \int_0^t d\tau \left[\frac{\xi^2}{2} (\partial e^{k\mathcal{D}_{H_{n,m}}^2} \Phi_{n+1})^2 \right.$$
$$- k \int_0^1 d\rho (\partial(-\xi^2 \mathcal{D}_{H_{n,m}}^2 + 1) e^{(2-\rho)k\mathcal{D}_{H_{n,m}}^2} \Phi_{n+1})$$
$$\left. \times (\partial e^{k\rho\mathcal{D}_{H_{n,m}}^2} \Phi_{n+1}) \right], \qquad (17d)$$

where initial iteration in m is taken as

$$\Phi_{n,0} = \Phi_n, \qquad H_{n,0} = H_n. \qquad (17e)$$

By V'^{-1} we denote a function inverse to V'. Note how subiteration $\Phi_{n,m}$ depends only on H_n, i.e. the whole step of subiterations for H. Note also that $H_{n,m}$ depends on Φ_{n+1}, this is essentially a way to accelerate convergence.

Intuitively this iteration procedure does the following. First fixing H_n it finds Φ_{n+1} using iteration process (17c) which is a natural generalization of a process which is described in [36–38] and for which convergence is proved analytically. Then H_{n+1} is obtained by iterating (17d) where Φ_{n+1} is fixed as computed in the previous step. Thus found H_{n+1} is used to find Φ_{n+2} and so on. This kind of multistep iterative procedure is known to provide rapid convergence. In our experiments this iterative procedure converges much more reliably as compared to a more naive iterative procedure where one does not do intermediate iterations.

In a similar fashion one can construct iterative procedures for systems with several scalar fields such as those studied in [35,39].

B. Exponentiating $\mathcal{D}_H^2$ by solving diffusion equation

Equations of motion as well as the iteration method discussed above require computation of action with operator $e^{k\rho\mathcal{D}_H^2}$ on a given function. Computationally this is a nontrivial problem by itself. We use the following method which beside giving mathematical definition for such an operator more importantly provides a corresponding computational method.

The result of the acting with operator $e^{\rho\mathcal{D}_H^2}$ on a given function $\varphi(t)$ is natural to define as a solution of the following diffusion partial differential equation with corresponding initial and boundary conditions:

$$\frac{\partial}{\partial\rho} f(\rho, t) = \mathcal{D}_H^2 f(\rho, t), \qquad (18)$$

$$f(0, t) = \varphi(t), \qquad (19)$$

$$f(\rho, \pm\infty) = \varphi(\pm\infty), \qquad (20)$$

where $-\infty < t < +\infty$, $\rho \geq 0$. Note that here evolution goes along the ρ axis, while t plays a role of space. The solution of (18) is a rigorous mathematical definition of the operator $e^{\rho\mathcal{D}_H^2}$ acting on function φ

$$e^{\rho\mathcal{D}_H^2}\varphi(t) = f(\rho, t).$$

Such a solution exists and is unique for wide classes of H and φ [40]. Since in standard settings such a diffusion problem is well defined, in particular, for H, $\varphi \in \mathcal{L}_2(\mathbb{R})$ which seem to suffice applications which are considered here, we will not discuss this topic any further.

C. Fixing potential and other parameters

So far we did not specify the exact shape of the potential for our system. In this paper we are interested in coupled to the gravity a tachyon field which appears as a lowest excitation in the level-truncation scheme for fermionic string field theory [31]. Thus all further discussion will be about quartic potential

$$V(\Phi) = \tfrac{1}{4}\Phi^4. \qquad (21)$$

We would like to note though that from a numerical perspective this potential can lead to difficulties as it results in solutions which are not differentiable at zero (they behave as $t^{1/3}$ in the vicinity of zero [41]). For the later reason we found it useful to study a similar problem in potential

$$V_\alpha(\Phi) = (1-\alpha)\tfrac{1}{2}\Phi^2 + \alpha\tfrac{1}{4}\Phi^4, \qquad 0 < \alpha < 1.$$

As $\alpha \to 1$ this potential tends to potential (21). The interesting property of this potential is that the solution has a finite derivative at zero and moreover there is a theorem which states convergence of our type of iterative procedure in case of Minkowski metric [42].

Let us return to the potential (21). Equation (15c) is invariant under shifts

$$\Phi(t) \to \Phi(t + t_0), \qquad H(t) \to H(t + t_0), \qquad (22)$$

and mirroring

$$\Phi(t) \to -\Phi(-t), \qquad H(t) \to -H(-t). \qquad (23)$$

Thus it is natural to look for odd solutions. Without loss of generality we can set $\Phi(0) = 0$.

Equation (15c) has the following constant solutions:

$$\Phi = 0, \pm 1.$$

Following the same logic as in the Minkowski case [31,36], since $V(\Phi = +1) = V(\Phi = -1)$ we can expect the existence of kink-type solutions where the scalar field interpolates between $\Phi = +1$ and $\Phi = -1$.

