

**Braneworld dynamics in Einstein-Gauss-Bonnet gravity**

Hideki Maeda\*

*Centro de Estudios Científicos (CECS), Arturo Prat 514, Valdivia, Chile  
and Department of Physics, International Christian University, 3-10-2 Osawa, Mitaka-shi, Tokyo 181-8585, Japan*

Varun Sahni†

*Inter-University Centre for Astronomy & Astrophysics, Post Bag 4, Pune 411 007, India*

Yuri Shtanov‡

*Bogolyubov Institute for Theoretical Physics, Kiev 03680, Ukraine*

(Received 27 August 2007; published 15 November 2007)

We discuss the cosmological evolution of a braneworld in five-dimensional Gauss-Bonnet gravity. Our discussion allows the fifth (bulk) dimension to be spacelike as well as timelike. The resulting equations of motion have the form of a cubic equation in the  $(H^2, (\rho + \sigma)^2)$  plane, where  $\sigma$  is the brane tension and  $\rho$  is the matter density. This allows us to conduct a comprehensive pictorial analysis of cosmological evolution for the Gauss-Bonnet brane. The many interesting properties of this braneworld include the possibility of accelerated expansion at late times. For a finite region in parameter space the accelerated expansion can be *phantomlike* so that  $w < -1$ . At late times, this branch approaches de Sitter space ( $w = -1$ ) and avoids the *big-rip* singularities usually present in phantom models. For a timelike extra dimension the Gauss-Bonnet brane can bounce and avoid the initial singularity.

DOI: [10.1103/PhysRevD.76.104028](https://doi.org/10.1103/PhysRevD.76.104028)

PACS numbers: 04.50.+h, 98.80.Cq

**I. INTRODUCTION**

Braneworld models of the universe—in which the observable universe is a four-dimensional timelike hypersurface (brane) embedded in a higher-dimensional (bulk) spacetime—have attracted much attention recently. This is partly due to the fact that superstring/M-theory seems to require the existence of extra dimensions and the braneworld approach may be one way of reconciling our (3 + 1)-dimensional universe with these higher-dimensional theories [1–3].

Another reason for the current popularity of the braneworld construct is the fact that brane cosmology is usually accompanied by new features and is therefore, in principle, falsifiable [4,5]. The simplest Randall-Sundrum (RS) braneworld, for instance, gives rise to an evolutionary equation for the brane which differs from standard general relativity (GR) at *early times* [3]. This leads to several interesting consequences. For instance, the very early universe expands as  $H \propto \rho$ , instead of the more familiar  $H \propto \sqrt{\rho}$  in standard cosmology. The changed expansion rate causes a scalar field to experience greater damping, which, in turn, allows inflation to occur for a broader class of initial conditions and potentials [6]. If the fifth dimension is timelike then the universe generically *bounces* and avoids the initial big-bang singularity which plagues standard cosmology [7]. (The behavior of anisotropies in the RS scenario can also be very different from that in standard general relativity [8].)

A complementary approach to braneworld cosmology pioneered by the DGP model [9] allows the universe to accelerate at *late times*, thus providing a geometrical answer to the riddle posed by dark energy. Models which unify the RS and DGP approaches also lead to several new features [5,10]. For instance, (i) the phenomenon of dark energy can be transient so that the universe accelerates for a while before settling back into matter dominated expansion, (ii) the effective equation of state of dark energy can be *phantomlike* ( $w_{\text{eff}} \leq -1$ ), and (iii) new cosmological singularities can arise in such models [11]. Such alternative cosmological models provide reasonable fits to the current cosmological data [12].

In this paper we address the issue of cosmological evolution on a brane in a theory of gravity whose action includes, in addition to the familiar Einstein term, a Gauss-Bonnet contribution. Gauss-Bonnet terms arise naturally in superstring theories [13], and their cosmological effects have been discussed in several papers [14–16]. The present paper deals with this issue in greater generality; we examine both cases: when the bulk dimension is spacelike as well as when it is timelike. We also develop a new pictorial method of analysis which provides qualitative insights into the evolution of the universe in this potentially important new model of gravity.

**II. BASIC EQUATIONS**

We begin with the following  $n$ -dimensional ( $n \geq 5$ ) action:

$$S = \int d^n x \sqrt{-g} \left[ \frac{1}{2\kappa_n^2} (R - 2\Lambda + \alpha L_{\text{GB}}) \right], \quad (1)$$

\*hideki@cecs.cl

†varun@iucaa.ernet.in

‡shtanov@bitp.kiev.ua

where  $R$  is the  $n$ -dimensional Ricci scalar,  $\Lambda$  is the  $n$ -dimensional cosmological constant, and  $\kappa_n := \sqrt{8\pi G_n}$ , where  $G_n$  is the  $n$ -dimensional gravitational constant. The Gauss-Bonnet term  $L_{\text{GB}}$  is a combination of the Ricci scalar, the Ricci tensor  $R_{\mu\nu}$ , and the Riemann tensor  $R^\mu{}_{\nu\rho\sigma}$ :

$$L_{\text{GB}} := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \quad (2)$$

The constant  $\alpha$  in (1) is the coupling constant of the Gauss-Bonnet term, and for  $\alpha \rightarrow 0$  our model reduces to the familiar Randall-Sundrum model [3]. The action (1) can be obtained in the low-energy limit of heterotic superstring theory [13], in which case  $\alpha$  can be regarded as the inverse string tension and is positive definite. We, therefore, assume  $\alpha > 0$  throughout this paper. (We shall explicitly be assuming  $n \geq 5$  since for  $n \leq 4$  the Gauss-Bonnet term is a topological invariant and does not contribute to the field equations.)

The gravitational equations which result from the action (1) are

$$G^\mu{}_\nu + \alpha H^\mu{}_\nu + \Lambda \delta^\mu{}_\nu = 0, \quad (3)$$

where

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \quad (4)$$

---


$$0 = r^2[2\alpha(n-3)(n-4)h - \varepsilon\{r^2 + 2\alpha k(n-3)(n-4)\}] \frac{d^2h}{dr^2} + 2(n-3)r[2\alpha(n-4)(n-5)h - \varepsilon\{r^2 + 2\alpha k(n-4)(n-5)\}] \frac{dh}{dr} + 2\alpha r^2(n-3)(n-4) \left(\frac{dh}{dr}\right)^2 + \alpha(n-3)(n-4)(n-5)(n-6)h^2 - \varepsilon(n-3)(n-4)h[r^2 + 2\alpha k(n-5)(n-6)] - 2\Lambda r^4 + k(n-3)(n-4)r^2 + \alpha k^2(n-3)(n-4)(n-5)(n-6), \quad (7)$$

$$0 = (n-2)r[2\alpha(n-3)(n-4)h - \varepsilon\{r^2 + 2\alpha k(n-3)(n-4)\}] \frac{dh}{dr} + \alpha(n-2)(n-3)(n-4)(n-5)h^2 - \varepsilon(n-2)(n-3)h[r^2 + 2\alpha k(n-4)(n-5)] - 2\Lambda r^4 + k(n-2)(n-3)r^2 + \alpha k^2(n-2)(n-3)(n-4)(n-5), \quad (8)$$

where the former is the  $(i, i)$  component of Eq. (3), while the latter is the  $(t, t)$  or  $(r, r)$  component acting as a constraint. The general solution of these equations is

$$h(r) = \varepsilon k + \frac{r^2}{2(n-3)(n-4)\alpha} \times \left( \varepsilon \mp \sqrt{1 + \frac{\alpha\mu}{r^{n-1}} + \frac{8(n-3)(n-4)}{(n-1)(n-2)}\alpha\Lambda} \right), \quad (9)$$

where  $\mu$  is a constant. Our solution for  $h(r)$  has two branches, which correspond to the two signs in front of the square root in Eq. (9). We call the family with the minus (plus) sign the minus-branch (plus-branch) solution.

