Cosmology with twisted tori

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We consider the cosmological role of the scalar fields generated by the compactification of 11-dimensional Einstein gravity on a 7-dimensional elliptic twisted torus, which has the attractive features of giving rise to a positive semidefinite potential, and partially fixing the moduli. This compactification is therefore relevant for low-energy *M*-theory, 11-dimensional supergravity. We find that there is no slow-roll inflation within a subclass of these twisted tori and give evidence that this result extends to a more general situation. Despite the lack of slow-roll, we find that there is a novel scaling solution in Friedmann cosmologies in which the massive moduli oscillate but maintain a constant energy density relative to the background barotropic fluid.

I. INTRODUCTION

Current attempts to unify forces and interactions are mainly based on superstring or *M*-theory with 10 or 11 space-time dimensions, with the extra dimensions taking the form of a small compact manifold. Finding such a manifold consistent with low-energy particle physics and cosmology presents us with several hard problems. A given geometry has a number of size and shape parameters, or moduli, which become scalar fields in the effective 4-dimensional theory, with a potential generated in the first instance by the Ricci scalar of the compact space. If the internal space is positively curved the potential is negative, leading to anti−de Sitter solutions, which are disastrous in cosmological terms without extra ingredients such as *D*-branes to cancel the negative cosmological constant [1].

Another problem with compactification is that some or all of the moduli are not fixed by the effective 4-dimensional potential, at least before quantum corrections are taken into account. As these moduli often affect parameters in the effective theory, we lose predictability, and also gain unwanted massless scalars with gravitational strength couplings. The moduli which do appear in the potential correspond to weakly coupled massive 4-dimensional fields, which we must be careful not to excite, otherwise they will dominate the energy density too early in a standard Friedmann cosmology.

A promising route addressing some of these problems is to make use of the flat-groups of Scherk and Schwarz [2], which have negative semidefinite curvature, and so lead to a theory with a positive semidefinite potential and Minkowski minima. They also fix some, but not all, of the moduli and so alleviate the moduli-fixing problem generic to flat compactifications. Another nice route to achieving interesting cosmologies from dimensional re-

duction was put forward in [3,4]. There it was noted that using compact, hyperbolic internal spaces leads naturally to accelerating cosmologies.

In this paper we study the cosmology of these flat-groups, or "twisted torus" compactifications, with an eye to finding inflationary behavior and scaling regimes, in which the energy density in the moduli fields remains a constant fraction of the background barotropic fluid. The work forms a natural continuation of the work presented in [5] where we studied the 7-dimensional cosets of compact Lie groups, classified in [6]. In that case one finds that the curvature of these internal spaces is unbounded and leads to a singular cosmology. Applying the same technology to the particular class of twisted tori introduced by Scherk and Schwarz we show first that slow-roll inflation is not possible with "diagonal" moduli fields (the no-go theorem of [7] does not apply here as the internal space is allowed to be time-dependent). Using numerical simulations we then give evidence that the more general, nondiagonal, case also does not satisfy the slow-roll conditions. We also find a novel scaling solution in which the massive moduli are oscillating, but rather than dominating the fluid-filled universe instead contributes a fixed fraction (of approximately $1/3$ for radiation) to the energy density. The alleviation of the cosmological moduli problem comes at a price: the volume of the internal space grows approximately as $t^{0.5}$ during the radiation era. As coupling constants generically depend on the volume of the internal space, a realistic cosmology exploiting this novel scaling regime appears problematic.

The structure of the paper is as follows. Sections II and III are reviews of dimensional reduction of Einstein gravity and scalar cosmology, with Sec. IV giving the relevant formalism for multifield cosmology and slow-roll behavior. Section V sees the introduction of twisted torus manifolds which are then applied in the cosmological context in Secs. VI and VII. The evolutionary behavior in terms of slow-roll and scaling is studied in Secs. VIII and IX, which

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we analyze in terms of effective degrees of freedom in Secs. X and XI.

II. DIMENSIONALLY REDUCED GRAVITY ON A COMPACT LIE MANIFOLD

We start with pure gravity in $d + D$ dimensions which is described by the Einstein-Hilbert action,

$$\hat{\mathcal{L}} = \frac{1}{2\hat{\kappa}^2} \int d^d x d^D y \sqrt{-\hat{g}} \hat{\mathcal{R}}, \qquad (1)$$

where the hatted quantities are $d + D$ dimensional quantities. We adopt a modified ansatz for the metric, given by the Scherk-Schwarz [2] form which guarantees a consistent truncation [8],

$$ds^2 = e^{2\psi(x)} ds^2_{(d)} + g_{ab}(x) \nu^a(y) \otimes \nu^b(y) \qquad (2)$$

$$\nu^a(y) = e^a(y) - \mathcal{A}^a(x), \qquad (3)$$

where $\mathcal{A}^a$ are gauge fields and the e^as are left-invariant one-forms on the internal manifold, which we take to be a compact Lie group. (Indices of the form $a, b, \ldots$ span the D extra dimensions.) Upon substituting this into the Riemann scalar of the action (Appendix A) we find an effective theory described by the scalars g_{ab}, charged under the non-Abelian gauge fields $\mathcal{A}^a$.

Performing the $d^D y$ integration over the unsquashed volume of the internal manifold allows us to make an identification of the form

$$\frac{1}{2\kappa^2} = \frac{\mathcal{V}_{\text{internal}}}{2\hat{\kappa}^2}, \qquad (4)$$

and gives a nice geometric interpretation of Newton's constant. The constant κ has the units of length, and the y coordinates are rescaled by κ such that they are dimensionless; this means that the g_{ab} components have dimensions of (length)^2.

The determinant of the full metric decomposes into $\sqrt{-\hat{g}} = e^{d\psi} \sqrt{-g_{(d)}} \sqrt{g_{ab}}$, and we can recover a pure d-dimensional gravity term in the effective action by fixing the gauge of ψ using

$$e^{(d-2)\psi} \sqrt{g_{ab}} = \kappa^D. \qquad (5)$$

Specializing to $d = 4$ and switching off the gauge fields (Appendix C) we recover the effective action

$$\mathcal{L}_{\text{eff4}} = \int d^4 x \sqrt{-g_4} \left[\frac{1}{2\kappa^2} \mathcal{R}_4 - K^{abcd} \nabla_\mu g_{ab} \nabla^\mu g_{cd} - V \right] \qquad (6)$$

$$K^{abcd} = \frac{1}{2\kappa^2} \left(\frac{1}{4} g^{ac} g^{bd} + \frac{1}{8} g^{ab} g^{cd} \right) \qquad (7)$$

$$V = -\frac{1}{2\kappa^2} \frac{\kappa^D}{\sqrt{g_{ab}}} \mathcal{R}_{\text{internal}}, \qquad (8)$$

where K^{abcd} can be interpreted as a field space metric for the noncanonical scalars g_{ab}, and V is the potential in which they sit. This is similar to [9] except that we do not make the restriction $\det(g_{ab}) = 1$ and thus avoid bringing in an additional scalar field.

For a general Lie group manifold the Ricci curvature can be shown to be [10]

$$\mathcal{R}_{\text{internal}} = \frac{1}{2} (f^c{}_{ab} f^a{}_{dc} g^{db}) - \frac{1}{4} (f^c{}_{ab} f^f{}_{de} g^{ad} g^{be} g_{cf}), \qquad (9)$$

where $f^a{}_{bc}$ are the structure constants (B4) for the group, and g_{ab} is the choice of metric on the manifold.

III. COSMOLOGY WITH A SCALAR

As we are interested in the cosmological dynamics of this theory with many noncanonical scalars, we start by reminding ourselves of how the standard single scalar picture works.

The simplest way of incorporating dark energy into a cosmological model is to allow the negative pressure to be provided by a single scalar field,

$$S = \int \sqrt{-g} \left[\frac{1}{2\kappa^2} \mathcal{R} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right]. \qquad (10)$$

We take a Friedman-Robertson-Walker (FRW) universe with homogeneity of the universe allowing the spacial gradients to be neglected, and a barotropic fluid in the background with energy density ρ_γ and pressure P_γ related by the equation of state

$$P_\gamma = (\gamma - 1)\rho_\gamma, \qquad (11)$$

with γ specifying the type of fluid ($\gamma = 4/3$ for radiation, $\gamma = 1$ for pressureless matter and $\gamma = 0$ for vacuum energy). The equations of motion for this system are

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{\kappa^2}{3} \left[\frac{1}{2} \dot{\phi}^2 + V + \rho_\gamma \right] \qquad (12)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{3} \left[\dot{\phi}^2 - V + \frac{1}{2}(3\gamma - 2)\rho_\gamma \right] \qquad (13)$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi} \qquad (14)$$

$$\dot{\rho}_\gamma = -3\gamma H \rho_\gamma. \qquad (15)$$

Inflation is defined as a period of evolution where $\ddot{a} > 0$, by which consideration we arrive at the so-called *slow-roll* conditions. The first of these, [11]

$$\epsilon \equiv \frac{1}{2} \frac{1}{3\kappa^2} (V'/V)^2 \ll 1, \qquad (16)$$

can be interpreted as saying that the slope is shallow enough for inflation while the second,

$$\eta \equiv \frac{1}{3} \frac{V''}{\kappa^2 V} \ll 1, \tag{17}$$

tells us that for inflation it should stay shallow for a while.

