Is the shell-focusing singularity of Szekeres space-time visible?

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The visibility of the shell-focusing singularity in Szekeres space-time—which represents quasispherical dust collapse—has been studied on numerous occasions in the context of the cosmic censorship conjecture. The various results derived have assumed that there exist radial null geodesics in the space-time. We show that such geodesics do not exist in general, and so previous results on the visibility of the singularity are not generally valid. More precisely, we show that the existence of a radial geodesic in Szekeres space-time implies that the space-time is axially symmetric, with the geodesic along the polar direction (i.e. along the axis of symmetry). If there is a second nonparallel radial geodesic, then the space-time is spherically symmetric, and so is a Lemaître-Tolman-Bondi space-time. For the case of the polar geodesic in an axially symmetric Szekeres space-time, we give conditions on the free functions (i.e. initial data) of the space-time which lead to visibility of the singularity along this direction. Likewise, we give a sufficient condition for censorship of the singularity. We point out the complications involved in addressing the question of visibility of the singularity both for nonradial null geodesics in the axially symmetric case and in the general (nonaxially symmetric) case, and suggest a possible approach.

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I. INTRODUCTION AND SUMMARY

The weak cosmic censorship hypothesis (CCH) maintains that realistic gravitational collapse leads to the formation of a black hole rather than a naked singularity. Among the different studies of the hypothesis, we mention two categories of interest. First, there are demonstrations of the validity of the hypothesis in specific circumstances (the prime example of this is Christodoulou's proof of the instability—and hence nonrealistic nature—of naked singularities in the spherical collapse of a minimally coupled scalar field [1]). The other category involves the construction of an example of a space-time which undergoes collapse from a regular configuration to a naked singularity. Many such examples have been constructed, but to date, none has been shown to involve both (i) a physically realistic matter model and (ii) stability in the initial data space of those space-times which give rise to naked singularities. It is probably fair to say that the main utility of the latter class of studies has been to refine and better understand the content of the CCH.

These examples have mainly involved spherically symmetric space-times, for example, the shell-crossing [2,3] and shell-focusing [4] singularities in Lemaître-Tolman-Bondi (LTB) spherical dust collapse, and the naked singularity solutions that arise at the threshold of black hole formation in scalar field and perfect fluid collapse [5]. A notable (although as we will see flawed) exception to this involves the various studies of the visibility of the shell-focusing singularity in Szekeres space-time [6–12]. First analyzed by Szekeres in [13], this class of space-times corresponds to solutions of the Einstein equations for dust, where the fluid flow vector is geodesic and nonrotating. The metric admits no Killing vector fields (in the general case) but for reasons described below is referred to as quasispherical. It can be understood as a nonspherical generalization of the LTB class of dust-filled space-times, and its evolutionary aspects are very closely related to those of the corresponding LTB models, and so are relatively straightforward. Hence analyzing the visibility or otherwise of singularities that arise in this model affords an opportunity to study cosmic censorship in nonspherical collapse.

Szekeres initiated the study of the singularities in this model, and noted the possibility that the shell-crossing singularity (see below) may be visible [14]. He also noted the occurrence of an apparent horizon (or in current terminology, a marginally trapped tube) that forms at least as early as the shell-focusing singularity (again, see below). As in the spherical case, the shell-crossing singularity is interpreted as being fundamentally nongravitational in origin and not of particular significance for cosmic censorship. Thus attention turned to the shell-focusing singularity, where in the spherical case, a more-or-less complete understanding of the visibility or otherwise of the singularity has been developed. In these studies [6–12], the question of visibility of the shell-focusing singularity is considered from the point of view of the existence or otherwise of future-pointing null geodesics with an additional simplifying property that allows one to refer to these geodesics as "radial" (see below). We show below that

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radial geodesics do not exist in a general Szekeres space-time and consequently, the analyses of the question of visibility of the singularity of [6–12] are not generally valid. Motivated by this observation, we revisit the question of the visibility of the shell-focusing singularity. We provide some preliminary results on this question, and, in particular, consider it in the case when the space-time is axially symmetric. In this case, a single radial geodesic direction exists, and the analysis of the visibility of the singularity along this direction is essentially the same as the spherically symmetric case. We emphasize that the question becomes considerably more difficult in the general (nonaxially symmetric) case.

In the next section, we review the basic properties of Szekeres space-time, and discuss what is meant by referring to this space-time as "quasispherical." We then discuss the formation of singularities in a collapsing Szekeres space-time, and discuss the conditions on the free metric functions that arise from the imposition of regularity conditions on the initial data. Here, we specialize to the so-called marginally bound case. We will indicate clearly when this restriction is in place, and when results apply generally.

In Sec. III, we analyze the geodesic equations of Szekeres space-time and show that the existence of a radial geodesic implies that the space-time is axially symmetric. Furthermore, we show that the existence of a second noncollinear radial geodesic implies that the space-time is spherically symmetric. The marginally bound assumption is *not* required in this section.

In Sec. IV, we derive some elementary results in the marginally bound case relating to the visibility of the shell-focusing singularity, which is spherically symmetric in the sense that it corresponds to a surface $t = t_c(r)$, $r \geq 0$. This singularity is always preceded by an apparent horizon given by $t = t_{\rm ah}(r)$: we have $t_{\rm ah}(r) \leq t_c(r)$ for all r with equality if and only if $r = 0$. The region of space-time with $t_{\rm ah}(r) < t < t_c(r)$ is trapped. Then intuitively, one expects that only the central singularity $(t, r) = (t_c(0), 0)$ can be visible. This is immediate in spherical symmetry, but is slightly nontrivial in the quasispherical case—the result is confirmed nonetheless. We derive the related result that a geodesic that emerges into the future from the central singularity must emerge into the untrapped region $t < t_{\rm ah}$, and hence show that the singularity is censored if $t'_{\rm ah}(r) < 0$ (this condition can be described in terms of the initial data of the collapse). We also show that for sufficiently small values of r, the apparent horizon is a one-way membrane for all future-pointing causal geodesics: such geodesics cannot leave the trapped region. Again, this is immediate in the spherical case, but requires checking in the quasispherical case.