- (i) For  $\varepsilon = 1$ , the minus-branch solution has the general-relativistic limit as  $\alpha \rightarrow 0$ , while there is no general-relativistic limit for the plus-branch solution. (The global structures of this solution were presented in [17].)

$$H_{\mu\nu} := 2[RR_{\mu\nu} - 2R_{\mu\alpha}R^\alpha{}_\nu - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} + R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}] - \frac{1}{2}g_{\mu\nu}L_{\text{GB}}. \quad (5)$$

### A. Bulk solution

The  $n$ -dimensional vacuum solution can be obtained as a product manifold  $M^n \approx M^2 \times K^{n-2}$  with the line element

$$ds_n^2 = -h(r)dt^2 + \varepsilon \frac{dr^2}{h(r)} + r^2\gamma_{ij}dx^i dx^j, \quad (6)$$

where  $K^{n-2}$  is an  $(n-2)$ -dimensional space of constant curvature with unit metric  $\gamma_{ij}$ . In the equations which follow,  $k$  denotes the curvature of  $K^{n-2}$  and takes the values 1 (positive curvature), 0 (zero curvature), and  $-1$  (negative curvature). The value of the constant  $\varepsilon$  determines whether the (bulk) fifth dimension is spacelike ( $\varepsilon = 1$ ) or timelike ( $\varepsilon = -1$ ). In the former case,  $M^2$  is a Lorentzian manifold, whereas in the latter case, it is a Euclidean manifold.

The basic equations of the theory under consideration are

- (ii) For  $\varepsilon = -1$ , the plus-branch solution has the general-relativistic limit as  $\alpha \rightarrow 0$ , while the minus-branch solution does not.

Hereafter, we shall be considering a five-dimensional bulk spacetime, for which the metric (6) reduces to

$$ds_5^2 = g_{\mu\nu}dx^\mu dx^\nu = -h(r)dt^2 + \varepsilon \frac{dr^2}{h(r)} + r^2[d\chi^2 + f_k(\chi)^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (10)$$

$$h(r) = \varepsilon k + \frac{r^2}{4\alpha} \left( \varepsilon \mp \sqrt{1 + \frac{\alpha\mu}{r^4} + \frac{4}{3}\alpha\Lambda} \right), \quad (11)$$

where  $f_0(\chi) = \chi$ ,  $f_1(\chi) = \sin\chi$ ,  $f_{-1}(\chi) = \sinh\chi$ , and  $\varepsilon = \pm 1$ .

In this spacetime, there are two classes of singularities when  $\mu \neq 0$ . One is the central singularity at  $r = 0$ , and the other is the *branch singularity* at  $r = r_b := [-\alpha\mu/(1 + 4\alpha\Lambda/3)]^{1/4} > 0$ , when the term inside the square root in Eq. (11) vanishes. The branch singularity exists if  $\mu$  is negative, or if  $1 + 4\alpha\Lambda/3 < 0$  for positive  $\mu$ .

### B. Friedmann equation on the brane

The position of the three-brane is described by the functions  $r = a(\tau)$  and  $t = T(\tau)$  parametrized by the proper time  $\tau$  on the brane. The tangent vector to the brane is written as

$$u^\mu \frac{\partial}{\partial x^\mu} = \dot{T} \frac{\partial}{\partial t} + \dot{a} \frac{\partial}{\partial r}, \quad (12)$$

where a dot denotes the differentiation with respect to  $\tau$ . The normalization condition  $u_\mu u^\mu = -1$  leads to

$$1 = h(a)\dot{T}^2 - \varepsilon \frac{\dot{a}^2}{h(a)}, \quad (13)$$

and the induced metric of the three-brane  $\bar{g}_{ab}$  is given by

$$\begin{aligned} ds_4^2 &= \bar{g}_{ab} dy^a dy^b \\ &= -d\tau^2 + a(\tau)^2 [d\chi^2 + f_k(\chi)^2 (d\theta^2 + \sin^2\theta d\phi^2)]. \end{aligned} \quad (14)$$

The unit normal one-form to the three-brane  $n_\mu$  is given by

$$n_\mu dx^\mu = \dot{a} dt - \dot{T} dr, \quad (15)$$

where  $n_\mu u^\mu = 0$  and  $n_\mu n^\mu = 1/\varepsilon$  are satisfied.

The extrinsic curvature of the three-brane is obtained from  $K_{ab} := n_{\mu;\nu} e_a^\mu e_b^\nu$ , where  $e_a^\mu := \partial x^\mu / \partial y^a$ . We have

$$e_a^0 dy^a = id\tau, \quad e_a^1 dy^a = \dot{a} d\tau, \quad e_a^i dy^a = \delta_j^i dy^j, \quad (16)$$

and

$$K_{ab} = -n_\mu e_{a,b}^\mu - \Gamma_{\mu\nu}^\kappa n_\kappa e_a^\mu e_b^\nu. \quad (17)$$

Then, we obtain the nonzero component of  $K^a_b$  as

$$K_\tau^\tau = -\frac{1}{h\dot{T}} \left( \ddot{a} + \frac{h'}{2\varepsilon} \right), \quad K_j^i = -\frac{h\dot{T}}{\varepsilon a} \delta_j^i, \quad (18)$$

where a prime denotes differentiation with respect to  $a$ .

The junction condition at the brane is given by [15,16]

$$\begin{aligned} [K^a_b]_\pm - \delta^a_b [K]_\pm + 2\alpha(3[J^a_b]_\pm \\ - \delta^a_b [J]_\pm - 2P^a_{dbf} [K^{df}]_\pm) = -\kappa_5^2 \tau^a_b, \end{aligned} \quad (19)$$

where

$$\begin{aligned} J_{ab} := & \frac{1}{3}(2KK_{ad}K^d_b + K_{df}K^{df}K_{ab} \\ & - 2K_{ad}K^{df}K_{fb} - K^2K_{ab}), \end{aligned} \quad (20)$$

$$P_{adbf} := R_{adbf} + 2h_{a[f}R_{b]d} + 2h_{d[b}R_{f]a} + Rh_{a[b}h_{f]d}. \quad (21)$$

The energy-momentum tensor  $\tau^a_b$  on the brane is given by

$$\tau^a_b = \text{diag}(-\rho, p, p, p) + \text{diag}(-\sigma, -\sigma, -\sigma, -\sigma), \quad (22)$$

where  $\rho$  and  $p$  are the energy density and pressure of a perfect fluid on the three-brane, and the constant  $\sigma$  is the brane tension. We have introduced the notation

$$[X]_\pm := X^+ - X^-, \quad (23)$$

where  $X^\pm$  is the quantity  $X$  evaluated either on the + or - side of the brane, and  $P_{adbf}$  is the divergence-free part of the Riemann tensor, i.e.,

$$D_a P^a_{dbf} = 0, \quad (24)$$

where  $D_a$  is the covariant derivative on the brane.

From the  $(\tau, \tau)$  and  $(i, i)$  components of Eqs. (13) and (19), we obtain

$$\begin{aligned} \frac{\kappa_5^4}{36} (\rho + \sigma)^2 &= \left( \frac{h(a)}{a^2} + \varepsilon H^2 \right) \\ &\times \left[ 1 + \frac{4\alpha}{3} \left( \frac{3k - h(a)}{a^2} + (3 - \varepsilon)H^2 \right) \right]^2, \end{aligned} \quad (25)$$

where  $H := \dot{a}/a$ . Here, we have assumed  $Z_2$  symmetry of reflection with respect to the brane. This generalized Friedmann equation reduces to that obtained by Davis [16] for  $\varepsilon = 1$ .

Differentiating Eq. (25) with respect to  $\tau$  and using Eq. (13) and the  $(\tau, \tau)$  and  $(i, i)$  components of Eq. (19), we obtain

$$\dot{\rho} = -3H(p + \rho), \quad (26)$$

which is the energy-conservation equation on the three-brane. Let us assume that the perfect fluid on the three-brane obeys

$$p = (\gamma - 1)\rho, \quad (27)$$

where we assume that the equation of state of matter on the brane lies within the Zeldovich interval  $0 < \gamma \leq 2$  due to the dominant energy condition (equivalently,  $-1 < w \leq 1$ , where  $w := p/\rho = \gamma - 1$  is the equation of state). From Eq. (26), we then obtain

$$\rho = \frac{\rho_0}{a^{3\gamma}}, \quad (28)$$

where  $\rho_0$  is a positive constant, so that  $\rho$  is a monotonically decreasing function of  $a$  for  $\gamma > 0$ .