Iterative procedure (17) requires initial iteration Φ_0, H_0 to be specified explicitly. To find such kink solutions it is natural to take

$$\Phi_0(t) = \text{sign}(t), \qquad H_0(t) = 0.$$

This initial iteration leads to a rapid convergence. It is interesting that our numerical experiments show that iterative procedure (17) converges to a kink-type solution if we start from any two bounded functions which are posi-

tive on a positive semiaxis and negative on a negative one. This observation provides a strong indication for uniqueness of the solutions (up to shifts and mirroring).

The value of ξ^2 is determined during a level-truncation procedure; its value is

$$\xi_{\text{phys}}^2 = -\frac{1}{4\log_3\frac{4}{3\sqrt{3}}} \approx 0.96. \qquad (24)$$

Nevertheless, since different values of ξ^2 already in the Minkowski case result in different physical pictures, we will systematically consider various values of ξ^2. Note also that the case $\xi^2 = 0$ corresponds to p-adic string with $p = 3$. The value of the nonlocal coupling constant is fixed by the same procedure; it is $k = \frac{1}{8}$ in our case.

IV. NUMERICAL RESULTS

A. Numerical solutions for Friedmann equations

In what follows for numerical computations we put $m_p = 1$. We used an iterative method presented in the previous section to solve Friedmann equations for our system. We observe the crucial role of the value of ξ^2; as it varies the following physical properties of the system are affected (see Fig. 1).

(i) For $\xi^2 = 0$ the solution in both Φ and H has a monotonic kink shape. This behavior is qualitatively similar to the case of the Minkowski metric [31,37,43].

(ii) There exists a critical value $\xi_{\text{osc}}^2 \approx 1.18$ which determines between types of the Φ component of the solution—kink or oscillatory with finite period. For $\xi^2 < \xi_{\text{osc}}^2$ the solution has a kink shape with exponentially decreasing oscillations around ± 1 as $t \to \pm\infty$. On the other hand if $\xi^2 > \xi_{\text{osc}}^2$ the solution converts to oscillations with finite period.[1] Similar behavior was observed in the case of the Minkowski metric, although in that case the critical value was higher ≈ 1.38 [31,37].

(iii) There is one more critical value $\xi_{\text{shape}}^2 \approx 0.42$ which determines a shape of a Hubble function (see Fig. 2). More precisely, as ξ^2 grows the Hubble function H gets a turning point at some positive time t_0. Moreover, for $\xi^2 > \xi_{\text{shape}}^2$ H ends up in tending a

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[1]Note that for the iterative method which we present in this paper it is essential that solution Φ has well-defined limits at infinite times. This fact is used in particular for the boundary condition (20). This means that this method is not suitable for finding oscillatory solutions with high precision. Oscillatory behavior reported in this paper was obtained by vastly enlarging the lattice in t; this did not allow finding critical value of ξ^2 with more then 3 significant digits. Reliable methods for finding oscillatory solutions for the type of equations considered in this paper is an open mathematical problem even for the Minkowski metric, see [37]. Note though that physically significant solutions in this model turned out to be of kink type where iterations rapidly converge.

negative (positive) value as $t \to \infty$ ($t \to -\infty$). This is a new type of behavior which was not present in the Minkowski case.

Note that the physical value of ξ^2 (24) is in the region

$$\xi_{\text{shape}}^2 < \xi_{\text{phys}}^2 < \xi_{\text{osc}}^2.$$

Figures 1 and 2 also show how dynamics of the scale factor $a(t)$

$$a(t) = a_0 \exp\left(\int_0^t H(\tau)d\tau\right) \qquad (25)$$

change as ξ^2 increases. For $\xi^2 = 0$ it has a has a minimum at the perturbative vacuum $\phi = 0$ and increases as ϕ tends to nonperturbative vacua ± 1. As ξ^2 increases the shape of a changes and for $\xi^2 > \xi_{\text{shape}}^2$ the scale factor decreases as $\phi \to \pm 1$ ($t \to \pm\infty$). For numerical computations we use initial condition $a_0 = 1$ in (25).

B. Two profiles of the Hubble function

To analyze the late time behavior of the Hubble function, let us use the so-called mechanical approximation which was studied, for example, in [14] and gives a realistic qualitative picture at least for the late time behavior. Mechanical approximation is constructed by keeping in the original nonlocal expression only terms with derivatives no higher than second order. For (16b) we get

$$H(t) \approx -\frac{1}{m_p^2}\int_0^t d\tau\left(\frac{\xi^2}{2}(\partial\phi)^2 - k(\partial\phi)^2\right)$$

$$= \frac{1}{m_p^2}\int_0^t d\tau\left(k - \frac{\xi^2}{2}\right)(\partial\phi)^2. \qquad (26)$$

We can see from the expression above that there exists a value of ξ^2 which determines the change of sign for the Hubble function, more precisely for $\xi^2 < 2k$ the Hubble function is positive for $t \to \infty$ (negative as $t \to -\infty$), while for $\xi^2 > 2k$ the reverse becomes true. Thus mechanical approximation gives us the critical value of $\xi^2 = 2k = 0.25$. As already described, numerical computations give us the value $\xi_{\text{shape}}^2 \approx 0.42$ which slightly differs from the just found approximate critical ξ^2. This difference manifests the influence of higher-order nonlocal terms. Nevertheless the analysis above clarifies the phenomenon which causes the change of sign for the asymptotic values of the Hubble function.