In Sec. V, again working in the marginally bound case, we consider the question of the visibility of the singularity for the polar null geodesic of the axially symmetric models. This problem is essentially the same as the corresponding spherically symmetric problem, and we can give sufficient conditions (in terms of the initial data) for the formation of a naked singularity.

We conclude by briefly considering the substantive, and crucially, open, question of the visibility of the shell-focusing singularity in quasispherical collapse. We point out how this question is considerably more difficult than in the spherical case and suggest an approach to its consideration. We set $8\pi G = c = 1$.

II. SZEKERES SPACE-TIME AND ITS SINGULARITIES

A comprehensive review of the properties of the Szekeres space-times representing nonaccelerating, irrotational dust is given in [15]; this review includes equivalent invariant characterizations of the class which involve some technicalities that will not play any role here. Suffice to say that these lead uniquely to the line element

$$ds^2 = -dt^2 + e^{2\alpha} dr^2 + e^{2\beta}(dx^2 + dy^2) \quad (1)$$

where

$$e^{\beta} = R(t, r) e^{\nu}, \quad (2)$$

$$e^{-\nu} = A(r)(x^2 + y^2) + B_1(r)x + B_2(r)y + C(r), \quad (3)$$

$$e^{\alpha} = \frac{R' + R\nu'}{\sqrt{1 + f(r)}}, \quad (4)$$

where the prime denotes differentiation with respect to r. The geodesic fluid flow vector is $\frac{\partial}{\partial t}$ and the coordinates (r, x, y) are comoving. The ranges of these coordinates are $(x, y) \in \mathbb{R}^2$, $r \geq 0$ and that of t will be discussed below. The free functions A, B_1, B_2 and C are related by

$$B_1^2 + B_2^2 - 4AC = -1. \quad (5)$$

(There are also solutions with 0 and +1 on the right hand side here: the choice −1 forms part of the input essential to the interpretation of these space-times as being quasispherical.) The remaining Einstein equations determine the evolution of R and define the energy density of the space-time:

$$\left(\frac{\partial R}{\partial t}\right)^2 = f(r) + \frac{F(r)}{R}, \quad (6)$$

$$\rho(t, r, x, y) = \frac{F' + 3F\nu'}{R^2(R' + R\nu')}. \quad (7)$$

$F \geq 0$ is a function of integration. We note that the coordinate freedom corresponds to rescalings of the comoving radial coordinate $r \to \hat{r}(r)$ (which must be a monotone mapping), the transformations in the (x, y) plane discussed in (8) below and trivial shifts of the origin of t.

The quasispherical interpretation arises as follows [14]: each 2-surface $S_{t,r}$ of constant t and r is a round 2-sphere with proper radius $R(t, r)$. To see this, we write the line element $dl^2_{(t,r)}$ of $S_{t,r}$ in the stereographic coordinates $\zeta = x + iy$:

$$dl^2_{(t,r)} = \frac{R^2(t, r)}{(A(r)\zeta\bar{\zeta} + B(r)\zeta + \bar{B}(r)\bar{\zeta} + C(r))^2} d\zeta d\bar{\zeta}$$

where $B = (B_1 - iB_2)/2$. The form of this line element is invariant under the fractional linear transformation

$$\zeta \rightarrow \xi = \frac{k\zeta + l}{m\zeta + n}, \qquad kn - lm = 1 \tag{8}$$

and for each fixed value of t and r, such a transformation can be found so that the line element has the form

$$dl^2_{(t,r)} = R^2(t, r)\frac{4d\xi d\bar{\xi}}{(1 + \xi\bar{\xi})^2},$$

which is the line element of the round 2-sphere with radius R. It should be noted that the condition (5) plays a crucial role in deriving the explicitly spherical form of $dl^2_{(t,r)}$. The transformation $(x, y) \rightarrow (\theta, \phi)$ given by

$$\zeta = e^{i\phi}\cot\frac{\theta}{2}$$

yields the more familiar spherical form

$$dl^2_{(t,r)} = R^2(t, r)(d\theta^2 + \sin^2\theta d\phi^2).$$

The fact that a different transformation (8) is required for each different $S_{t,r}$ indicates that in each 3-space of constant t, these 2-spheres are not concentric.

It will be useful to consider the form of the 4-dimensional line element using the spherical coordinates (θ, ϕ). This yields

$$ds^2 = -dt^2 + e^{2\alpha}dr^2 + e^{2\beta}(d\theta^2 + \sin^2\theta d\phi^2), \tag{9}$$

where now

$$e^{\beta} = R(t, r)e^{\mu}, \tag{10}$$

$$e^{-\mu} = a\cos\theta + b_1\sin\theta\cos\phi + b_2\sin\theta\sin\phi + c, \tag{11}$$

$$e^{\alpha} = \frac{R' + R\mu'}{\sqrt{1 + f}}. \tag{12}$$

The functions a, b_i, c are related to A, B_i, C by

$$a = A - C, \qquad b_i = B_i, \qquad c = A + C. \tag{13}$$

The condition (5) reads

$$c^2 - a^2 - b_1^2 - b_2^2 = 1. \tag{14}$$

Before proceeding to discuss gravitational collapse and the formation and nature of singularities in these space-times, we note that the spherical limit arises when and only when $B_1 = B_2 = 0$ and A and C are constant and equal [and so by (5) both equal to $1/2$]. Furthermore, it is clear from (9) and (11) that the space-time is axially symmetric with Killing vector $\frac{\partial}{\partial\phi}$ when $B_1 = B_2 = 0$.