### C. The Randall-Sundrum limit

In this paper, we shall consider only those solutions of (11) and (25) which possess the general-relativistic limit

since other solutions may describe physically inadmissible evolution of our brane. The minus- and plus-branch solutions in (11) have the general-relativistic limits for  $\varepsilon = 1$  and  $-1$  in (25), respectively.

As mentioned earlier, the action (1) contains the Randall-Sundrum model as a subclass. Setting  $\alpha \rightarrow 0$  in Eq. (25), one gets the generalized RS model

$$\frac{\kappa_5^4}{36}(\rho + \sigma)^2 = \frac{\varepsilon}{a^2} \left( k - \frac{\mu}{8a^2} - \frac{1}{6} \Lambda a^2 \right) + \varepsilon H^2. \quad (29)$$

From this equation and from Eq. (28), we obtain

$$H^2 = \frac{\kappa_5^4}{36\varepsilon} \left( \frac{\rho_0}{a^{3\gamma}} + \sigma \right)^2 - \frac{k}{a^2} + \frac{\mu}{8a^4} + \frac{1}{6} \Lambda. \quad (30)$$

The Randall-Sundrum model corresponds to  $\varepsilon = 1$ , while the dual model with  $\varepsilon = -1$  was discussed in [7]. Permitted values of the expansion factor must clearly satisfy  $H^2 \geq 0$ . An interesting consequence of (30) is the possibility of singularity-free solutions when  $\varepsilon = -1$  [7].

### III. PICTORIAL ANALYSIS OF COSMOLOGICAL EVOLUTION

We saw in the previous section that the evolution equation for the Gauss-Bonnet brane can be quite complicated and, therefore, difficult to analyze analytically. In this section, we present a general method of analysis which allows one to study pictorially the behavior arising from the generic cosmological equation (25).

We notice that Eq. (25), describing the cosmological evolution of the Gauss-Bonnet brane, always has the form of a *cubic curve* in the  $(H^2, \rho_{\text{tot}}^2)$  plane:

$$C\rho_{\text{tot}}^2 = (A \pm H^2)(B + H^2)^2, \quad (31)$$

where  $\rho_{\text{tot}} := \rho + \sigma$ ,  $A$  and  $B$  are functions of  $a$ ,  $C$  is a positive constant, and the  $\pm$  sign corresponds to  $\varepsilon = \pm 1$ . The value of the cosmological constant  $\sigma$  can be positive, negative, or zero. The right-hand side of Eq. (31) has exactly three real zeros in  $H^2$ , two of which coincide, namely,  $(H^2)_1 = \mp A$  and  $(H^2)_{2,3} = -B$ . Only part of this cubic curve lies in the physical domain  $H^2 \geq 0$ ,  $\rho_{\text{tot}}^2 \geq 0$ , and it is in this domain that the evolution of the brane takes place. Consequently, the evolution of our brane universe can be pictured as a point moving along this cubic curve in the physical domain  $\rho_{\text{tot}}^2 \geq 0$ ,  $H^2 \geq 0$ .

This pictorial representation is very useful in appreciating the full gamut of possibilities for cosmic evolution of this brane. For comparison, it is helpful to note that cosmological evolution in GR is described by

$$H^2 = \rho_{\text{tot}} - \frac{k}{a^2}, \quad (32)$$

where we have set the proportionality term  $8\pi G/3$  to unity. Equation (32) describes a *quadratic curve* in the  $(H^2, \rho_{\text{tot}}^2)$  plane. Another example is the Randall-Sundrum brane, which, for every value of  $a$ , is described by a *straight*

*line* in the  $(H^2, \rho_{\text{tot}}^2)$  plane:

$$H^2 = \frac{\rho_{\text{tot}}^2}{\varepsilon} + \frac{\Lambda}{6} - \frac{k}{a^2} + \frac{\mu}{8a^4}, \quad (33)$$

where  $\varepsilon = \pm 1$ , and we have set the term  $\kappa_5^4/36$  in (30) to unity.

Before commencing our discussion on the subtleties of cosmological evolution on the Gauss-Bonnet brane, it will be helpful to first consider the different evolutionary possibilities in a spatially flat universe described by the more familiar general-relativistic equation (32) with  $k = 0$ , where  $\sigma$  acts as a cosmological constant. In this case the expansion of the universe can proceed in three distinct ways, corresponding to the cases  $\sigma > 0$ ,  $\sigma < 0$ , and  $\sigma = 0$ . All three possibilities correspond to motion along the quadratic curve in Fig. 1.

Notice that expansion along the *entire curve* from the top (early times) to the origin (late times) takes place only if  $\sigma \leq 0$ . In the absence of a cosmological constant ( $\sigma = 0$ ) the origin  $(H^2, \rho_{\text{tot}}^2) = (0, 0)$  is reached at  $\tau \rightarrow \infty$ . In contrast, for  $\sigma < 0$ , the origin is reached in a finite interval of time when the matter density has dropped to  $\rho = |\sigma|$ . At this point  $H = 0$ ; in other words, expansion ceases and the universe begins to contract. Evolution thereafter proceeds upward along the same curve—in *reverse* fashion.

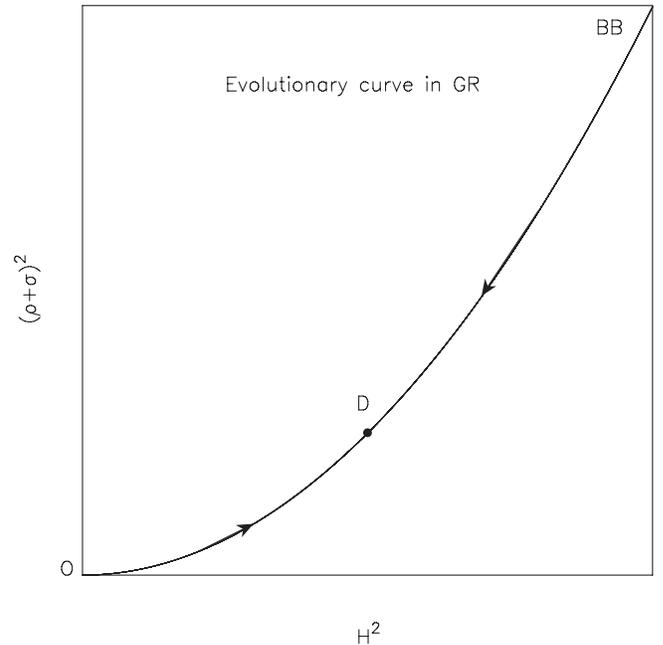


FIG. 1. The evolution of a spatially flat Friedmann-Robertson-Walker universe in GR proceeds along this curve. The downward arrow indicates *expansion* while the upward arrow indicates *contraction*. The latter is only possible if  $\sigma < 0$ . For  $\sigma > 0$  the expansion of the universe terminates at the point  $D$  at which  $\rho = 0$ . At this point, the universe expands exponentially. For  $\sigma = 0$ , the origin ( $H = 0$ ,  $\rho = 0$ ) marks the end point of evolution. The scale of the  $x$  and  $y$  axes is arbitrary.

Finally, if  $\sigma > 0$ , evolution does not proceed all the way to the origin but terminates at some point  $D$  along the curve. At this point,  $\rho \rightarrow 0$  so that  $\rho_{\text{tot}} = \sigma$  and  $H^2 = \sigma$ . The end point  $D$  of evolution corresponds to the universe's asymptotic approach towards de Sitter space. (This, for instance, would be the case for a spatially flat  $\Lambda$ CDM universe which accelerates at late times.)