IV. EFFECTIVE THEORY

We can now compute the equations of motion of the theory of Sec. II. The degrees of freedom of the internal metric become scalar fields under dimensional reduction, with an equation of motion

$$\ddot{g}_{ab} + 3 \frac{\dot{a}}{a} \dot{g}_{ab} = -\left(2 g_{ac} g_{bd} - \frac{2}{D+2} g_{ab} g_{cd}\right)$$
$$\times \kappa^2 \partial V / \partial g_{cd} + g^{cd} \dot{g}_{ac} \dot{g}_{bd} \tag{18}$$

corresponding to (14). The scale factor equations of motion,

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{\kappa^2}{3}\left[\frac{1}{2} K + V + \rho_\gamma\right] \tag{19}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{3}\left[K - V + \frac{1}{2}(3\gamma - 2)\rho_\gamma\right] \tag{20}$$

$$\dot{\rho}_\gamma = -3\gamma H \rho_\gamma, \tag{21}$$

take the same form as (12), (13), and (15) where we define

$$K = K(g, \dot{g}) = 2 K^{abcd} \dot{g}_{ab} \dot{g}_{cd}$$
$$= \frac{1}{4\kappa^2}\left(g^{ac} g^{bd} + \frac{1}{2} g^{ab} g^{cd}\right) \dot{g}_{ab} \dot{g}_{cd}, \tag{22}$$

which serves as the analogue of $\dot{\phi}^2$ in the single field case such that the kinetic energy in the scalar sector is $\frac{1}{2} K$. Expressed like this it is apparent that although the theory is described by many noncanonical scalars, the overall dynamics could be considered to be driven by a single effective scalar with $\dot{\phi}_{\text{eff}} = \sqrt{(K)}$, at least instantaneously.

We are also interested in the inflationary behavior of the system, and therefore it is useful to derive a form of the first slow-roll condition for this kind of noncanonical system of scalars,

$$\epsilon \equiv \frac{1}{6V^2}\left[g_{ac} g_{bd} - \frac{1}{D+2} g_{ab} g_{cd}\right] \frac{\partial V}{\partial g_{ab}} \frac{\partial V}{\partial g_{cd}} \ll 1. \tag{23}$$

With this we will be able to probe the field space numerically to see whether there are any regions in moduli space in which inflationary behavior is possible.

V. TWISTED TORI

In an earlier paper [5] a model similar to this one was studied where the internal manifold was chosen as the 7-dimensional cosets of continuous Lie groups G/H, classified in [6]. It was discovered there that these manifolds

gave singular cosmologies due to the ability of the curvature to change sign, and leading to potentials which were unbounded from below.

We turn our attention now to the twisted tori manifolds of Scherk and Schwarz [2]. These so-called "flat-groups" can be made compact by forming a coset G/Γ of a particular Lie group G divided out by one of its discrete subgroups Γ, and was shown there to allow a consistent dimensional reduction of the type discussed here. Most important for us however is that these cosets have negative curvature leading to positive semidefinite potentials and therefore effective theories with Minkowski minima.

These manifolds are related to the *duality twists* of [9,12], where they are identified as *elliptic* twisted tori and related to the Lie group $ISO(N)$. Two further groups which should fulfil the needs of our model are also presented there, the *hyperbolic* and *parabolic* classes, respectively, related to the Lie group $SO(P, Q)$ and the Heisenberg group. For purposes of demonstration we consider here only the elliptic groups.

The elliptic groups are related to $ISO(N)$, the isometries of a space of N dimensions, in the following way. They are noncompact Lie groups with a Lie algebra formed of N generators, T_p, corresponding to the translations and $N(N-1)/2$ generators, R_q, corresponding to rotations. To form the elliptic group the rotational generators are coupled together to form a single generator $R = \sum_q m_q R_q$, which when combined with the translational generators gives an $N+1$ dimensional subspace. For such a group the structure constants are given by

$$f^i{}_{0j} = M^i{}_j, \tag{24}$$

where $i, j = 1, 2, \ldots, N-1$, and the 0 direction corresponding to the rotational generator. It is clear that this should be the case when one considers that translations commute, and that the commutator of a translation and a rotation will always be equivalent to a translation in a different direction. The resulting M is a skew symmetric, real $N \times N$ matrix populated by the m_qs, but we can always choose a basis where M takes the form

$$M = \begin{pmatrix} 0 & m_1 & 0 & 0 & 0 \\ -m1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & -m_2 & 0 & 0 \\ 0 & & 0 & 0 & \ddots \end{pmatrix}. \tag{25}$$

This group is noncompact, and for the Kaluza-Klein ansatz to hold we need a compact manifold; however this can be achieved by identifying two separated points in each of the translation directions to form a torus, making a coset of the group with a discrete subgroup, G/Γ (see Appendix B). It is this resulting manifold which is the elliptic twisted torus.

All that is left is to consider the form of the metric, g_{ab}, which is permissible to put on the manifold. Unlike the coset manifolds of [5] there are no further symmetries with which to restrict the metric, and so we are at liberty to consider all of the degrees of freedom in our cosmological analyses. However there are still some gauge freedoms available which we can evoke to set $g_{0i} = 0$ with no loss of generality, and so we will work in this gauge from this point onwards. We refer the reader to Appendix C for the details of this gauge fixing.

VI. CANONICAL SCALARS

For particle physics reasons we will assume that the internal manifold is 7-dimensional so that the entire space before compactification is of 11 dimensions, descending from the pure gravity sector of 11-dimensional supergravity, or M-theory, for example. We are at liberty to place any metric we want on the manifold, but for comparison with a system of canonical scalars we start our investigation with the purely diagonal form

$$g_{ab} = \kappa^2 \text{diag}(A_0^2, A_1^2, \ldots, A_6^2), \qquad (26)$$

substituting this into (9) and finding that the curvature is given by

$$\mathcal{R} = -\frac{1}{2\kappa^2 A_0^2}\left[\frac{(A_1^2 - A_2^2)^2}{A_1^2 A_2^2}m_1^2 + \frac{(A_3^2 - A_4^2)^2}{A_3^2 A_4^2}m_2^2 + \frac{(A_5^2 - A_6^2)^2}{A_5^2 A_6^2}m_3^2\right]. \qquad (27)$$

We can immediately see that as expected the curvature is negative semidefinite, and this provides us via (8) with a positive semidefinite potential, and a Minkowski minima for the theory. We also observe that the metric structure follows that of the underlying geometry, with the A_0 direction parametrizing the rotational twist inherent in the geometry, and the remaining directions pairing up into three independent planes in which the translation generators operate.

The metric is diagonal and so we are free to redefine these scalars such that we normalize the kinetic terms and bring them into canonical form, which is a useful exercise if only to compare the potential of the theory to other theories that we may know. We therefore define a set of canonical scalars φ_a related to the A_as in the following way,

$$A_0 = e^{\kappa((1/3)\sqrt{2/7}\varphi_1 - (\varphi_2/\sqrt{2}) - (\varphi_3/\sqrt{6}) - (\varphi_4/2\sqrt{3}) - (\varphi_5/2\sqrt{5}) - (\varphi_6/\sqrt{30}) - (\varphi_7/\sqrt{42}))}$$

$$A_1 = e^{\kappa((1/3)\sqrt{2/7}\varphi_1 + (\varphi_2/\sqrt{2}) - (\varphi_3/\sqrt{6}) - (\varphi_4/2\sqrt{3}) - (\varphi_5/2\sqrt{5}) - (\varphi_6/\sqrt{30}) - (\varphi_7/\sqrt{42}))}$$

$$A_2 = e^{\kappa((1/3)\sqrt{2/7}\varphi_1 + \sqrt{2/3}\varphi_3 - (\varphi_4/2\sqrt{3}) - (\varphi_5/2\sqrt{5}) - (\varphi_6/\sqrt{30}) - (\varphi_7/\sqrt{42}))}$$

$$A_3 = e^{\kappa((1/3)\sqrt{2/7}\varphi_1 + (\sqrt{3}\varphi_4/2) - (\varphi_5/2\sqrt{5}) - (\varphi_6/\sqrt{30}) - (\varphi_7/\sqrt{42}))}$$

$$A_4 = e^{\kappa((1/3)\sqrt{2/7}\varphi_1 + (2\varphi_5/\sqrt{5}) - (\varphi_6/\sqrt{30}) - (\varphi_7/\sqrt{42}))}$$

$$A_5 = e^{\kappa((1/3)\sqrt{2/7}\varphi_1 + \sqrt{5/6}\varphi_6 - (\varphi_7/\sqrt{42}))}$$

$$A_6 = e^{\kappa((1/3)\sqrt{2/7}\varphi_1 + \sqrt{6/7}\varphi_7)}. \qquad (28)$$

With these new scalars the effective potential and kinetic terms in (6) become,

$$V(\varphi_a) = \frac{1}{4\kappa^4}e^{-3\sqrt{2/7}\kappa\varphi_1}\left[e^{-2\sqrt{2/3}\kappa\varphi_3 + (\kappa\varphi_4/\sqrt{3}) + (\kappa\varphi_5/\sqrt{5}) + \sqrt{2/15}\kappa\varphi_6 + \sqrt{2/21}\kappa\varphi_7}(e^{\sqrt{2}\kappa\varphi_2} - e^{\sqrt{6}\kappa\varphi_3})^2 m_1^2 \right.$$

$$+ e^{\sqrt{2}\kappa\varphi_2 + \sqrt{2/3}\kappa\varphi_3 - (2\kappa\varphi_4/\sqrt{3}) - (4\kappa\varphi_5/\sqrt{5}) + \sqrt{2/15}\kappa\varphi_6 + \sqrt{2/21}\kappa\varphi_7}(e^{\sqrt{3}\kappa\varphi_4} - e^{\sqrt{5}\kappa\varphi_5})^2 m_2^2$$

$$\left. + e^{\sqrt{2}\kappa\varphi_2 + \sqrt{2/3}\kappa\varphi_3 + (\kappa\varphi_4/\sqrt{3}) + (\kappa\varphi_5/\sqrt{5}) - 4\sqrt{2/15}\kappa\varphi_6 - 2\sqrt{6/7}\kappa\varphi_7}(e^{\sqrt{10/3}\kappa\varphi_6} - e^{\sqrt{14/3}\kappa\varphi_7})^2 m_3^2\right] \qquad (29)$$

$$\frac{1}{2}K(\varphi_a) = \frac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a, \qquad (30)$$

and the theory is in canonical form.

