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Newtonian limit of $f(R)$ gravity

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A general analytic procedure is developed to deal with the Newtonian limit of $f(R)$ gravity. A discussion comparing the Newtonian and the post-Newtonian limit of these models is proposed in order to point out the differences between the two approaches. We calculate the post-Newtonian parameters of such theories without any redefinition of the degrees of freedom, in particular, without adopting some scalar fields and without any change from Jordan to Einstein frame. Considering the Taylor expansion of a generic $f(R)$ theory, it is possible to obtain general solutions in terms of the metric coefficients up to the third order of approximation. In particular, the solution relative to the g_{tt} component gives a gravitational potential always corrected with respect to the Newtonian one of the linear theory $f(R) = R$. Furthermore, we show that the Birkhoff theorem is not a general result for $f(R)$ gravity since time-dependent evolution for spherically symmetric solutions can be achieved depending on the order of perturbations. Finally, we discuss the post-Minkowskian limit and the emergence of massive gravitational wave solutions.

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I. INTRODUCTION

In recent years, the effort to give a physical explanation to the today observed cosmic acceleration [1–3] has attracted a good amount of interest in $f(R)$ gravity, considered as a viable mechanism to explain the cosmic acceleration by extending the geometric sector of field equations [4–6]. There are several physical and mathematical motivations to enlarge general relativity (GR) by these theories. For a comprehensive review, see [7–9].

Specifically, cosmological models coming from $f(R)$ gravity were first introduced by Starobinsky [10] in the early 1980s to build up a feasible inflationary model where geometric degrees of freedom had the role of the scalar field ruling the inflation and the structure formation.

On the other side, dealing with such extended gravity models at shorter astrophysical scales (Galaxy and Solar System), one faces the emergence of corrected gravitational potentials with respect to the Newton one coming out from GR. This result has been well known for a long time [11], and recently it has been pursued to carry out the possibility of explaining the flatness of spiral galaxies rotation curves without the addition of a huge amount of dark matter. In particular, the rotation curves of a wide sample of low-surface-brightness spiral galaxies have been successfully fitted by these corrected potentials [12], and reliable results are also expected for other galaxy types [13].

Other issues as, for example, the observed Pioneer anomaly problem [14] can be framed into the same approach [15] and then, apart from the cosmological dynamics, a systematic analysis of such theories urges at short scale and in the low-energy limit.

In this paper, we are going to discuss, without specifying the form of the theory, the Newtonian limit of $f(R)$ gravity pointing out the differences and the relations with respect to the post-Newtonian and the post-Minkowskian limits. In literature, there are several definitions and several claims in this direction but clear statements and discussion on these approaches are in order to find out definite results to be tested by experiments [16].

The discussion about the short-scale behavior of higher-order gravity has been quite vivacious in the last years since GR shows its best predictions just at the Solar System level. As matter of fact, measurements coming from weak field limit tests like the bending of light, the perihelion shift of planets, and frame dragging experiments represent inescapable tests for whatever theory of gravity. Actually, in our opinion, there are sufficient theoretical predictions to state that higher-order theories of gravity can be compatible with Newtonian and post-Newtonian prescriptions. In other papers [17] we have shown that this result can be achieved by means of the analogy of $f(R)$ models with scalar-tensor gravity.

Nevertheless, up to now, the discussion on the weak field limit of $f(R)$ theories is far from definitive and there are several papers claiming opposite results [18,19], or stating that no progress has been reached in the last 40 due to several common misconceptions in the various theories of gravity [16].

In particular, people approached the weak limit issue following different schemes and developing different parametrizations which, in some cases, turn out to be not necessarily correct.

The purpose is to take part in the debate, building up a rigorous formalism which deals with the formal definition

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of weak field and small velocities limit applied to fourth-order gravity. In a series of papers, our aim is to pursue a systematic discussion involving: i) the Newtonian limit of $f(R)$ gravity (the present paper); ii) spherically symmetric solutions versus the weak field limit in $f(R)$ gravity [20]; and, finally, iii) general fourth-order theories where invariants such as $R_{\mu\nu}R^{\mu\nu}$ or $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ are also considered, [21].