### A. Spacelike extra dimension ( $\varepsilon = 1$ )

Let us now discuss the evolution on the Gauss-Bonnet brane in greater detail. For a spacelike extra dimension, the cosmological equation (25) has the form

$$C\rho_{\text{tot}}^2 = (A + H^2)(B + H^2)^2, \quad (34)$$

where

$$C := \frac{\kappa_5^2}{36} \left( \frac{3}{8\alpha} \right)^2 > 0, \quad (35)$$

and

$$A := \frac{1}{4\alpha} \left( 1 \mp \sqrt{1 + \frac{\alpha\mu}{a^4} + \frac{4}{3}\alpha\Lambda} \right), \quad (36)$$

$$B := \frac{1}{8\alpha} \left( 2 \pm \sqrt{1 + \frac{\alpha\mu}{a^4} + \frac{4}{3}\alpha\Lambda} \right) = \frac{3}{8\alpha} - \frac{A}{2},$$

in general, are functions of the scale factor  $a$ .

As mentioned earlier, Eq. (34) has the form of a cubic curve in the  $(H^2, \rho_{\text{tot}}^2)$  plane. The two signs in (36) correspond to the two different ways of embedding the brane in the bulk space. In this paper we only consider the upper sign, which has the general-relativistic limit.

As discussed in the previous section, the evolution of the braneworld is described by a point moving along the cubic curve in the  $(H^2, \rho_{\text{tot}}^2)$  plane, in the physical domain  $H^2 \geq 0$ ,  $\rho_{\text{tot}}^2 \geq 0$ , with the parameters of the cubic curve simultaneously changing with time due to the dependence of  $A$

and  $B$  on the scale factor (see below). The evolution can proceed in three distinct ways which are summarized below. All three cases correspond to  $B > 0$  in Eq. (36), and the first two also have  $A < 0$ .

- (1) The behavior of the universe is shown in the left panel of Fig. 2. The point  $P$  corresponds to  $H^2 = -A$ . During the course of expansion, the motion along the curve is initially downwards from the initial big-bang singularity towards  $P$ . However, for  $P$  to be reachable in a finite time interval, the brane tension  $\sigma$  must be negative since only then is  $(\rho + \sigma)^2 = 0$  permitted. The point  $P$  marks a turning point for the evolution along the curve: after this point, the energy density of the universe keeps decreasing while the quantity  $(\rho + \sigma)^2$  is increasing. In the case  $\mu = 0$ , we also have  $\dot{H} = 0$  at the point  $P$ . In this case, the Hubble parameter passes through an inflection point at  $P$ . Since

$$\frac{\ddot{a}}{a} = \dot{H} + H^2, \quad (37)$$

it follows that  $\ddot{a} = aH^2 > 0$  when  $\rho = |\sigma|$ . In other words,  $\dot{H} > 0$  for some length of time during the upward motion along the curve away from  $P$ . Thus the universe *accelerates at late times*. Note that  $\dot{H} = -4\pi G\rho$  in  $\Lambda$ CDM and  $\dot{H} > 0$  is usually associated with a *phantom* equation of state  $w < -1$  in dark-energy models. (This qualitative behavior will take place also for sufficiently small values of  $\alpha\mu/a^4$  reached in the course of expansion in the neighborhood of the point  $P$ , which will make  $A$  almost constant in time.)

The growth in  $H$ , however, cannot continue indefinitely since  $\rho \rightarrow 0$  eventually, and  $(\rho + \sigma)^2 \rightarrow \sigma^2$  (corresponding to the point  $D$ ), which implies  $H^2 \rightarrow \text{const}$ . This means that the universe approaches a de Sitter-like attractor ( $D$ ) at very late times. We there-

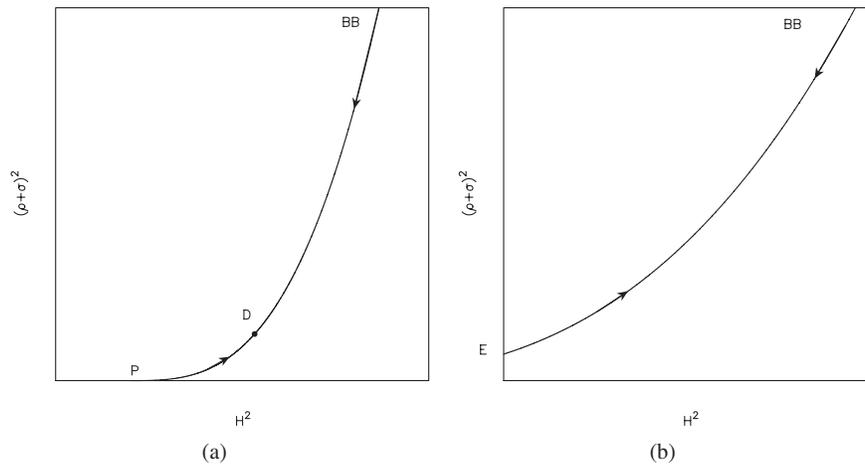


FIG. 2. Spacelike extra dimension:  $A < 0$ ,  $B > 0$  (left panel) and  $A > 0$ ,  $B > 0$  (right panel) in (34). The point  $P$  is the turning point, and the point  $E$  is the point of recollapse.

fore conclude that our Gauss-Bonnet brane can display phantomlike features and super-accelerate at late times, before approaching  $w \rightarrow -1$  in the distant future. Note that the big-rip future singularity (at which  $H \rightarrow \infty$ ) is absent in this case, which is one of the appealing features of this scenario. (Other braneworld models with this property have been discussed in [10].)

- (2) For  $\sigma > 0$ , the point  $P$  can never be reached, and expansion proceeds along  $BB \rightarrow D$ , culminating in de Sitter-like asymptotic expansion at  $D$ .
- (3) The case with  $B > 0$  and  $A > 0$  is shown in the right panel of Fig. 2. The point  $E$  is the point of recollapse. At this point the (spatially flat) universe ceases to expand and begins to contract. The point  $E$  is reachable either if the brane tension is negative, or if it is positive with the value of  $\sigma^2$  lying below the point  $E$ .

One should note that, theoretically, the scale-factor dependent parameter  $A(a)$  can change sign during the course of evolution, so that the curve along which the evolution takes place can continuously evolve from that in the right panel of Fig. 2 to that in its left panel, and vice versa. This introduces an obvious modification to the description of the evolution, which does not change in any significant way.

The complete set of figures showing the  $(H^2, \rho_{\text{tot}}^2)$  plane are shown in Fig. 7 of the Appendix.

**B. Timelike extra dimension ( $\epsilon = -1$ )**

Also in this case, there is only one branch of the generic cosmological equation (25) having the general-relativistic limit which has the form (31), namely,

$$C\rho_{\text{tot}}^2 = (A - H^2)(B + H^2)^2, \tag{38}$$

where

$$C := \frac{\kappa_5^2}{36} \left( \frac{3}{16\alpha} \right)^2 > 0, \tag{39}$$

and

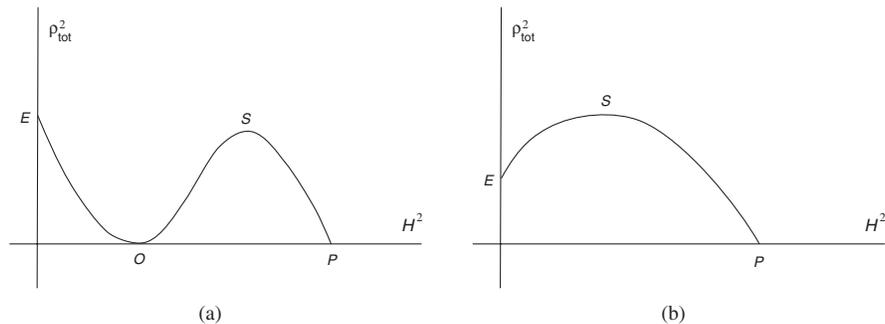


FIG. 3. Timelike extra dimension:  $B < 0$  (left panel) and  $0 < B < 2A$  (right panel). The points  $E$  (bounce or recollapse) and  $P$  are the turning points of the evolution, while the point  $S$  corresponds to a sudden or quiescent singularity.

$$A := \frac{1}{4\alpha} \left( \sqrt{1 + \frac{\alpha\mu}{a^4} + \frac{4}{3}\alpha\Lambda} + 1 \right), \tag{40}$$

$$B := \frac{1}{4\alpha} \left( 1 - \frac{1}{4} \sqrt{1 + \frac{\alpha\mu}{a^4} + \frac{4}{3}\alpha\Lambda} \right) = \frac{3}{16\alpha} - \frac{A}{4}.$$

Clearly, the theory makes sense only for  $A > 0$  (for which the branch singularity does not appear). Then, typically,  $B < 0$ , in which case we obtain the graph shown in the left panel of Fig. 3. The point  $P$  corresponds to  $H^2 = A$ , and the point  $O$  corresponds to  $H^2 = -B$ .

