Covariant approach for perturbations of rotationally symmetric spacetimes

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We present a covariant decomposition of Einstein's field equations which is particularly suitable for perturbations of spherically symmetric—and general locally rotationally symmetric—spacetimes. Based upon the utility of the 1 + 3 covariant approach to perturbation theory in cosmology, the semi-tetrad, 1 + 1 + 2 approach presented here should be useful for analyzing perturbations of a variety of systems in a covariant and gauge-invariant manner. Such applications range from stellar objects to cosmological models such as the spherically symmetric Lemaître-Tolman-Bondi solutions or the class of locally rotationally symmetric Bianchi models.

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I. INTRODUCTION

Tetrad formalisms in general relativity have played a pivotal role in its development as well as our understanding of the subject. These range from the complex null tetrad of Newmann and Penrose, to the 1 + 3 approach of Ehlers, Ellis, and others, which includes both a full tetrad approach as well as a partial "covariant" approach where only one timelike tetrad vector is chosen (see [1] for a review and references). These techniques formulate the equations of general relativity as first-order differential equations in the physical curvature and dynamic variables of the covariant derivatives of the tetrad vectors, as opposed to the more usual coordinate approach involving second-order partial differential equations in functions appearing in the metric. The differential operators which appear are convective derivatives along the tetrad vectors as opposed to partial derivatives with respect to particular coordinates. Much of their utility arises in spacetimes with special symmetry. For example, the 1 + 3 covariant approach is perfect for cosmology because it covariantly factorizes out the essential coordinate—time—leaving all the background field equations as covariant scalar equations. Under perturbations all 3-vectors and tensors (which must vanish in the background due to homogeneity and isotropy) become gauge-invariant first-order quantities making a Fourier analysis easy [1,2].

We formulate here an approach which involves a semi-tetrad: we keep the timelike threading vector field of the 1 + 3 approach and introduce one spatial vector. The remaining two dimensions are left untouched, rather like the "3" in the 1 + 3 approach. Indeed the formalism presented here may be considered as halfway between the 1 + 3 tetrad and covariant approaches. A similar approach has been discussed before in [3–6], and we expand on this considerably here by presenting the full system of 1 + 1 + 2 equations.

It is expected that this approach may find use in perturbations of spacetimes with a preferred spatial direction at each point—so-called locally rotationally symmetric spacetimes [7]. These include the spherically symmetric Lemaître-Tolman-Bondi models, many classes of Bianchi models, as well as forming the background for most stellar models. In this paper we provide the algorithm of how to calculate gravitational perturbations in any locally rotationally symmetric (LRS) spacetime in a covariant and gauge-invariant (GI) way.

As an example of its utility, such a covariant perturbative scheme was applied to the Schwarzschild solution in [8]. Despite being a well-understood problem, it was shown using the 1 + 1 + 2 approach how both the axial and polar degrees of freedom may be unified into a single transverse-traceless tensor which obeys the tensorial form of the Regge-Wheeler equation [8,9],

$$- \ddot{W}_{ab} + \hat{\hat{W}}_{ab} + \mathcal{A}\hat{W}_{ab} - \phi^2 W_{ab} + \delta^2 W_{ab} = 0, \quad (1)$$

where the Regge-Wheeler tensor W_{ab} is a gauge- and frame-invariant transverse traceless tensor, defined in [8] (other variables are defined below), and $\dot{}$, $\hat{}$, and δ are time, radial, and angular derivatives, respectively. This tensor contains in compact form the curved space generalization of the two flat space gravitational wave polarizations h_+ and $h_\times$ [10] (see also [11] for an extension of this work). The approach here also has been used to study scalar and electromagnetic perturbations of LRS spacetimes, and generalized Regge-Wheeler equations were found [12,13]. Furthermore, it has been used to study the interaction of magnetic fields and gravitational waves around a black hole—a process which produces electromagnetic radiation mirroring the gravitational waves [10].

In Sec. II we discuss the 1 + 1 + 2 approach in full generality, and then in Sec. III we discuss the perturbation procedure for LRS spacetimes, before summarizing in Sec. IV.

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II. FORMALISM

In the $1+3$ approach, a timelike threading vector field u^a ($u^a u_a = -1$) is introduced, representing the observers' congruence. Given this vector field, the projection tensor $h_a{}^b = g_a{}^b + u_a u^b$ is introduced, which projects all vectors and tensors orthogonal to u^a. Using h_{ab}, any 4-vector may be split into a ($1+3$ scalar) part parallel to u^a and a (3-vector) part orthogonal to u^a. Any second rank tensor may be covariantly and irreducibly split into scalar, vector, and projected, symmetric, trace-free (PSTF) 3-tensor parts, which requires the alternating tensor $\varepsilon_{abc} = u^d \eta_{dabc}$ [1]. Tensors of higher rank may be similarly split, but are rarely used (an important exception being cosmic microwave background physics [14,15]). These are the fundamental quantities describing the spacetime, after the introduction of u^a.

We now introduce another vector field and perform another split, but this time of the $1+3$ equations. The "$1+1+2$" decomposition we develop here has been partially studied before, mostly in the context of symmetries of solutions of the Einstein field equation [5,16]. It was introduced by [3] and further developed in [4,8,11–13]. However, there are important differences with the work presented here. In the following we assume the $1+3$ covariant split of the equations (as given in [1], for example), with all tensors split into scalars, vectors, and PSTF tensors with respect to u^a.

Take a unit vector n^a orthogonal to u^a: $n^a n_a = 1$, $u^a n_a = 0$, and define the projection tensor,

$$N_a{}^b \equiv h_a{}^b - n_a n^b = g_a{}^b + u_a u^b - n_a n^b, \tag{2}$$

which project vectors orthogonal to n^a (and u^a): $n^a N_{ab} = 0 = u^a N_{ab}$, onto 2-surfaces ($N_a{}^a = 2$) which we refer to as the sheet. This is also the screen space of the null vector $k^a \propto u^a + n^a$.

Any 3-vector ψ^a can now be irreducibly split into a scalar, Ψ, which is the part of the vector parallel to n^a, and a vector, Ψ^a, lying in the sheet orthogonal to n^a;

$$\psi^a = \Psi n^a + \Psi^a, \qquad \text{where } \Psi \equiv \psi_a n^a, \qquad \text{and} \quad \Psi^a \equiv N^{ab}\psi_b \equiv \psi^{\bar{a}}, \tag{3}$$

where we use a bar over an index to denote projection with N_{ab} on that index. Similarly, any PSTF tensor, ψ_{ab}, can now be split into scalar, vector, and tensor (which are PSTF with respect to n^a) parts:

$$\psi_{ab} = \psi_{\langle ab\rangle} = \Psi\left(n_a n_b - \tfrac{1}{2}N_{ab}\right) + 2\Psi_{(a}n_{b)} + \Psi_{ab}, \tag{4}$$

where

$$\Psi \equiv n^a n^b \psi_{ab} = -N^{ab}\psi_{ab}, \qquad \Psi_a \equiv N_a{}^b n^c \psi_{bc} = \Psi_{\bar{a}},$$
$$\Psi_{ab} \equiv \psi_{\{ab\}} \equiv \left(N_{(a}{}^c N_{b)}{}^d - \tfrac{1}{2}N_{ab}N^{cd}\right)\psi_{cd}. \tag{5}$$

We use curly brackets to denote the PSTF with respect to n^a part of a tensor. Note that for 2nd-rank tensors in the $1+1+2$ formalism "PSTF" is precisely equivalent to "transverse-traceless." Note also that $h_{\{ab\}} = 0$, $N_{\langle ab\rangle} = -n_{\langle a}n_{b\rangle} = N_{ab} - \frac{2}{3}h_{ab}$.

We also define the alternating Levi-Cività 2-tensor

$$\varepsilon_{ab} \equiv \varepsilon_{abc}n^c = u^d \eta_{dabc} n^c, \tag{6}$$

so that $\varepsilon_{ab}n^b = 0 = \varepsilon_{(ab)}$, and

$$\varepsilon_{abc} = n_a \varepsilon_{bc} - n_b \varepsilon_{ac} + n_c \varepsilon_{ab}, \tag{7}$$

$$\varepsilon_{ab}\varepsilon^{cd} = N_a{}^c N_b{}^d - N_a{}^d N_b{}^c, \tag{8}$$

$$\varepsilon_a{}^c \varepsilon_{bc} = N_{ab}, \qquad \varepsilon^{ab}\varepsilon_{ab} = 2. \tag{9}$$

Note that for a 2-vector Ψ^a, ε_{ab} may be used to form a vector orthogonal to Ψ^a but of the same length.

With these definitions we may split any object into scalars, 2-vectors in the sheet, and transverse-traceless 2-tensors, also defined in the sheet. These three types of objects are the only objects which appear, after a complete decomposition. Hereafter, we will assume such a split has been made, and "vector" will generally refer to a vector projected orthogonal to u^a and n^a, and "tensor" will generally mean transverse-traceless tensor, defined by Eq. (5).

There are two new derivatives of interest now, which n^a defines, for any object $\psi_{\cdots}{}^{\cdots}$:

$$\hat{\psi}_{a\cdots b}{}^{c\cdots d} \equiv n^e \mathrm{D}_e \psi_{a\cdots b}{}^{c\cdots d}, \tag{10}$$

$$\delta_e \psi_{a\cdots b}{}^{c\cdots d} \equiv N_e{}^j N_a{}^f \cdots N_b{}^g N_h{}^c \cdots N_i{}^d \mathrm{D}_j \psi_{f\cdots g}{}^{h\cdots i}. \tag{11}$$

The hat derivative is the derivative along the vector field n^a in the surfaces orthogonal to u^a. This definition represents a conceptual divergence from the $1+3$ tetrad approach, in which the basis vectors appear on an equal footing [i.e., with ∇_a rather than D_a in Eq. (10)]. As a result, the congruence u^a retains the primary importance it has in the $1+3$ covariant approach. (We choose to think of $\mathcal{A} \equiv u^a n^b \nabla_a u_b = -u^a u^b \nabla_a n_b$ as the radial component of the acceleration of u^a, rather than the time component of $\dot{n}^a$.) The δ-derivative, defined by Eq. (11) is a projected derivative on the sheet, with projection on every free index.