In order to model gravitational collapse using a Szekeres space-time, we choose the negative root of (6). The resulting equation, and the analysis of its consequences, are greatly simplified by taking $f(r) = 0$. By analogy with the spherical case, this is referred to as the marginally bound case. The equation is then easily integrated and the solution can be written in the form

$$R^3 = \frac{9}{4}F(t_c(r) - t)^2, \tag{15}$$

where t_c is a function of integration that describes the time at which the 2-sphere $S_{t,r}$ collapses to zero radius. This is called the shell-focusing singularity: all the "shells" $S_{t,r}$ collapse to zero radius at this surface.

It is convenient to exploit the freedom in the comoving radial coordinate r to set $R = r$ on an initial surface: by an allowed shift of the origin of t, we can take this to be $t = 0$ without loss of generality. Thus $R(0, r) = r$ and hence $R'(0, r) = 1$. With this choice, we have

$$t_c(r) = \frac{2}{3}\sqrt{\frac{r^3}{F}}, \tag{16}$$

which leads to the following convenient form of (15):

$$\left(\frac{R}{r}\right)^3 = \left(1 - \frac{t}{t_c}\right)^2. \tag{17}$$

It can be shown that the Kretschmann scalar of (1) diverges if and only if the density (7) does so. Thus scalar curvature singularities can be discussed in terms of the latter quantity (i.e. the density—or equivalently the Ricci scalar) alone. As well as the shell-focusing singularity at $t = t_c(r)$ for which $R = 0$, there may be singularities when $R' + R\nu'$ vanishes. By analogy with the spherical case, these are referred to as shell-crossing singularities. [The analogy is perhaps not quite appropriate: shells of Szekeres space-time, i.e. the 2-spheres $S_{t,r}$ will cross if we encounter $R(t, r_1) = R(t, r_2)$ for some $r_1 \neq r_2$ and some $t > 0$. A necessary and sufficient condition for this to occur is $R'(t, r) = 0$ at some $t > 0$, $r > 0$.] A crucial task is to rule out the occurrence of both types of singularity on the initial slice. To this end, we note that the initial density is given by

$$\rho_0(r) := \rho(0, r) = \frac{F' + 3F\nu'}{r^2(1 + r\nu')}.$$

We will require this term to be non-negative and finite for all $r \geq 0$. Thus we impose

$$F' + 3F\nu' \geq 0, \tag{18}$$

$$1 + r\nu' > 0 \tag{19}$$

for all $r \geq 0$ and all $(x, y) \in \mathbb{R}^2$. We note that we also impose $e^{-\nu} > 0$. Identical conditions hold with ν replaced by μ. These conditions—i.e. (19) and $e^{-\mu} > 0$—have been considered by Szekeres [14], and it is worth repeating the result here as we will use the corresponding conditions on a, b_i and c below:

Lemma 1: (i) Let the condition (5) hold. Then $e^{-\nu} = A(x^2 + y^2) + B_1 x + B_2 y + C$ is positive for all $(x, y) \in \mathbb{R}^2$ and all $r \geq 0$ if and only if

$$A(r) > 0 \quad \text{for all } r \geq 0. \tag{20}$$

(ii) Assuming the conditions (5) and $e^{-\nu} > 0$,

$$1 + r\nu' > 0$$

for all $(x, y) \in \mathbb{R}^2$ and all $r \geq 0$ if and only if

$$A - rA' > 0 \quad \text{for all } r \geq 0 \tag{21}$$

and

$$A'C' - B'\bar{B}' > -\frac{4}{r^2} \quad \text{for all } r \geq 0 \tag{22}$$

■

Requiring that ρ_0 be finite in the limit as $r \to 0$, Szekeres [14] also derives the condition

$$F(r) = O(r^3), \qquad r \to 0 \tag{23}$$

which we shall assume henceforth.

Next, we will consider conditions that rule out the occurrence of a shell-crossing singularity ($R' + R\nu' = 0$) prior to the occurrence of the shell-focusing singularity. That is, we seek conditions on the initial data functions so that

$$R' + R\nu' > 0, \quad \text{for all } 0 \leq t < t_c(r),\ r \geq 0.$$

We note that this condition follows by (19) *if* the inequality $R' - R/r \geq 0$ holds. But using (17), this latter condition is equivalent to $t_c' \geq 0$. Hence the condition

$$F' - 3\frac{F}{r} \leq 0, \qquad r \geq 0, \tag{24}$$

which is equivalent to $t_c' \geq 0$, is a condition that can be imposed on the initial data and that guarantees the absence of shell-crossing singularities. It is worth noting that for $t > 0$, (24) is *equivalent* to the condition

$$R' \geq \frac{R}{r}. \tag{25}$$

In a sense, this is the best bound that can be imposed. First, it includes all cases of interest: $t_c' \geq 0$ is a necessary condition for the visibility of the shell-focusing singularity. (To see this, we simply note that t must increase along a future-pointing causal geodesic emerging from the singularity.) Second, if the bound (24) is violated, then we can find examples of ν (which is constructed entirely from initial data functions) for which the corresponding space-time will contain shell-crossing singularities in its evolution.

To summarize, we assume the conditions on A, B_i and C of Lemma 1 and the conditions (23) and (24) on F. These guarantee that the collapse proceeds from a regular state to the shell-focusing singularity, and that no shell-crossing singularities occur prior to the shell-focusing singularity.