C. Two regimes of the solution

In this section we will address the mechanism which forms numerically found exponentially decreasing oscillations and the existence of critical value ξ_{osc}^2. We will also compare this behavior with results obtained without a gravitational term in action [37].
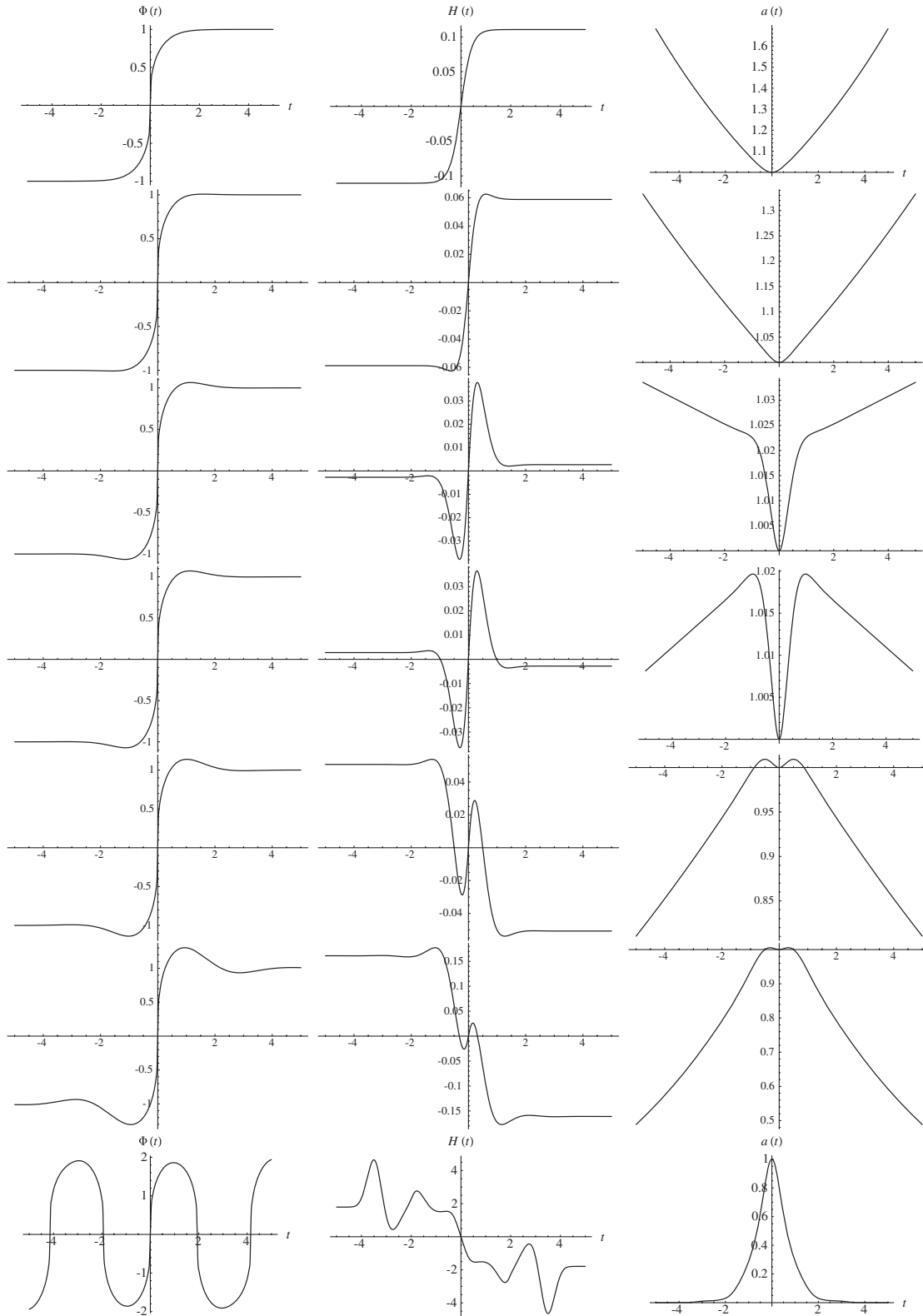
FIG. 1.   Solutions of the Friedmann equations Φ, H, and a (left to right) for different values of the parameter $\xi^2 = 0, 0.2, 0.41, 0.43$, 0.6, 0.96, 2 (top to bottom). Note that