These derivatives then affect our projection tensor N_{ab} and Levi-Cività tensor as follows:

$$\dot{N}_{ab} = 2u_{(a}\dot{u}_{b)} - 2n_{(a}\dot{n}_{b)} = 2u_{(a}\mathcal{A}_{b)} - 2n_{(a}\alpha_{b)}, \tag{12}$$

$$\hat{N}_{ab} = -2n_{(a}\hat{n}_{b)}, \tag{13}$$

$$\delta_c N_{ab} = 0, \tag{14}$$

$$\dot{\varepsilon}_{ab} = -2u_{[a}\varepsilon_{b]c}\mathcal{A}^c + 2n_{[a}\varepsilon_{b]c}\alpha^c, \tag{15}$$

$$\hat{\varepsilon}_{ab} = 2n_{[a}\varepsilon_{b]c}a^c, \quad (16)$$

$$\delta_c \varepsilon_{ab} = 0. \quad (17)$$

We now decompose the covariant derivative of n^a orthogonal to u^a into its irreducible form:

$$\mathrm{D}_a n_b = n_a a_b + \tfrac{1}{2}\phi N_{ab} + \xi\varepsilon_{ab} + \zeta_{ab}, \quad (18)$$

where

$$a_a \equiv n^c \mathrm{D}_c n_a = \hat{n}_a, \quad (19)$$

$$\phi \equiv \delta_a n^a, \quad (20)$$

$$\xi \equiv \tfrac{1}{2}\varepsilon^{ab}\delta_a n_b, \quad (21)$$

$$\zeta_{ab} \equiv \delta_{\{a}n_{b\}}. \quad (22)$$

We may interpret these as follows: traveling along n^a, ϕ represents the sheet expansion, ζ_{ab} is the shear of n^a (distortion of the sheet), and a^a its acceleration, while ξ represents a "twisting" of the sheet—the rotation of n^a [4]. The other derivative of n^a is its change along u^a,

$$\dot{n}_a = \mathcal{A}u_a + \alpha_a, \quad \text{where } \alpha_a \equiv \dot{n}_{\bar{a}} \quad \text{and} \quad \mathcal{A} = n^a \dot{u}_a. \quad (23)$$

The new variables a_a, ϕ, ξ, ζ_{ab}, and α_a are fundamental objects in the spacetime, and their dynamics gives us information about the spacetime geometry. They are treated on the same footing as the kinematical variables of u^a in the 1 + 3 approach (which also appear here).

For any vector Ψ^a orthogonal to n^a and u^a (i.e., $\Psi^a = \Psi^{\bar{a}}$), we may decompose the different parts of its spatial derivative:

$$\mathrm{D}_a \Psi_b = -n_a n_b \Psi_c a^c + n_a \hat{\Psi}_{\bar{b}} - n_b[\tfrac{1}{2}\phi\Psi_a + (\xi\varepsilon_{ac} + \zeta_{ac})\Psi^c] + \delta_a \Psi_b. \quad (24)$$

Similarly, for a tensor Ψ_{ab}: $\Psi_{ab} = \Psi_{\{ab\}}$, we have

$$\mathrm{D}_a \Psi_{bc} = -2n_a n_{(b}\Psi_{c)d}a^d + n_a \hat{\Psi}_{bc} - 2n_{(b}[\tfrac{1}{2}\phi\Psi_{c)a} + \Psi_{c)}{}^d(\xi\varepsilon_{ad} + \zeta_{ad})] + \delta_a \Psi_{bc}. \quad (25)$$

Note that for a scalar, we have $\mathrm{D}_a \Psi = \hat{\Psi} n_a + \delta_a \Psi$.

We take n^a to be arbitrary at this point, and then split the usual 1 + 3 kinematical and Weyl quantities into the irreducible set $\{\theta, \mathcal{A}, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \mathcal{A}^a, \Sigma^a, \mathcal{E}^a, \mathcal{H}^a, \Sigma_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}\}$ using (3) and (4) as follows:

$$\dot{u}^a = \mathcal{A}n^a + \mathcal{A}^a, \quad (26)$$

$$\omega^a = \Omega n^a + \Omega^a, \quad (27)$$

$$\sigma_{ab} = \Sigma(n_a n_b - \tfrac{1}{2}N_{ab}) + 2\Sigma_{(a}n_{b)} + \Sigma_{ab}, \quad (28)$$

$$E_{ab} = \mathcal{E}(n_a n_b - \tfrac{1}{2}N_{ab}) + 2\mathcal{E}_{(a}n_{b)} + \mathcal{E}_{ab}, \quad (29)$$

$$H_{ab} = \mathcal{H}(n_a n_b - \tfrac{1}{2}N_{ab}) + 2\mathcal{H}_{(a}n_{b)} + \mathcal{H}_{ab}. \quad (30)$$

The shear scalar, σ, for example, may be expressed in the form

$$\sigma^2 \equiv \tfrac{1}{2}\sigma_{ab}\sigma^{ab} = \tfrac{3}{4}\Sigma^2 + \Sigma_a\Sigma^a + \tfrac{1}{2}\Sigma_{ab}\Sigma^{ab}. \quad (31)$$

Similarly we may split the fluid variables q^a and π_{ab},

$$q^a = Qn^a + Q^a, \quad (32)$$

$$\pi_{ab} = \Pi(n_a n_b - \tfrac{1}{2}N_{ab}) + 2\Pi_{(a}n_{b)} + \Pi_{ab}. \quad (33)$$

Having described the splitting of the 1 + 3 variables to obtain their 1 + 1 + 2 parts, and the introduction of the new 1 + 1 + 2 variables corresponding to the irreducible parts of $\nabla_a n_b$, it only remains to apply this decomposition procedure to the 1 + 3 equations themselves, as well as the Ricci identities for n^a. We give these equations in Sec. III.

A. Commutation relations

In general, the three derivatives we now have defined, "$\cdot$," "$\hat{}$," and "δ_a" do not commute. Instead, when acting on a scalar ψ, they satisfy

$$\hat{\dot{\psi}} - \dot{\hat{\psi}} = -\mathcal{A}\dot{\psi} + (\tfrac{1}{3}\theta + \Sigma)\hat{\psi} + (\Sigma_a + \varepsilon_{ab}\Omega^b - \alpha_a)\delta^a\psi, \quad (34)$$

$$\delta_a\dot{\psi} - N_a{}^b(\delta_b\psi)^{\cdot} = -\mathcal{A}_a\dot{\psi} + (\alpha_a + \Sigma_a - \varepsilon_{ab}\Omega^b)\hat{\psi} + (\tfrac{1}{3}\theta - \tfrac{1}{2}\Sigma)\delta_a\psi + (\Sigma_{ab} + \Omega\varepsilon_{ab})\delta^b\psi, \quad (35)$$

$$\delta_a\hat{\psi} - N_a{}^b(\widehat{\delta_b\psi}) = (\Sigma_a - \varepsilon_{ab}\Omega^b)\dot{\psi} + a_a\hat{\psi} + \tfrac{1}{2}\phi\delta_a\psi + (\zeta_{ab} + \xi\varepsilon_{ab})\delta^b\psi, \quad (36)$$

$$\delta_a\delta_b\psi - \delta_b\delta_a\psi = 2\varepsilon_{ab}(\Omega\dot{\psi} - \xi\hat{\psi}) + 2a_{[a}\delta_{b]}\psi. \quad (37)$$

The commutation relations for 2-vectors ψ_a are

$$\hat{\dot{\psi}}_{\bar{a}} - \dot{\hat{\psi}}_{\bar{a}} = -\mathcal{A}\dot{\psi}_{\bar{a}} + (\tfrac{1}{3}\theta + \Sigma)\hat{\psi}_{\bar{a}} + (\Sigma_b + \varepsilon_{bc}\Omega^c - \alpha_b)\delta^b\psi_a + \mathcal{A}_a(\Sigma_b + \varepsilon_{bc}\Omega^c)\psi^b + \mathcal{H}\varepsilon_{ab}\psi^b, \quad (38)$$

$$\delta_a\dot{\psi}_b - N_a{}^c N_b{}^d(\delta_c\psi_d)^{\cdot} = -\mathcal{A}_a\dot{\psi}_b + (\alpha_a + \Sigma_a - \varepsilon_{ac}\Omega^c)\hat{\psi}_{\bar{b}} + (\tfrac{1}{3}\theta - \tfrac{1}{2}\Sigma)(\delta_a\psi_b + \psi_a\mathcal{A}_b) + \mathcal{H}_a\varepsilon_{bc}\psi^c + (\Sigma_{ac} + \Omega\varepsilon_{ac})(\delta^c\psi_b + \psi^c\mathcal{A}_b) + \tfrac{1}{2}(\psi_a Q_b - N_{ab}\psi^c Q_c) - (\tfrac{1}{2}\phi N_{ac} + \xi\varepsilon_{ac} + \zeta_{ac})\psi^c\alpha_b, \tag{39}$$

$$\delta_a\hat{\psi}_b - N_a{}^c N_b^d\widehat{(\delta_c\psi_d)} = (\Sigma_a - \varepsilon_{ac}\Omega^c)\dot{\psi}_{\bar{b}} + a_a\hat{\psi}_{\bar{b}} + \tfrac{1}{2}\phi(\delta_a\psi_b - \psi_a a_b) + (\zeta_{ac} + \xi\varepsilon_{ac})(\delta^c\psi_b - \psi^c a_b) + N_{ab}\psi^c(\tfrac{1}{2}\Pi_c + \mathcal{E}_c) - \psi_a(\tfrac{1}{2}\Pi_b + \mathcal{E}_b), \tag{40}$$

$$\begin{aligned}\delta_a\delta_b\psi_c - \delta_b\delta_a\psi_c = {}& 2\varepsilon_{ab}(\Omega\dot{\psi}_{\bar{c}} - \xi\hat{\psi}_{\bar{c}}) + 2[(\tfrac{1}{3}\theta - \Sigma)^2 - \tfrac{1}{4}\phi^2 + \tfrac{1}{2}\Pi + \mathcal{E} - \tfrac{1}{3}(\mu + \Lambda)]\psi_{[a}N_{b]c} \\ &- 2\psi_{[a}[-(\tfrac{1}{3}\theta - \tfrac{1}{2}\Sigma)(\Sigma_{b]c} + \Omega\varepsilon_{b]c}) + \tfrac{1}{2}\phi(\zeta_{b]c} + \xi\varepsilon_{b]c}) + \tfrac{1}{2}\Pi_{b]c} + \mathcal{E}_{b]c}] \\ &+ 2N_{[ac}[-(\tfrac{1}{3}\theta - \tfrac{1}{2}\Sigma)(\Sigma_{b]d} + \Omega\varepsilon_{b]d}) + \tfrac{1}{2}\phi(\zeta_{b]d} + \xi\varepsilon_{b]d}) + \tfrac{1}{2}\Pi_{b]d} + \mathcal{E}_{b]d}]\psi^d \\ &+ 2[-(\Sigma_{[ac} + \Omega\varepsilon_{[ac})(\Sigma_{b]d} + \Omega\varepsilon_{b]d}) + (\zeta_{[ac} + \xi\varepsilon_{[ac})(\zeta_{b]d} + \xi\varepsilon_{b]d})]\psi^d.\end{aligned} \tag{41}$$

These relations are more complicated for tensors. These last two equations in the case of scalars are the decomposition of the $1+3$ commutation relation

$$\mathrm{curl}\,\mathrm{D}_a\psi = 2\dot{\psi}\omega_a. \tag{42}$$

From Eq. (37), we see that our sheet will be a genuine 2-surface in the spacetime (and, in particular, that the derivative δ_a will be a true covariant derivative on this surface) if and only if $\xi = \Omega = a^a = 0$. (Recall that the $1+3$ spatial metric h_{ab} corresponds to a genuine 3-surface when $\omega^a = 0$.) Otherwise, the sheet is really just a collection of tangent planes. In addition, the two vectors u^a and n^a are 2-surface forming if and only if the commutator $[u, n]$ in (34) has no component in the sheet: that is, when Greenberg's vector

$$\Sigma^a + \varepsilon^{ab}\Omega_b - \alpha^a \tag{43}$$

vanishes [16]—see Eq. (34).