Our analysis is based on the metric approach, developed in the Jordan frame, assuming that the observations are performed in it, without resorting to any conformal transformation as done in several cases [22]. This point of view is adopted in order to avoid dangerous variable changes which could compromise the correct physical interpretation of the results.

We will show that the corrections induced on the gravitational potentials can be suitable to explain relevant astrophysical behaviors or can be related with some relevant physical issues.

As a preliminary analysis, we will concentrate on the vacuum case with the aim to build up a further rigorous formalism for the Newtonian and post-Newtonian limit of $f(R)$ theories in the presence of matter. As we will see, it is possible to deduce an effective estimation of the post-Newtonian parameter γ by considering the second-order solutions of the metric coefficient in the vacuum case. For the sake of completeness we will treat the problem also by imposing the harmonic gauge on the field equations.

The paper is organized as follows: in Sec. II, the general formalism concerning the spherically symmetric background in fourth-order gravity is introduced. Section III is devoted to a discussion of the post-Newtonian approximation considering the differences with respect to GR: in this theory not all order of perturbations can be consistently achieved if conservation laws are taken into account; in $f(R)$ gravity this shortcoming can be, in principle, avoided. In Sec. IV, the analytic approach to the weak field in $f(R)$ gravity is developed. In particular, we achieve the gravitational potential (related to the g_{tt} component of the metric) which is always corrected with respect to the Newtonian one of the linear $f(R) = R$ theory. Besides, we show that the Birkhoff theorem is not a general result for $f(R)$ gravity since time-dependent evolution for spherically symmetric solutions can be achieved depending on the order of perturbations. In Sec. V, the post-Minkowskian limit is discussed considering also the possibility of obtaining gravitational waves solutions. Sec. VI is devoted to the discussion and conclusions.

II. $f(R)$ GRAVITY IN SPHERICALLY SYMMETRIC SPACETIME

The action for $f(R)$ gravity reads:

$$\mathcal{A} = \int d^4x\sqrt{-g}[f(R) + \mathcal{X}\mathcal{L}_m], \tag{1}$$

where $f(R)$ is an analytic function of the Ricci scalar, $\mathcal{X} = \frac{16\pi G}{c^4}$ is the coupling constant, and $\mathcal{L}_m$ describes the ordinary matter Lagrangian. Such an action is the straightforward generalization of the Hilbert-Einstein action of GR where $f(R) = R$ is assumed.

By varying (1) with respect to the metric, one obtains the fourth-order field equations:

$$f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - f'_{;\mu\nu} + g_{\mu\nu}\Box f' = \frac{\mathcal{X}}{2}T_{\mu\nu}, \tag{2}$$

with $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ and $f' = \frac{df(R)}{dR}$. The trace is

$$3\Box f' + f'R - 2f = \frac{\mathcal{X}}{2}T, \tag{3}$$

and such an expression can be read as a Klein-Gordon equation, where the effective field is f', if $f(R)$ is nonlinear in R [10].

As said, we are interested in investigating the Newtonian and the post-Newtonian limit of $f(R)$ gravity in a spherically symmetric background. Solutions can be obtained considering the metric (see also [23,24])

$$ds^2 = g_{\sigma\tau}dx^\sigma dx^\tau = A(x^0, r)dx^{02} - B(x^0, r)dr^2 - r^2d\Omega \tag{4}$$

where $x^0 = ct$; A and B are generic functions depending on time and coordinate radius; $d\Omega$ is the angular element. The field equations (2) turn out to be

$$\begin{aligned} H_{\mu\nu} &= f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} + \mathcal{H}_{\mu\nu} = \frac{\mathcal{X}}{2}T_{\mu\nu} \\ H &= g^{\sigma\tau}H_{\sigma\tau} = f'R - 2f + \mathcal{H} = \frac{\mathcal{X}}{2}T \end{aligned} \tag{5}$$