We ran numerical simulations with both matter and radiation present in the barotropic fluid and a wide range of random initial field conditions, and sampled from the results those which ran for the longest. This was achieved using a fourth-order Runge-Kutta scheme, using the Friedmann equation to monitor the accuracy of the simulation and allowing us to verify that the dynamical variables had remained true to the constraint surface. In Fig. 1 we present typical examples of the evolution of the fields.[1]

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[1]We take $m_1 = m_2 = m_3 = 1$.

One of the fields encodes for the volume of the internal space which grows throughout the simulation, while the remaining fields evolve in the fashion demonstrated until settling down into an oscillation around the minimum of the potential at $A_1^2 = A_2^2$, $A_3^2 = A_4^2$, $A_5^2 = A_6^2$.

The expansion of the volume demonstrates that the internal manifold is in the process of decompactifying, which would suggest that as the system evolves we would also need to consider nonzero Kaluza-Klein modes in the effective action. However, in keeping with earlier studies [13] we shall not consider these modes, and note that if decompactification happens slowly enough we can delay analysis of these additional modes until much later in the history of the universe and avoid spoiling any of the results here.
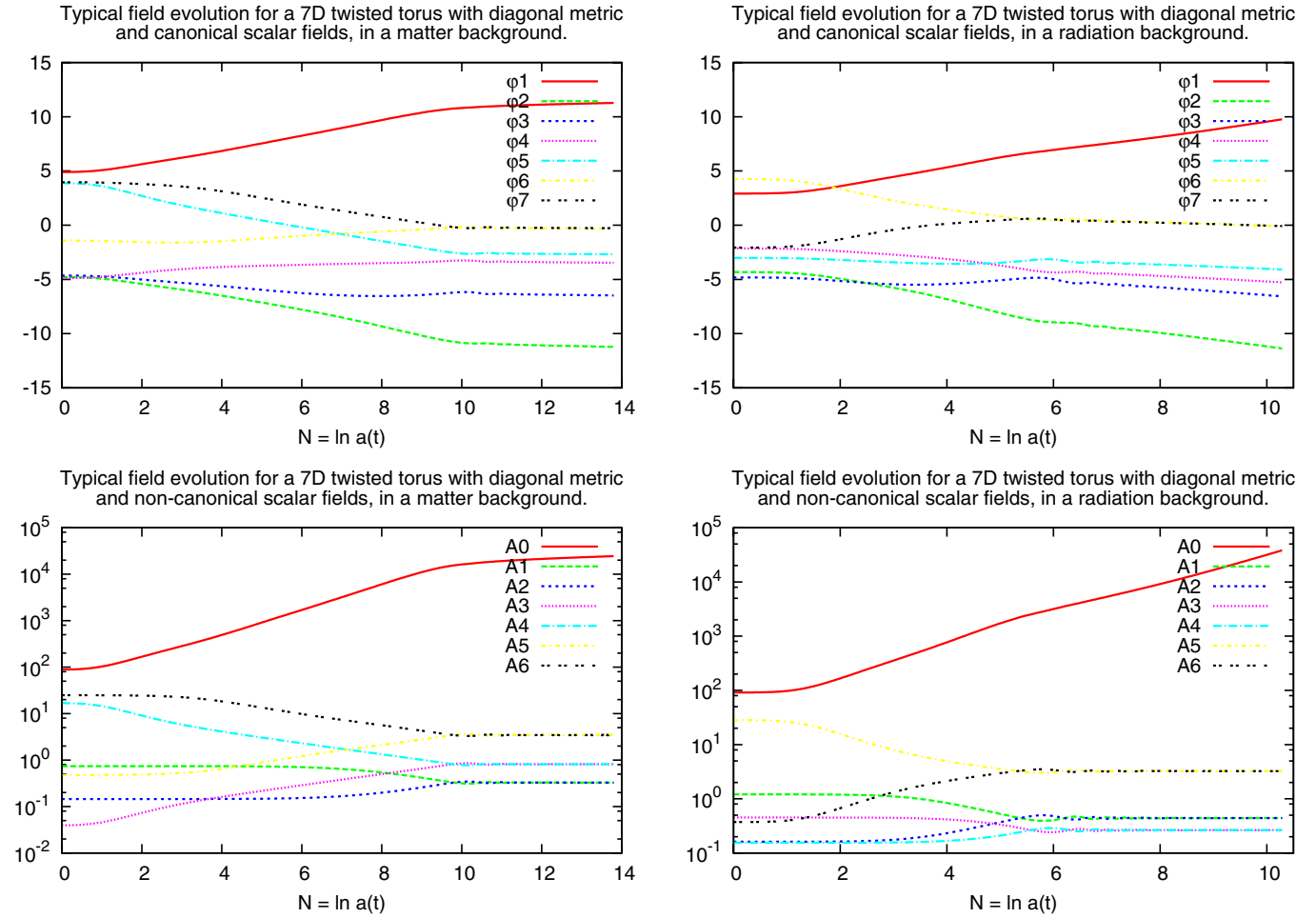
FIG. 1 (color online).   Typical evolution of a system of canonical scalar fields evolving in the effective potential of (29) along with a barotropic fluid in the overall energy density. On the left is the evolution with matter ($\gamma = 1$) as the fluid component, and on the right with radiation ($\gamma = 4/3$). The upper plots are of the canonical scalars φ_a, and the lower plots are of the noncanonical fields A_a, which are the diagonal components of the internal metric.

VII. NONCANONICAL EVOLUTION

As stated earlier this twisted torus places no restrictions on the form of the internal metric, and so we should say something about the evolution of the system with all the degrees of freedom switched on. The 7-dimensional internal metric has 28 degrees of freedom, of which 7 can be gauged away (Appendix C). We choose to gauge away 6 of these allowing us to set $g_{0i} = 0$; however that still leaves 22 scalar fields evolving in the potential

$$V = \frac{1}{2\kappa^2} \frac{\kappa^D}{\sqrt{g_{ab}}} g^{00}(M^i{}_j M^j{}_i + M^i{}_k M^j{}_l g^{kl} g_{ij}). \quad (31)$$

A typical numerical realization of this full noncanonical system is presented in Fig. 2, in this case with matter in the background fluid. As with the diagonal metric we see that all the scalars dynamically evolve towards oscillating solutions where presumably the minimum of the potential is reached; however this time the dependence of the potential
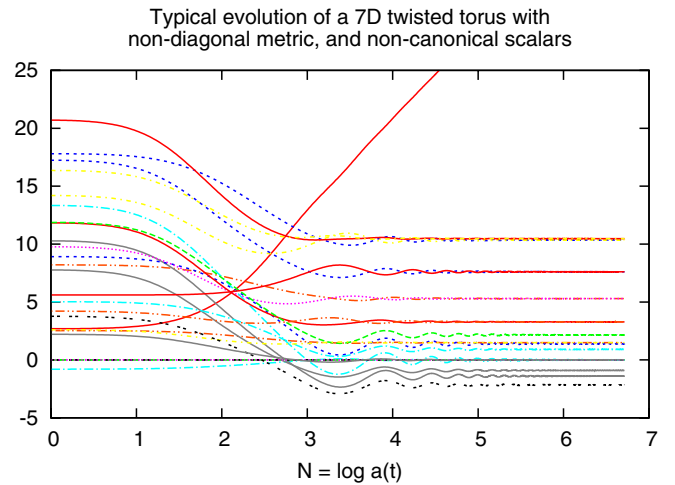

FIG. 2 (color online).   Typical evolution of a system of noncanonical scalar fields, g_{ab}, evolving according to Eqs. (6)–(8).

upon the inverse metric makes it difficult to determine where the minimum lies in field space.

In order to make some sense of this we decomposed the step-wise values of the metric into its associated eigensystem. To our surprise we discovered that the evolution of the eigenvalues, as shown in the left-hand side of Fig. 3, evolve in essentially the same fashion as the canonical fields. Does this imply that only 7 degrees of freedom are involved in the dynamics?