III. THE EQUATIONS

Once the vector n^a has been introduced it is possible, and necessary, to augment the $1+3$ equations with the Ricci identities for n^a; without these we do not have enough equations to determine the new $1+1+2$ variables. The Ricci identities for n^a are

$$R_{abc} \equiv 2\nabla_{[a}\nabla_{b]}n_c - R_{abcd}n^d = 0, \tag{44}$$

where R_{abcd} is the Riemann curvature tensor. This third-rank tensor may be covariantly split using the two vector fields u^a and n^a, and gives dynamical equations for the covariant parts of the derivative of n^a (namely α_a, a_a, ϕ, ξ, and ζ_{ab}) in the form of *evolution* equations, involving dot derivatives of these variables, and *propagation* equations, involving hat derivatives. In order to facilitate the calculation of these Ricci identities, which appear in the following section, we give here the expression for the full covariant derivative of n^a in terms of the relevant $1+1+2$ variables:

$$\begin{aligned}\nabla_a n_b = {}& -\mathcal{A}u_a u_b - u_a\alpha_b + (\Sigma + \tfrac{1}{3}\theta)n_a u_b \\ &+ (\Sigma_a - \varepsilon_{ac}\Omega^c)u_b + n_a a_b + \tfrac{1}{2}\phi N_{ab} \\ &+ \xi\varepsilon_{ab} + \zeta_{ab},\end{aligned} \tag{45}$$

which may be inserted into Eq. (44). The full decomposition of the covariant derivative of u^a is

$$\begin{aligned}\nabla_a u_b = {}& -u_a(\mathcal{A}n_b + \mathcal{A}_b) + n_a n_b(\tfrac{1}{3}\theta + \Sigma) \\ &+ n_a(\Sigma_b + \varepsilon_{bc}\Omega^c) + (\Sigma_a - \varepsilon_{ac}\Omega^c)n_b \\ &+ N_{ab}(\tfrac{1}{3}\theta - \tfrac{1}{2}\Sigma) + \Omega\varepsilon_{ab} + \Sigma_{ab},\end{aligned} \tag{46}$$

which in turn implies the useful relation

$$\hat{u}_a = (\tfrac{1}{3}\theta + \Sigma)n_a + \Sigma_a + \varepsilon_{ab}\Omega^b. \tag{47}$$

We have now assembled all the tools necessary to provide the full system of equations for the $1+1+2$ formalism. This consists of evolution equations, propagation equations, mixtures of both, and constraints. Formulas which are useful for splitting $1+3$ equations are given in Appendix A.

A. Evolution equations

We find evolution equations for the $1+1+2$ variables ϕ, ξ, and ζ_{ab} from the projection $u^a R_{abc}$.

$u^a N^{bc} R_{abc}$:

$$\begin{aligned}\dot{\phi} = {}& (\tfrac{2}{3}\theta - \Sigma)(\mathcal{A} - \tfrac{1}{2}\phi) + 2\xi\Omega + \delta_a\alpha^a + \mathcal{A}^a(\alpha_a - a_a) \\ &+ (a^a - \mathcal{A}^a)(\Sigma_a - \varepsilon_{ab}\Omega^b) - \zeta^{ab}\Sigma_{ab} + Q,\end{aligned} \tag{48}$$

$u^a\varepsilon^{bc}R_{abc}$:

$$\begin{aligned}\dot{\xi} = {}& (\tfrac{1}{2}\Sigma - \tfrac{1}{3}\theta)\xi + (\mathcal{A} - \tfrac{1}{2}\phi)\Omega \\ &+ \tfrac{1}{2}(a^a + \mathcal{A}^a)[\Omega_a + \varepsilon_{ab}(\alpha^b + \Sigma^b)] + \tfrac{1}{2}\varepsilon_{ab}\delta^a\alpha^b \\ &- \tfrac{1}{2}\varepsilon_{ca}\zeta^c_b\Sigma^{ab} + \tfrac{1}{2}\mathcal{H},\end{aligned} \tag{49}$$

$u^c R_{c\{ab\}}$:

$$\dot{\zeta}_{\{ab\}} = (\tfrac{1}{2}\Sigma - \tfrac{1}{3}\theta)\zeta_{ab} + \Omega\varepsilon_{c\{a}\zeta_{b\}}{}^c + (\mathcal{A} - \tfrac{1}{2}\phi)\Sigma_{ab} - \xi\varepsilon_{c\{a}\Sigma_{b\}}{}^c - \zeta_{c\{a}\Sigma_{b\}}{}^c + \delta_{\{a}\alpha_{b\}} + (\mathcal{A}_{\{a} - a_{\{a)})\alpha_{b\}} - (\mathcal{A}_{\{a} + a_{\{a)})(\Sigma_{b\}} - \varepsilon_{b\}d}\Omega^d) - \varepsilon_{c\{a}\mathcal{H}_{b\}}{}^c. \quad (50)$$

Then a 1 + 1 + 2 decomposition of the standard 1 + 3 evolution equations gives us the remaining evolution equations, which cannot be found from R_{abc}.

Vorticity evolution equation:

$$\dot{\Omega} = \tfrac{1}{2}\varepsilon_{ab}\delta^a\mathcal{A}^b + \mathcal{A}\xi + \Omega(\Sigma - \tfrac{2}{3}\theta) + \Omega_a(\Sigma^a + \alpha^a). \quad (51)$$

Shear evolution:

$$\dot{\Sigma}_{\{ab\}} = \delta_{\{a}\mathcal{A}_{b\}} + \mathcal{A}_{\{a}\mathcal{A}_{b\}} - \Sigma_{\{a}[\Sigma_{b\}} + 2\alpha_{b\}}] - \Omega_{\{a}\Omega_{b\}} + \mathcal{A}\zeta_{ab} - (\tfrac{2}{3}\theta + \tfrac{1}{2}\Sigma)\Sigma_{ab} - \Sigma_{c\{a}\Sigma_{b\}}{}^c - \mathcal{E}_{ab} + \tfrac{1}{2}\Pi_{ab}. \quad (52)$$

B. Mixture of propagation and evolution

$u^a n^b R_{ab\bar{c}} = n^a u^b R_{ab\bar{c}}$:

$$\hat{\alpha}_{\bar{a}} - \dot{a}_{\bar{a}} = -(\tfrac{1}{2}\phi + \mathcal{A})\alpha_a - \xi\varepsilon_{ab}\alpha^b + (\tfrac{1}{3}\theta + \Sigma)(\mathcal{A}_a - a_a) + (\tfrac{1}{2}\phi - \mathcal{A})(\Sigma_a + \varepsilon_{ab}\Omega^b) - \xi(\varepsilon_{ab}\Sigma^b - \Omega_a) + \zeta_{ab}(-\alpha^b + \Sigma^b + \varepsilon^{bc}\Omega_c) + \tfrac{1}{2}Q_a - \varepsilon_{ab}\mathcal{H}^b, \quad (53)$$

$u^a n^b u^c R_{abc} = -n^a u^b u^c R_{abc}$:

$$\hat{\mathcal{A}} - \tfrac{1}{3}\dot{\theta} - \dot{\Sigma} = -\mathcal{A}^2 + (\tfrac{1}{3}\theta + \Sigma)^2 - 2\alpha_a\Sigma^a + \Sigma_a\Sigma^a - \Omega_a\Omega^a - a_a\mathcal{A}^a + \varepsilon_{ab}\alpha^a\Omega^b + \tfrac{1}{6}(\mu + 3p - 2\Lambda) + \mathcal{E} - \tfrac{1}{2}\Pi. \quad (54)$$

Raychaudhuri equation:

$$\hat{\mathcal{A}} - \dot{\theta} = -\delta_a\mathcal{A}^a - (\mathcal{A} + \phi)\mathcal{A} + (a_a - \mathcal{A}_a)\mathcal{A}^a + \tfrac{1}{3}\theta^2 + \tfrac{3}{2}\Sigma^2 - 2\Omega^2 + 2\Sigma_a\Sigma^a - 2\Omega_a\Omega^a + \Sigma_{ab}\Sigma^{ab} + \tfrac{1}{2}(\mu + 3p) - \Lambda. \quad (55)$$

Vorticity evolution:

$$\dot{\Omega}_{\bar{a}} + \tfrac{1}{2}\varepsilon_{ab}\hat{\mathcal{A}}^b = -(\tfrac{2}{3}\theta + \tfrac{1}{2}\Sigma)\Omega_a + \Omega(\Sigma_a - \alpha_a) + \tfrac{1}{2}\xi\mathcal{A}_a + \tfrac{1}{2}\varepsilon_{ab}[-\mathcal{A}a^b + \delta^b\mathcal{A} - \tfrac{1}{2}\phi\mathcal{A}^b] - \tfrac{1}{2}\varepsilon_{ab}\zeta^{bc}\mathcal{A}_c + \Sigma_{ab}\Omega^b. \quad (56)$$

