Kaluza-Klein gravity and scalar-tensor theories

Bertrand Chauvineau
Observatoire de la Côte d'Azur, UMR 6162 (ARTEMIS), Avenue Copernic, 06130 Grasse, France
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In this paper, we propose a Kaluza-Klein approach to gravity in $\Delta = 4 + n_1 + n_2 + \ldots$ dimensions, where $n_1, n_2, \ldots$ are the dimensions of independent internal spaces. One is interested in the case where each internal metric depends on the four-dimensional coordinates by a conformal factor. If all these conformal factors depend on the four-dimensional coordinates through a common scalar function Ψ, the induced effective four-dimensional gravity theory turns out to be of general scalar-tensor type. One shows that, if there are at least two internal spaces, the theory is not ruled out by experimental tests on gravitation, even if there is no massive scalar-potential term in the effective four-dimensional Lagrangian (contrary to what happens if there is only one internal space, in which case ω is of order unity, whatever the dimension of this internal space).

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I. INTRODUCTION

While general relativity (GR) currently passes successfully all observational and experimental tests on gravity [1], one observes a surge of interest in scalar-tensor (ST) gravity theories [2–10], the prototype of which is the Brans-Dicke (BD) theory [2]. From a phenomenological point of view, this is related to the facts that (i) ST solutions are arbitrarily close to GR solutions for arbitrarily large values of ω (but not necessarily with the same matter content) [11–18], and (ii) the scalar field is naturally driven to a value for which ω is infinite by the cosmic expansion for a large class of ST theories [19,20]. From a fundamental point of view, this is related to the fact that the (four-dimensional) gravity theories induced by more fundamental theories (string theory, multidimensional theories, ...) often involve a Brans-Dicke scalar field accompanying the metric tensor (see, for instance, [9,10] for recent reviews). This gives to the induced gravity a structure closer to ST theory (in fact BD in most of the cases) than to GR.

However, if one considers the previous fundamental motivation, it could be strongly tempered by the fact that the induced Brans-Dicke parameter is generally of order unity, a value by far ruled out by present experimental data, which require $\omega > 4 \times 10^4$ at present epoch, at least in the cases where there is no scalar potential with a large effective mass [1,21]. For instance, the low energy limit of the string theory leads to a BD theory, with $\omega = -1$ [9,10,22–24]. In modern versions of Kaluza-Klein gravity, the extra-dimensional part of the metric has a conformal scale factor depending on the four space-time coordinates. This conformal factor results in the presence of a BD scalar field in the effective four-dimensional action [9,10]. In this case, the parameter ω is related to the number n of compactified extra dimensions, in such a way that it is of order unity, whatever n ($\omega = -1 + 1/n$). Hence, both string theory and Kaluza-Klein approaches lead to effective BD gravity, but in a version which necessitates a sufficiently massive scalar field in order to survive the current gravitational tests. In the case of a nonmassive induced scalar field (or scalar field with weak effective mass), the corresponding ST theory is ruled out by observations and experiments.

Other examples of theories resulting in effective ST four-dimensional gravity are presented in [9,10]. These theories generally lead to the same incompatibility when confronted to experimental data.

Besides, in both the string theory and Kaluza-Klein approaches, a coupling of the scalar field with matter generically appears, resulting in weak equivalence principle violation [9,10]. Such a violation does not occur in original BD and ST theories [2–6], since it is explicitly prescribed from the start that the scalar field does not enter the matter part of the action (in Jordan's representation of the theory) [8].

In this paper, a Kaluza-Klein approach to gravity is considered, but in which several independent internal spaces are attached to the four-dimensional space-time. It is shown that this generalization leads to four-dimensional gravity of general ST type. Besides, it turns out that the value of ω derived from this approach is not constrained to be of order of unity, and (arbitrarily) large values are allowed. Hence, while current fundamental theories can pass the experimental gravity tests only if the induced scalar field is sufficiently massive, the proposed version of Kaluza-Klein approach to gravity is not ruled out by observations and experiments, even in the massless or weakly massive cases.