As a step towards making some sense of this we remind ourselves how the fields are related to the metric of the internal space. At a each point on a manifold there is a frame of vectors, $\{e_a\}$, which forms a basis for the tangent space there. The metric can be thought of as the "dot product" of these vectors,

$$g_{ab} = e_a \cdot e_b = |e_a||e_b| \cos\theta_{ab}, \qquad (32)$$

where $|e_a|$ is the length of the basis vector e_a and $0 < \theta_{ab} < \pi$ is the angle between the pair of basis vectors e_a and e_b. The diagonal entries have $g_{aa} = |e_a|^2$ and so encode for the lengths of the frame vectors, whereas the off-diagonal terms are also proportional to the cosine of the angle between a pair of frame vectors.

Through the Kaluza-Klein ansatz we have integrated over the internal space removing its coordinate dependence and so effectively there is only a single point, a purely space-time one, at which this internal frame is located. The scalar fields of the dimensionally reduced theory are the manifestation of the D lengths and $D(D-1)/2$ angles between this set of D internal frame vectors. That the evolution of the fields appears to be dependent upon just the eigenvalues tells us only the lengths of the internal frame vectors appear to play a role in the dynamics. What then of the angles, what role do they play?

It is well known that a real symmetric matrix is guaranteed to have an orthogonal set of eigenvectors, that is every frame can be rewritten in terms of a unique set of orthogonal vectors. We recover the angles associated with the directions of these vectors via

$$\theta_{ab} \propto \cos^{-1}(e_a \cdot \gamma_b) \qquad (33)$$

where the $\{\gamma_a\}$ are a canonical set of orthogonal vectors satisfying $\gamma_a \cdot \gamma_b = \delta_{ab}$, and we plot how these evolve in the right-hand side of Fig. 3. It does appear that the frame is rotating as the fields evolve; however we are unclear what role these rotations play in the theory.

VIII. SLOW-ROLL INFLATIONARY BEHAVIOR

We now turn our attention to the inflationary behavior of the system. We initially hoped that there might be a slow-roll regime, however the first slow-roll condition (23) was never satisfied in any of the numerical simulations. To understand this result we now turn our attention to an analysis of the slow-roll condition for our twisted torus. We start by exploring the behavior of the 3-dimensional system with the diagonal metric $g_{ab} = \kappa^2 \text{diag}(A^2, B^2, C^2)$. Expanding (23) we obtain

$$\epsilon = 1 + \frac{4}{3} \frac{(B^2 + C^2)^2}{(B^2 - C^2)^2}, \qquad (34)$$

which overtly fails the slow-roll condition and leads us to conclude that the canonical 3-dimensional system cannot ever slow-roll, which matches our observations under simulation. Further we find that adding the nondiagonal degree of freedom,

$$g_{ab} = \begin{pmatrix} A^2 & 0 & 0 \\ 0 & B^2 & D^2 \\ 0 & D^2 & C^2 \end{pmatrix}, \qquad (35)$$

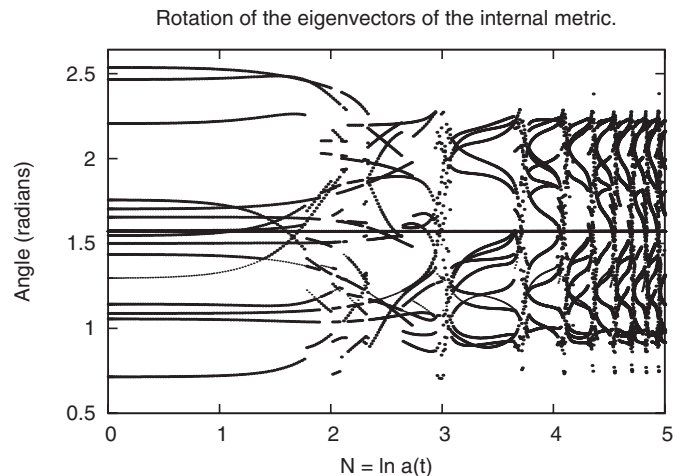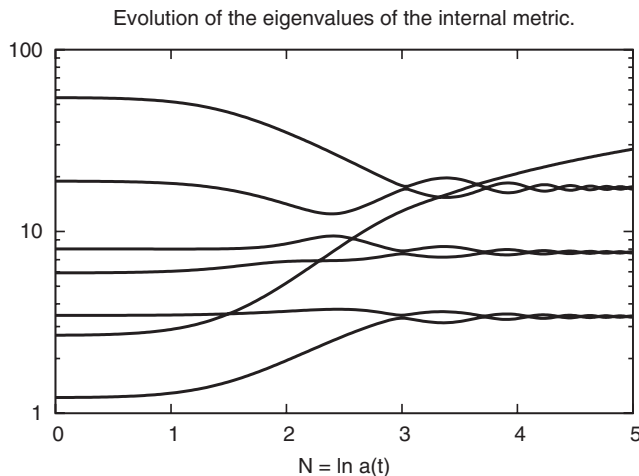does not improve the situation. Now slow-roll is controlled by

Evolution of the eigenvalues of the internal metric.

Rotation of the eigenvectors of the internal metric.

FIG. 3. Evolution of the eigensystem related to the scalar fields, g_{ab}. On the left we have the eigenvalues, and on the right the angles of the eigenvectors as measured relative to a canonical orthogonal frame γ_a.

$$\epsilon = 1 + \frac{4}{3} \frac{(B^2 + C^2)^2}{((B^2 - C^2)^2 + 4D^4)}, \tag{36}$$

for which we also have $\epsilon \geq 1$, and so once again there is no slow-roll regime here either, and the additional degree of freedom does not change the picture significantly.

The same pattern appears with the 5-dimensional system; with $g_{ab} = \kappa^2 \mathrm{diag}(A^2, B^2, C^2, D^2, E^2)$ we find

$$\epsilon = 1 + \frac{4}{3} \frac{((B^4 - C^4)^2 D^4 E^4 m_1{}^4 + B^4 C^4 (D^4 - E^4)^2 m_2{}^4)}{((B^2 - C^2)^2 D^2 E^2 m_1{}^2 + B^2 C^2 (D^2 - E^2)^2 m_2{}^2)^2}, \tag{37}$$

where the second term is once again overtly positive definite. Finally the diagonal 7-dimensional system of Sec. VI with metric $g_{ab} = \kappa^2 \mathrm{diag}(A^2, B^2, C^2, D^2, E^2, F^2, G^2)$ reveals this pattern as being generic,

$$\epsilon = 1 + \frac{4}{3} \frac{(F^4 G^4 ((B^4 - C^4)^2 D^4 E^4 m_1{}^4 + B^4 C^4 (D^4 - E^4)^2 m_2{}^4) + B^4 C^4 D^4 E^4 (F^4 - G^4)^2 m_3{}^4)}{(F^2 G^2 ((B^2 - C^2)^2 D^2 E^2 m_1{}^2 + B^2 C^2 (D^2 - E^2)^2 m_2{}^2) + B^2 C^2 D^2 E^2 (F^2 - G^2)^2 m_3{}^2)^2}. \tag{38}$$

We predict that a full noncanonical system with generic nondiagonal metric will also fail to slow-roll, pointing to the results of Sec. VII and (36) as evidence, and conclude that with elliptic twisted tori in the internal manifold there can be no slow-roll inflation. To back up this claim we performed a Markov-Chain Monte Carlo analysis of the field space, looking for regions where the slow-roll parameter (23) was less than unity. We undertook this analysis with the full, nondiagonal metric g_{ab}. In more than 750 000 samples we were unable to discover any regions with a slow-roll parameter less than 3.01. Given that there is no evidence for the first slow-roll parameter (23) we have not analyzed the conditions for the other slow-roll parameters.

IX. SCALING SOLUTIONS

One of our motivations for studying the model of this paper was to determine whether this choice of internal manifold could permit regimes of scaling behavior in the effective theory, and so it is to this question that we now turn.

To study such behavior it is usual to reformulate the equations of motion in terms of an autonomous system [14–16]. The Friedmann Eq. (19) describes the energy of the system and can be recast in terms of the variables,

$$x \equiv \frac{\kappa \sqrt{K}}{\sqrt{6} H} \qquad y \equiv \frac{\kappa \sqrt{V}}{\sqrt{3} H} \qquad z \equiv \frac{\kappa \sqrt{\rho_\gamma}}{\sqrt{3} H}, \tag{39}$$

which become constant in a scaling regime, and in terms of these the expression of energy conservation can be rewritten more simply as

$$x^2 + y^2 + z^2 = 1. \tag{40}$$

In a cosmological scaling solution the various components of the energy density evolve in constant ratio to each other, causing the effective equation of state parameter in the scalar sector,

$$\gamma_\varphi \equiv \frac{\rho_\varphi + P_\varphi}{\rho_\varphi} = \frac{2K}{K + V} = \frac{2x^2}{\Omega_\varphi}, \tag{41}$$

to track that of the background fluid (11), where ρ_φ and P_φ

are, respectively, the energy density and pressure in the scalar sector, and $\Omega_\varphi \equiv \kappa^2 \rho_\varphi / 3H^2 = x^2 + y^2$ is the total energy density of the scalar sector. For these kinds of systems the evolution of the scale factor can be shown to evolve according to $a(t) \propto t^p$, where $p = 2/3\gamma_\varphi$.

To see whether the system exhibits scaling we need to study the evolution of the autonomous system variables (39). In Fig. 4 we have a plot of these scaling quantities for the evolutions of Fig. 1, and we do indeed see that the system settles into what looks like a scaling regime, at least for a short period, before transitioning into a long phase where the energy density components appear to be oscillating around constant values. This oscillating behavior appears to continue generically with the amplitude reducing very slowly, and it does therefore appear to be in an *effective scaling* regime, oscillating around a true scaling solution. We also note the plots show that each contribution to the energy density is nowhere vanishing.