Shear evolution:

$$\dot{\Sigma} - \tfrac{2}{3}\hat{\mathcal{A}} = \tfrac{1}{3}(2\mathcal{A} - \phi)\mathcal{A} - (\tfrac{2}{3}\theta + \tfrac{1}{2}\Sigma)\Sigma - \tfrac{2}{3}\Omega^2 - \tfrac{1}{3}\delta_a\mathcal{A}^a + \Sigma_a[2\alpha^a - \tfrac{1}{3}\Sigma^a] - \tfrac{1}{3}\mathcal{A}_a[2a^a - \mathcal{A}^a] + \tfrac{1}{3}\Omega_a\Omega^a + \tfrac{1}{3}\Sigma_{ab}\Sigma^{ab} - \mathcal{E} + \tfrac{1}{2}\Pi \quad (57)$$

$$\dot{\Sigma}_{\bar{a}} - \tfrac{1}{2}\hat{\mathcal{A}}_{\bar{a}} = \tfrac{1}{2}\delta_a\mathcal{A} + (\mathcal{A} - \tfrac{1}{4}\phi)\mathcal{A}_a - (\tfrac{2}{3}\theta + \tfrac{1}{2}\Sigma)\Sigma_a + \tfrac{1}{2}\mathcal{A}a_a - \tfrac{3}{2}\Sigma\alpha_a - \Omega\Omega_a - \tfrac{1}{2}(\xi\varepsilon_{ab} + \zeta_{ab})\mathcal{A}^b + \Sigma_{ab}(\alpha^b - \Sigma^b) - \mathcal{E}_a + \tfrac{1}{2}\Pi_a. \quad (58)$$

Energy conservation:

$$\dot{\mu} + \hat{Q} = -\delta_a Q^a - \theta(\mu + p) - (\phi + 2\mathcal{A})Q - \tfrac{3}{2}\Sigma\Pi + (a_a - 2\mathcal{A}_a)Q^a - 2\Sigma_a\Pi^a - \Sigma_{ab}\Pi^{ab}. \quad (59)$$

Momentum conservation:

$$\dot{Q} + \hat{p} + \hat{\Pi} = -\delta_a\Pi^a - (\tfrac{3}{2}\phi + \mathcal{A})\Pi - (\tfrac{4}{3}\theta + \Sigma)Q - (\mu + p)\mathcal{A} + (\alpha_a - \Sigma_a + \varepsilon_{ab}\Omega^b)Q^a + (2a_a - \mathcal{A}_a)\Pi^a + \zeta_{ab}\Pi^{ab} \quad (60)$$

$$\dot{Q}_{\bar{a}} + \hat{\Pi}_{\bar{a}} = -\delta_a p + \tfrac{1}{2}\delta_a\Pi - \delta^b\Pi_{ab} - Q(\alpha_a + \Sigma_a + \varepsilon_{ab}\Omega^b) - \tfrac{3}{2}\Pi a_a - (\tfrac{4}{3}\theta - \tfrac{1}{2}\Sigma)Q_a + \Omega\varepsilon_{ab}Q^b - (\tfrac{3}{2}\phi + \mathcal{A})\Pi_a + \xi\varepsilon_{ab}\Pi^b - (\mu + p - \tfrac{1}{2}\Pi)\mathcal{A}_a - \Sigma_{ab}Q^b - \zeta_{ab}\Pi^b + \Pi_{ab}(a^b - \mathcal{A}^b). \quad (61)$$

Electric Weyl evolution:

$$\dot{\mathcal{E}} + \tfrac{1}{2}\dot{\Pi} + \tfrac{1}{3}\hat{Q} = +\varepsilon_{ab}\delta^a\mathcal{H}^c + \tfrac{1}{6}\delta_a Q^a + (\tfrac{3}{2}\Sigma - \theta)\mathcal{E} - \tfrac{1}{2}(\tfrac{1}{3}\theta + \tfrac{1}{2}\Sigma)\Pi + \tfrac{1}{3}(\tfrac{1}{2}\phi - 2\mathcal{A})Q + 3\xi\mathcal{H} - \tfrac{1}{2}(\mu + p)\Sigma + (2\alpha_a + \Sigma_a - \varepsilon_{ab}\Omega^b)\mathcal{E}^a + (\alpha_a - \tfrac{1}{6}\Sigma_a - \tfrac{1}{2}\varepsilon_{ab}\Omega^b)\Pi^a + \tfrac{1}{3}(a_a + \mathcal{A}_a)Q^a + 2\varepsilon_{ab}\mathcal{A}^a\mathcal{H}^c - \Sigma_{ab}(\mathcal{E}^{ab} + \tfrac{1}{2}\Pi^{ab}) + \varepsilon_{ab}\mathcal{H}^{bc}\zeta^a{}_c \quad (62)$$

$$\dot{\mathcal{E}}_{\bar{a}} + \tfrac{1}{2}\varepsilon_{ab}\hat{\mathcal{H}}^b + \tfrac{1}{2}\dot{\Pi}_{\bar{a}} + \tfrac{1}{4}\hat{Q}_{\bar{a}} = \tfrac{3}{4}\varepsilon_{ab}\delta^b\mathcal{H} + \tfrac{1}{2}\varepsilon_{bc}\delta^b\mathcal{H}^c{}_a - \tfrac{1}{4}\delta_a Q - \tfrac{1}{2}(\mu + p - \tfrac{3}{2}\mathcal{E} + \tfrac{1}{4}\Pi)\Sigma_a + \tfrac{3}{4}(\mathcal{E} + \tfrac{1}{2}\Pi)\varepsilon_{ab}\Omega^b - \tfrac{1}{2}Q\mathcal{A}_a + \tfrac{3}{2}\mathcal{H}\varepsilon_{ab}\mathcal{A}^b - \tfrac{3}{2}(\mathcal{E} + \tfrac{1}{2}\Pi)\alpha_a - \tfrac{1}{4}Qa_a - \tfrac{3}{4}\mathcal{H}\varepsilon_{ab}a^b + (\tfrac{3}{4}\Sigma - \theta)\mathcal{E}_a - \tfrac{1}{2}\Omega\varepsilon_{ab}\mathcal{E}^b + \tfrac{5}{2}\xi\mathcal{H}_a - (\tfrac{1}{4}\phi + \mathcal{A})\varepsilon_{ab}\mathcal{H}^b + \tfrac{1}{2}(\tfrac{1}{4}\phi - \mathcal{A})Q_a + \tfrac{1}{4}\xi\varepsilon_{ab}Q^b - \tfrac{1}{2}(\tfrac{1}{3}\theta + \tfrac{1}{4}\Sigma)\Pi_a - \tfrac{1}{4}\Omega\varepsilon_{ab}\Pi^b + \tfrac{1}{2}\Sigma_{ab}(3\mathcal{E}^b - \tfrac{1}{2}\Pi^b) + \tfrac{1}{2}(3\mathcal{E}_{ab} - \tfrac{1}{2}\Pi_{ab})\Sigma^b - (\mathcal{E}_{ab} + \tfrac{1}{2}\Pi_{ab})(\alpha^b + \tfrac{1}{2}\varepsilon^{bc}\Omega_c) + \tfrac{1}{2}\zeta_{ab}(\varepsilon^{bc}\mathcal{H}_c + Q^b) - \mathcal{H}_{ab}\varepsilon^{bc}\mathcal{A}_c \quad (63)$$

$$\dot{\mathcal{E}}_{\{ab\}} - \varepsilon_{c\{a}\hat{\mathcal{H}}_{b\}}{}^c + \tfrac{1}{2}\dot{\Pi}_{\{ab\}} = -\varepsilon_{c\{a}\delta^c\mathcal{H}_{b\}} - \tfrac{1}{2}\delta_{\{a}Q_{b\}} - \tfrac{1}{2}(\mu + p + 3\mathcal{E} - \tfrac{1}{2}\Pi)\Sigma_{ab} - \tfrac{1}{2}Q\zeta_{ab} - \tfrac{3}{2}\mathcal{H}\varepsilon_{c\{a}\zeta_{b\}}{}^c - (\theta + \tfrac{3}{2}\Sigma)\mathcal{E}_{ab} + \Omega\varepsilon_{c\{a}\mathcal{E}_{b\}}{}^c - (\tfrac{1}{6}\theta - \tfrac{1}{4}\Sigma)\Pi_{ab} + \tfrac{1}{2}\Omega\varepsilon_{c\{a}\Pi_{b\}}{}^c + \xi\mathcal{H}_{ab} + (\tfrac{1}{2}\phi + 2\mathcal{A})\varepsilon_{c\{a}\mathcal{H}_{b\}}{}^c - \mathcal{A}_{\{a}Q_{b\}} - (\alpha_{\{a} + \tfrac{1}{2}\varepsilon_{c\{a}\Omega^c)(2\mathcal{E}_{b\}} + \Pi_{b\}}) + \Sigma_{\{a}(3\mathcal{E}_{b\}} - \tfrac{1}{2}\Pi_{b\}}) + 2\varepsilon_{c\{a}\mathcal{H}_{b\}}(a^c - \mathcal{A}^c) + \Sigma_{c\{a}(3\mathcal{E}_{b\}}{}^c - \tfrac{1}{2}\Pi_{b\}}{}^c) + \varepsilon_{c\{a}\mathcal{H}_{b\}d}\zeta^{cd}. \quad (64)$$

Magnetic Weyl evolution:

$$\dot{\mathcal{H}} = -\varepsilon_{ab}\delta^a\mathcal{E}^b + \tfrac{1}{2}\varepsilon_{ab}\delta^a\Pi^b - 3\xi\mathcal{E} + (\tfrac{3}{2}\Sigma - \theta)\mathcal{H} + \Omega Q + \tfrac{3}{2}\xi\Pi - 2\varepsilon_{ab}\mathcal{A}^a\mathcal{E}^b + (2\alpha_a + \Sigma_a - \varepsilon_{ab}\Omega^b)\mathcal{H}^a - \tfrac{1}{2}(\Omega_a + \varepsilon_{ab}\Sigma^b)Q^a - \Sigma_{ab}\mathcal{H}^{ab} - \tfrac{1}{2}\varepsilon_{ab}\mathcal{E}^{bc}\zeta^a{}_c \quad (65)$$