There is also a small range of parameters where  $B > 0$ . For this, we require

$$A < \frac{3}{4\alpha} \quad \text{or} \quad \sqrt{1 + \frac{\alpha\mu}{a^4} + \frac{4}{3}\alpha\Lambda} < 4. \tag{41}$$

A typical graph illustrating the case  $0 < B < 2A$  (equivalently  $1/12\alpha < A < 3/4\alpha$ ) is shown in the right panel of Fig. 3.

The graph corresponding to  $B > 2A$ , which is equivalent to

$$A < \frac{1}{12\alpha} \quad \text{or} \quad \sqrt{1 + \frac{\alpha\mu}{a^4} + \frac{4}{3}\alpha\Lambda} < \frac{4}{3}, \tag{42}$$

is shown in Fig. 4.

The end points  $E$  and  $P$  in all graphs are the reverse points of evolution. The point  $S$  is the position of a sudden “quiescent” singularity of the type described in [11,18]. Indeed, the evolution of the universe cannot be continued beyond this point because the quantity  $\rho_{\text{tot}}^2$  should change in the same direction (decrease), which is physically impossible. Note that the value of  $H$  is finite and nonzero at this point, while  $\dot{H}$  is divergent. [This can easily be seen by writing  $d(H^2)/d\tau = d(H^2)/d\rho_{\text{tot}} \cdot d\rho_{\text{tot}}/d\tau$  where  $d\rho_{\text{tot}}/d\tau = -3H\rho$  and noting that  $d(H^2)/d\rho_{\text{tot}} \rightarrow \infty$  at  $S$ .] The Kretschmann invariant on the brane  $K := R_{abcd}R^{abcd}$  is given by

$$K = 12[H^4 + (\dot{H} + H^2)^2], \tag{43}$$

and diverges as the *quiescent* singularity is approached.

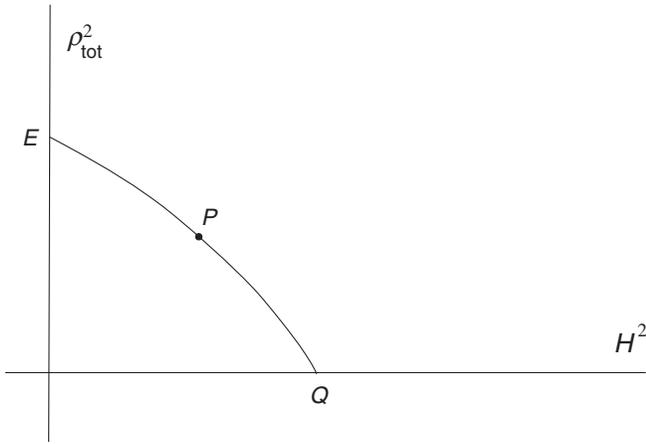


FIG. 4. Bouncing Gauss-Bonnet brane with a timelike extra dimension and  $B > 2A$ . The Gauss-Bonnet brane contracts from a de Sitter-like initial stage at  $P$  to  $Q$  at which  $\rho = |\sigma|$  and  $\rho_{\text{tot}} := \rho + \sigma = 0$  (since  $\sigma < 0$ ). Further contraction takes the universe from  $Q$  to  $E$ , and along this segment both  $\rho$  and  $\rho + \sigma$  increase. At  $E$  the density of the universe has reached its maximum value, while the Hubble parameter has declined to zero. The universe therefore bounces at  $E$ , then reexpands and evolves in reverse fashion along  $E \rightarrow Q \rightarrow P$ . Note that  $P$  marks the beginning and end point of evolution.

Consider first the evolution of the Gauss-Bonnet brane suggested by the left panel in Fig. 3. The big-bang singularity which featured prominently in Fig. 2 has effectively been replaced by the sudden singularity  $S$ . The following three possibilities for evolution immediately suggest themselves:

- (1) Expansion commences at  $S$  and proceeds via  $O$  to  $E$  which marks a turning point at which  $H = 0$ . Thereafter the universe ceases to expand and begins to contract. The contracting trajectory ends (via  $O$ ) at  $S$ . The sudden singularity at  $S$  marks both the beginning and end of evolution in this scenario. (The possibility that quantum effects might modify cosmological evolution in the vicinity of such a singularity has been discussed in [19]; see also [20].)

- (2) The universe *contracts* from the singularity at  $S$  until it reaches  $E$ , where it bounces, then expands via  $O$  to  $S$ .

In both cases, the brane tension must be negative ( $\sigma < 0$ ) since that is a necessary condition for reaching the point  $O$ . In the vicinity of  $\rho = |\sigma|$ , the map  $(\rho + \sigma)^2 \rightarrow \rho$  is bivalued (see Fig. 5), which allows  $(\rho + \sigma)^2$  to increase both when  $\rho$  increases as well as decreases. This ambiguity is responsible for the two possibilities discussed above.

- (3) The trajectory  $S \rightarrow P$  describes a super-accelerating universe expanding from the singularity  $S$ , since it suggests that  $H^2$  increases while  $\rho$  decreases. (In fact,  $\dot{H} \rightarrow \infty$  at the point  $S$ .) If  $\mu = 0$ , then  $\dot{H} > 0$  throughout this phase, and it is unlikely that  $SP$  in this case can describe the real universe. If  $\mu > 0$ ,

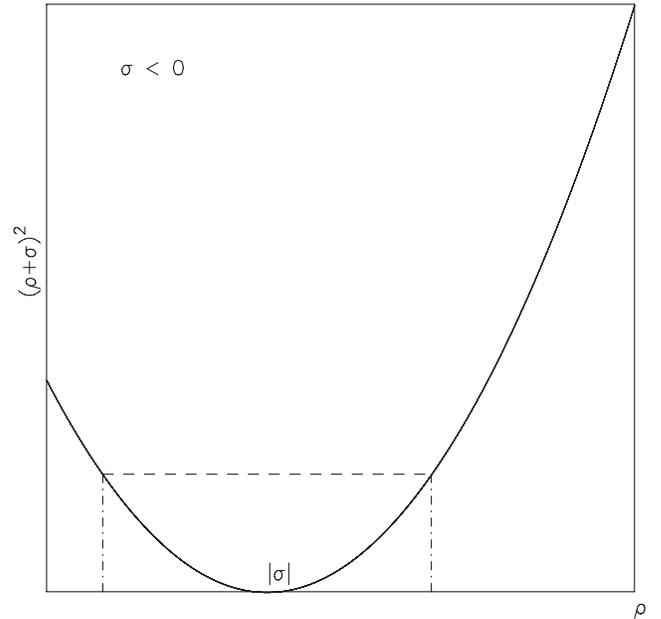


FIG. 5. For negative values of the brane tension ( $\sigma < 0$ ) two values of the matter density  $\rho$  map onto a single value of  $\rho_{\text{tot}} = \rho + \sigma$ , as illustrated in this figure.

then, in the course of the evolution, super-acceleration may be replaced by the “usual” acceleration. If the brane tension is negative, then the point  $P$  is reached, after which the evolution turns back to the  $PS$  path. Then, depending on the value of the brane tension, it either reaches the singularity  $S$  again or asymptotically approaches the de Sitter state at an intermediate point between  $S$  and  $P$ .

Of course, we have described only the main and most interesting possibilities of evolution. We remember that during the evolution the parameters  $A$  and  $B$  describing the cubic curve change their values, and it may happen that the curve changes its shape during the evolution, that some critical points leave the physical domain  $H^2 \geq 0$ ,  $\rho_{\text{tot}}^2 \geq 0$  or, on the contrary, enter this domain. All such possibilities are quite easy to investigate case by case, but we will not do so in this paper.