II. SCALAR CURVATURE AND LAGRANGIAN IN $\Delta = 4 + n$ DIMENSIONS

Let us consider a variety with $\Delta = 4 + n$ dimensions, corresponding to the four usual space-time dimensions, and n extra spacelike dimensions. This variety is supposed to have an attached metric tensor g_{AB} on the form

$$(g_{AB}) = \begin{pmatrix} g_{\alpha\beta}(x^\gamma) & 0 \\ 0 & g_{\mu\nu}(x^\alpha, x^\lambda) \end{pmatrix} \tag{1}$$

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where the capital Latin indices $A, B, C\ldots$ run over the whole set of values $0, 1, \ldots, \Delta - 1$. The Greek indices $\alpha, \beta, \gamma, \ldots$ run from 0 to 3 (the usual four-dimensional space-time) while the Greek indices $\lambda, \mu, \nu, \ldots$ run from 4 to $\Delta - 1$ (the n extra-dimensions). Since this paper is only interested in the gravitational sector, there is no $g_{\alpha\mu}$ component in (1). A quantity X computed in the whole space-time will be noted ${}^{(\Delta)}X$, while ${}^{(4)}X$ and ${}^{(n)}X$ denote the analogous quantities, computed in the four-dimensional space-time and in the extra-dimensional space, respectively. In order to simplify notations, when X contains indices (tensors, connection, ...), we will sometimes use the following conventions:

(i) If at least one of the indices is $A, B, C, \ldots$, the superscript (Δ) is suppressed ($\Gamma^C_{AB} = {}^{(\Delta)}\Gamma^C_{AB}$, $R_{\alpha A} = {}^{(\Delta)}R_{\alpha A}$);

(ii) If all the indices are $\alpha, \beta, \gamma, \ldots$, the superscript (4) is suppressed if and only if the four-dimensional quantity is considered ($R_{\alpha\beta} = {}^{(4)}R_{\alpha\beta}$, but $\neq {}^{(\Delta)}R_{\alpha\beta}$);

(iii) If all the indices are $\lambda, \mu, \nu, \ldots$, the superscript (n) is suppressed if and only if the extra-dimensional quantity is considered ($R_{\mu\nu} = {}^{(n)}R_{\mu\nu}$, but $\neq {}^{(\Delta)}R_{\mu\nu}$);

(iv) Since it is not necessary, no superscript is written for the covariant components of the metric tensor. The block diagonal form (1) allows one to use the same convention for the contravariant components.

One finds:

$$\begin{aligned} {}^{(\Delta)}\Gamma^\gamma_{\alpha\beta} &= \Gamma^\gamma_{\alpha\beta} & {}^{(\Delta)}\Gamma^\alpha_{\mu\nu} &= -\tfrac{1}{2}g^{\alpha\beta}\partial_\beta g_{\mu\nu} \\ {}^{(\Delta)}\Gamma^\lambda_{\alpha\mu} &= \tfrac{1}{2}g^{\lambda\sigma}\partial_\alpha g_{\mu\sigma} & {}^{(\Delta)}\Gamma^\lambda_{\mu\nu} &= \Gamma^\lambda_{\mu\nu} \end{aligned} \tag{2}$$

all the other connection components being zero. Defining

$$\Gamma_A = \Gamma^B_{AB} \qquad H^A = g^{BC}\Gamma^A_{BC} \tag{3}$$

it turns out that

$$\begin{aligned} {}^{(\Delta)}\Gamma_\alpha &= \Gamma_\alpha + \tfrac{1}{2}\partial_\alpha \ln({}^{(n)}g) & {}^{(\Delta)}\Gamma_\mu &= \Gamma_\mu \\ {}^{(\Delta)}H^\alpha &= H^\alpha - \tfrac{1}{2}g^{\alpha\beta}\partial_\beta \ln({}^{(n)}g) & {}^{(\Delta)}H^\mu &= H^\mu \end{aligned} \tag{4}$$

where, as usual, g stands for the determinant of the covariant components of the metric tensor.

The scalar curvature

$${}^{(\Delta)}R = g^{AB}\partial_C\Gamma^C_{AB} - g^{AB}\partial_B\Gamma_A + H^A\Gamma_A - g^{AB}\Gamma^D_{AC}\Gamma^C_{BD} \tag{5}$$

turns out to write

$$\begin{aligned} {}^{(\Delta)}R = {}^{(4)}R + \tfrac{1}{4}g^{\alpha\beta}\partial_\alpha g_{\mu\nu}\partial_\beta g^{\mu\nu} - \tfrac{1}{4}(\partial \ln{}^{(n)}g)^2 \\ - {}^{(4)}\Box \ln{}^{(n)}g + {}^{(n)}R \end{aligned} \tag{6}$$