Further evidence that the system is in scaling can be seen from a plot of the equation of state parameter for the scalar field (41). It is well known from the autonomous system analysis [14] that in the fluid-dominated case there is a tracker solution in which the equation of state parameter of the scalar sector mimics that of the background barotropic fluid. In Fig. 5 we see γ_φ does indeed appear to oscillate around γ, with the center of oscillation appearing to approach the value of the background fluid asymptotically. We take this as further evidence that this system of scalars is effectively scaling.

This behavior was found to be characteristic of this system of scalars; in all numerical simulations there was period where the system appeared to converge on a scaling solution which, although pronounced in the data presented here, was present in all the evolution runs we examined. This was always followed by a phase change into oscillating behavior around another scaling solution. It appears that this generic behavior with nonvanishing potential and kinetic energy could carry on for some time, although in our runs we had to terminate the simulations early as the oscillations lead to exponential memory requirements and limited the ability to run for much longer than the examples shown here.
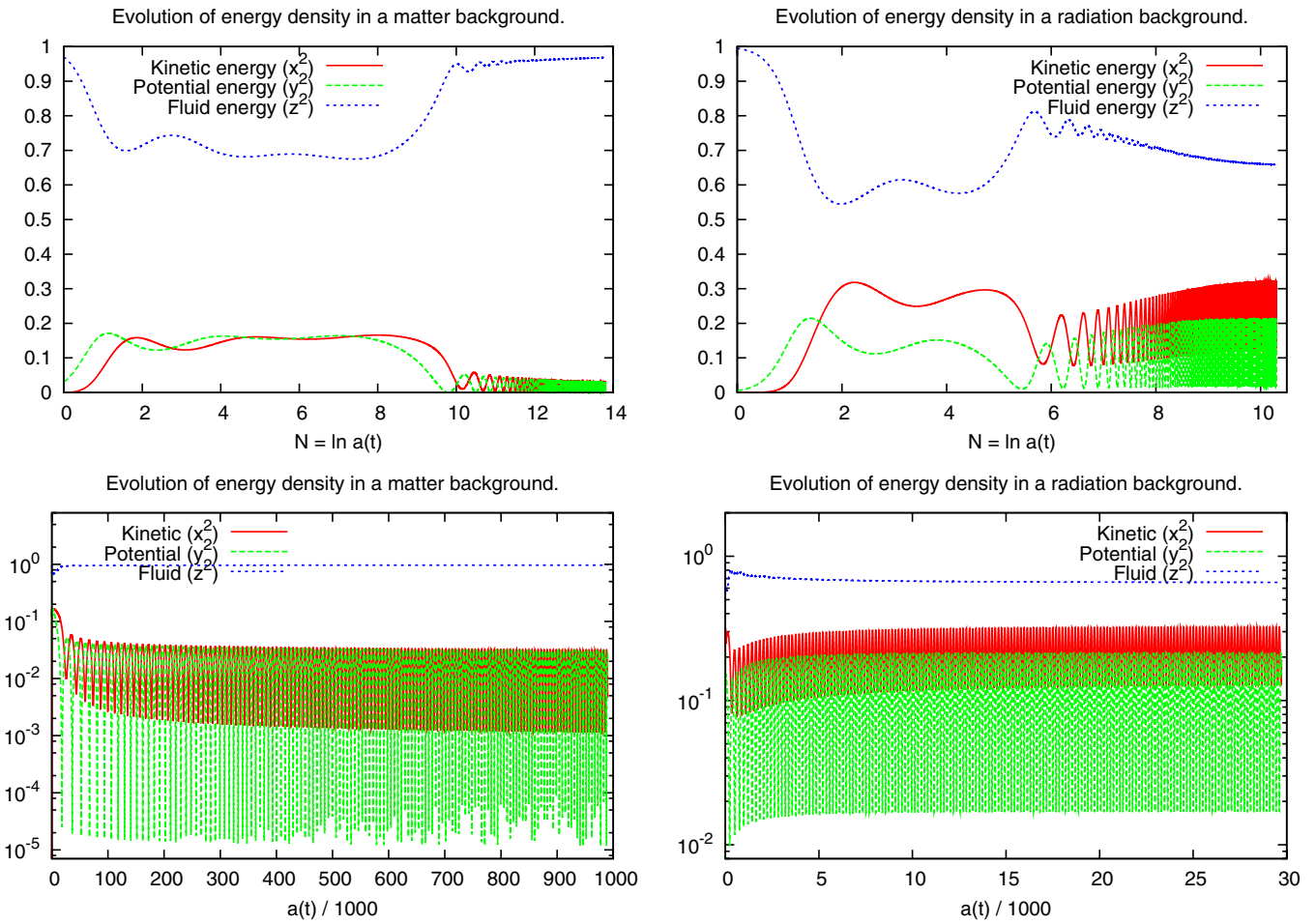
FIG. 4 (color online).   Evolution of the energy densities of the components of the fluid, for the simulations presented in Fig. 1. On the left is the evolution with matter ($\gamma = 1$) as the fluid component, and on the right that with radiation ($\gamma = 4/3$). Comparison between the top and bottom graphs shows that initially there is a period of scaling before an oscillating phase takes over.
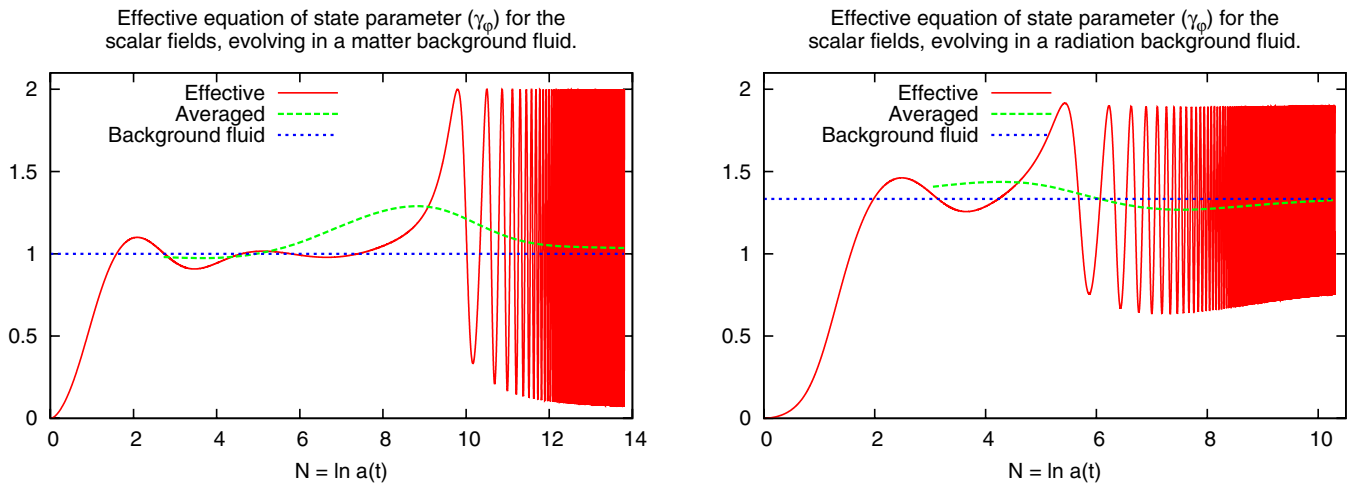

FIG. 5 (color online).   The evolution of the equation of state parameter for the scalar fields, given by $\gamma_\varphi = \frac{2K}{K+V}$, is shown along with that of the background barotropic fluid. The midpoint of γ_φ during the oscillating phase is also shown, which clearly indicates the oscillations asymptote to the same equation of state as the fluid. This suggests that the system is scaling.

X. EFFECTIVE SCALAR BEHAVIOR

Many workers have studied scaling behavior in cosmologies with scalar fields and exponential potentials [14–16]. Our potential (29) is also of exponential form but with too many terms to be understood in terms of these analyses; however it may be possible to find an effective potential which broadly behaves in the same manner. This is motivated by the observation that in (19) the quantity $\sqrt{K}$ could be considered to be analogous to the time derivative of an effective scalar field $\dot{\phi}_{\text{eff}}$. Can the dynamics alternatively be described in terms of the evolution of this effective scalar, and if so what form might its effective potential take?

The simplest exponential potential of a single field that we can write down is

$$V_{\text{eff}}(\phi_{\text{eff}}) = V_0 e^{-\lambda_{\text{eff}} \phi_{\text{eff}}}, \tag{42}$$

of which the asymptotic scaling behavior is well known. We see from Fig. 4 that the energy density is entirely dominated by the fluid component, for which analytically an attractor solution for the effective scalar exists with $\gamma_{\varphi} = \gamma$. Assuming the dynamics of the full system is caught in this attractor we are lead directly to a value for λ_{eff}; in terms of the autonomous system variables x and y we find

$$\lambda_{\text{eff}} = \sqrt{\frac{3}{2}} \frac{\gamma_{\varphi}}{x} = \sqrt{\frac{3}{2}} \frac{\sqrt{(2 - \gamma_{\varphi})\gamma_{\varphi}}}{y}, \tag{43}$$

which we show calculated both ways in Fig. 6.