III. GEODESICS, RADIAL GEODESICS AND SYMMETRY

With a clear description of the shell-focusing singularity in place, we can now consider its causal nature or more accurately, the question of its visibility. Of course this requires the analysis of geodesics, and the determination of whether or not there are future-pointing causal geodesics that emerge from the singularity. We will use a subscript (α_t etc.) to denote partial derivatives with respect to t, x and y. We retain the prime for derivatives with respect to r, and an overdot will represent differentiation along the geodesic (e.g. with respect to an affine parameter for null geodesics). We also note that we can drop the assumption that the space-time is marginally bound. The geodesic equations for the line element (1) are

$$\ddot{t} + \alpha_t e^{2\alpha}\dot{r}^2 + \beta_t e^{2\beta}(\dot{x}^2 + \dot{y}^2) = 0, \tag{26}$$

$$\ddot{r} + \alpha'\dot{r}^2 + 2\alpha_t\dot{t}\,\dot{r} + 2\alpha_x\dot{x}\,\dot{r} + 2\alpha_y\dot{y}\,\dot{r} - \beta' e^{2\beta-2\alpha}(\dot{x}^2 + \dot{y}^2) = 0, \tag{27}$$

$$\ddot{x} + \beta_x(\dot{x}^2 - \dot{y}^2) + 2\beta_y\dot{x}\,\dot{y} + 2\beta_t\dot{t}\,\dot{x} + 2\beta'\dot{r}\,\dot{x} - \alpha_x e^{2\alpha-2\beta}\dot{r}^2 = 0, \tag{28}$$

$$\ddot{y} + \beta_y(\dot{y}^2 - \dot{x}^2) + 2\beta_x\dot{x}\,\dot{y} + 2\beta_t\dot{t}\,\dot{y} + 2\beta'\dot{r}\,\dot{y} - \alpha_y e^{2\alpha-2\beta}\dot{r}^2 = 0, \tag{29}$$

and we have the first integral

$$-\dot{t}^2 + e^{2\alpha}\dot{r}^2 + e^{2\beta}(\dot{x}^2 + \dot{y}^2) = \epsilon, \tag{30}$$

where $\epsilon = 0, +1, -1$ for null, spacelike and timelike geodesics, respectively.

In [6–12], radial geodesics are defined to be those along which x and y have constant values. From (28) and (29), we see that along such a geodesic, we must have

$$\frac{\partial\alpha}{\partial x}e^{2\alpha-2\beta}\dot{r}^2 = \frac{\partial\alpha}{\partial y}e^{2\alpha-2\beta}\dot{r}^2 = 0. \tag{31}$$

We rule out $e^{\alpha-\beta} = 0$, as this corresponds to a singularity (and the geodesics must reside in the space-time rather than on its singular boundary). Both equations of (31) are satisfied if we take $\dot{r} = 0$. The only possible solutions of the geodesic equations then have $\dot{t}^2 = 1 = -\epsilon$: these are the fluid flow lines and we will refer to these as trivial radial geodesics. In particular, there are no *null* geodesics that satisfy the condition $\dot{r} = 0$. The only possibility that remains in (31) is that

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \alpha}{\partial y} = 0$$

along the geodesic. It is clear that this is a restriction on the metric functions. We can determine the exact geometric nature of this restriction.

In order to do so, we note first that

$$\frac{\partial \alpha}{\partial x} = \frac{Re^{-\alpha}}{\sqrt{1+f}} \frac{\partial^2 \nu}{\partial r \partial x},$$

with a similar result holding for α_y. Then a straightforward calculation shows that the vanishing of α_x is equivalent to the vanishing of

$$Q_1 := (AB_1' - A'B_1)(x^2 - y^2) + 2(AB_2' - A'B_2)xy + 2(AC' - A'C)x + (B_1B_2' - B_1'B_2)y + (B_1C' - B_1'C), \quad (32)$$

while vanishing of α_y is equivalent to vanishing of

$$Q_2 := (AB_2' - A'B_2)(y^2 - x^2) + 2(AB_1' - A'B_1)xy + 2(AC' - A'C)y + (B_2B_1' - B_2'B_1)x + (B_2C' - B_2'C). \quad (33)$$

We can now prove the following result.

Proposition 1: If a Szekeres space-time admits a nontrivial radial geodesic, then it also admits a Killing vector field generating an axial isometry.

Proof: Suppose that there is a nontrivial radial geodesic along which $(x, y) = (x_0, y_0)$ is constant. Using a transformation of the form (8),

$$\zeta = x + iy \rightarrow \xi = u + iv = \frac{k\zeta + l}{m\zeta + n}$$

we can assume without loss of generality that $(x_0, y_0) = (0, 0)$. The necessary conditions $Q_1 = Q_2 = 0$ for the existence of a nontrivial radial geodesic then yield

$$B_1 = \lambda_1 C, \qquad B_2 = \lambda_2 C$$

for some constants λ_1, λ_2. [We note that (5) implies that $C \neq 0$.] If we consider a further coordinate transformation of the form (8)—but with $l = 0$ to preserve the origin—we find that [with $B = (B_1 - iB_2)/2$)]

$$B \rightarrow B_* = k\bar{n}B + m\bar{n}C = \bar{n}(k\lambda + m)C$$

where $\lambda = (\lambda_1 - i\lambda_2)/2$. Since $kn - lm = kn = 1$, we have $k \neq 0 \neq \bar{n}$, and so we can choose $m = -k\lambda$ to get $B_* = 0$. Thus by using the coordinate freedom in the stereographic coordinates (x, y), we can write the line element (1) in a form in which $B_1 = B_2 = 0$. As seen in Sec. II above, this is a sufficient condition for the space-time to be axially symmetric, with axial Killing field given by $\frac{\partial}{\partial \phi} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. ■

Remark 1: In this proposition, the radial geodesic is normal to each of the 2-spheres and emerges from the point with stereographic coordinates $x = y = 0$. In spherical coordinates, this corresponds to $\theta = \pi$: the south pole in the standard configuration. The north pole ($\theta = 0$) corresponds to the point at infinity in stereographic coordinates, and so is not covered by the coordinate patch $(x, y) \in \mathbb{R}^2$. However this point can be included by using an additional coordinate patch and it is then clear that there is also a radial geodesic emerging from the north pole. We will refer the these collinear radial geodesics as the polar geodesics.