$$\dot{\mathcal{H}}_{\bar{a}} - \tfrac{1}{2}\varepsilon_{ab}\hat{\mathcal{E}}^b + \tfrac{1}{4}\varepsilon_{ab}\hat{\Pi}^b = -\tfrac{3}{4}\varepsilon_{ab}\delta^b\mathcal{E} + \tfrac{3}{8}\varepsilon_{ab}\delta^b\Pi - \tfrac{1}{2}\varepsilon_{bc}\delta^b\mathcal{E}^c{}_a + \tfrac{1}{4}\varepsilon_{bc}\delta^b\Pi^c{}_a + \tfrac{3}{4}\mathcal{H}\Sigma_a + \tfrac{1}{4}Q\varepsilon_{ab}\Sigma^b + \tfrac{3}{4}Q\Omega_a + \tfrac{3}{4}\mathcal{H}\varepsilon_{ab}\Omega^b - \tfrac{3}{2}\mathcal{E}\varepsilon_{ab}\mathcal{A}^b - \tfrac{3}{2}\mathcal{H}\alpha_a + \tfrac{3}{4}(\mathcal{E} - \tfrac{1}{2}\Pi)\varepsilon_{ab}a^b - \tfrac{5}{2}\xi\mathcal{E}_a + (\tfrac{1}{4}\phi + \mathcal{A})\varepsilon_{ab}\mathcal{E}^b + (\tfrac{3}{4}\Sigma - \theta)\mathcal{H}_a - \tfrac{1}{2}\Omega\varepsilon_{ab}\mathcal{H}^b + \tfrac{3}{4}\Omega Q_a - \tfrac{3}{8}\Sigma\varepsilon_{ab}Q^b + \tfrac{5}{4}\xi\Pi_a - \tfrac{1}{8}\phi\varepsilon_{ab}\Pi^b + \Sigma_{ab}(\tfrac{3}{2}\mathcal{H}^b + \tfrac{1}{4}\varepsilon^{bc}Q_c) + \tfrac{3}{2}\varepsilon_{ab}\zeta^{bc}(\mathcal{E}_c - \tfrac{1}{2}\Pi_c + \tfrac{2}{3}\mathcal{A}_c) + \mathcal{H}_{ab}(\alpha^b + \tfrac{3}{2}\Sigma^b - \tfrac{1}{2}\varepsilon^{bc}\Omega_c) \quad (66)$$

$$\dot{\mathcal{H}}_{\{ab\}} + \varepsilon_{c\{a}\hat{\mathcal{E}}_{b\}}{}^c - \tfrac{1}{2}\varepsilon_{c\{a}\hat{\Pi}_{b\}}{}^c = \varepsilon_{c\{a}\delta^c\mathcal{E}_{b\}} - \tfrac{1}{2}\varepsilon_{c\{a}\delta^c\Pi_{b\}} - \tfrac{3}{2}\mathcal{H}\Sigma_{ab} + \tfrac{1}{2}Q\varepsilon_{c\{a}\Sigma_{b\}}{}^c + \tfrac{3}{2}(\mathcal{E} - \tfrac{1}{2}\Pi)\varepsilon_{c\{a}\zeta_{b\}}{}^c - \xi\mathcal{E}_{ab} - (\tfrac{1}{2}\phi + 2\mathcal{A})\varepsilon_{c\{a}\mathcal{E}_{b\}}{}^c - (\theta + \tfrac{3}{2}\Sigma)\mathcal{H}_{ab} - \Omega\varepsilon_{c\{a}\mathcal{H}_{b\}}{}^c + \tfrac{1}{2}\xi\Pi_{ab} + \tfrac{1}{4}\phi\varepsilon_{c\{a}\Pi_{b\}}{}^c + \Sigma_{\{a}(3\mathcal{H}_{b\}} - \varepsilon_{b\}c}Q^c) + \Omega_{\{a}\tfrac{3}{2}Q_{b\}} - \varepsilon_{b\}c}H^c) - 2\alpha_{\{a}\mathcal{H}_{b\}} + \mathcal{E}_{\{a}2\varepsilon_{b\}c}(a^c + \mathcal{A}^c) - \Pi_{\{a}\varepsilon_{b\}c}a^c + 3\Sigma_{c\{a}\mathcal{H}_{b\}}{}^c - \varepsilon_{c\{a}\zeta^{cd}(\mathcal{E}_{b\}d} - \tfrac{1}{2}\Pi_{b\}d}). \quad (67)$$

C. Propagation equations

Propagation and constraint equations are formed from either projecting R_{abc} as indicated, or from projections of the 1 + 3 constraint equations, denoted C_i, as given in [1].

$n^aN^{bc}R_{abc}$:

$$\hat{\phi} = -\tfrac{1}{2}\phi^2 + 2\xi^2 + (\tfrac{1}{3}\theta + \Sigma)(\tfrac{2}{3}\theta - \Sigma) + \delta_a a^a - a_a a^a - \zeta_{ab}\zeta^{ab} + 2\varepsilon_{ab}\alpha^a\Omega^b - \Sigma_a\Sigma^a + \Omega_a\Omega^a - \tfrac{2}{3}(\mu + \Lambda) - \tfrac{1}{2}\Pi - \mathcal{E}, \quad (68)$$

$n^a\varepsilon^{bc}R_{abc}$:

$$\hat{\xi} = -\phi\xi + (\tfrac{1}{3}\theta + \Sigma)\Omega + \tfrac{1}{2}\varepsilon_{ab}\delta^a a^b + \tfrac{1}{2}\varepsilon_{ab}\Sigma^a a^b + (\tfrac{1}{2}a_a + \alpha_a)\Omega^a \quad (69)$$

$n^aR_{a\{bc\}}$:

$$\hat{\zeta}_{\{ab\}} = -\phi\zeta_{ab} - \zeta^c{}_{\{a}\zeta_{b\}c} + \delta_{\{a}a_{b\}} - a_{\{a}a_{b\}} + 2\alpha_{\{a}\varepsilon_{b\}c}\Omega^c - \Omega_{\{a}\Omega_{b\}} - \Sigma_{\{a}\Sigma_{b\}} + (\tfrac{1}{3}\theta + \Sigma)\Sigma_{ab} - \tfrac{1}{2}\Pi_{ab} - \mathcal{E}_{ab}. \quad (70)$$

Shear divergence $(C_1)^a n_a$:

$$\hat{\Sigma} - \tfrac{2}{3}\hat{\theta} = -\tfrac{3}{2}\phi\Sigma - 2\xi\Omega - \delta_a\Sigma^a - \varepsilon_{ab}\delta^a\Omega^b + 2\Sigma_a a^a - 2\varepsilon_{ab}\mathcal{A}^a\Omega^b + \Sigma_{ab}\zeta^{ab} - Q \quad (71)$$

and $(C_1)_{\bar{a}}$:

$$\hat{\Sigma}_{\bar{a}} - \varepsilon_{ab}\hat{\Omega}^b = \tfrac{1}{2}\delta_a\Sigma + \tfrac{2}{3}\delta_a\theta - \varepsilon_{ab}\delta^b\Omega - \tfrac{3}{2}\phi\Sigma_a + \xi\varepsilon_{ab}\Sigma^b - \xi\Omega_a + (\phi + 2\mathcal{A})\varepsilon_{ab}\Omega^b - \tfrac{3}{2}\Sigma a_a + \Omega\varepsilon_{ab}[a^b - 2\mathcal{A}^b] - \delta^b\Sigma_{ab} - \zeta_{ab}\Sigma^b + \Sigma_{ab}a^b + \varepsilon_{ab}\zeta^{bc}\Omega_c - Q_a. \tag{72}$$

Vorticity divergence equation (C_2):

$$\hat{\Omega} = -\delta_a\Omega^a + (\mathcal{A} - \phi)\Omega + (a_a + \mathcal{A}_a)\Omega^a \tag{73}$$

$(C_3)_{\{ab\}}$:

$$\hat{\Sigma}_{\{ab\}} = \delta_{\{a}\Sigma_{b\}} - \varepsilon_{c\{a}\delta^c\Omega_{b\}} - \tfrac{1}{2}\phi\Sigma_{ab} + \xi\varepsilon_{c\{a}\Sigma_{b\}}{}^c + \tfrac{3}{2}\Sigma\zeta_{ab} - \Omega\varepsilon_{c\{a}\zeta_{b\}}{}^c - 2\Sigma_{\{a}a_{b\}} - 2\varepsilon_{c\{a}\mathcal{A}^c\Omega_{b\}} - \Sigma_{c\{a}\zeta_{b\}}{}^c - \varepsilon_{c\{a}\mathcal{H}_{b\}}{}^c. \tag{74}$$

Electric Weyl divergence $(C_4)^a n_a$:

$$\hat{\mathcal{E}} - \tfrac{1}{3}\hat{\mu} + \tfrac{1}{2}\hat{\Pi} = -\delta_a\mathcal{E}^a - \tfrac{1}{2}\delta_a\Pi^a - \tfrac{3}{2}\phi(\mathcal{E} + \tfrac{1}{2}\Pi) + (\tfrac{1}{2}\Sigma - \tfrac{1}{3}\theta)Q + 3\Omega\mathcal{H} + (2\mathcal{E}_a + \Pi_a)a^a + \tfrac{1}{2}\Sigma_a Q^a + 3\Omega_a\mathcal{H}^a - \tfrac{3}{2}\varepsilon_{ab}\Omega^a Q^b + \varepsilon_{ab}\Sigma^{ac}\mathcal{H}_c{}^b + (\mathcal{E}_{ab} + \tfrac{1}{2}\Pi_{ab})\zeta^{ab} \tag{75}$$

$(C_4)_{\bar{a}}$:

$$\hat{\mathcal{E}}_{\bar{a}} + \tfrac{1}{2}\hat{\Pi}_{\bar{a}} = \tfrac{1}{2}\delta_a\mathcal{E} + \tfrac{1}{3}\delta_a\mu + \tfrac{1}{4}\delta_a\Pi - \delta^b\mathcal{E}_{ab} - \tfrac{1}{2}\delta^b\Pi_{ab} + \tfrac{1}{2}Q\Sigma_a + \mathcal{H}\varepsilon_{ab}\Sigma^b - \tfrac{3}{2}\mathcal{H}\Omega_a - \tfrac{3}{2}Q\varepsilon_{ab}\Omega^b - \tfrac{3}{2}(\mathcal{E} + \tfrac{1}{2}\Pi)a_a - \tfrac{3}{2}\phi(\mathcal{E}_a + \tfrac{1}{2}\Pi_a) + \xi\varepsilon_{ab}(\mathcal{E}^b + \tfrac{1}{2}\Pi^b) + 3\Omega\mathcal{H}_a - \Sigma\varepsilon_{ab}\mathcal{H}^b - (\tfrac{1}{3}\theta + \tfrac{1}{4}\Sigma)Q_a + \tfrac{3}{2}\Omega\varepsilon_{ab}Q^b + \tfrac{1}{2}\Sigma_{ab}Q^b - \zeta_{ab}(\mathcal{E}^b + \tfrac{1}{2}\Pi^b) + (\mathcal{E}_{ab} + \tfrac{1}{2}\Pi_{ab})a^b + 3\mathcal{H}_{ab}\Omega^b. \tag{76}$$