The single field approximation appears to be a fairly good one, at least initially, with the system evolving around a scaling solution with $\lambda_{\text{eff}} \approx 3$ before the transition into the oscillating phase. After this transition however it is unclear what this approach can tell us; with radiation it
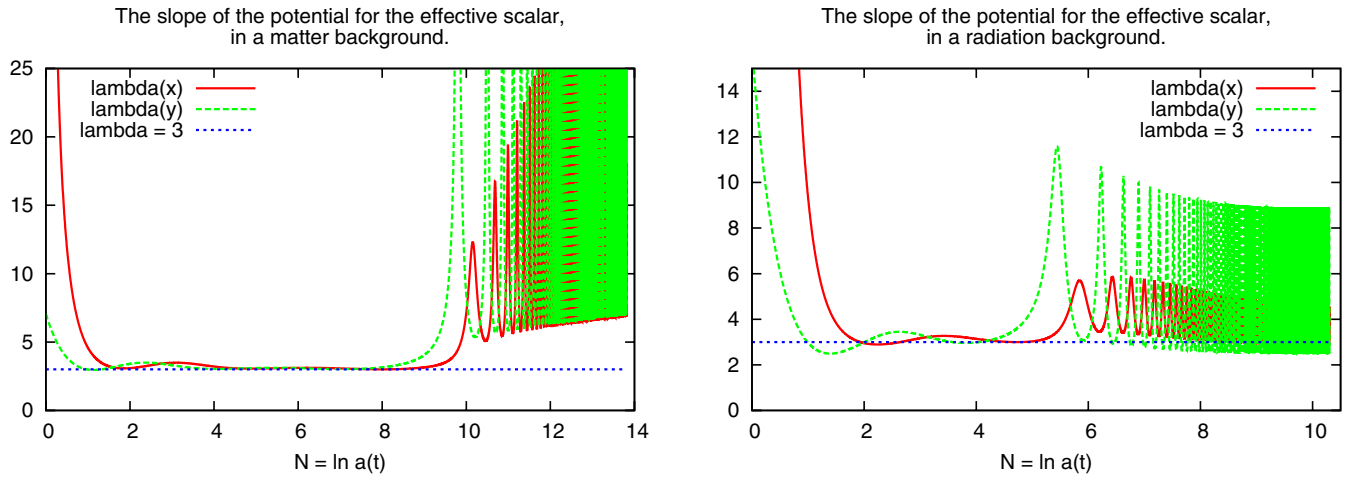

FIG. 6 (color online). Values of the slope λ_{eff} of the effective potential (42), calculated in terms of the autonomous system variables (39) and the effective equation of state parameter for the scalar fields (41).
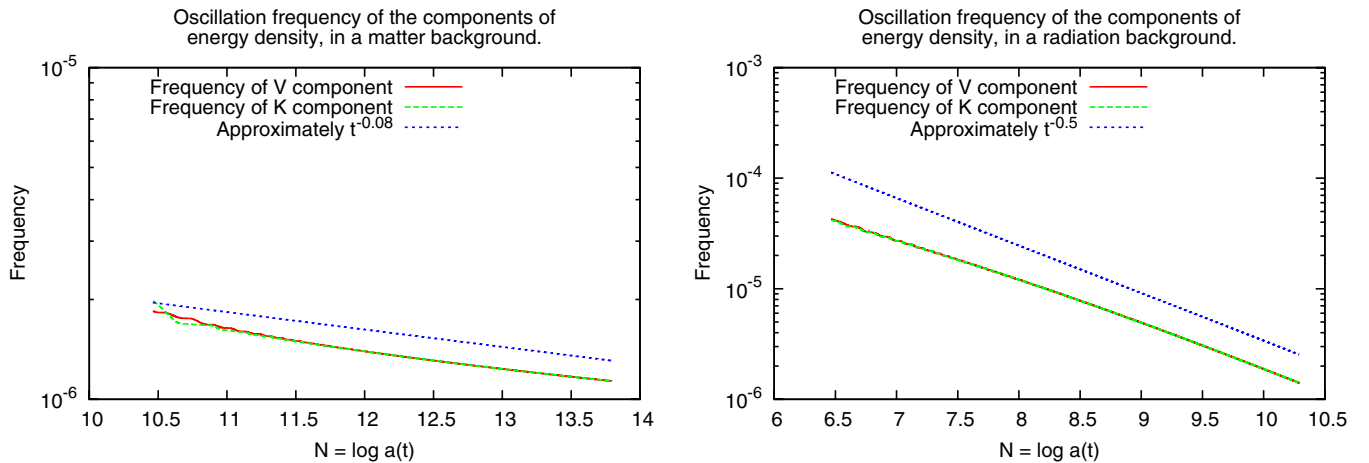

FIG. 7 (color online). The effective frequency of oscillations of the x and y autonomous system variables (39).

appears that there could be some effective lambda value, probably different from the initial one, that the system is evolving around asymptotically, however this is less clear with matter.

The autonomous system analysis we have used is not designed to function in this kind of oscillating picture; however if we imagine that it did and that the oscillations are caused by a second transverse massive scalar, we might be interested in how the frequency associated with that scalar is evolving. We can calculate that from the numerical results utilizing an idea from [17] by which we determine the pointwise wavelength, $T = 2\pi/\omega$, of a given simply oscillating function by calculating the value of T for which the integral

$$\int_{t-t_0/2}^{t+t_0/2} \left[f\left(t - \frac{T}{2}\right) - f\left(t + \frac{T}{2}\right) \right]^2 dt \qquad (44)$$

is minimized, over some suitable width t_0 around the point in question. In this way we calculate the frequency of the oscillations in the x and y autonomous variables, picking t_0 to be the difference between the pair of maxima adjacent to each considered point, and minimizing for T to arrive at the frequency plots in Fig. 7.

XI. EVOLUTION NEAR THE VACUUM: SCALING WITH OSCILLATIONS

The plots for A_a (Fig. 1) make clear that the later evolution of the system consists of shape oscillations around the vacuum $A_1 = A_2$, $A_3 = A_4$, $A_5 = A_6$, with a monotonic increase in the volume modulus φ_1. In a suitable basis of fields we may write an effective Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial \varphi_1)^2 - \frac{1}{2}(\partial \chi_A)^2 - \frac{1}{2}\mu_A^2 \chi_A^2 e^{-\lambda \kappa \varphi_1}, \quad (45)$$

where $\lambda = 3\sqrt{2/7}$ and the index A runs over three dynamical shape moduli which can be thought of as the ratio of the repeat lengths in the three toroidally compactified planes. In a FRW background we therefore arrive at the following field equations for these scalars:

$$\ddot{\varphi}_1 + 3H\dot{\varphi}_1 = \frac{1}{2}\lambda \kappa \mu_A^2 \chi_A^2 e^{-\lambda \kappa \varphi_1}, \qquad (46)$$

$$\ddot{\chi}_A + 3H\dot{\chi}_A = -\mu_A^2 e^{-\lambda \kappa \varphi_1} \chi_A. \qquad (47)$$

This effective Lagrangian describes shape moduli oscillations with frequency $\omega_A = \mu_A e^{-\lambda \kappa \varphi_1/2}$, which contribute an effective potential to the volume modulus of $\frac{1}{2}\lambda \mu_A^2 \langle \chi_A^2 \rangle e^{-\lambda \kappa \varphi_1}$. Here $\langle \chi_A^2 \rangle$ is the time average of the shape moduli oscillations, which are assumed to have a much shorter period than the time scales set by H^{-1} or $\varphi_1/\dot{\varphi}_1$.

Let us assume that the amplitude of the shape moduli oscillations decay as $t^{-\sigma}$ from some time t_i; that the dynamics behave according to a scaling regime so that

each term in the equation of motion scales with the same power of t; and that the energy density in the fluid is scaling along with the scalar so that $H = 2/3\gamma t$. In this case there is an approximate solution to (46) $\varphi_1 = (\alpha/\kappa)\ln(t/t_i) + \varphi_{1,i}$, with

$$\alpha = \frac{2\lambda}{3\gamma(2 - \gamma)} \frac{\kappa^2 \mu_A^2 \langle \chi_{Ai}^2 \rangle e^{-\lambda \kappa \varphi_{1,i}}}{3H_i^2}, \qquad (48)$$

$$\sigma = 1 - \frac{1}{2}\alpha \lambda. \qquad (49)$$

Given that the energy density of the scalar fields is

$$\rho_s = \frac{1}{2}\dot{\varphi}_1^2 + \frac{1}{2}\dot{\chi}_A^2 + \frac{1}{2}\mu_A^2 \chi_A^2 e^{-\lambda \kappa \varphi_1}, \qquad (50)$$

we can take a time average to remove the effects of the oscillations, using $\langle \dot{\chi}_A^2 \rangle = \omega_A^2 \langle \chi_A^2 \rangle$

$$\langle \rho_s \rangle = \frac{\alpha^2}{2\kappa^2 t^2} + \mu_A^2 \langle \chi_A^2 \rangle e^{-\lambda \kappa \varphi_1}. \qquad (51)$$

Using our approximate solution for φ_1, as well as its equation of motion (46), this can be reexpressed as

$$\langle \rho_s \rangle = \frac{\alpha^2}{2\kappa^2 t^2} + \frac{2(2 - \gamma)}{\lambda \gamma} \frac{\alpha}{\kappa^2 t^2}, \qquad (52)$$

and hence for small Ω_ϕ, the kinetic energy of φ_1 is $O(\alpha)$ down from the oscillatory contributions and can indeed be neglected. We then find that we can express the parameter α describing the rate of change of φ_1 as

$$\alpha = \frac{2\lambda}{3\gamma(2 - \gamma)}\Omega_\phi, \qquad (53)$$

and we see that the time-averaged energy scales as t^{-2} which is indeed a scaling solution and our assumptions are consistent.