Corollary 1: If a Szekeres space-time admits two noncollinear nontrivial radial geodesics, then the space-time is spherically symmetric.

Proof: From the previous proposition, we may situate the first nontrivial geodesic in the direction $(x, y) = (0, 0)$. Then as we have seen $B_1 = B_2 = 0$ and the space-time is axially symmetric. A second noncollinear nontrivial radial geodesic has $(x, y) = (x_0, y_0) \neq (0, 0)$ constant along the geodesic, and as we have seen, a necessary condition for the existence of such a geodesic is that Q_1 and Q_2 vanish along the geodesic. As $B_1 = B_2 = 0$ and x_0 and y_0 are not both zero, (32) and (33) yield

$$AC' - A'C = 0.$$

The condition (5) in the present case gives $AC = 1/4$. Combining this with the previous relation shows that A and C are both constant. A transformation of the form (8) (with $l = 0$ to preserve the origin) can then be used to set $2A = 2C = 1$, and so the line element is spherically symmetric. ■

Remark 2: We can give a geometric interpretation of this corollary. Proposition 1 shows that to each nontrivial radial geodesic of Szekeres there corresponds an axis of symmetry. Then the existence of a second noncollinear nontrivial radial geodesic implies that the space-time admits two nonparallel axes of symmetry and so must be spherically symmetric.

Remark 3: The results of this section imply that the analysis of [6–12], which were carried out for radial null geodesics, can only be valid for the polar geodesic of an axially symmetric Szekeres space-time, or for a spherically symmetric Szekeres space-time—i.e. LTB space-time. Thus the question of the visibility of the shell-focusing singularity in a general Szekeres space-time remains open. It is worth noting that the imposition of the assumption of axial symmetry, equivalent to setting $B_1 = B_2$, implies that the number of free functions (A, B_1, B_2, f, F) has been reduced from 5 to 3. Therefore this sector of the Szekeres class is highly specialized—one could impose a topology on the space of free functions in which the nonaxially symmetric solutions comprise an open dense subset of the whole space—and so the results pertaining to the polar geodesics cannot be assumed to reflect the general behavior. It is possible that the naked singularities found in the axially symmetric case along the polar direction are (i) not visible from any other direction and/or (ii) are not present in the nonsymmetric case.

IV. SOME BASIC RESULTS IN THE MARGINALLY BOUND CASE

We have argued above that the question of cosmic censorship in Szekeres is open. In the remainder of this paper, we seek to address this question. In this section, we will derive some results, valid in the marginally bound case only, which provide some useful preliminary results for the study of the visibility of the shell-focusing singularity. These results share the feature that they are trivial in the spherically symmetric case: in the quasispherical case, some checking is required.

To begin, we recall from the work of Szekeres [14] that an apparent horizon forms at the hypersurface $R(t, r) = F(r)$. That is, the outgoing future-pointing null geodesic normals to each 2-sphere $S_{t,r}$ have zero expansion on this hypersurface. By outgoing, we mean that r increases with the affine parameter along the geodesic. Note also that while these geodesics are initially and instantaneously radial, this condition is immediately violated as the geodesic moves away from the 2-sphere to which is was normal—see (28) and (29).

From (17), we can show that the apparent horizon is given by

$$t = t_{ah}(r) = t_c(r) - \frac{2}{3}F(r). \tag{34}$$

Thus the apparent horizon precedes the shell-focusing singularity ($F > 0$ for $r > 0$), and the condition (24) implies that the apparent horizon and the shell-focusing singularity meet at the central singularity $(t, r) = (t_c(0), 0)$. As shown in [14], the region $t_{ah}(r) < t < t_c(r)$, $r \geq 0$ is trapped. That is, the 2-spheres $S_{t,r}$ are closed trapped surfaces in this region. The region $t < t_{ah}(r)$, $r \geq 0$ is untrapped.

First, we point out that the portion $r > 0$ of the shell-focusing singularity is censored.

Proposition 2: There are no future-pointing causal geodesics of a marginally bound Szekeres space-time with past end point on the surface $t = t_c(r)$ for any $r > 0$.

Proof: We consider a future-pointing ($\dot{t} > 0$) outgoing ($\dot{r} > 0$) causal geodesic of (1). Either the geodesic remains outgoing, or we encounter a value τ_0 of the geodesic parameter τ (affine parameter or proper time) for which $\dot{r}(\tau_0) = 0$. But then (27) shows that this is a local minimum ($\ddot{r} > 0$) of r along the geodesic. (To see this, we note that

$$\frac{\partial \beta}{\partial r} = (R' + R\nu')e^{\nu},$$

which is positive by the no-shell-crossing singularity condition.) As all stationary points must be local minima, there can in fact be only one local minimum. Thus $\dot{r}(\tau) < 0$ for all $\tau < \tau_0$. Suppose that this geodesic were to meet the singularity $t = t_c(r)$. Now along the geodesic we have

$$\dot{R} = R_{,t}\dot{t} + R'\dot{r} = -\sqrt{\frac{F}{R}}\dot{t} + R'\dot{r}. \tag{35}$$

Since $\dot{t} > 0$, $R' \geq 0$ and $\dot{r} < 0$, this must by negative in the approach to the singularity. But the singularity occurs at $R = 0$, and so R cannot increase into the past ($\dot{R} < 0$) to reach the singularity, and so we get a contradiction.