Magnetic Weyl divergence $(C_5)^a n_a$:

$$\hat{\mathcal{H}} = -\delta_a\mathcal{H}^a - \tfrac{1}{2}\varepsilon_{ab}\delta^a Q^b - \tfrac{3}{2}\phi\mathcal{H} - (3\mathcal{E} + \mu + p - \tfrac{1}{2}\Pi)\Omega - Q\xi + 2\mathcal{H}_a a^a - 3\Omega_a(\mathcal{E}^a - \tfrac{1}{6}\Pi^a) + \zeta_{ab}\mathcal{H}^{ab} - \varepsilon_{ab}\Sigma^a{}_c(\mathcal{E}^{bc} + \tfrac{1}{2}\Pi^{bc}) \tag{77}$$

$(C_5)_{\bar{a}}$:

$$\hat{\mathcal{H}}_{\bar{a}} - \tfrac{1}{2}\varepsilon_{ab}\hat{Q}^b = \tfrac{1}{2}\delta_a\mathcal{H} - \delta^b\mathcal{H}_{ab} - \tfrac{1}{2}\varepsilon_{ab}\delta^b Q - \tfrac{3}{2}(\mathcal{E} + \tfrac{1}{2}\Pi)\varepsilon_{ab}\Sigma^b - (-\tfrac{3}{2}\mathcal{E} + \mu + p + \tfrac{1}{4}\Pi)\Omega_a - \tfrac{3}{2}\mathcal{H}a_a + Q\varepsilon_{ab}a^b - 3\Omega\mathcal{E}_a + \tfrac{3}{2}\Sigma\varepsilon_{ab}\mathcal{E}^b - \tfrac{3}{2}\phi\mathcal{H}_a + \xi\varepsilon_{ab}\mathcal{H}^b - \tfrac{1}{2}\xi Q_a + \tfrac{1}{4}\phi\varepsilon_{ab}Q^b + \tfrac{1}{2}\Omega\Pi_a + \tfrac{3}{4}\Sigma\varepsilon_{ab}\Pi^b + \mathcal{H}_{ab}a^b - \zeta_{ab}\mathcal{H}^b - 3(\mathcal{E}_{ab} - \tfrac{1}{6}\Pi_{ab})\Omega^b + \tfrac{1}{2}\varepsilon_{ab}\zeta^{bc}Q_c. \tag{78}$$

D. Constraint

$\varepsilon^{ab}u^c R_{abc}$:

$$\delta_a\Omega^a + \varepsilon_{ab}\delta^a\Sigma^b = (2\mathcal{A} - \phi)\Omega - 3\xi\Sigma + \varepsilon_{ab}\zeta^{ac}\Sigma^b{}_c + \mathcal{H} \tag{79}$$

$N^{bc}R_{\bar{a}bc}$:

$$\tfrac{1}{2}\delta_a\phi - \varepsilon_{ab}\delta^b\xi - \delta^b\zeta_{ab} = -2\xi\varepsilon_{ab}a^b - \Omega(\Omega_a + \varepsilon_{ab}\Sigma^b - 2\varepsilon_{ab}\alpha^b) - (\tfrac{1}{3}\theta - \tfrac{1}{2}\Sigma)(\Sigma_a - \varepsilon_{ab}\Omega^b) - (\Sigma^b - \varepsilon^{bc}\Omega_c)\Sigma_{ab} - \tfrac{1}{2}\Pi_a - \mathcal{E}_a. \tag{80}$$

From $(C_3)_{ab}n^b$ and $(C_1)_{\bar{a}}$, or $n^a u^c R_{a\bar{b}c}$

$$\delta_a\Sigma - \tfrac{2}{3}\delta_a\theta + 2\varepsilon_{ab}\delta^b\Omega + 2\delta^b\Sigma_{ab} = -\phi(\Sigma_a - \varepsilon_{ab}\Omega^b) - 2\xi(\Omega_a - 3\varepsilon_{ab}\Sigma^b) - 4\Omega\varepsilon_{ab}\mathcal{A}^b + 2\zeta_{ab}\Sigma^b + 2\varepsilon_{ab}\zeta^{bc}\Omega_c + \Sigma_{ab}a^b - 2\varepsilon_{ab}\mathcal{H}^b - Q_a. \tag{81}$$

Finally, we note that the equation formed from $(C_3)_{ab}n^a n^b$ is equivalent to Eqs. (79) and (109).

It is worth noting that one of Eqs. (54), (55), and (57) is redundant since (54) $= \tfrac{1}{3}$(55)–(57). Also, note that there are no evolution equations for $\mathcal{A}$, $\mathcal{A}_a$, α_a, and there is no propagation equation for a_a; these all must be determined by specifying a choice of frame.

E. Maxwell's equations

For completeness we also give the decomposition of Maxwell's equations, previously given in [10]. We decompose the electric and magnetic field vectors as

$$E^a = \mathscr{E}n^a + \mathscr{E}^a, \tag{82}$$

$$B^a = \mathscr{B}n^a + \mathscr{B}^a, \tag{83}$$

while the 3-current may be written as

$$j^a = \mathcal{J} n^a + \mathcal{J}^a. \quad (84)$$

Maxwell's equations then become

$$\hat{\mathcal{E}} + \delta_a \mathcal{E}^a = -\phi \mathcal{E} + \mathcal{E}_a a^a + 2\Omega \mathcal{B} + 2\Omega^a \mathcal{B}_a + \mu_0 \rho_e, \quad (85)$$

$$\hat{\mathcal{B}} + \delta_a \mathcal{B}^a = -\phi \mathcal{B} + \mathcal{E}_a a^a - 2\Omega \mathcal{E} - 2\Omega^a \mathcal{E}_a, \quad (86)$$

$$\dot{\mathcal{E}} - \varepsilon_{ab}\delta^a \mathcal{B}^b = 2\xi \mathcal{B} + \mathcal{E}^a \alpha_a - (\tfrac{2}{3}\theta - \Sigma)\mathcal{E} + \Sigma^a \mathcal{E}_a + \varepsilon_{ab}(\mathcal{A}^a \mathcal{B}^b + \Omega^a \mathcal{E}^b) - \mu_0 \mathcal{J}, \quad (87)$$

$$\dot{\mathcal{B}} + \varepsilon_{ab}\delta^a \mathcal{E}^b = -2\xi \mathcal{E} + \mathcal{B}^a \alpha_a - (\tfrac{2}{3}\theta - \Sigma)\mathcal{B} + \Sigma^a \mathcal{B}_a - \varepsilon_{ab}(\mathcal{A}^a \mathcal{E}^b - \Omega^a \mathcal{B}^b), \quad (88)$$

$$\begin{aligned}\dot{\mathcal{E}}_{\bar{a}} + \varepsilon_{ab}(\hat{\mathcal{B}}^b - \delta^b \mathcal{B}) = {} & \xi \mathcal{B}_a - (\tfrac{1}{2}\phi + \mathcal{A})\varepsilon_{ab}\mathcal{B}^b \\ & - (\tfrac{2}{3}\theta + \tfrac{1}{2}\Sigma)\mathcal{E}_a - \Omega \varepsilon_{ab}\mathcal{E}^b \\ & + \mathcal{E}(-\alpha_a + \Sigma_a + \varepsilon_{ab}\Omega^b) \\ & + \mathcal{B}\varepsilon_{ab}(\mathcal{A}^b - a^b) + \Sigma_{ab}\mathcal{E}^b \\ & - \varepsilon_{ab}\zeta^{bc}\mathcal{B}_c - \mu_0 \mathcal{J}_a, \end{aligned} \quad (89)$$

$$\begin{aligned}\dot{\mathcal{B}}_{\bar{a}} - \varepsilon_{ab}(\hat{\mathcal{E}}^b - \delta^b \mathcal{E}) = {} & -\xi \mathcal{E}_a + (\tfrac{1}{2}\phi + \mathcal{A})\varepsilon_{ab}\mathcal{E}^b \\ & - (\tfrac{2}{3}\theta + \tfrac{1}{2}\Sigma)\mathcal{B}_a - \Omega \varepsilon_{ab}\mathcal{B}^b \\ & + \mathcal{B}(-\alpha_a + \Sigma_a + \varepsilon_{ab}\Omega^b) \\ & - \mathcal{E}\varepsilon_{ab}(\mathcal{A}^b - a^b) + \Sigma_{ab}\mathcal{B}^b \\ & + \varepsilon_{ab}\zeta^{bc}\mathcal{E}_c. \end{aligned} \quad (90)$$

Here, SI units are used (μ_0), and ρ_e is the charge density. The first two equations arise from the constraint Maxwell equations (ME), while the rest are the evolution ME. In flat space in the absence of currents and charges the rhs of these equations vanish (for a static "natural" choice of frame). Thus, gravity modifies ME in the form of generalized currents. Note how the rotation terms ξ, Ω, and Ω^a flip the parities of the electromagnetic fields.

IV. PERTURBATIONS OF SPHERICALLY SYMMETRIC AND LRS SPACETIMES

The utility of the approach presented here is that for LRS spacetimes, for which all quantities are rotationally symmetric about a preferred spatial direction (i.e., they admit a one-dimensional isotropy group), all the nonzero 1 + 1 + 2 variables are scalars. This direction may be specified, for example, by a nondegenerate eigenvector of the electric Weyl tensor, or by the vorticity vector. A full discussion of LRS spacetimes in the covariant approach is given in [7]; see their Table 1 for a summary of the different cases which can occur, in a notation similar to that presented here.

The fact that background quantities are scalars in LRS spacetimes means that under linear perturbations, all vector and tensor quantities are automatically gauge invariant, by the Stewart-Walker lemma [17]. We shall now give an overview of how to set up the perturbation equations.