The complete set of figures showing the  $(H^2, \rho_{\text{tot}}^2)$  plane are shown in Fig. 8 of the Appendix.

### C. Bouncing braneworld

It is clear that the only bouncing point during the universe contraction can be the point  $E$  in Figs. 3(a) and 4. For the bounce to be possible, we need the inverse monotonic dependence between  $H^2$  and  $\rho_{\text{tot}}^2$  in the neighborhood of the bouncing point  $E$ .

In order to better address the issue of a bounce in the Gauss-Bonnet brane, let us first consider this issue within the context of the Randall-Sundrum model (which presents a limiting case of our braneworld). As mentioned earlier, cosmological evolution of the RS brane is described by

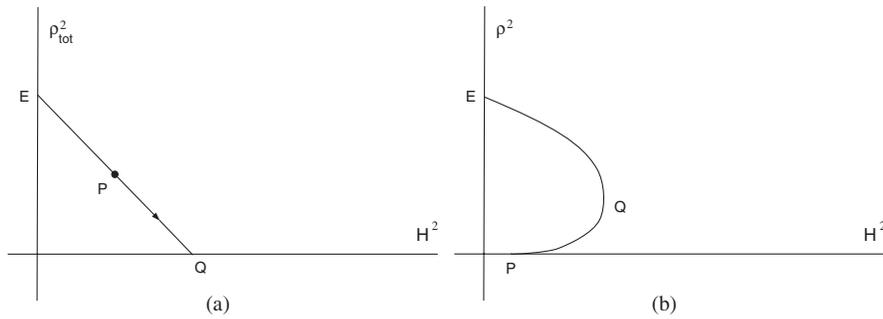


FIG. 6. The RS brane with a timelike extra dimension. Note that the left panel shows  $\rho_{\text{tot}} \equiv (\rho + \sigma)^2$  as a function of  $H^2$ , while the right panel shows  $\rho^2$  as a function of  $H^2$ . Since the map  $(\rho + \sigma)^2 \rightarrow \rho$  is bivalued (see Fig. 5), it is easier to discern the salient features of the bounce in the right panel than in the left.

Eq. (33) which represents a straight line in the  $(H^2, \rho_{\text{tot}}^2)$  plane. We show this line in the left panel of Fig. 6 for a timelike extra dimension ( $\varepsilon = -1$ ).

One can see a close qualitative similarity between the curve on the left panel of this figure and the curve in Fig. 4. The bouncing scenario in Figs. 4 and 6 proceeds as follows: the universe begins to contract from the point  $P$  at which  $\rho = 0$  and  $H = \text{const}$ . In other words, both the starting point and end point of evolution correspond to de Sitter space. An increase in the value of the matter density brings us to the point  $Q$  at which  $\rho = |\sigma|$  and  $\rho_{\text{tot}} := \rho + \sigma = 0$ . (Note that  $\sigma < 0$  is a prerequisite of this model, since otherwise the point  $Q$  cannot be reached.) The universe contracts further from  $Q$  to  $E$ , and in this segment both  $\rho$  and  $\rho + \sigma$  increase. At  $E$  the density has reached a finite maximum value, while the Hubble parameter has declined to zero. The universe therefore bounces at  $E$ , then reexpands and evolves in reverse fashion along  $E \rightarrow Q \rightarrow P$ .

#### IV. DISCUSSION

Braneworld cosmology has attracted considerable interest during the past decade. This is partly due to the fact that such models may play an important role in the low-energy limit of M-theory/string theory. Another reason for the growing interest in brane dynamics is associated with the new features which some of these models possess and which, in turn, can lead to new cosmological predictions and scenarios. Our attempt in this paper has been to develop a completely general qualitative approach to determine the salient features of a brane embedded in a five-dimensional bulk and evolving according to the precepts of Einstein-Gauss-Bonnet gravity. For this purpose we show that the  $(3 + 1)$ -dimensional equations of motion of several popular cosmological models can be depicted as simple curves in the  $(H^2, (\rho + \sigma)^2)$  plane. (Here  $H$  is the Hubble parameter,  $\rho$  the density, and  $\sigma$  the brane tension.)

For instance, the spatially flat Friedmann-Robertson-Walker universe in GR has the form of a *quadratic curve*, while the Randall-Sundrum model describes a *straight line* in the  $(H^2, (\rho + \sigma)^2)$  plane. The Gauss-Bonnet brane, on the other hand, describes a *cubic curve* in the  $(H^2, (\rho +$

$\sigma)^2)$  plane—see Eqs. (31)–(33). This *pictorial depiction of dynamics* permits us to discover the salient features of cosmic evolution very simply. Applying this approach to the Gauss-Bonnet brane, we discover the following interesting properties:

- (1) For a finite region in parameter space the Gauss-Bonnet brane *accelerates* at late times. Acceleration can be *phantomlike* ( $w < -1$ ) but does not lead to the eventual destruction of the universe in a *big-rip* future singularity. Instead, at very late times the expansion of the universe approaches de Sitter space and becomes exponential (i.e.  $w \rightarrow -1$ ). (The possibility that the current expansion of the universe may be phantomlike has evoked much recent interest and discussion; see [10,21] for a nonexhaustive list of papers discussing this issue, and [22] for a summary of recent observational results.)
- (2) The expansion of the universe may commence from or terminate in a “sudden” quiescent singularity, at which the Hubble parameter and the density of matter remain finite but  $\dot{H}$  diverges.
- (3) The universe can evade the initial big-bang singularity and *bounce*. (This possibility is realized if the fifth dimension is timelike.)

Whether any of these properties of the Gauss-Bonnet braneworld is realized in practice is currently an open question which can be answered by the following: (i) a deeper understanding of the embedding of this cosmology within a more fundamental theoretical framework and (ii) a comparison with observations.

#### ACKNOWLEDGMENTS

H. M. would like to thank IUCAA, where this work was conceived and formulated, for warm hospitality. The work of H. M. was supported by Grant No. 1071125 from FONDECYT (Chile) and the Grant-in-Aid for Young Scientists (B), No. 18740162, from the Scientific Research Fund of the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. CECS is funded in part by an institutional grant from the