We can now assume that $\dot{r} > 0$ for all $\tau \leq \tau_0$ where τ_0 is some arbitrary initial value for τ. This being the case, we can use r as a parameter along the geodesic. Then along the geodesic we may use (30) to write

$$\frac{dt}{dr} = e^{\alpha}\left[1 - \frac{\epsilon}{\dot{r}^2}e^{-2\alpha} + e^{2\beta-2\alpha}\left(\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2\right)\right]^{1/2}. \tag{36}$$

The fact that the geodesic is both future pointing and outgoing indicates that the correct (positive) root has been taken here. The change of R along a future-pointing geodesic that emerges from the shell-focusing singularity at some $r > 0$ satisfies the following:

$$\begin{aligned}\frac{dR}{dr} &= R_{,t}\frac{dt}{dr} + R' \\ &= -\sqrt{\frac{F}{R}}\frac{dt}{dr} + R' < -\frac{dt}{dr} + R' < -e^{\alpha} + R' \\ &= -R\nu' < \frac{R}{r}.\end{aligned} \tag{37}$$

The second line comes from the field Eq. (6) in the marginally bound case. The third line arises due to the fact that for sufficiently small R, the geodesic must be in the trapped region for which $F > R$. The fourth line follows from (36) above and the definition (4). The last line comes from the initial regularity condition (19). Integrating the overall inequality proves the stated result: R cannot reach zero (its value on the shell-focusing singularity) unless r also drops to zero. ■

Corollary 2: A future-pointing causal geodesic with past end point on the shell-focusing singularity must have its past end point on the *central singularity* $(t, r) = (t_c(0), 0)$. ■

Corollary 3: A future-pointing null geodesic with past end point on the central singularity must emerge into the untrapped region of space-time.

Proof: Suppose on the contrary that the geodesic emerges into the trapped region, i.e. there exists $\delta > 0$ such that $R(t(r), r) < F(r)$ for values of R along the geodesic and for all $0 < r < \delta$. Letting $r_0 \in (0, \delta)$ and integrating (37) from r to r_0 shows that in the R-r plane, R stays above the line $R = \frac{R_0}{r_0}r$ for all $r \leq r_0$. Using (23), this implies that $R > F$ for sufficiently small values of r, yielding a contradiction. ■

Proposition 3: If $t'_{ah}(r) < 0$ on $[0, \delta)$ for some $\delta > 0$, then the central singularity is censored.

Proof: The proof follows immediately from Corollary 3: if a geodesic were to emerge from the central singularity, then it must emerge into the untrapped region and must have $t'(r)$ non-negative for all sufficiently small values of r. This cannot happen if t'_{ah} is negative in a neighborhood of the singularity. ■

Remark 4: We note that this repeats in the Szekeres case a result that holds in some generality in spherical symmetry [16], and provides a sufficient condition, in terms of initial data, for the singularity to be censored.

Finally in this section, we prove another result that mirrors precisely the situation in the spherical case. As in that case, the proof relies crucially on some of our assumptions about the regularity of the initial data.

Proposition 4: For sufficiently small values of r, the apparent horizon $t = t_{ah}(r)$ acts as a one-way membrane: a future-pointing null geodesic cannot cross the horizon from the trapped to the untrapped region.

Proof: Let p be a point of space-time on the apparent horizon with $r > 0$ and consider a future-pointing null geodesic at p. If $\dot{r}|_p = 0$, then the fact that

$$t'_{ah}(r) = t'_c(r) - \frac{2}{3}F'(r)$$

is finite for all $r > 0$ and that $\dot{t}|p > 0$ proves the stated result. Suppose then that $\dot{r}|_p \neq 0$. Then there is a neighborhood $I \ni s_0$ such that $\dot{r}(s) \neq 0$ for all $s \in I$ with $s|_p = s_0$ where s is the parameter along the geodesic in question. Then for points on the geodesic corresponding to I, we can take r to be the parameter along the geodesic. Along a future-pointing outgoing null geodesic, we then have, using (36),

$$\frac{dt}{dr} > e^{\alpha} > R' - \frac{R}{r} \tag{38}$$

where we have used (4) and the no-shell-crossing condition (25). Using (17), we can show that

$$\left(R' - \frac{R}{r}\right)\bigg|_{t=t_{ah}(r)} = \frac{t_{ah}}{t_c}t'_c. \tag{39}$$

Then using $t'_{ah} = t'_c - \frac{2}{3}F'$, we can show that

$$\left(R' - \frac{R}{r}\right)\bigg|_{t=t_{ah}(r)} > t'_{ah} \Leftrightarrow F' - \frac{F}{r} > 0.$$

The initial regularity condition (23) indicates that this last inequality holds for sufficiently small values of r. Thus when projected onto the r-t plane, for sufficiently small values of r, an outgoing null geodesic can only cross the apparent horizon from below. The same result is immediate for ingoing null geodesics as we can show that subject to (23), the slope of the apparent horizon t'_{ah} is positive for small values of r. ■

Remark 5: We note that the result above is equivalent to stating that the apparent horizon is spacelike for small values of r. Globally, there is no restriction: the horizon may be null or timelike for larger values of r, and can change character. Thus the apparent horizon is not always a one-way membrane in Szekeres space-time.