where

$$\begin{aligned} \mathcal{H}_{\mu\nu} = &-f''\{R_{,\mu\nu} - \Gamma^0_{\mu\nu}R_{,0} - \Gamma^r_{\mu\nu}R_{,r} \\ &- g_{\mu\nu}[(g^{00}{}_{,0} + g^{00}\ln\sqrt{-g}_{,0})R_{,0} \\ &+ (g^{rr}{}_{,r} + g^{rr}\ln\sqrt{-g})R_{,r} + g^{00}R_{,00} + g^{rr}R_{,rr}]\} \\ &- f'''[R_{,\mu}R_{,\nu} - g_{\mu\nu}(g^{00}R_{,0}{}^2 + g^{rr}R_{,r}{}^2)] \\ \mathcal{H} = &\, g^{\sigma\tau}\mathcal{H}_{\sigma\tau} \\ = &\, 3f''[(g^{00}{}_{,0} + g^{00}\ln\sqrt{-g}_{,0})R_{,0} \\ &+ (g^{rr}{}_{,r} + g^{rr}\ln\sqrt{-g}_{,r})R_{,r} + g^{00}R_{,00} + g^{rr}R_{,rr}] \\ &+ 3f'''[g^{00}R_{,0}{}^2 + g^{rr}R_{,r}{}^2] \end{aligned} \tag{6}$$

are the higher-than-second-order terms of the theory. We are adopting the convention $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ for the Ricci tensor and $R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \ldots$, for the Riemann tensor. Connections are Levi-Civita:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\rho}(g_{\alpha\rho,\beta} + g_{\beta\rho,\alpha} - g_{\alpha\beta,\rho}). \tag{7}$$

III. GENERAL REMARKS ON THE NEWTONIAN AND THE POST-NEWTONIAN APPROXIMATION

At this point, it is worth discussing some general issues on the Newtonian and post-Newtonian limits. Basically there are some general features one has to take into account when approaching these limits, whatever the underlying theory of gravitation is.

If one considers a system of gravitationally interacting particles of mass $\bar{M}$, the kinetic energy $\frac{1}{2}\bar{M}\bar{v}^2$ will be, roughly, of the same order of magnitude as the typical potential energy $U = G\bar{M}^2/\bar{r}$, with $\bar{M}$, $\bar{r}$, and $\bar{v}$ the typical average values of masses, separations, and velocities of these particles. As a consequence:

$$\bar{v}^2 \sim \frac{G\bar{M}}{\bar{r}}, \tag{8}$$

(for instance, a test particle in a circular orbit of radius r about a central mass M will have velocity v given in Newtonian mechanics by the exact formula $v^2 = GM/r$.)

The post-Newtonian approximation can be described as a method for obtaining the motion of the system to higher than the first order (approximation which coincides with the Newtonian mechanics) with respect to the quantities $G\bar{M}/\bar{r}$ and $\bar{v}^2$ assumed small with respect to the squared light speed c^2. This approximation is sometimes referred to as an expansion in inverse powers of the light speed.

The typical values of the Newtonian gravitational potential U are nowhere larger than 10^{-5} in the Solar System (in geometrized units, U/c^2 is dimensionless). On the other hand, planetary velocities satisfy the condition $\bar{v}^2 \lesssim U$[1], while the matter pressure p experienced inside the Sun and the planets is generally smaller than the matter gravitational energy density ρU, in other words[2] $p/\rho \lesssim U$. Furthermore, one must consider that even other forms of energy in the Solar System (compressional energy, radiation, thermal energy, etc.) have small intensities and the specific energy density Π (the ratio of the energy density to the rest-mass density) is related to U by $\Pi \lesssim U$ (Π is $\sim 10^{-5}$ in the Sun and $\sim 10^{-9}$ in the Earth [25]). As matter of fact, one can consider that these quantities, as function of the velocity, give second-order contributions:

$$U \sim v^2 \sim p/\rho \sim \Pi \sim O(2). \tag{9}$$

Therefore, the velocity v gives O(1) terms in the velocity expansions, U^2 is of order O(4), Uv of O(3), $U\Pi$ is of O(4), and so on. Considering these approximations, one has