In the background, which we shall take as a general LRS spacetime, all vector and tensor equations are automatically zero, resulting in the set

$$\dot{\phi} = (\tfrac{2}{3}\theta - \Sigma)(\mathcal{A} - \tfrac{1}{2}\phi) + 2\xi\Omega + Q, \quad (91)$$

$$\dot{\xi} = (\tfrac{1}{2}\Sigma - \tfrac{1}{3}\theta)\xi + (\mathcal{A} - \tfrac{1}{2}\phi)\Omega + \tfrac{1}{2}\mathcal{H}, \quad (92)$$

$$\dot{\Omega} = +\mathcal{A}\xi + \Omega(\Sigma - \tfrac{2}{3}\theta), \quad (93)$$

$$\hat{\mathcal{A}} - \dot{\theta} = -(\mathcal{A} + \phi)\mathcal{A} + \tfrac{1}{3}\theta^2 + \tfrac{3}{2}\Sigma^2 - 2\Omega^2 + \tfrac{1}{2}(\mu + 3p) - \Lambda, \quad (94)$$

$$\dot{\Sigma} - \tfrac{2}{3}\hat{\mathcal{A}} = \tfrac{1}{3}(2\mathcal{A} - \phi)\mathcal{A} - (\tfrac{2}{3}\theta + \tfrac{1}{2}\Sigma)\Sigma - \tfrac{2}{3}\Omega^2 - \mathcal{E} + \tfrac{1}{2}\Pi, \quad (95)$$

$$\dot{\mu} + \hat{Q} = -\theta(\mu + p) - (\phi + 2\mathcal{A})Q - \tfrac{3}{2}\Sigma\Pi, \quad (96)$$

$$\dot{Q} + \hat{p} + \hat{\Pi} = -(\tfrac{3}{2}\phi + \mathcal{A})\Pi - (\tfrac{4}{3}\theta + \Sigma)Q - (\mu + p)\mathcal{A}, \quad (97)$$

$$\dot{\mathcal{E}} + \tfrac{1}{2}\dot{\Pi} + \tfrac{1}{3}\hat{Q} = +(\tfrac{3}{2}\Sigma - \theta)\mathcal{E} - \tfrac{1}{2}(\tfrac{1}{3}\theta + \tfrac{1}{2}\Sigma)\Pi + \tfrac{1}{3}(\tfrac{1}{2}\phi - 2\mathcal{A})Q + 3\xi\mathcal{H} - \tfrac{1}{2}(\mu + p)\Sigma, \quad (98)$$

$$\dot{\mathcal{H}} = -3\xi\mathcal{E} + (\theta + \tfrac{3}{2}\Sigma)\mathcal{H} + \Omega Q + \tfrac{3}{2}\xi\Pi, \quad (99)$$

$$\hat{\phi} = -\tfrac{1}{2}\phi^2 + 2\xi^2 + (\tfrac{1}{3}\theta + \Sigma)(\tfrac{2}{3}\theta - \Sigma) - \tfrac{2}{3}(\mu + \Lambda) - \tfrac{1}{2}\Pi - \mathcal{E}, \quad (100)$$

$$\hat{\xi} = -\phi\xi + (\tfrac{1}{3}\theta + \Sigma)\Omega, \quad (101)$$

$$\hat{\Sigma} - \tfrac{2}{3}\hat{\theta} = -\tfrac{3}{2}\phi\Sigma - 2\xi\Omega - Q, \quad (102)$$

$$\hat{\Omega} = +(\mathcal{A} - \phi)\Omega, \quad (103)$$

$$\hat{\mathcal{E}} - \tfrac{1}{3}\hat{\mu} + \tfrac{1}{2}\hat{\Pi} = -\tfrac{3}{2}\phi(\mathcal{E} + \tfrac{1}{2}\Pi) + (\tfrac{1}{2}\Sigma - \theta)Q + 3\Omega\mathcal{H}, \quad (104)$$

$$\hat{\mathcal{H}} = -\tfrac{3}{2}\phi\mathcal{H} - (3\mathcal{E} + \mu + p - \tfrac{1}{2}\Pi)\Omega - Q\xi, \quad (105)$$

$$0 = (2\mathcal{A} - \phi)\Omega - 3\xi\Sigma + \mathcal{H}. \quad (106)$$

These equations were first presented in this form in [12] for LRS class II models (which satisfy $\xi = \Omega = 0 \Rightarrow \mathcal{H} = 0$), and were shown to be consistent with the commutation relation (34).

It is perhaps easier to think of these in matrix form. Let

$$\boldsymbol{X} = \begin{pmatrix} \phi \\ \theta \\ \Sigma \\ \mathcal{A} \\ \Omega \\ \xi \\ \mathcal{E} \\ \mathcal{H} \\ \mu \\ p \\ Q \\ \Pi \end{pmatrix} \quad (107)$$

be the column matrix of all nonzero scalar quantities. Depending on the LRS model in question $\boldsymbol{X}$ will not be this big. For example, for the Schwarzschild solution we have just $\boldsymbol{X} = (\phi, \mathcal{A}, \mathcal{E})^T$. Then, in general, this system of equations may be cast in the form

$$\boldsymbol{\alpha}\dot{\boldsymbol{X}} + \boldsymbol{\beta}\hat{\boldsymbol{X}} = \boldsymbol{\Gamma}\boldsymbol{X} + \boldsymbol{X}^T\boldsymbol{\Delta}\boldsymbol{X}, \quad (108)$$

where the constant matrices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\Gamma}$, $\boldsymbol{\Delta}$ may be read off from the above equations.

We can now set up the perturbative procedure schematically as follows:

(1) *Find a complete set of gauge-invariant perturbation variables.*—This may be achieved by defining

$$\boldsymbol{\Psi}_a = \delta_a \boldsymbol{X}; \quad (109)$$

i.e., by taking angular derivatives of the background variables we find a new set of gauge-invariant variables. The remaining GI variables are all the 1 + 1 + 2 vectors and tensors: $\boldsymbol{\chi}_a = (\mathcal{E}_a, a_a, \ldots)$, $\boldsymbol{\chi}_{ab} = (\zeta_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}, \ldots)$, which obey linearized versions of the above 1 + 1 + 2 equations. Under perturbations Eq. (108) becomes

$$\boldsymbol{\alpha}\dot{\boldsymbol{X}} + \boldsymbol{\beta}\hat{\boldsymbol{X}} = \boldsymbol{\Gamma}\boldsymbol{X} + \boldsymbol{X}^T\boldsymbol{\Delta}\boldsymbol{X} + \boldsymbol{A}\delta^a\boldsymbol{\chi}_a + \boldsymbol{B}\varepsilon_{ab}\delta^a\boldsymbol{\chi}^b, \quad (110)$$

where the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ have constant coefficients. Evolution and propagation equations for the new GI variables $\boldsymbol{\Psi}_a$ may be found by taking the angular derivative of Eq. (110), and using the commutation relations (35) and (36), giving

$$\begin{aligned}\boldsymbol{\alpha}\dot{\boldsymbol{\Psi}}_a + \boldsymbol{\beta}\hat{\boldsymbol{\Psi}}_a &= [\boldsymbol{\Gamma} + (\tfrac{1}{2}\Sigma - \tfrac{1}{3}\theta - \tfrac{1}{2}\phi)\boldsymbol{\alpha}]\boldsymbol{\Psi}_a \\ &\quad - (\Omega\boldsymbol{\alpha} + \xi\boldsymbol{\beta})\varepsilon_{ab}\boldsymbol{\Psi}^b + \boldsymbol{X}^T\boldsymbol{\Delta}\boldsymbol{\Psi}_a \\ &\quad + \boldsymbol{\Psi}_a^T\boldsymbol{\Delta}\boldsymbol{X} + \boldsymbol{A}\delta_a\delta^b\boldsymbol{\chi}_b \\ &\quad + \boldsymbol{B}\varepsilon_{bc}\delta_a\delta^b\boldsymbol{\chi}^c. \end{aligned} \quad (111)$$

These equations replace the corresponding system (110) in the 1 + 1 + 2 equations.

(2) *Harmonic analysis.*—Two parities of harmonics may be introduced, generalizing the axial and polar modes for spherical symmetry. These were first defined in [8,12], and are discussed in Appendix B. These are analogous to the scalar-vector-tensor decomposition in Friedman-Lemaître-Robertson-Walker models. After this, all variables become scalars, which are functions of two affine parameters associated with u^a and n^a.

(3) *Master variables.*—At this stage the governing system of equations is linear in the perturbation variables $\boldsymbol{\Phi}$ and $\bar{\boldsymbol{\Phi}}$, which are the column vectors containing all the even and odd harmonically decomposed variables, and splits into two parities. We then have two linear systems of equations looking like

$$\boldsymbol{\gamma}\dot{\boldsymbol{\Phi}} + \boldsymbol{\lambda}\hat{\boldsymbol{\Phi}} = \boldsymbol{\Xi}\boldsymbol{\Phi}, \quad (112)$$

where $\boldsymbol{\Xi}$ is a matrix with coefficients depending only on the background parameters (as well as the harmonic index k), and $\boldsymbol{\gamma}$, $\boldsymbol{\lambda}$ are constant matrices. The true degrees of freedom of this system will be governed by a reduced set of frame independent master variables, which will obey a closed set of wave equations. Finding these can be tricky. All other variables are related to the master variables by quadrature, plus frame degrees of freedom. See [8] for the full details in the Schwarzschild case.

These are the key steps required given a particular LRS model is chosen. Steps 1 and 2 are algorithmic; step 3 can be very difficult.

V. SUMMARY

We have presented a new semi-tetrad approach to analyzing Einstein's field equations. By introducing a single spacelike vector into the 1 + 3 approach we decomposed the 1 + 3 equations into a system of evolution, propagation, and constraint equations. These were supplemented by a 1 + 1 + 2 decomposition of the Ricci equations for the spatial vector. Although presented in restricted form elsewhere, the full system was presented here for the first time.

A key feature of the approach is that under a complete decomposition all objects are covariantly defined scalars, 2-vectors in the sheet and transverse-traceless 2-tensors, also in the sheet. In an LRS spacetime, provided the spatial vector is chosen appropriately, all the vectors and tensors vanish, leaving just scalars. Under perturbations all indexed objects are first-order ensuring that there are no tensorial products; this ensures that we can introduce natural harmonic functions on the background which remove all tensorial properties of the equations. Finally, we are left with a system of gauge-invariant and covariant first-order partial differential equations to manipulate. The solution of this system provides the solution of the perturbation problem.

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Note added in proof.—After this work was submitted, the articles in Ref. [18] appeared which discuss using this 1 + 1 + 2 approach for perturbations of LRS spacetimes, and elaborate in an elegant way some of the issues discussed in Sec. IV.