V. POLAR GEODESICS IN THE AXIALLY SYMMETRIC CASE

As we have seen in Sec. III, the only case in which radial geodesics exist in Szekeres space-time is when the space-time is axially symmetric and that furthermore the only radial geodesics that can emerge are in the polar direction. In this situation, the analysis of the visibility of the singularity is essentially the same as that for the spherically symmetric case. We show here that there are choices of the initial data for which the central singularity is visible along the polar direction. We follow the treatment of the spherically symmetric (LTB) case given in [17]. However as with the previous section, some care must be taken to account for the minor differences between the present case and the spherically symmetric case.

The axially symmetric case is obtained by setting $B_1 = B_2 = 0$ in (3), and the polar geodesic corresponds to $x = y = 0$. Then the null geodesic equations (26)–(30) reduce to

$$\dot{t}^2 - e^{2\alpha}\dot{r}^2 = 0, \tag{40}$$

$$\ddot{t} + \alpha_t \dot{t}^2 = 0, \tag{41}$$

$$\ddot{r} + (\alpha' + 2\alpha_t e^{\alpha})\dot{r}^2 = 0. \tag{42}$$

Our first step is to show that we can replace the affine parameter s by the coordinate r, and consider the projection of the geodesic into the r-t plane. To see this, we note that from (42), if $\dot{r}(s_0) = 0$ for some s_0, then $\dot{r}(s) = 0$ for all s, and the geodesic reduces to a single point. So we can assume that $\dot{r} \neq 0$ along the geodesic. Hence a polar null geodesic that is initially outgoing $\dot{r}(s_0) > 0$ remains outgoing for all s. Consequently, apart from the question of the maximal s-interval of existence of the geodesic, all information regarding the outgoing polar geodesic is contained in the single equation

$$\frac{dt}{dr} = e^{\alpha} = R' + R\nu'. \tag{43}$$

Along such a geodesic γ, we have

$$\nu'|_{\gamma} = -\frac{C'}{C}.$$

In the axially symmetric case, the conditions (5) and (20)–(22) reduce to

$$AC = \frac{1}{4}, \tag{44}$$

$$A > 0, \tag{45}$$

$$A - rA' > 0, \tag{46}$$

$$A'C' > -\frac{1}{4r^2}. \tag{47}$$

From these we obtain

$$-\frac{1}{r} < \nu'|_\gamma < \frac{1}{r}. \tag{48}$$

Thus the additional symmetry in the problem yields a useful additional bound on ν' [cf. (19)].

To proceed, we make an additional mild assumption on the structure of the function F. We define the number f_0 and the function F_1 by

$$F = r^3(f_0 + F_1(r)), \qquad F_1(0) = 0.$$

Our mild assumption is that $f_0 > 0$: this corresponds to the initial central density being strictly positive. The no-shell-crossing condition corresponds to $F_1' < 0$ for $r > 0$, and so $f_1 := F_1'(0) \leq 0$.

Proposition 5: If $f_1 < 0$, then there is a future-pointing outgoing polar geodesic with past end point on the central singularity.

Proof: For $f_1 < 0$, we can use (34) to write

$$t_{\rm ah} = \frac{2}{3} f_0^{-1/2} - \frac{1}{3} f_0^{-3/2} f_1 r + O(r^2),$$

where this and all other asymptotic relations in the present proof refer to the limit $r \to 0$. For constant κ with $0 < \kappa < -\frac{1}{3} f_0^{-3/2} f_1 =: \lambda$, define

$$t_\kappa(r) = \frac{2}{3} f_0^{-1/2} + \kappa r.$$

Then for sufficiently small δ_1, there is a nonempty region

$$\Omega[\delta_1, \kappa] = \{(t, r) \colon t_\kappa(r) < t < t_{\rm ah}(r), 0 < r < \delta_1\}.$$

Note that $t_\kappa' = \kappa > 0$. Along a future-pointing outgoing polar geodesic γ, we have

$$\frac{dt}{dr} = R' + R\nu' < R' + \frac{R}{r},$$

and a straightforward calculation using (17) then yields

$$\frac{dt}{dr}\bigg|_\gamma < \left(\frac{9}{4}\right)^{1/3} f_0^{1/3} (\lambda - \kappa)^{2/3} \left(2 - \frac{2f_1}{3 f_0^{3/2}(\lambda - \kappa)}\right) r^{2/3} + O(r^{5/2}).$$

Since $f_1 < 0$ and $0 < \kappa < \lambda$, the leading coefficient here is positive, and so there is a $\delta_2 > 0$ such that for all $r < \delta_2$, we have

$$\frac{dt}{dr}\bigg|_\gamma < t_\kappa'.$$

Hence by taking δ to be sufficiently small to allow use of Proposition 4 and to minimize δ_1 and δ_2, we see that a future-pointing outgoing polar null geodesic in the region $\Omega[\delta, \kappa]$ as defined above cannot leave this region as we extend back into the past. Hence the geodesic must extend back to the central singularity $r = 0$, $t = t_c(0) = t_{\rm ah}(0) = t_\kappa(0)$. ■

Remark 6: We note that as in the spherically symmetric case, it is possible to consider the case where $f_1 = 0$. This requires an additional assumption on the differentiability of F and on the value of the coefficients of a Taylor expansion of the function around $r = 0$. There is nothing to indicate that the results obtained in this way would differ from those obtained in the spherically symmetric case.

We consider next the important question of whether or not these geodesics meet the singularity at some finite affine parameter value in the past, or if $s \to -\infty$ as $r \to 0$, $t \to t_c(0)$. In order to do this, we study the sign of α_t in the region $\Omega[\delta, \kappa]$ introduced in the proof of Proposition 5 above.