on the figures for scale factor $a(t)$ axes cross at the point ($t = 0$, $a = 1$). We can see that when ξ^2 increases the shape of the scale factor changes from parabolic type [$a(t) \geq 1$ for all times] to lump type [$a(t) \leq 1$ for all times]; transition between these shape types is illustrated in Fig. 2.
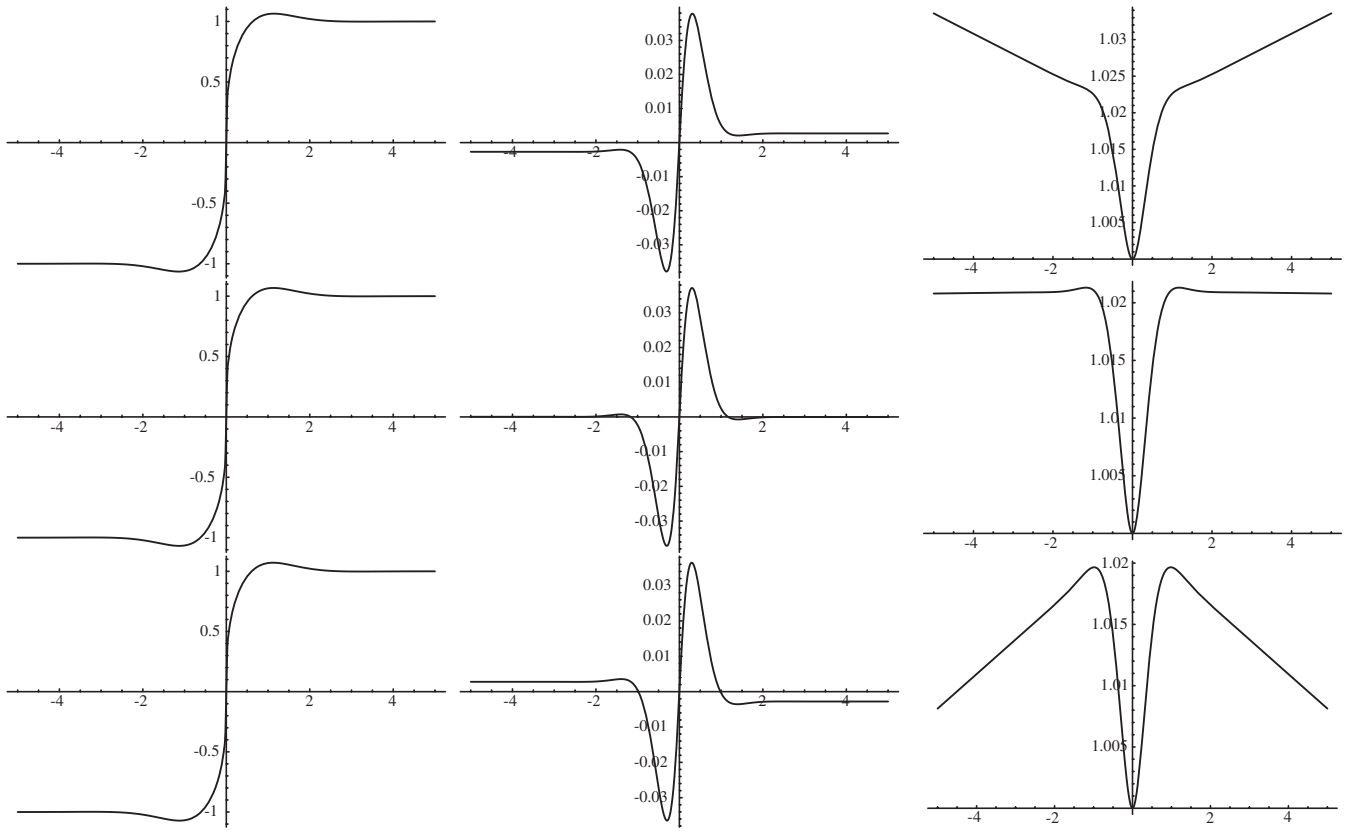
FIG. 2.   Functions Φ, H, and a (left to right) for $\xi^2 = 0.41, 0.42, 0.43$ (top to bottom). Note how asymptotic behavior changes as we go through $\xi^2 = 0.42$.