APPENDIX A: USEFUL RELATIONS FOR DECOMPOSING EQUATIONS

Given any 1 + 3 vectors and tensors, we may decompose them as

$$x^a = Xn^a + X^a, \tag{A1}$$

$$y^a = Yn^a + Y^a, \tag{A2}$$

$$\psi_{ab} = \psi_{\langle ab\rangle} = \Psi(n_a n_b - \tfrac{1}{2}N_{ab}) + 2\Psi_{(a}n_{b)} + \Psi_{ab}, \tag{A3}$$

$$\phi_{ab} = \phi_{\langle ab\rangle} = \Phi(n_a n_b - \tfrac{1}{2}N_{ab}) + 2\Phi_{(a}n_{b)} + \Phi_{ab}. \tag{A4}$$

Then we have the following expansions from 1 + 3 quantities → 1 + 1 + 2 variables:

$$x_a x^a = X^2 + X_a X^a, \tag{A5}$$

$$\eta_{abc}x^b y^c = (\varepsilon_{bc}X^b Y^c)n_a + \varepsilon_{ab}(YX^b - XY^b), \tag{A6}$$

$$x_{\langle a}y_{b\rangle} = \tfrac{1}{3}(2XY - X_c Y^c)(n_a n_b - \tfrac{1}{2}N_{ab}) + [XY_{(a} + YX_{(a}]n_{b)} + X_{\{a}Y_{b\}}, \tag{A7}$$

$$\psi_{ab}x^b = (X\Psi + X_b\Psi^b)n_a - \tfrac{1}{2}\Psi X_a + X\Psi_a + \Psi_{ab}X^b, \tag{A8}$$

$$\eta_{cd\langle a}x^c\psi_{b\rangle}{}^d = \varepsilon_{cd}X^c\Psi^d(n_a n_b - \tfrac{1}{2}N_{ab}) + [(X\Psi^c - \tfrac{3}{2}\Psi X^c)\varepsilon_{c(a} + \varepsilon_{cd}X^c\Psi^d{}_{(a}]n_{b)} + X\varepsilon_{c\{a}\Psi_{b\}}{}^c - X^c\varepsilon_{c\{a}\Psi_{b\}}, \tag{A9}$$

$$\psi_{ab}\psi^{ab} = \tfrac{3}{2}\Psi^2 + 2\Psi_a\Psi^a + \Psi_{ab}\Psi^{ab}, \tag{A10}$$

$$\psi_{c\langle a}\phi_{b\rangle}{}^c = (\tfrac{1}{2}\Psi\Phi + \tfrac{1}{3}\Psi_c\Phi^c - \tfrac{1}{3}\Psi_{cd}\Phi^{cd})(n_a n_b - \tfrac{1}{2}N_{ab}) + [\tfrac{1}{2}\Psi\Phi_{(a} + \tfrac{1}{2}\Phi\Psi_{(a} + \Psi^c\Phi_{c(a} + \Phi^c\Psi_{c(a}]n_{b)} - \tfrac{1}{2}\Psi\Phi_{ab} - \tfrac{1}{2}\Phi\Psi_{ab} + \Psi_a\Phi_b + \Psi_{ca}\Phi_{b\}}{}^c, \tag{A11}$$

$$\eta_{abc}\psi^b{}_d\phi^{dc} = n_a\varepsilon_{bc}\Psi^b{}_d\Phi^{dc} + \tfrac{3}{2}\varepsilon_{ab}(\Phi\Psi^b - \Psi\Phi^b). \tag{A12}$$

For 1 + 3 derivatives we find

$$\dot{x}_{\langle a\rangle} = (\dot{X} - X_b\alpha^b)n_a + X\alpha_a + \dot{X}_{\bar{a}}, \tag{A13}$$

$$\dot{\psi}_{\langle ab\rangle} = (\dot{\Psi} - 2\Psi_c\alpha^c)n_a n_b - \tfrac{1}{2}\dot{\Psi}N_{ab} + [3\Psi\alpha_{(a} + 2\dot{\Psi}_{(\bar{a}} - 2\alpha^c\Psi_{c(a}]n_{b)} + 2\Psi_{(a}\alpha_{b)} + \dot{\Psi}_{\{ab\}}, \tag{A14}$$

$$\mathrm{D}_a x^a = \hat{X} + X\phi - X_a a^a + \delta_a X^a, \tag{A15}$$

$$\eta_{abc}\mathrm{D}^b x^c = (2X\xi + \varepsilon_{bc}\delta^b X^c)n_a + \xi X_a + \varepsilon_{ab}[-Xa^b + \delta^b X - \hat{X}^b - \tfrac{1}{2}\phi X^b - \zeta^{bc}X_c], \tag{A16}$$

$$\mathrm{D}_{\langle a}x_{b\rangle} = \tfrac{1}{3}[2\hat{X} - \phi X - 2X_c a^c - \delta_c X^c](n_a n_b - \tfrac{1}{2}N_{ab}) + [Xa_{(a} + \delta_{(a}X + \hat{X}_{(\bar{a}} - \tfrac{1}{2}\phi X_{(a} + X^c(\xi\varepsilon_{c(a} - \zeta_{c(a)})]n_{b)} + X\zeta_{ab} + \delta_{\{a}X_{b\}}, \tag{A17}$$

$$\mathrm{D}^b\psi_{ab} = (\hat{\Psi} + \tfrac{3}{2}\phi\Psi - 2\Psi_b a^b + \delta_b\Psi^b - \Psi_{bc}\zeta^{bc})n_a + \hat{\Psi}_{\bar{a}} + \tfrac{3}{2}\phi\Psi_a + \tfrac{3}{2}\Psi a_a - \tfrac{1}{2}\delta_a\Psi - \Psi_{ab}a^b + [-\xi\varepsilon_{ab} + \zeta_{ab}]\Psi^b + \delta^b\Psi_{ab}, \tag{A18}$$

$$\eta_{cd\langle a}\mathrm{D}^c\psi_{b\rangle}{}^d = (3\xi\Psi + \varepsilon_{cd}\delta^c\Psi^d - \varepsilon_{cd}\Psi^{de}\zeta^c{}_e) \times (n_a n_b - \tfrac{1}{2}N_{ab}) + \{[-\tfrac{3}{2}\delta^c\Psi + \tfrac{3}{2}\Psi a^c + \hat{\Psi}^c + \tfrac{1}{2}\phi\Psi^c + 2\Psi_d\zeta^{cd}]\varepsilon_{c(a} + 5\xi\Psi_{(a} + \varepsilon^{cd}[\Psi_d\zeta_{c(a} + \delta_c\Psi_{d(a)}]\}n_{b)} - \varepsilon_{c\{a}\delta^c\Psi_{a\}} + 2\varepsilon_{c\{a}a^c\Psi_{b\}} + \varepsilon_{c\{a}\hat{\Psi}^c{}_{b\}} + \tfrac{1}{2}\phi\varepsilon_{c\{a}\Psi^c{}_{b\}} - \tfrac{3}{2}\Psi\varepsilon_{c\{a}\zeta^c{}_{b\}} + \xi\Psi_{ab} + \varepsilon_{c\{a}\Psi_{b\}d}\zeta^{cd}. \tag{A19}$$

Given any relation in 1 + 3 notation, these relations may be substituted directly to aid decomposition.

APPENDIX B: HARMONIC FUNCTIONS

We introduce dimensionless harmonic functions Q, defined on any LRS background, as eigenfunctions of the 2-dimensional Laplace-Beltrami operator:

$$\delta^2 Q = -\frac{k^2}{r^2} Q, \qquad \hat{Q} = 0 = \dot{Q} \qquad (0 \le k^2). \tag{B1}$$

The function r is, up to an irrelevant constant, covariantly defined by

$$\frac{\hat{r}}{r} \equiv \frac{1}{2}\phi, \qquad \frac{\dot{r}}{r} \equiv \frac{1}{3}\theta - \frac{1}{2}\Sigma, \qquad \delta_a r \equiv 0. \tag{B2}$$

While we have not chosen a specific basis for Q, we can now expand any first-order scalar ψ in terms of these functions schematically as

$$\psi = \sum_k \psi_{\mathsf{S}}^{(k)} Q^{(k)} = \psi_{\mathsf{S}} Q, \tag{B3}$$

where the sum (or integral) over k is implicit in the last equality. The S subscript reminds us that ψ is a scalar, and that a harmonic expansion has been made.

We also need to expand vectors in harmonics. We therefore define the *even* (electric) parity vector harmonics as

$$Q_a^{(k)} = r\delta_a Q^{(k)} \Rightarrow \hat{Q}_{\bar{a}} = 0 = \dot{Q}_{\bar{a}}, \qquad \delta^2 Q_a = (1-k^2) r^{-2} Q_a; \tag{B4}$$

where the (k) superscript is implicit, and we define *odd* (magnetic) parity vector harmonics as

$$\bar{Q}_a^{(k)} = r\varepsilon_{ab}\delta^b Q^{(k)} \Rightarrow \hat{\bar{Q}}_{\bar{a}} = 0 = \dot{\bar{Q}}_{\bar{a}}, \qquad \delta^2 \bar{Q}_a = (1-k^2) r^{-2} \bar{Q}_a. \tag{B5}$$

Note that $\bar{Q}_a = \varepsilon_{ab} Q^b \Leftrightarrow Q_a = -\varepsilon_{ab}\bar{Q}^b$, so that ε_{ab} is a parity operator. The crucial difference between these two types of vector harmonics is that $\bar{Q}_a$ is solenoidal, so

$$\delta^a \bar{Q}_a = 0, \tag{B6}$$

while

$$\delta^a Q_a = -k^2 r^{-1} Q. \tag{B7}$$

Note also that

$$\varepsilon_{ab}\delta^a Q^b = 0, \qquad \text{and} \qquad \varepsilon_{ab}\delta^a \bar{Q}^b = +k^2 r^{-1} Q. \tag{B8}$$

The harmonics are orthogonal: $Q^a \bar{Q}_a = 0$ (for each k), which implies that any first-order vector ψ_a can now be written

$$\psi_a = \sum_k \psi_{\mathsf{V}}^{(k)} Q_a^{(k)} + \bar{\psi}_{\mathsf{V}}^{(k)} \bar{Q}_a^{(k)} = \psi_{\mathsf{V}} Q_a + \bar{\psi}_{\mathsf{V}} \bar{Q}_a. \tag{B9}$$

Again, we implicitly assume a sum over k in the last equality, and the V subscript reminds us that ψ_a is a vector expanded in harmonics.

Similarly we define even and odd tensor spherical harmonics as

$$Q_{ab} = r^2 \delta_{\{a}\delta_{b\}} Q \Rightarrow \hat{Q}_{ab} = 0 = \dot{Q}_{ab}, \tag{B10}$$

$$\bar{Q}_{ab} = r^2 \varepsilon_{c\{a}\delta^c \delta_{b\}} Q \Rightarrow \hat{\bar{Q}}_{ab} = 0 = \dot{\bar{Q}}_{ab}, \tag{B11}$$

which are orthogonal: $Q_{ab}\bar{Q}^{ab} = 0$, and are parity inversions of one another: $Q_{ab} = -\varepsilon_{c\{a}\bar{Q}_{b\}}{}^c \Leftrightarrow \bar{Q}_{ab} = \varepsilon_{c\{a}Q_{b\}}{}^c$. Any first-order tensor may be expanded:

$$\Psi_{ab} = \sum_k \Psi_{\mathsf{T}}^{(k)} Q_{ab}^{(k)} + \bar{\Psi}_{\mathsf{T}}^{(k)} \bar{Q}_{ab}^{(k)} = \Psi_{\mathsf{T}} Q_{ab} + \bar{\Psi}_{\mathsf{T}} \bar{Q}_{ab}. \tag{B12}$$

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