Lemma 2: There are values of $\kappa \in (0, -\frac{1}{3} f_0^{-3/2} f_1)$ and $\delta > 0$ such that $\alpha_t > 0$ for all $(t, r) \in \Omega[\delta, \kappa]$.

Proof: We have

$$\alpha_t = \frac{R_t' + R_t \nu'}{R' + R\nu'}$$

and so using the no-shell-crossing condition $R' + R\nu'$, this is positive if and only if the numerator is positive. From (48), we have

$$R_t' + R_t \nu' > R_t' + \frac{R_t}{r}$$

(recall that $R_t < 0$). The latter term is positive if and only if

$$\frac{1}{2}\left(\frac{R}{r}\right)^{-3/2}\left(\frac{1}{r} - \frac{1}{3}\frac{F'}{F}\right) > \frac{1}{r} + \frac{F'}{3F} = \frac{2}{r} + O(1).$$

We note that

$$\frac{1}{r} - \frac{1}{3}\frac{F'}{F} = -\frac{1}{3}\frac{f_1}{f_0} + O(r) > 0.$$

For $t > t_\kappa$, we have

$$\left(\frac{R}{r}\right)^{-3/2} > \left(1 - \frac{t_\kappa}{t_c}\right)^{-1} = -t_c(0)(\kappa - \lambda)^{-1} r^{-1} + O(r^2).$$

We note that the coefficient of r^{-1} here is positive. Hence if we can choose κ so that

$$\frac{t_c(0)}{6}\frac{f_1}{f_0}(\kappa - \lambda)^{-1} = \frac{\lambda}{3(\lambda - \kappa)} > 2,$$

then there exists $\delta > 0$ so that $\alpha_t > 0$ on $\Omega[\delta, \kappa]$ as required. This choice entails $\kappa > \frac{5}{6}\lambda$, which can always be made. ■

Proposition 6: Let κ and δ have the values required by Lemma 2. Then any future-pointing polar null geodesic that enters the region $\Omega[\delta, \kappa]$ extends back to the central singularity in finite affine parameter time.

Proof: It only remains to check the statement regarding finiteness of the value of the affine parameter s at which the geodesic meets the singularity. By Lemma 2 and (41), we see that $\ddot{t} < 0$ in the limit as the singularity is approached. If the limit $t \to t_c(0)$ is reached only as $s \to -\infty$, then we would have

$$\lim_{s \to -\infty} \ddot{t} \geq 0,$$

contradicting the statement above. ∎

VI. DISCUSSION

We have revisited the issue of the visibility of the shell-focusing singularity in quasispherical dust collapse, motivated by the observation that previous results have incorrectly assumed that there exist radial null geodesics in such space-times. As we have seen, this is not generally the case: the existence of such geodesics implies an additional symmetry of the space-time. It is worth noting that our discussion has been restricted to the 4-dimensional case, whereas there have been several studies carried out in higher $n + 2 = D \geq 5$ dimensional Szekeres space-times. We suspect that an analogous result applies: the existence of a radial geodesic in the space-time implies the existence of $(n - 1)$ Killing fields, leaving just $2 + 1$ nonignorable coordinates. This conjecture is based on the structure of the derivates $\frac{\partial \nu}{\partial x_i}$, $1 \leq i \leq n$ in the higher dimensional case. However, we have not been able to determine the structure of the group of transformations for the higher dimensional case that corresponds to the transformations (8) that play a crucial role in the proof of Proposition 1.

It is perhaps worth pointing out that with the benefit of hindsight, it is not surprising to see a connection between the existence of radial geodesics and symmetry. First, and on general grounds, it would be unusual to see a situation in which one had a conserved quantity (the values of the angles in this case) without the presence of some form of symmetry. Second and more specifically for this quasi-spherical situation, it is hard to envisage a geodesic emerging orthogonally from one 2-sphere and remaining orthogonal to other 2-spheres that it meets—unless the centers of those 2-spheres are aligned, with the alignment direction forming an axis of symmetry of the space-time. This is exactly what we see happening in Proposition 1.

In the axially symmetric case, there is one direction along which radial geodesics exist: the polar direction. We have looked briefly at the issue of the visibility of the singularity along this direction in the marginally bound case, and find no difference between the present case and the spherically symmetric case. However, we cannot generalize our finding to either nonradial geodesics in the axially symmetric case, or to geodesics in the general case. The fundamental difficulty in doing so is that one *cannot* project the geodesic onto the r-t plane and retain all information required. The geodesic equations in the general case (no symmetry and hence no radial assumption allowed) form a second order nonlinear dynamical system with singular coefficients. The presence of the singular coefficients—which correspond to the points of space-time we are interested in analyzing—mean that standard methods of smooth dynamical systems do not offer means of approaching the problem. One possible approach is to rescale the dynamical system by multiplying through by the most strongly vanishing denominator in the singular coefficients. One then absorbs this coefficient into a rescaled affine parameter, to obtain a smooth system. When this is done carefully, it is possible to convert the singular point of the original system to a stationary point of the rescaled system. However, this procedure typically yields nonhyperbolic equilibrium points and spurious equilibrium sets requiring the use of center manifold analysis. Nonetheless, it has yielded useful results in a different context where similar problems (singular dynamical systems) arise [18]. It would be of interest to see if this approach could be used to study general geodesics in the nonaxially symmetric Szekeres space-time, or indeed the nonradial (nonpolar) geodesics of the axially symmetric Szekeres space-time. The existence of additional bounds on the metric functions [like (48)] would be of considerable use in this case.

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