$$\frac{\partial}{\partial x^0} \sim \mathbf{v} \cdot \nabla, \tag{10}$$

and

$$\frac{|\partial/\partial x^0|}{|\nabla|} \sim O(1). \tag{11}$$

Now, particles move along geodesics:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\sigma\tau}\frac{dx^\sigma}{ds}\frac{dx^\tau}{ds} = 0, \tag{12}$$

which can be written in detail as

$$\frac{d^2x^i}{dx^{02}} = -\Gamma^i_{00} - 2\Gamma^i_{0m}\frac{dx^m}{dx^0} - \Gamma^i_{mn}\frac{dx^m}{dx^0}\frac{dx^n}{dx^0} + \left[\Gamma^0_{00} + 2\Gamma^0_{0m}\frac{dx^m}{dx^0} + 2\Gamma^0_{mn}\frac{dx^m}{dx^0}\frac{dx^n}{dx^0}\right]\frac{dx^i}{dx^0}. \tag{13}$$

In the Newtonian approximation, that is, vanishingly small velocities and only first-order terms in the difference between $g_{\mu\nu}$ and the Minkowski metric $\eta_{\mu\nu}$, one obtains that the particle motion equations reduce to the standard result:

$$\frac{d^2x^i}{dx^{02}} \simeq -\Gamma^i_{00} \simeq -\frac{1}{2}\frac{\partial g_{00}}{\partial x^i}. \tag{14}$$

The quantity $1 - g_{00}$ is of order $G\bar{M}/\bar{r}$, so that the Newtonian approximation gives $\frac{d^2x^i}{dx^{02}}$ to the order $G\bar{M}/\bar{r}^2$, that is, to the order $\bar{v}^2/r$. As a consequence if we would like to search for the post-Newtonian approximation, we need to compute $\frac{d^2x^i}{dx^{02}}$ to the order $\bar{v}^4/\bar{r}$. Because of the equivalence principle and the differentiability of the spacetime manifold, we expect that it should be possible to find a coordinate system in which the metric tensor is nearly equal to the Minkowski one $\eta_{\mu\nu}$, the correction being expandable in powers of $G\bar{M}/\bar{r} \sim \bar{v}^2$. In other words one has to consider the metric developed as follows:

$$\begin{cases} g_{00}(x^0, \mathbf{x}) \simeq 1 + g^{(2)}_{00}(x^0, \mathbf{x}) + g^{(4)}_{00}(x^0, \mathbf{x}) + O(6) \\ g_{0i}(x^0, \mathbf{x}) \simeq g^{(3)}_{0i}(x^0, \mathbf{x}) + O(5) \\ g_{ij}(x^0, \mathbf{x}) \simeq -\delta_{ij} + g^{(2)}_{ij}(x^0, \mathbf{x}) + O(4) \end{cases} \tag{15}$$

where δ_{ij} is the Kronecker delta, and for the controvariant form of $g_{\mu\nu}$, one has

$$\begin{cases} g^{00}(x^0, \mathbf{x}) \simeq 1 + g^{(2)00}(x^0, \mathbf{x}) + g^{(4)00}(x^0, \mathbf{x}) + O(6) \\ g^{0i}(x^0, \mathbf{x}) \simeq g^{(3)0i}(x^0, \mathbf{x}) + O(5) \\ g^{ij}(x^0, \mathbf{x}) \simeq -\delta^{ij} + g^{(2)ij}(x^0, \mathbf{x}) + O(4). \end{cases} \tag{16}$$

In evaluating $\Gamma^\mu_{\alpha\beta}$ we must take into account that the scales of distance and time, in our systems, are, respectively, set by $\bar{r}$ and $\bar{r}/\bar{v}$, thus the space and time derivatives should be regarded as being of order

$$\frac{\partial}{\partial x^i} \sim \frac{1}{\bar{r}}, \qquad \frac{\partial}{\partial x^0} \sim \frac{\bar{v}}{\bar{r}}. \tag{17}$$

Using the above approximations (15) and (16), we have,

[1] We consider here the velocity v in units of the light speed c.