where ${}^{(4)}\Box$ stands for the four-dimensional Dalembertian operator $[{}^{(4)}\Box \ln{}^{(n)}g = g^{\alpha\beta}(\partial_\alpha\partial_\beta \ln{}^{(n)}g - \Gamma^\gamma_{\alpha\beta}\partial_\gamma \ln{}^{(n)}g)]$. $(\partial \ln{}^{(n)}g)^2$ stands for $g^{\alpha\beta}\partial_\alpha \ln{}^{(n)}g\partial_\beta \ln{}^{(n)}g$. Let us point out that ${}^{(n)}R$ $(= g^{\mu\nu(n)}R_{\mu\nu})$ depends explicitly on four-dimensional coordinates. Equation (6) can be equivalently rewritten

$$\begin{aligned} {}^{(\Delta)}R = {}^{(4)}R + \frac{1}{4}g^{\alpha\beta}\partial_\alpha g_{\mu\nu}\partial_\beta g^{\mu\nu} + \frac{1}{{}^{(n)}g}(\partial\sqrt{{}^{(n)}g})^2 \\ - \frac{2}{\sqrt{{}^{(n)}g}}{}^{(4)}\Box\sqrt{{}^{(n)}g} + {}^{(n)}R. \end{aligned} \tag{7}$$

The related Δ-dimensional Lagrangian density writes

$${}^{(\Delta)}\mathcal{L} = {}^{(\Delta)}R\sqrt{|{}^{(\Delta)}g|} = {}^{(\Delta)}R\sqrt{-{}^{(4)}g}\sqrt{{}^{(n)}g} \tag{8}$$

which gives, using (7)

$$\begin{aligned} \frac{{}^{(\Delta)}\mathcal{L}}{\sqrt{-{}^{(4)}g}} = \sqrt{{}^{(n)}g}{}^{(4)}R + \frac{1}{4}\sqrt{{}^{(n)}g}g^{\alpha\beta}\partial_\alpha g_{\mu\nu}\partial_\beta g^{\mu\nu} \\ + \frac{1}{\sqrt{{}^{(n)}g}}(\partial\sqrt{{}^{(n)}g})^2 - 2{}^{(4)}\Box\sqrt{{}^{(n)}g} + \sqrt{{}^{(n)}g}{}^{(n)}R. \end{aligned} \tag{9}$$

III. CASE OF SEVERAL INDEPENDENT INTERNAL SPACES

Let us consider the case where the extra-dimensional space is composed by a (finite) number of different independent ''internal spaces,'' and where each internal metric depends on the four-dimensional space-time coordinates through a conformal factor. This means that the extra-dimensional part of the metric in (1) has the block diagonal form

$$(g_{\mu\nu}(x^\alpha, x^\lambda)) = \begin{pmatrix} A_1(x^\alpha)\tilde{g}_{\mu_1\nu_1}(x^{\lambda_1}) & 0 & 0 \\ 0 & A_2(x^\alpha)\tilde{g}_{\mu_2\nu_2}(x^{\lambda_2}) & 0 \\ 0 & 0 & \ldots \end{pmatrix} \tag{10}$$

with obvious notations for the indices. The dimension of the kth internal space is noted n_k, in such a way that

$$n_1 + n_2 + \ldots = n. \tag{11}$$

Let us define $\Phi(x^\alpha)$ (> 0) by

$$\Phi^2 = A_1^{n_1}A_2^{n_2}\ldots. \tag{12}$$

The Lagrangian (9) can be rewritten

$$\frac{{}^{(\Delta)}\mathcal{L}}{\sqrt{-{}^{(4)}g}} = \left[\Phi^{(4)}R - \frac{\Phi}{4}\left(n_1\frac{(\partial A_1)^2}{A_1^2} + \ldots\right) + \frac{1}{\Phi}(\partial\Phi)^2 - 2^{(4)}\Box\Phi\right]\sqrt{\tilde{g}} + \Phi\sqrt{\tilde{g}}\left(\frac{{}^{(n_1)}\tilde{R}}{A_1} + \ldots\right) \tag{13}$$

where ${}^{(n_k)}\tilde{R}$ is the scalar curvature of the kth internal space, computed using the metric $\tilde{g}_{\mu_k\nu_k}$, and $\tilde{g}$ stands for ${}^{(n_1)}\tilde{g}^{(n_2)}\tilde{g}\ldots$.