Now we turn to the solution for χ_A, which we have assumed takes the form

$$\chi_A \propto \Re\left[t^{-\sigma} \exp\left(-i \int^t \omega_A(t')dt'\right) \right]. \qquad (54)$$

Substituting into Eq. (47), and comparing real and imaginary parts, we find

$$\frac{\sigma(1 - 2/\gamma) + \sigma^2}{t^2} - \omega_A^2 + \mu_A^2 e^{-\lambda \kappa \varphi_1} = 0, \qquad (55)$$

$$\frac{2(\sigma - 1/\gamma)\omega_A}{t} - \dot{\omega}_A = 0. \qquad (56)$$

Hence we recover $\omega_A \simeq \mu_A e^{-\lambda \kappa \varphi_1/2}$ which implies $\dot{\omega}_A = -\frac{1}{2}\lambda \alpha \omega_A/t$, and thus find that

$$\sigma = \frac{1}{\gamma} - \frac{1}{4}\lambda \alpha. \qquad (57)$$

Equations (49) and (57) are in contradiction unless

$$\alpha = 4(\gamma - 1)/\gamma\lambda, \tag{58}$$

and so therefore α should be approximately $1/\lambda$ in the radiation era and vanish in the matter era. Computing Ω_ϕ we find

$$\rho_s \simeq \frac{8(\gamma - 1)}{\lambda^2 \gamma^2} \frac{1}{\kappa^2 t^2}, \tag{59}$$

and hence

$$\Omega_s \simeq \frac{6(\gamma - 1)}{\lambda^2}. \tag{60}$$

In the case of radiation we therefore expect that $\Omega_s \simeq 0.7$. Note that about $1/3$ of the energy density in the scalar fields comes from the kinetic energy of φ_1, which is non-oscillatory. Looking at the radiation era plot in Fig. 4 we see that there is a substantial nonoscillatory component and that $\Omega_\phi \simeq 0.33$, which is tantalizingly out by a factor of 2 from the analysis, but at least in the right order of magnitude.

The frequency plot is also roughly consistent: the prediction is that the oscillation frequency should decrease as $t^{-\alpha\lambda/2}$, which should be $t^{-(1/2)}$ in the radiation era, not far from the slope at the end of the run which was measured to be $t^{-0.5}$.

In the matter era the simulations show that $\Omega_\phi \simeq 0.03$, although the prediction is that it should vanish, which we put down to a second-order effect of unknown origin. Similarly, the frequency should decay as t^0 and we measured $t^{-0.08}$.

XII. CONCLUSIONS

In this paper we have investigated some cosmological aspects of compactifications of 11-dimensional Einstein gravity on an elliptic twisted torus, which have the nice property of possessing a positive semidefinite potential with 4-dimensional Minkowski minima, and partial fixing of the moduli. We find that slow-roll inflation using the potential is not possible, as the inflationary ϵ parameter is always greater than unity. This can be shown analytically in the case of a diagonal internal metric (Sec. VIII), and using Monte Carlo techniques for the general case. We also find that there are novel scaling solutions in Friedmann cosmologies in which the energy density in the massive

moduli tracks that of the background barotropic fluid, while the volume modulus increases as approximately $t^{0.5}$ in the radiation era and a very small power of t in the matter era.

It is an interesting question to ask whether such a scaling solution could be used to alleviate the cosmological moduli problem. This seems phenomenologically difficult at first sight, as coupling constants in the low-energy theory typically depend on some power of the volume modulus, and are quite tightly constrained at and after nucleosynthesis. It would also be interesting to explore compactifications with fluxes [18], where a Freund-Rubin flux in the space-time or wrapping some of the internal dimensions may change our conclusions.

APPENDIX A: REDUCING THE RICCI SCALAR

We choose a higher dimensional metric to consist of a space-time part and an internal Lie group manifold part according to

$$
\begin{aligned}
ds^2 &= e^{2\psi(x)} ds^2_{(1,d-1)} + g_{ij}(x) e^i \otimes e^j, \\
&= e^{2\psi(x)} \eta_{\mu\nu} e^\mu \otimes e^\nu + g_{ij}(x) e^i \otimes e^j = \hat{g}_{\hat\mu\hat\nu} e^{\hat\mu} \otimes e^{\hat\nu},
\end{aligned} \tag{A1}
$$

with the coordinates on space-time being represented by x and those on the internal space by y; $\psi(x)$ represents a freedom to choose the space-time coordinates. In the following we shall analyze this space using the frame $e^{\hat\mu} = (e^\mu, e^i)$; note that this is not an orthonormal frame. In order to find the connection one-forms, $\omega^{\hat\mu}{}_{\hat\nu}$, we need to solve

$$
\begin{aligned}
d\hat{g}_{\hat\mu\hat\nu} - \omega^{\hat\rho}{}_{\hat\mu} g_{\hat\rho\hat\nu} - \omega^{\hat\rho}{}_{\hat\nu} g_{\hat\mu\hat\rho} &= 0 \\
de^{\hat\mu} + \omega^{\hat\mu}{}_{\hat\nu} \wedge e^{\hat\nu} &= 0,
\end{aligned} \tag{A2}
$$

and the curvature two-forms follow from

$$\hat{R}^{\hat\mu}{}_{\hat\nu} = d\omega^{\hat\mu}{}_{\hat\nu} + \omega^{\hat\mu}{}_{\hat\rho} \wedge \omega^{\hat\rho}{}_{\hat\nu}. \tag{A3}$$

We find that the Ricci tensor is given by

$$
\begin{aligned}
\hat{\mathcal{R}}_{\mu\nu} &= \mathcal{R}_{(d)\mu\nu} - (d-2)\nabla_\mu\nabla_\nu\psi - \eta_{\mu\nu}\nabla_\rho\nabla^\rho\psi - (d-2)\eta_{\mu\nu}\nabla_\rho\psi\nabla^\rho\psi + (d-2)\nabla_\mu\psi\nabla_\nu\psi - \frac{1}{4}\nabla_\mu g^{ij}\nabla_\nu g_{ij} \\
&\quad - \frac{1}{2}g^{ij}\nabla_\mu\nabla_\nu g_{ij} + \frac{1}{2}g^{ij}(\nabla_\mu g_{ij}\nabla_\nu\psi + \nabla_\nu g_{ij}\nabla_\mu\psi) - \frac{1}{2}\eta_{\mu\nu}g^{ij}\nabla_\rho g_{ij}\nabla^\rho\psi \\
\hat{\mathcal{R}}_{\mu j} &= -\frac{1}{2}g^{kl}\nabla_\mu g_{km} f^m{}_{lj} \\
\hat{\mathcal{R}}_{ij} &= \hat{R}_{ij} + e^{-2\psi}\left(\frac{1}{2}g^{kl}\nabla_\mu g_{ik}\nabla^\mu g_{jl} - \frac{1}{2}\nabla_\mu\nabla^\mu g_{ij} + \frac{1}{4}g_{kl}\nabla_\mu g^{kl}\nabla^\mu g_{ij} - \frac{1}{2}(d-2)\nabla_\mu\psi\nabla^\mu g_{ij}\right).
\end{aligned} \tag{A4}
$$

In deriving this we have used the fact that compact Lie groups are unimodular, giving $f^I_{\ IJ} = 0$ [2,19,20]. $\tilde{\mathcal{R}}_{ij}$ denotes the curvature of the internal space, treating the g_{ij} as constant and the covariant derivatives; ∇_μ are for the metric $ds^2_{(1,d-1)}$ with their indices raised by $\eta^{\mu\nu}$. Given the Ricci curvatures above we can see one of the issues related

to the consistency of truncation, namely, that there is nothing to source $\mathcal{R}_{\mu j}$, and so it must vanish by the 11-dimensional equations of motion. For the cases we consider, we find that this term does vanish.