Let us present the solution of Eq. (15c) for the quartic potential $V(\Phi) = \frac{1}{4}\Phi^4$

$$(-\xi^2 \mathcal{D}_H^2 + 1)e^{2k\mathcal{D}_H^2}\Phi = \Phi^3 \tag{27}$$

as a sum

$$\Phi(t) = \Phi_0(t) + \chi(t), \tag{28}$$

where Φ_0 denotes the solution of Eq. (15c) for $\xi^2 = 0$. Substituting (28) to (15c) and leaving only linear terms in χ we get

$$(-\xi^2 \mathcal{D}_H^2 + 1)e^{2k\mathcal{D}_H^2}(\Phi_0 + \chi) = \Phi_0^3 + 3\Phi_0^2\chi. \tag{29}$$

Using the fact that Φ_0 is the solution of Eq. (15c) for $\xi^2 = 0$ we can write equation for $\chi(t)$ which in linear approximation has the form

$$(-\xi^2 \mathcal{D}_H^2 + 1)e^{2k\mathcal{D}_H^2}\chi = 3\Phi_0^2\chi + \xi^2 \mathcal{D}_H^2 e^{2k\mathcal{D}_H^2}\Phi_0. \tag{30}$$

Using the fact that operator $e^{2k\mathcal{D}_H^2}$ acts as an identity on constants, we can write large t approximation

$$(-\xi^2 \mathcal{D}_H^2 + 1)e^{2k\mathcal{D}_H^2}\chi = 3\chi, \tag{31}$$

were we used asymptotical properties of solution Φ_0.

To carry out harmonic analysis for Eq. (31) we will consider eigenfunctions of the D'Alembertian operator $\mathcal{D}_H^2 v_\lambda = -\lambda v_\lambda$. Using that $e^{2k\mathcal{D}_H^2}v_\lambda = e^{-2k\lambda}v_\lambda$ and expanding χ in v_λ, Eq. (31) leads us to

$$(\xi^2 \lambda + 1)e^{-2k\lambda}v_\lambda = 3v_\lambda. \tag{32}$$

Our goal is to analyze which values of ξ^2 allow real-valued λ, i.e. nondamping oscillations take place. Considering Eq. (32) as an equation for complex variable λ with parameter ξ^2 we obtain that there is a minimum value of $\xi_0^2 \approx 1.77$ for which λ is real. So for $\xi^2 < \xi_0^2$ we have solutions with nonzero imaginary parts which result in vanishing solutions while $\xi^2 > \xi_0^2$ leads to the oscillatory regime. Equation (32) has exactly the same form in the case of Minkowski space which was considered earlier [37] (functions v_λ are different as they depend on H). As we can see the method discussed in this section provides us only a qualitative explanation of the changing of the regimes; the value of critical ξ^2 found here only approximately reproduces numerically obtained $\xi_{osc}^2 \approx 1.18$. It is interesting to note that the Minkowski case leads to a higher value of critical ξ^2 [37], i.e. it appears that "friction" $H(t)$ does not damp oscillations but otherwise increases them.

V. COSMOLOGICAL PROPERTIES

It is interesting to note that initially we considered a model with potential $V(\phi) = -\frac{1}{2}\phi^2 + \frac{1}{4}\phi^4 + \frac{1}{4}$ [see Fig. 3(a)]. A nonlocal term contributes to a kinetic term, most importantly it changes its sign [11], or equivalently changes the sign of potential (kinetic term is positive in this case), see Fig. 3. In this context it is interesting to study the already-mentioned mechanical approximation which can illustrate such an effect.

Solutions of (15) lead to interesting cosmological properties. Let us consider them for physically interesting values of ξ^2 and see what kind of physical behavior they result in. We would like to study dynamics of the following physical quantities. The state parameter which is defined as usual

$$w = \frac{p}{\rho}, \tag{33}$$

or in the terms of Hubble parameter

$$w = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}, \tag{34}$$

will be presented on the figures below along with a deceleration parameter defined as

$$q(t) = -\frac{\ddot{a}a}{\dot{a}}. \tag{35}$$

A. Effective mechanical potential

In order to better understand which behavior of the system we might expect, let us first study Eq. (9) in local approximations.

If we ignore the nonlocal operator $e^{2k\mathcal{D}_H^2}$ in (9) we get

$$(-\xi^2 \mathcal{D}_H^2 + 1)\Phi = \Phi^3. \tag{36}$$

Let us reproduce the action which leads to the equation above

$$S = \int d^4x \sqrt{-g}\left(\frac{m_p^2}{2}R + \frac{\xi^2}{2}\Phi\Box_g\Phi + \frac{1}{2}\Phi^2 - \frac{1}{4}\Phi^4\right). \tag{37}$$

This action describes a particle moving in the system with friction $H(t)$

$$\xi^2 \partial^2\Phi = \Phi - \Phi^3 - 3H\xi^2\partial\Phi, \tag{38a}$$

$$\dot{H} = -\frac{\xi^2}{m_p^2}(\partial\Phi)^2. \tag{38b}$$

As we can see here, the Hubble function is explicitly negative and thus we obtained a system which is unusual from a mechanical point of view—it is an anharmonic oscillator with *negative* friction (see Fig. 4). Such a behavior looks similar with the behavior for nonlocal systems studied in [14,15] in the Minkowski background, when the trajectories of the scalar field exceed the extremum of the potential, meanwhile the energy is conserved.