IV. EFFECTIVE FOUR-DIMENSIONAL LAGRANGIAN AND ST GRAVITY

A. Effective four-dimensional Lagrangian

As usual (see, for instance, [9]), let us define the effective four-dimensional Lagrangian ${}^{(4)}\mathcal{L}$ by

$$\int d^n x^{\lambda (\Delta)}\mathcal{L} = {}^{(4)}\mathcal{L}\int d^n x^\lambda.\sqrt{\tilde{g}}. \tag{14}$$

Let us also define the scalar constants Λ_k, one per internal space, by

$$\Lambda_k = \frac{\int\sqrt{{}^{(n_k)}\tilde{g}}{}^{(n_k)}\tilde{R}d^{n_k}x^{\lambda_k}}{\int\sqrt{{}^{(n_k)}\tilde{g}}d^{n_k}x^{\lambda_k}}. \tag{15}$$

One obtains

$${}^{(4)}\mathcal{L} = \sqrt{-{}^{(4)}g}\left[\Phi^{(4)}R - \frac{\Phi}{4}\left(n_1\frac{(\partial A_1)^2}{A_1^2} + \ldots\right) + \frac{1}{\Phi}(\partial\Phi)^2\right] + \Phi\sqrt{-{}^{(4)}g}\left(\frac{\Lambda_1}{A_1} + \ldots\right) \tag{16}$$

up to a four-dimensional divergence term.

B. ST gravity

Let us now consider the case where the scalar factors $A_k(x^\alpha)$ depend on the four-dimensional space-time coordinates through a common scalar function $\Psi(x^\alpha)$. This means that there are one-variable functions F_k such that

$$A_k(x^\alpha) = F_k[\Psi(x^\alpha)]. \tag{17}$$

At this level, there are no obvious fundamental motivations for such a choice. But let us adopt a phenomenological point of view, and examine the consequences of this hypothesis. In this case, Φ also depends on Ψ, and (16) writes

$${}^{(4)}\mathcal{L} = \sqrt{-{}^{(4)}g}\left[\Phi^{(4)}R - \frac{\omega(\Phi)}{\Phi}(\partial\Phi)^2\right] + \Phi\sqrt{-{}^{(4)}g}\left(\frac{\Lambda_1}{F_1} + \ldots\right) \tag{18}$$

with

$$\omega(\Phi) = \frac{\Phi^2}{4}\left(\frac{d\Psi}{d\Phi}\right)^2\left(n_1\frac{F_1'^2}{F_1^2} + \ldots\right) - 1 \tag{19}$$

where $F_k' = dF_k/d\Psi$. Since all the F_k depend on a same scalar field Ψ, one can take for Ψ one of the functions F_k, the field Φ or any well-suited function of the F_k. One does not make an explicit choice for the moment.

It turns out that the Lagrangian (18) has the formal structure of the general ST gravity theories Lagrangian, with a scalar dependent potential $\lambda(\Phi)$ [8]

$$\mathcal{L} = \sqrt{-g}\left[\Phi R - \frac{\omega(\Phi)}{\Phi}(\partial\Phi)^2 + 2\Phi\lambda(\Phi)\right] \tag{20}$$

since the F_k in (18) depends on Ψ, hence on Φ. Let us also point out that the resulting $\omega(\Phi)$ is necessarily ≥ -1, which excludes values $< -3/2$ from the start (recall that $\omega < -3/2$ leads to unstable solutions [25,26]).

V. DISCUSSION

As a particular case, let us first reconsider the case where there is only one internal space, i.e. $n_1 = n$ and $n_2 = n_3 = \ldots = 0$. One recovers that $\omega(\Phi)$ is a constant term, written ω_{BD}, and that the effective Lagrangian (18) takes the well-known form (setting $\Lambda = \Lambda_1$)

$${}^{(4)}\mathcal{L} = \sqrt{-{}^{(4)}g}\left[\Phi^{(4)}R - \frac{\omega_{\mathrm{BD}}}{\Phi}(\partial\Phi)^2\right] + \Lambda\Phi^{1-2/n}\sqrt{-{}^{(4)}g} \tag{21}$$

with

$$\omega_{\mathrm{BD}} = \frac{1}{n} - 1. \tag{22}$$

This is the Brans-Dicke theory Lagrangian, plus a cosmological potential term. In the case $n = 2$, one recovers that the parameter ω_{BD} takes the value $-1/2$, while the potential term reduces to a cosmological constant term [9].