[2] Typical values of p/ρ are $\sim 10^{-5}$ in the Sun and $\sim 10^{-10}$ in the Earth [25].

from the definition (7),

$$\begin{cases}\Gamma^{(3)0}{}_{00}=\frac{1}{2}g^{(2),0}_{00} & \Gamma^{(2)i}{}_{00}=\frac{1}{2}g^{(2),i}_{00}\\ \Gamma^{(2)i}{}_{jk}=\frac{1}{2}(g^{(2),i}{}_{jk}-g^{(2)i}{}_{j,k}-g^{(2)i}{}_{k,j}) & \Gamma^{(3)0}{}_{ij}=\frac{1}{2}(g^{(3)0}{}_{i,j}+g^{(3)0}{}_{j,i}-g^{(3),0}{}_{ij})\\ \Gamma^{(3)i}{}_{0j}=\frac{1}{2}(g^{(3),i}{}_{0j}-g^{(3)i}{}_{0,j}-g^{(2)i}{}_{j,0}) & \Gamma^{(4)0}{}_{0i}=\frac{1}{2}(g^{(4)0}{}_{0,i}+g^{(2)00}g^{(2)}_{00,i})\\ \Gamma^{(4)i}{}_{00}=\frac{1}{2}(g^{(4),i}{}_{00}+g^{(2)im}g^{(2)}{}_{00,m}-2g^{(3)i}{}_{0,0}) & \Gamma^{(2)0}{}_{0i}=\frac{1}{2}g^{(2)0}{}_{0,i}\end{cases}. \tag{18}$$

The Ricci tensor components are

$$\begin{cases}R^{(2)}_{00}=\frac{1}{2}\triangle g^{(2)}_{00}\\ R^{(4)}_{00}=\frac{1}{2}\triangle g^{(4)}_{00}-\frac{1}{2}g^{(2)mn}{}_{,m}g^{(2)}_{00,n}-\frac{1}{2}g^{(2)mn}g^{(2)}_{00,mn}+\frac{1}{2}g^{(2)m}{}_{m,00}-\frac{1}{4}g^{(2)0,m}{}_{0}g^{(2)}_{00,m}-\frac{1}{4}g^{(2)m,n}{}_{m}g^{(2)}_{00,n}-g^{(3)m}{}_{0,m0}\\ R^{(3)}_{0i}=\frac{1}{2}\triangle g^{(3)}_{0i}-\frac{1}{2}g^{(2)m}{}_{i,m0}-\frac{1}{2}g^{(3)m}{}_{0,mi}+\frac{1}{2}g^{(2)m}{}_{m,0i}\\ R^{(2)}_{ij}=\frac{1}{2}\triangle g^{(2)}_{ij}-\frac{1}{2}g^{(2)m}{}_{i,mj}-\frac{1}{2}g^{(2)m}{}_{j,mi}-\frac{1}{2}g^{(2)0}{}_{0,ij}+\frac{1}{2}g^{(2)m}{}_{m,ij}\end{cases} \tag{19}$$

and assuming the harmonic gauge $g^{\rho\sigma}\Gamma^{\mu}_{\rho\sigma}=0$ (see the Appendix for details), one can rewrite these last expressions as

$$\begin{cases}R^{(2)}_{00}=\frac{1}{2}\triangle g^{(2)}_{00}\\ R^{(4)}_{00}=\frac{1}{2}\triangle g^{(4)}_{00}-\frac{1}{2}g^{(2)mn}g^{(2)}_{00,mn}-\frac{1}{2}g^{(2)0}{}_{0,00}-\frac{1}{2}|\nabla_\eta g^{(2)}_{00}|^2\\ R^{(3)}_{0i}=\frac{1}{2}\triangle g^{(3)}_{0i}\\ R^{(2)}_{ij}=\frac{1}{2}\triangle g^{(2)}_{ij}\end{cases} \tag{20}$$

with $\triangle$ and ∇, respectively, the Laplacian and the gradient in flat space. The Ricci scalar reads