Now let us consider mechanical approximation by accounting in Eq. (9) only derivatives no higher than second order; we get

$$((2k - \xi^2)\partial^2 + 1)\Phi = \Phi^3 - 3(2k - \xi^2)H(t)\partial\Phi. \tag{39}$$

Equation (39) could be obtained from the following action:

$$S = \int d^4x \sqrt{-g}\left(\frac{m_p^2}{2}R + \frac{(\xi^2 - 2k)}{2}\Phi\Box_g\Phi + \frac{1}{2}\Phi^2 - \frac{1}{4}\Phi^4\right), \tag{40}$$

which leads to the following local Friedmann equations:

$$\partial^2\Phi = -\frac{\Phi - \Phi^3}{(2k - \xi^2)} - 3H\partial\Phi, \tag{41a}$$

$$\dot{H} = \frac{(2k - \xi^2)}{m_p^2}(\partial\Phi)^2. \tag{41b}$$

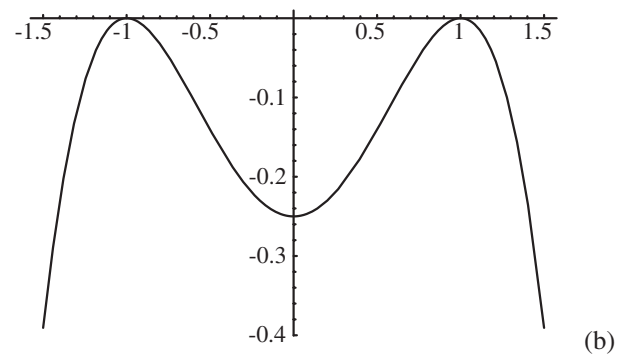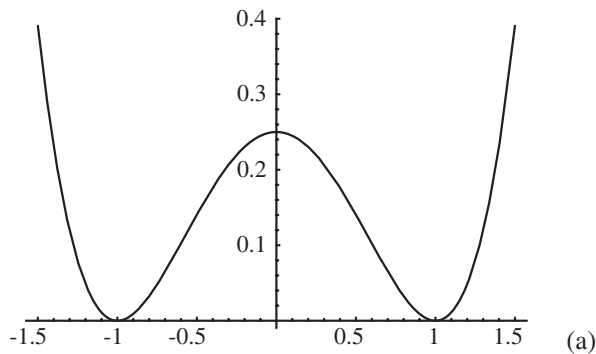This system of equations describes particles moving in the following potential:

(a)

(b)

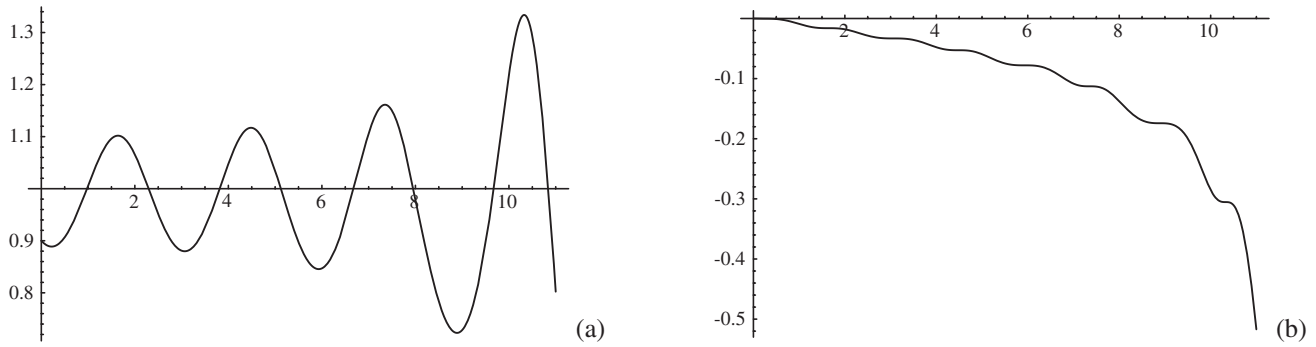FIG. 3. W and $-W$-shaped potentials.

FIG. 4.   Anharmonic oscillator with W-shape potential; (a) periodic scalar field trajectory located around $\Phi = 1$; (b) the solution for the Hubble function $H(t)$, $\xi^2 = 0.4$.

$$V(\Phi) = \frac{2\Phi^2 - \Phi^4}{4(2k - \xi^2)}.$$

As we can see for $\xi^2 < 2k$, a flip of the potential takes place. In particular, for $\xi^2 = 0$ we get an equation describing a particle moving in the $-W$-shape potential [see Fig. 3(b)]. For $\xi^2 > 2k$ the potential has W-shape [see Fig. 3(a)]. This simple remark leads to important physical consequences: we might expect two different regimes of the solution. Solutions for the scalar field and Hubble function for different values of ξ^2 are presented on Fig. 5. We see two regimes—decreasing oscillations with positive Hubble function and increasing oscillations with positive Hubble function.

It is important to note that while mechanical approximation helps us to illustrate or explain qualitatively some physical consequences, it is a rather rough approximation and does not necessarily capture some nonlocal effects. This situation is already depicted by the fact that in a full nonlocal model and its mechanical approximation the change of regime of the solution corresponding to W and $-W$ shapes of the potential happens for different values of ξ^2, they are of the same order though.