We may now trace the above to find the following Ricci scalar

$$\hat{\mathcal{R}} = \mathcal{R}_{\text{int}} + e^{-2\psi}[\mathcal{R}_{(d)} - 2(d-1)\nabla^2\psi - (d-1)(d-2)\nabla_\mu\psi\nabla^\mu\psi - g^{ij}\nabla^2 g_{ij}$$
$$- \frac{3}{4}\nabla_\mu g^{ij}\nabla_\mu g_{ij} - (d-2)g^{ij}\nabla_\mu g^{ij}\nabla^\mu\psi - \frac{1}{4}g^{ij}\nabla_\mu g_{ij}g^{kl}\nabla^\mu g_{kl}]. \tag{A5}$$

Making use of the gauge freedom we choose

$$e^{(d-2)\psi}\sqrt{g_{ij}} = \kappa^D \tag{A6}$$

showing that the physical volume of the internal space is given by

$$\mathcal{V}_{\text{phys}} = +\mathcal{V}_{\text{int}}e^{(2-d)\psi}. \tag{A7}$$

This gauge choice enables us to write

$$\hat{\mathcal{R}}_{\mu\nu} = \mathcal{R}_{(d)\mu\nu} + \frac{1}{2(d-2)}g^{ij}\nabla_\sigma\nabla^\sigma g_{ij}\eta_{\mu\nu} + \frac{1}{4}\nabla_\mu g^{ij}\nabla_\nu g_{ij} + \frac{1}{2(d-2)}\eta_{\mu\nu}\nabla_\sigma g^{ij}\nabla^\sigma g_{ij} - \frac{1}{4(d-2)}g^{ij}\nabla_\mu g_{ij}g^{kl}\nabla_\nu g_{kl}$$

$$= \mathcal{R}_{(d)\mu\nu} + \frac{1}{4}\nabla_\mu g^{ij}\nabla_\nu g_{ij} - \eta_{\mu\nu}\nabla^2\psi - (d-2)\nabla_\mu\psi\nabla_\nu\psi$$

$$\hat{\mathcal{R}}_{ij} = \tilde{\mathcal{R}}_{ij} + \frac{1}{2}e^{-2\psi}[g^{kl}\nabla_\mu g_{ik}\nabla^\mu g_{jl} - \nabla_\mu\nabla^\mu g_{ij}]$$

$$\hat{\mathcal{R}} = e^{-2\psi}[\mathcal{R}_{(d)} - 2(d-1)\nabla^2\psi - g^{ij}\nabla^2 g_{ij} - \frac{3}{4}\nabla_\mu g^{ij}\nabla^\mu g_{ij} - \frac{1}{4(d-2)}g^{ij}\nabla_\mu g_{ij}g^{kl}\nabla^\mu g_{kl}] + \mathcal{R}_{\text{int}}. \tag{A8}$$

APPENDIX B: CONSTRUCTION OF THE ELLIPTIC TWISTED TORUS

These groups [12] are related to the $ISO(N)$ group of isometries in N-dimensional space, and are described by the structure constants $f^a_{\ bc}$, where the indices $a, b, c = 0, 1, 2...$ and the nonzero components are given by

$$f^i_{\ 0j} = M^i_{\ j}, \tag{B1}$$

with $i, j = 1, 2, ...$ The matrix M is skew symmetric and real, and we can choose a basis in full generality in which it takes the form

$$M = \begin{pmatrix} 0 & m_1 & 0 & 0 & 0 \\ -m1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & -m_2 & 0 & 0 \\ 0 & & 0 & 0 & \ddots \end{pmatrix}. \tag{B2}$$

In terms of left-invariant one-forms we have

$$de^a = -\frac{1}{2}f^a_{\ bc}e^b \wedge e^c, \tag{B3}$$

and in terms of generators we have

$$[T_b, T_c] = f^a_{\ bc}T_a. \tag{B4}$$

To understand the algebra first consider the 3-dimensional group,

$$[T_0, T_1] = -m_1 T_2, \qquad [T_0, T_2] = m_1 T_1,$$
$$[T_1, T_2] = 0 \tag{B5}$$

$$de^0 = 0, \qquad de^1 = -m_1 e^0 \wedge e^2, \qquad de^2 = m_1 e^0 \wedge e^1, \tag{B6}$$

which can be represented with the matrices

$$T_0 = \begin{pmatrix} 0 & m_1 & 0 \\ -m1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{B7}$$

which we recognize as the algebra of translations and rotations in 2-dimensional Euclidean space, and give the group element

$$g(x, y, \theta) = \begin{pmatrix} \cos(m_1\theta) & \sin(m_1\theta) & x/\kappa \\ -\sin(m_1\theta) & \cos(m_1\theta) & y/\kappa \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(B8)}$$

$$0 < \theta < 2\pi/m_1$$

from which we may calculate the left-invariant one-forms

$$e_L^0 = d\theta \quad \text{(B9)}$$

$$e_L^1 = \kappa^{-1}[\cos(m_1\theta)dx - \sin(m_1\theta)dy] \quad \text{(B10)}$$

$$e_L^2 = \kappa^{-1}[\sin(m_1\theta)dx + \cos(m_1\theta)dy] \quad \text{(B11)}$$

and the right-invariant one-forms

$$e_R^0 = d\theta \quad \text{(B12)}$$

$$e_R^1 = \kappa^{-1}[dx - myd\theta] \quad \text{(B13)}$$

$$e_R^2 = \kappa^{-1}[dy + mxd\theta]. \quad \text{(B14)}$$

Under left-translations, $g(\theta, x, y) \rightarrow g(\alpha, a, b)g(\theta, x, y)$, we have

$$\theta \rightarrow \theta + \alpha \quad \text{(B15)}$$

$$x \rightarrow a + x\cos(m_1\theta) + y\sin(m_1\theta) \quad \text{(B16)}$$

$$y \rightarrow b - x\sin(m_1\theta) + y\cos(m_1\theta) \quad \text{(B17)}$$

while under right translation, $g(\theta, x, y) \rightarrow g(\theta, x, y)g(\alpha, a, b)$, we have

$$\theta \rightarrow \theta + \alpha \quad \text{(B18)}$$

$$x \rightarrow x + a\cos(m_1\theta) + b\sin(m_1\theta) \quad \text{(B19)}$$

$$y \rightarrow y - a\sin(m_1\theta) + b\cos(m_1\theta) \quad \text{(B20)}$$

which allows us to make identifications on the space, thereby making it compact.

The adjoint action on the generators is

$$h^{-1}(0, a, b)T_a h(0, a, b) = D(h)_a{}^b T_b \quad \text{(B21)}$$

$$h^{-1}(0, a, b)T_0 h(0, a, b) = T_0 - mbT_1 + maT_2 \quad \text{(B22)}$$

$$h^{-1}(0, a, b)T_1 h(0, a, b) = T_1 \quad \text{(B23)}$$

$$h^{-1}(0, a, b)T_2 h(0, a, b) = T_2 \quad \text{(B24)}$$

giving

$$D(h)_0{}^0 = 1, \quad D(h)_0{}^1 = -mb, \quad D(h)_0{}^2 = ma, \quad \text{(B25)}$$

$$D(h)_1{}^1 = 1 \quad \text{(B26)}$$

$$D(h)_2{}^2 = 1. \quad \text{(B27)}$$

Under right-translations by $g(0, a, b)$ the left-invariant one-forms change by

$$e_L^0 \rightarrow e_L^0 \quad \text{(B28)}$$

$$e_L^1 \rightarrow e_L^1 + \kappa^{-1}bme_L^0 \quad \text{(B29)}$$

$$e_L^2 \rightarrow e_L^2 - \kappa^{-1}ame_L^0 \quad \text{(B30)}$$

which is the adjoint action of $e^a \rightarrow e^b D(h^{-1})_b{}^a$.

For our case of interest, 7-dimensional, we need the structure constants

$$f^1{}_{02} = m_1, \qquad f^3{}_{04} = m_2, \qquad f^5{}_{06} = m_3 \quad \text{(B31)}$$

with the rest either vanishing or given by antisymmetry. Note that the zero index still resides on the 7-dimensional space, and is not the time direction. Also note that this algebra is not semisimple because the Killing metric, $G_{cd} = f^a{}_{bc}f^b{}_{ad}$, is not invertible. Indeed, the only nonzero component is given by

$$G_{00} = \text{Tr}(M^2). \quad \text{(B32)}$$

In terms of the internal space ansatz, $g_{ab}e^a \otimes e^b$, if the space was being formed by a coset with a continuous subgroup we would expect to find restrictions on the available degrees of freedom due to invariance and consistency requirements [10]; however there is no such restriction here when dividing by a discrete subgroup as detailed above.

APPENDIX C: SCHERK-SCHWARZ ANSATZ AND GAUGE FIXING

We adopt a modified ansatz for the metric, given by the the Scherk-Schwarz form

$$ds^2 = e^{2\psi(x)}ds_{(4)}^2 + g_{ab}\nu^a \otimes \nu^b \quad \text{(C1)}$$

$$\nu^a = e^a - A^a \quad \text{(C2)}$$

where A^a are gauge fields. Upon substituting this into the Riemann scalar of the action we find an effective theory described by the scalars g_{ab}, charged under the non-Abelian gauge fields A^a. The gauge transformations are

$$\delta A^a = d\xi^a - f^a{}_{bc}\xi^b A^c \quad \text{(C3)}$$

$$\delta g_{ab} = -f^c{}_{ad}g_{cb}\xi^d - f^c{}_{bd}g_{ca}\xi^d, \quad \text{(C4)}$$

and the covariant derivative of the scalars is

$$Dg_{ab} = dg_{ab} + g_{ac}f^c{}_{bd}A^d + g_{bc}f^c{}_{ad}A^d. \quad \text{(C5)}$$

Within Scherk-Schwarz compactification we have the following gauge transformations:

$$\delta g_{00} = -2M^i{}_j g_{0i}\xi^j \quad \text{(C6)}$$

$$\delta g_{0i} = -M^j{}_k g_{ji} \xi^k + M^j{}_i g_{0j} \xi^0 \tag{C7}$$

$$\delta g_{ij} = (M^k{}_i g_{kj} + M^k{}_j g_{ki}) \xi^0. \tag{C8}$$

We may use the ξ^i gauge parameters to set the $g_{0i} = 0$

gauge, leaving us one more gauge degree of freedom which we shall not fix. However this does require some assumptions on the rank of M.

This all amounts to allowing us to use the gauge $g_{0i} = 0$, with no loss of generality.

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