$$\begin{cases}R^{(2)}=R^{(2)0}{}_0-R^{(2)m}{}_m=\frac{1}{2}\triangle g^{(2)0}{}_0-\frac{1}{2}\triangle g^{(2)m}{}_m\\ R^{(4)}=R^{(4)0}{}_0+g^{(2)00}R^{(2)}_{00}+g^{(2)mn}R^{(2)}_{mn}=\frac{1}{2}\triangle g^{(4)0}{}_0-\frac{1}{2}g^{(2)0,0}{}_{0,0}-\frac{1}{2}g^{(2)mn}[g^{(2)0}{}_{0,mn}-\triangle g^{(2)}_{mn}]-\frac{1}{2}|\nabla g^{(2)0}{}_0|^2+\frac{1}{2}g^{(2)00}\triangle g^{(2)}_{00}\end{cases}. \tag{21}$$

The inverse of the metric tensor is defined by means of the equation

$$g^{\alpha\rho}g_{\rho\beta}=\delta^\alpha_\beta \tag{22}$$

with δ^α_β the Kronecker delta. The relations among the higher-than-first-order terms turn out to be

$$\begin{cases}g^{(2)00}(x_0,\mathbf{x})=-g^{(2)}_{00}(x_0,\mathbf{x})\\ g^{(4)00}(x_0,\mathbf{x})=g^{(2)}_{00}(x_0,\mathbf{x})^2-g^{(4)}_{00}(x_0,\mathbf{x})\\ g^{(3)0i}=g^{(3)}_{0i}\\ g^{(2)ij}(x_0,\mathbf{x})=-g^{(2)}_{ij}(x_0,\mathbf{x})\end{cases}. \tag{23}$$

Finally, the Lagrangian of a particle in presence of a gravitational field can be expressed as proportional to the invariant distance $ds^{1/2}$, thus we have:

$$\begin{aligned}L&=\left(g_{\rho\sigma}\frac{dx^\rho}{dx^0}\frac{dx^\sigma}{dx^0}\right)^{1/2}=(g_{00}+2g_{0m}v^m+g_{mn}v^mv^n)^{1/2}\\ &=(1+g^{(2)}_{00}+g^{(4)}_{00}+2g^{(3)}_{0m}v^m-\mathbf{v}^2+g^{(2)}_{mn}v^mv^n)^{1/2},\end{aligned} \tag{24}$$

which, to the O(2) order, reduces to the classic Newtonian Lagrangian of a test particle $L_{\rm New}=(1+g^{(2)}_{00}-\mathbf{v}^2)^{1/2}$, where $\mathbf{v}=\frac{dx^m}{dx^0}\frac{dx_m}{dx^0}$. As matter of fact, post-Newtonian physics has to involve higher-than-O(4)-order terms in the Lagrangian.

An important remark concerns the odd-order perturbation terms O(1) or O(3). Since, these terms contain odd powers of velocity $\mathbf{v}$ or of time derivatives, they are related to the energy dissipation or absorption by the system. Nevertheless, the mass-energy conservation prevents the energy and mass losses and, as a consequence, prevents, in the Newtonian limit, terms of O(1) and O(3) orders in the Lagrangian. If one takes into account contributions higher than O(4) order, different theories give different predictions. GR, for example, due to the conservation of post-Newtonian energy, forbids terms of O(5) order; on the other hand, terms of O(7) order can appear and are related to the energy lost by means of the gravitational radiation.

IV. THE NEWTONIAN LIMIT OF $f(R)$ GRAVITY IN SPHERICALLY SYMMETRIC BACKGROUND VS. POST-NEWTONIAN LIMIT

Exploiting the formalism of post-Newtonian approximation described in the previous section, we can develop a systematic analysis in the limit of weak field and small velocities for $f(R)$ gravity. We are going to assume, as background, a spherically symmetric spacetime and we are going to investigate the vacuum case. Considering the metric (4), assuming, unless not specified, $c = 1$ and then $x^0 = ct \to t$, we have, for a given $g_{\mu\nu}$:

$$\begin{cases} g_{tt}(t, r) = A(t, r) \simeq 1 + g^{(2)}_{tt}(t, r) + g^{(4)}_{tt}(t, r) \\ g_{rr}(t, r) = -B(t, r) \simeq -1 + g^{(2)}_{rr}(t, r) \\ g_{\theta\theta}(t, r) = -r^2 \\ g_{\phi\phi}(t, r) = -r^2\sin^2\theta \end{cases}, \tag{25}$$

while the approximations for $g^{\mu\nu}$ are

$$\begin{cases} g^{tt} = A(t, r)^{-1} \simeq 1 - g^{(2)}_{tt} + [g^{(2)2}_{tt} - g^{(4)}_{tt}] \\ g^{rr} = -B(t, r)^{-1} \simeq -1 - g^{(2)}_{rr} \end{cases}. \tag{26}$$