Let us now consider the case of two independent extra dimensions. In this case, $n_1 = n_2 = 1$ and $n_3 = \ldots = 0$. Since a one-dimensional variety has no intrinsic curvature, it is not restrictive to set

$$\tilde{g}_{44} = \tilde{g}_{55} = 1 \tag{23}$$

and one has $\Lambda_k = 0$. Let us remark that these one-dimensional spaces, while locally flat in the geometrical sense, can be compactified in the sense that they can have the topology of the circle, with arbitrarily small "radius." As remarked before, it is not restrictive to put $\Psi = F_2$. Making this choice in this case, one has (setting $F = F_1$)

$$\Phi^2 = \Psi F(\Psi) \tag{24}$$

and the Lagrangian (18) takes the classical ST form (without potential)

$${}^{(4)}\mathcal{L} = \sqrt{-{}^{(4)}g}\left[\Phi^{(4)}R - \frac{\omega(\Phi)}{\Phi}(\partial\Phi)^2\right] \tag{25}$$

where the function $\omega(\Phi)$ writes

$$\omega(\Phi) = -2\frac{X}{(X+1)^2} \quad \text{where } X = \frac{d\ln F}{d\ln\Psi}. \tag{26}$$

The Brans-Dicke case (ω constant, noted ω_{BD}) is recovered if and only if F is a power law function

$$F(\Psi) \propto \Psi^m \tag{27}$$

where m is any constant. In this case, the relation between ω_{BD} and m writes

$$\omega_{BD} = -2\frac{m}{(m+1)^2}. \tag{28}$$

If $m = 1$, i.e. $F_1 = F_2$, the value $\omega_{BD} = -1/2$ is recovered [one (flat) internal space with two dimensions]. On the other hand, if m is (arbitrarily) close to the value -1, ω_{BD} is positive and arbitrarily large. Consequently, it is always possible to choose m in such a way that the resulting ω_{BD} is larger than any *a priori* fixed real number, hence in such a way that it satisfies the experimental gravitational tests constraint. From (24), the value $m = -1$ leads to a constant scalar field Φ (and reciprocally), in accordance with $\omega_{BD} = \infty$.

Considering the general case, Eq. (19) shows that this property extends to the general ST case, since

$$(\omega \to \infty) \Leftrightarrow \left(\frac{d\Phi}{d\Psi} \to 0\right). \tag{29}$$

This is realized if, for instance, the function $\Phi(\Psi)$ possesses a local extremum. From Eq. (12), this means that the product $A_1^{n_1} A_2^{n_2} \ldots. = F_1^{n_1} F_2^{n_2} \ldots.$ possesses an extremum for the value of the scalar field Ψ. If it is the case, this limit is naturally reached for a large class of ST theories as a consequence of the cosmic expansion. A ST theory (18) and (19) belongs to this class if the functions F_k and the dimensions n_k give to the resulting coupling function $\omega(\Phi)$ well-suited properties. These properties are discussed in [19,20] if there is no cosmological potential. In particular, $\omega(\Phi)$ has to exhibit a local extremum (note that this is not the case for BD theory). An effective cosmological potential is (generically) present in (18) if the internal spaces have more than one dimension and have intrinsic curvatures. The attractor mechanism has been investigated in the potential case by some authors [27–29]. In particular, a noticeable result obtained by some authors is that the presence of a potential makes the attractor mechanism more efficient, for both a flat and an open universe [29].

The property (17), which results in an effective four-dimensional gravity theory of ST type, is a more constraining property. It requires the conformal factors to depend on a common four-dimensional space-time coordinates function. Let us point out that this property is naturally fulfilled if these conformal factors are depending on time only, a property which is coherent with the presupposed space homogeneity in Robertson-Walker cosmology.

VI. CONCLUSION

One has shown that the modern approach of Kaluza-Klein gravity results in general scalar-tensor effective four-dimensional gravity, as soon as at least two independent internal spaces are considered, and when a specific assumption for the internal metrics is made. When matter is included, the scalar field couples with matter, generically leading to weak equivalence principle violation, as in the currently known case where only one internal space is present. However, as far as the purely gravitational sector is concerned, it appears that it is possible to choose the conformal factors in such a way that the resulting scalar-tensor theory agrees with experimental and observational data, even in the case of a massless or light scalar field. In the cosmological context, this choice could be naturally reached for a large class of induced scalar-tensor theories by the cosmic expansion.

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