B. Rolling tachyon in the FRW background and the effective cosmological constant Λ'

Let us study the value of effective cosmological constant Λ'. Formally Λ' enters action as a correction of $D3$-brane tension T. For a rolling solution in Minkowski background it is known [11,31,44] that the value of T is essential for calculating the stress tensor, but as it does not enter equations of motion it is not essential for the existence of the solution. Moreover, according to Sen's conjecture [11,31,44]

$$T_{\text{Minkowski}} = -V(\Phi = \pm 1) = \tfrac{1}{4}.$$

Such value of T in Minkowski background corresponds to a zero cosmological constant in the nonperturbative vacuum $\Phi = \pm 1$.

In the case of the FRW metric we represent $D3$-brane tension as $T = T_{\text{Minkowski}} + \Lambda'$, where Λ' is determined uniquely for each field configuration (situation is the same in local theories). Note that Λ' does not enter Eqs. (16); in fact we determine its value from (15a). Different values of Λ' for different ξ are presented on Table I. Note that we worked with dimensionless variables.
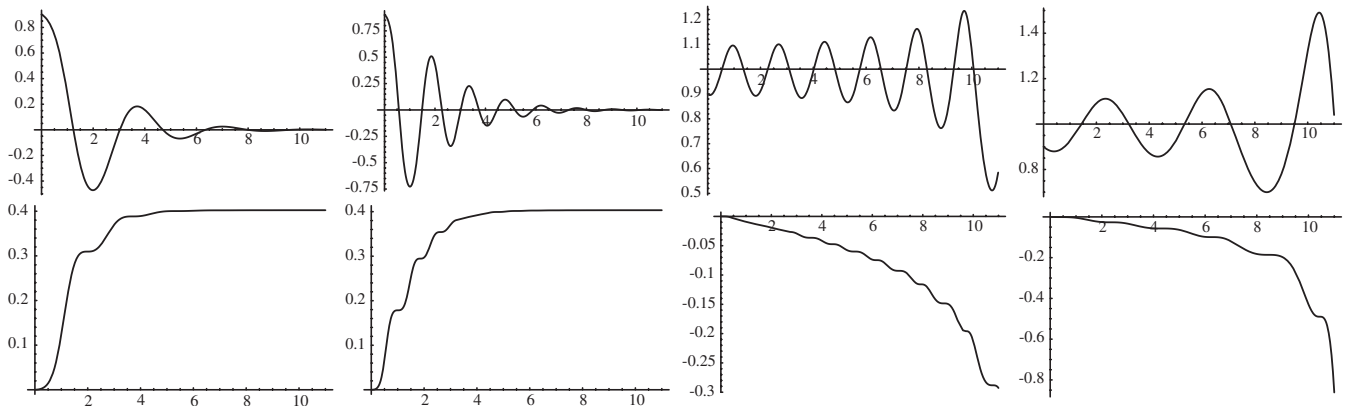

FIG. 5.   Scalar field $\Phi(t)$ (upper row) and Hubble function $H(t)$ (lower row) for the system (41) for different values for ξ^2 parameter $\xi^2 = 0, 0.2, 0.4, 1$ from left to right correspondingly.

TABLE I. Values of Λ' for different ξ^2.

ξ^2	Λ'
0	0.036 938
0.2	0.010 379
0.41	0.000 022
0.42	0
0.43	0.000 023
0.6	0.007 766
0.96	0.077 848

If we return to initial notations and take into account physical constants in action (6) we can get a realistic value of Λ because generally speaking the string scale does not coincide with the Plank mass [11,17,19].

C. *P*-adic cosmological model, $\xi^2 = 0$

Recently p-adic string models started to attract a lot of attention being used as toy cosmological models. In [19] the p-adic string model [31] was considered in the context of inflation and approximate solutions of the fully nonlocal p-adic string theory coupled to gravity were constructed. In Sec. IVA we presented a new algorithm for construction of precise solutions for these models. While an iterative procedure of Sec. IVA converges for any odd p (for rigorous mathematical proof for the simpler Minkowski case see [43]) here we will analyze cosmological properties of the solution for the case $p = 3$. This case is specifically interesting since as mentioned earlier it corresponds to the zero mass of the tachyon field, $\xi^2 = 0$, in level-truncated fermionic string field theory (SFT).
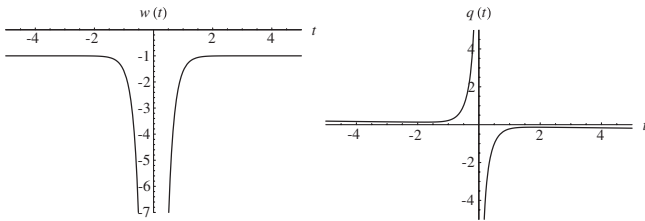

FIG. 6. The state and deceleration parameters for the case of the p-adic cosmological model ($\xi^2 = 0$).