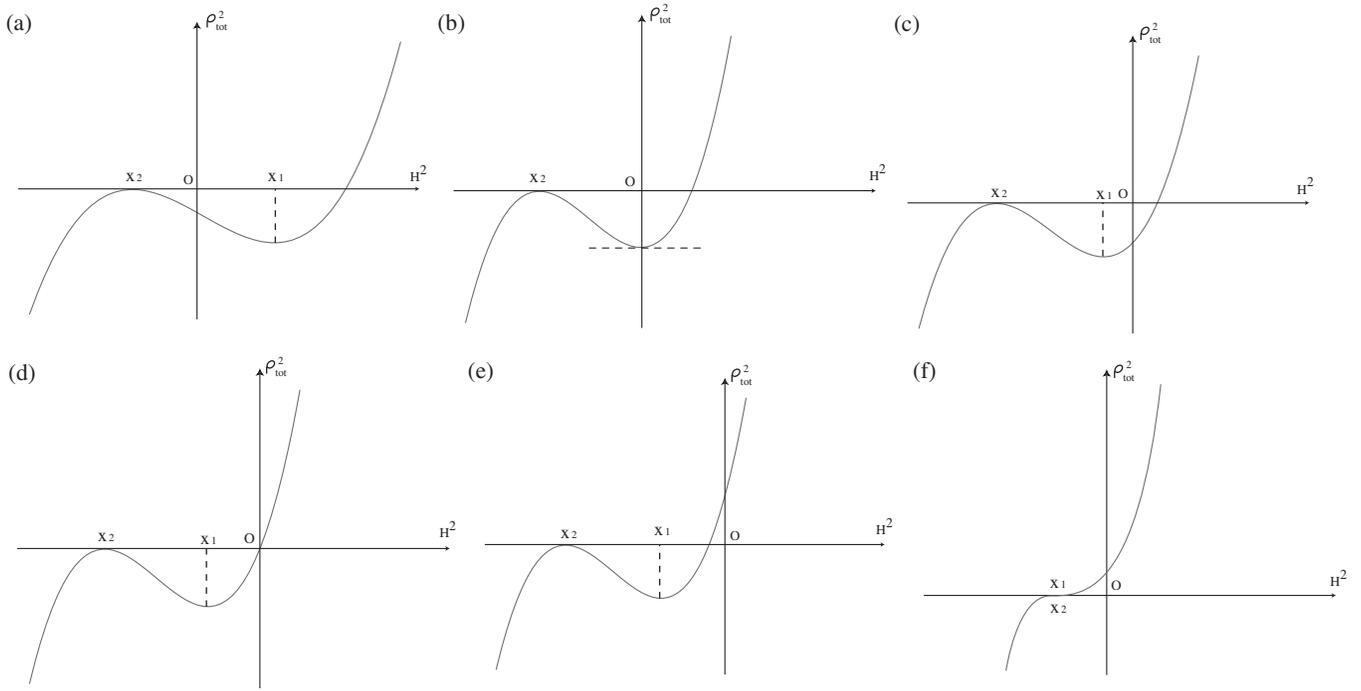


FIG. 7. The  $(H^2, \rho_{\text{tot}}^2)$  plane is shown for the GB brane with a spacelike extra dimension for the following values of the parameters  $A, B$  in (36) with the upper sign: (a)  $A < -B/2$ , (b)  $A = -B/2$ , (c)  $-B/2 < A < 0$ , (d)  $A = 0$ , (e)  $0 < A \neq B$ , and (f)  $A = B$  (at the branch singularity). Here,  $x_1 := -(B + 2A)/3$  and  $x_2 := -B$ . Note that the region with  $H^2 < 0$  or  $\rho_{\text{tot}}^2 < 0$  is nonphysical.

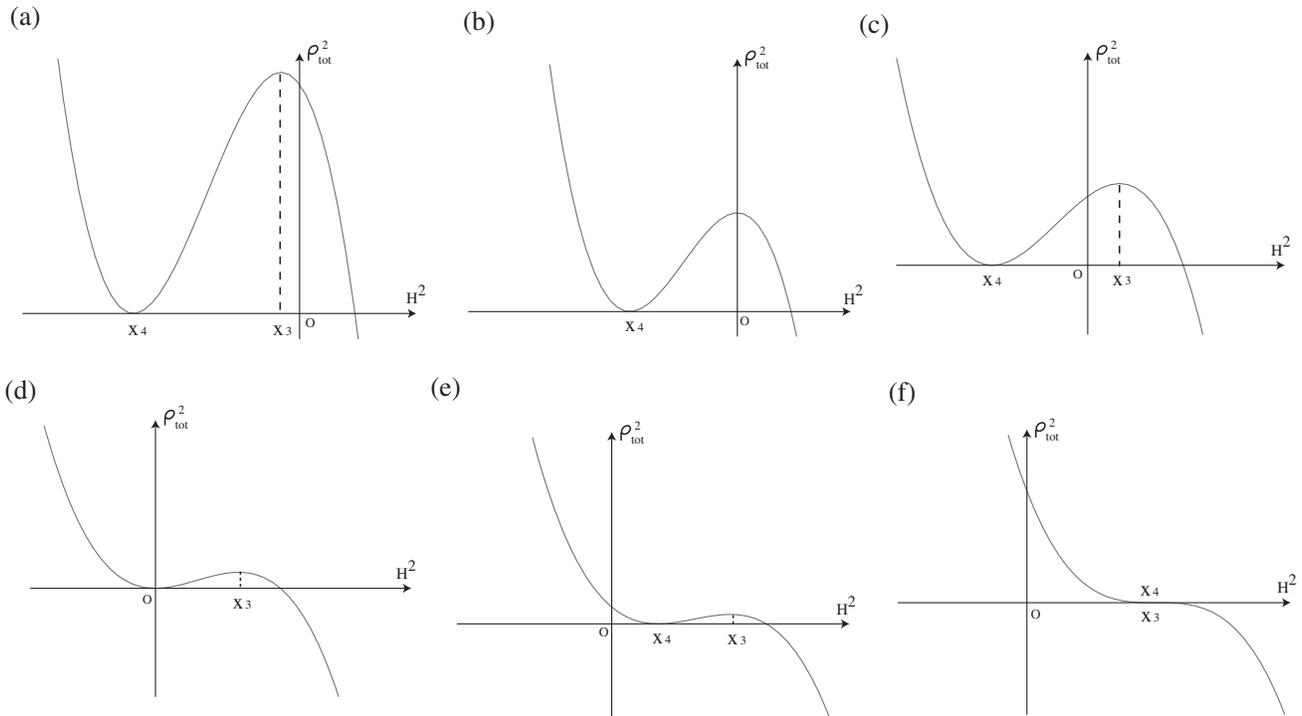


FIG. 8. The  $(H^2, \rho_{\text{tot}}^2)$  plane is shown for the GB brane with a timelike extra dimension for the following values of the parameters  $A, B$  in (40): (a)  $B > 2A$ , (b)  $B = 2A$ , (c)  $0 < B < 2A$ , (d)  $B = 0$ , (e)  $-A < B < 0$ , and (f)  $B = -A$  (at the branch singularity). The strange singularity characterized by  $A = 0$  and  $B = 3/(16\alpha)$  appears in the extremal case of (a). Here,  $x_3 := (2A - B)/3$  and  $x_4 := -B$ . Note that the region with  $H^2 < 0$  or  $\rho_{\text{tot}}^2 < 0$  is nonphysical.

Millennium Science Initiative, Chile, and the generous support to CECS from Empresas CMPC is gratefully acknowledged. V. S. and Yu. S. acknowledge the support of the Indo-Ukrainian program of cooperation in science and technology sponsored by the Department of Science and Technology of India and the Ministry of Education and Science of Ukraine. Yu. S. is also supported in part by Grant No. 5-20 of the ‘‘Cosmomicrophysics’’ programme and by the Program of Fundamental Research of the Physics and Astronomy Division of the National

Academy of Sciences of Ukraine, by Grant No. F16-457-2007 of the State Foundation of Fundamental Research of Ukraine, and by the INTAS Grant No. 05-1000008-7865.

## APPENDIX

Figures 7 and 8 describe the evolution of our brane with a spacelike and a timelike extra dimension, respectively. These figures supplement those appearing in the main body of the paper.