We can see in Figs. 1 and 6 that scalar field and background solutions are monotonic functions which tend to constants as $t \to \pm\infty$. Since these solutions do not have turning points (oscillations) there is no crossing of the cosmological state parameter barrier ($w = -1$); it is negative and approaches to -1 from below (Fig. 6). For these solutions we have a nonsingular accelerating Universe with a bounce. As an illustration we can note that behavior of the scale factor obtained could be well approximated by $a_{\text{approx}}(t) = e^{H_0 t}$ for large $t > 0$ and by $a_{\text{approx}}(t) = e^{-H_0 t}$ for large $t < 0$, where $H_0 = \lim_{t \to \infty} H(t)$. Thus we get a de Sitter type solution for flat FRW background for $t \to +\infty$ and anti–de Sitter as $t \to -\infty$. It might be useful to remark that bouncing cosmology is a subject of a recent investigation which includes the ekpyrotic [34], pre-big-bang [45], and higher derivative modification of Einstein gravity scenarios [46].

Deceleration parameter q is negative and approaches -1 from below as $t \to \pm\infty$.

D. SFT cosmological model, $\xi^2 = \xi^2_{\text{phys}}$

Today SFT is the strongest candidate for real physical theory of nature. As we mentioned earlier level truncation in nonsypersymmetric fermionic SFT leads to action (7) with potential (21) with $\xi^2 = \xi^2_{\text{phys}}$ (24) [11,31,33]. The algorithm presented in Sec. IVA allowed us to study this model coupled to the FRW backgroud without explicit approximations.

As shown in Figs. 1 and 7 in this case scalar field solution has turning points (decreasing oscillations) and tends to stationary solutions ± 1 as $t \to \pm\infty$. The shape of the Hubble function is essentially different as compared to the case $\xi^2 = 0$; indeed it has a clear maximum and tends to negative constant for $t > 0$, $\lim_{t \to \pm\infty} H(t) = \mp 0.15$. Such behavior is in agreement with analysis presented in the previous section since $\xi^2_{\text{phys}} > \xi^2_{\text{osc}}$.

The state parameter $w(t)$ tends to -1 as $t \to \pm\infty$ and has exponentially decreasing oscillations around the $w = -1$ barrier. Note that models which allow for crossing of the $w = -1$ barrier are the subject of many recent investigations [17,47–53]. Here we observe this behavior in a nonlocal model with only one scalar field as in [17]. We also
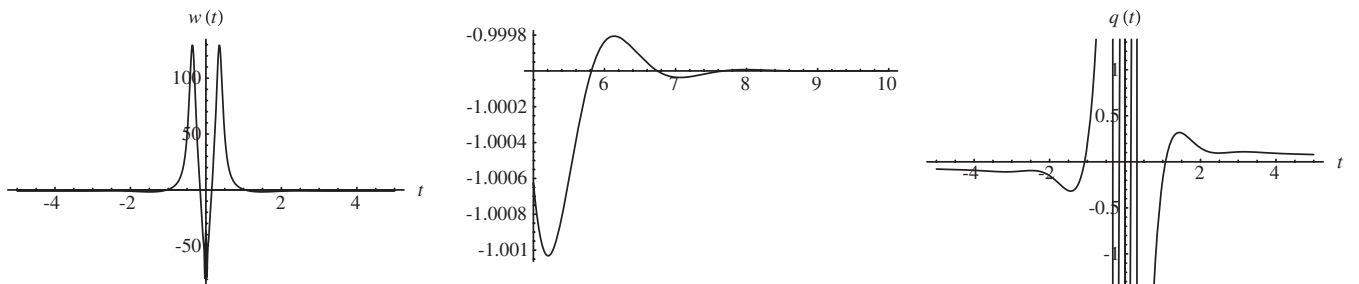

FIG. 7. The state parameter, its fine structure, and deceleration parameter for the case of the SFT cosmological model ($\xi^2 = \xi^2_{\text{phys}}$).

see that we have a decelerating phase at late times; this property was not predicted by approximate methods [11].

VI. CONCLUSIONS

In this paper we have studied the dynamics of nonlocal cosmological models driven by string field theory. These models have an infinite number of derivatives and are characterized by positive constants k, ξ, and Λ' which determine the shape of the solution. We have developed a new method for solving nonlocal Friedmann equations. Such equations contain an infinite number of derivatives and form a new class of equations in mathematical physics which recently started to be discussed in literature [15,18,19,37,41–43,54].

To study cosmological properties we have investigated the behavior of nonlocal models in the FRW background. We are especially interested in two physical cases: p-adic cosmological models and the level-truncated SFT model. In the p-adic cosmological model we obtained a nonsin-gular bouncing solution with the w parameter approaching the -1 barrier from below. The SFT model has a non-singular solution for which the w parameter crosses the -1 barrier. We also discussed the possibility of obtaining a realistic cosmological constant from the above considered nonlocal cosmological models.

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