- 
- [1] P. Hořava and E. Witten, Nucl. Phys. **B475**, 94 (1996); A. Lukas, B. A. Ovrut, K. S. Stelle, and D. Waldram, Phys. Rev. D **59**, 086001 (1999); A. Lukas, B. A. Ovrut, and D. Waldram, Phys. Rev. D **60**, 086001 (1999).
- [2] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **436**, 257 (1998).
- [3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).
- [4] R. Maartens, Living Rev. Relativity **7**, 7 (2004).
- [5] V. Sahni, ‘‘Cosmological Surprises from Braneworld Models of Dark Energy,’’ in Proceedings of the 14th Workshop on General Relativity and Gravitation, Kyoto, Japan, 2004, edited by W. Hikida, M. Sasaki, T. Tanaka, and T. Nakamura, pp. 95–115.
- [6] E. J. Copeland, A. R. Liddle, and J. E. Lidsey, Phys. Rev. D **64**, 023509 (2001).
- [7] Yu. Shtanov and V. Sahni, Phys. Lett. B **557**, 1 (2003).
- [8] R. Maartens, V. Sahni, and T. D. Saini, Phys. Rev. D **63**, 063509 (2001).
- [9] G. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B **485**, 208 (2000); G. Dvali and G. Gabadadze, Phys. Rev. D **63**, 065007 (2001); G. Dvali, G. Gabadadze, and M. Shifman, Phys. Rev. D **67**, 044020 (2003).
- [10] V. Sahni and Yu. Shtanov, J. Cosmol. Astropart. Phys. **11** (2003) 014; Int. J. Mod. Phys. D **11**, 1515 (2002).
- [11] Yu. Shtanov and V. Sahni, Classical Quantum Gravity **19**, L101 (2002).
- [12] U. Alam and V. Sahni, Phys. Rev. D **73**, 084024 (2006); R. Maartens and E. Majerotto, Phys. Rev. D **74**, 023004 (2006); R. Lazkoz, R. Maartens, and E. Majerotto, Phys. Rev. D **74**, 083510 (2006); R. Lazkoz and E. Majerotto, J. Cosmol. Astropart. Phys. **07** (2007) 015.
- [13] D. J. Gross and J. H. Sloan, Nucl. Phys. **B291**, 41 (1987); M. C. Bento and O. Bertolami, Phys. Lett. B **368**, 198 (1996).
- [14] S. Nojiri and S. D. Odintsov, J. High Energy Phys. **07** (2000) 049; H. M. Lee, arXiv:hep-th/0010193; B. Abdesselam and N. Mohammedi, Phys. Rev. D **65**, 084018 (2002); J. E. Lidsey, S. Nojiri, and S. D. Odintsov, J. High Energy Phys. **06** (2002) 026; S. Nojiri, S. D. Odintsov, and S. Ogushi, Phys. Rev. D **65**, 023521 (2001); Int. J. Mod. Phys. A **17**, 4809 (2002); I. P. Neupane, Classical Quantum Gravity **19**, 1167 (2002); **19**, 5507 (2002); C. Germani and C. F. Sopuerta, Phys. Rev. Lett. **88**, 231101 (2002); J. E. Lidsey and N. J. Nunes, Phys. Rev. D **67**, 103510 (2003); C. Charmousis and J. Dufaux, Phys. Lett. B **562**, 118 (2003); S. C. Davis, Phys. Rev. D **67**, 024030 (2003); J. P. Gregory and A. Padilla, Classical Quantum Gravity **20**, 4221 (2003); N. Deruelle and C. Germani, Nuovo Cimento Soc. Ital. Fis. B **118**, 977 (2003); N. Deruelle and J. Madore, arXiv:gr-qc/0305004; C. Barcelo, C. Germani, and C. F. Sopuerta, Phys. Rev. D **68**, 104007 (2003); K. Maeda and T. Torii, Phys. Rev. D **69**, 024002 (2004); B. C. Paul and M. Sami, Phys. Rev. D **70**, 027301 (2004); G. Kofinas and E. Papantonopoulos, J. Cosmol. Astropart. Phys. **12** (2004) 011; M. Sami, N. Savchenko, and A. Toporensky, Phys. Rev. D **70**, 123528 (2004); M. Sami and V. Sahni, Phys. Rev. D **70**, 083513 (2004); S. Nojiri, S. D. Odintsov, and M. Sasaki, Phys. Rev. D **71**, 123509 (2005); G. Calcagni, S. Tsujikawa, and M. Sami, Classical Quantum Gravity **22**, 3977 (2005); G. Calcagni, B. de Carlos, and A. De Felice, Nucl. Phys. **B752**, 404 (2006); L. Amendola, C. Charmousis, and S. C. Davis, J. Cosmol. Astropart. Phys. **12** (2006) 020; B. M. N. Carter and I. P. Neupane, J. Cosmol. Astropart. Phys. **06** (2006) 004; T. Kobayashi, T. Shiromizu, and N. Deruelle, Phys. Rev. D **74**, 104031 (2006); **74**, 104031 (2006); T. Koivisto and D. F. Mota, Phys. Rev. D **75**, 023518 (2007); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, Phys. Rev. D **75**, 086002 (2007); R. A. Brown, arXiv:gr-qc/0701083; M. Heydari-Fard and H. R. Sepangi, Phys. Rev. D **75**, 064010 (2007); B. M. Leith and I. P. Neupane, J. Cosmol. Astropart. Phys. **05** (2007) 019; L. Amendola, C. Charmousis, and S. C. Davis, arXiv:0704.0175; P. S. Apostolopoulos, N. Brouzakis, N. Tetradis, and E. Tzavara, Phys. Rev. D **76**, 084029 (2007); E. Elizalde, J. Q. Hurtado, and H. I. Arcos, arXiv:0708.0591; S.-F. Wu, A. Chatrabhuti, G.-H. Yang, and P.-M. Zhang, arXiv:0708.1038.
- [15] E. Gravanis and S. Willison, Phys. Lett. B **562**, 118 (2003).
- [16] S. C. Davis, Phys. Rev. D **67**, 024030 (2003).
- [17] T. Torii and H. Maeda, Phys. Rev. D **71**, 124002 (2005).
- [18] J. D. Barrow, Classical Quantum Gravity **21**, L79 (2004).
- [19] P. Tretyakov, A. Toporensky, Yu. Shtanov, and V. Sahni, Classical Quantum Gravity **23**, 3259 (2006).

- [20] S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **595**, 1 (2004).
- [21] A. A. Starobinsky, *JETP Lett.* **68**, 757 (1998); T. Chiba, T. Okabe, and M. Yamaguchi, *Phys. Rev. D* **62**, 023511 (2000); S. Hellerman, N. Kaloper, and L. Susskind, *J. High Energy Phys.* 06 (2001) 003; R. R. Caldwell, *Phys. Lett. B* **545**, 23 (2002); B. McInnes, *J. High Energy Phys.* 08 (2002) 029; R. R. Caldwell, M. Kamionkowski, and N. N. Weinberg, *Phys. Rev. Lett.* **91**, 071301 (2003); S. M. Carroll, M. Hoffman, and M. Trodden, *Phys. Rev. D* **68**, 023509 (2003); P. Frampton, *Phys. Lett. B* **555**, 139 (2003); P. Frampton and T. Takahashi, *Phys. Lett. B* **557**, 135 (2003); P. Singh, M. Sami, and N. K. Dadhich, *Phys. Rev. D* **68**, 023522 (2003); V. B. Johri, *Phys. Rev. D* **70**, 041303(R) (2004); M. P. Dąbrowski, T. Stachowiak, and M. Szydłowski, *Phys. Rev. D* **68**, 103519 (2003); S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **562**, 147 (2003); *Phys. Rev. D* **70**, 103522 (2004); M. C. B. Abdalla, S. Nojiri, and S. D. Odintsov, *Classical Quantum Gravity* **22**, L35 (2005); C. Csaki, N. Kaloper, and J. Terning, *Ann. Phys. (N.Y.)* **317**, 410 (2005); S. K. Srivastava, *Phys. Lett. B* **619**, 1 (2005); S. Nesseris and L. Perivolaropoulos, *Phys. Rev. D* **70**, 123529 (2004); A. Lue and G. D. Starkman, *Phys. Rev. D* **70**, 101501(R) (2004); J. M. Cline, S. Jeon, and G. D. Moore, *Phys. Rev. D* **70**, 043543 (2004); J. Sola and H. Stefancic, *Phys. Lett. B* **624**, 147 (2005); R.-G. Cai, H.-S. Zhang, and A. Wang, *Commun. Theor. Phys.* **44**, 948 (2005); R. A. Brown, *Gen. Relativ. Gravit.* **39**, 477 (2007); A. A. Andrianov, F. Cannata, and A. Yu. Kamenshchik, *J. Phys. A* **39**, 9975 (2006); L. P. Chimento, R. Lazkoz, R. Maartens, and I. Quiros, *J. Cosmol. Astropart. Phys.* 09 (2006) 004; L. P. Chimento and R. Lazkoz, *Phys. Lett. B* **639**, 591 (2006); F. Cannata and A. Yu. Kamenshchik, arXiv:gr-qc/0603129; J. C. Fabris and S. V. B. Goncalves, *Phys. Rev. D* **74**, 027301 (2006); H. Wei and R.-G. Cai, *Phys. Lett. B* **634**, 9 (2006); S. Cotsakis and I. Klaoudatou, *Geom. Phys.* **57**, 1303 (2007); E. I. Guendelman and A. B. Kaganovich, *Phys. Rev. D* **75**, 083505 (2007); arXiv:gr-qc/0703079.
- [22] V. Sahni and A. A. Starobinsky, *Int. J. Mod. Phys. D* **15**, 2105 (2006); U. Alam, V. Sahni, and A. A. Starobinsky, *J. Cosmol. Astropart. Phys.* 02 (2007) 011.