The determinant reads

$$g \simeq r^4\sin^2\theta\{-1 + [g^{(2)}_{rr} - g^{(2)}_{tt}] + [g^{(2)}_{tt}g^{(2)}_{rr} - g^{(4)}_{tt}]\}. \tag{27}$$

As a consequence, the Christoffel's symbols are

$$\begin{cases} \Gamma^{(3)t}{}_{tt} = \frac{g^{(2)}_{tt,t}}{2} & \Gamma^{(2)r}{}_{tt} + \Gamma^{(4)r}{}_{tt} = \frac{g^{(2)}_{tt,r}}{2} + \frac{g^{(2)}_{rr}g^{(2)}_{tt,r} + g^{(4)}_{tt,r}}{2} \\ \Gamma^{(3)r}{}_{tr} = -\frac{g^{(2)}_{rr,t}}{2} & \Gamma^{(2)t}{}_{tr} + \Gamma^{(4)t}{}_{tr} = \frac{g^{(2)}_{tt,r}}{2} + \frac{g^{(4)}_{tt,r} - g^{(2)}_{tt}g^{(2)}_{tt,r}}{2} \\ \Gamma^{(3)t}{}_{rr} = -\frac{g^{(2)}_{rr,t}}{2} & \Gamma^{(2)r}{}_{rr} + \Gamma^{(4)r}{}_{rr} = -\frac{g^{(2)}_{rr,r}}{2} - \frac{g^{(2)}_{rr}g^{(2)}_{rr,r}}{2} \\ \Gamma^r_{\phi\phi} = \sin^2\theta\Gamma^r_{\theta\theta} & \Gamma^{(0)r}{}_{\theta\theta} + \Gamma^{(2)r}{}_{\theta\theta} + \Gamma^{(4)r}{}_{\theta\theta} = -r - rg^{(2)}_{rr} - rg^{(2)2}_{rr} \end{cases}. \tag{28}$$

Let us even display the Ricci's tensor components

$$\begin{cases} R_{tt} \simeq R^{(2)}_{tt} + R^{(4)}_{tt} \\ R_{tr} \simeq R^{(3)}_{tr} \\ R_{rr} \simeq R^{(2)}_{rr} \\ R_{\theta\theta} \simeq R^{(2)}_{\theta\theta} \\ R_{\phi\phi} \simeq \sin^2\theta R^{(2)}_{\theta\theta} \end{cases} \tag{29}$$

where

$$\begin{cases} R^{(2)}_{tt} = \frac{rg^{(2)}_{tt,rr} + 2g^{(2)}_{tt,r}}{2r} \\ R^{(4)}_{tt} = \frac{-rg^{(2)2}_{tt,r} + 4g^{(4)}_{tt,r} + rg^{(2)}_{tt,r}g^{(2)}_{rr,r} + 2g^{(2)}_{rr}[2g^{(2)}_{tt,r} + rg^{(2)}_{tt,rr}] + 2rg^{(4)}_{tt,rr} + 2rg^{(2)}_{rr,tt}}{4r} \\ R^{(3)}_{tr} = -\frac{g^{(2)}_{rr,t}}{r} \\ R^{(2)}_{rr} = -\frac{rg^{(2)}_{tt,rr} + 2g^{(2)}_{rr,r}}{2r} \\ R^{(2)}_{\theta\theta} = -\frac{2g^{(2)}_{rr} + r[g^{(2)}_{tt,r} + g^{(2)}_{rr,r}]}{2} \end{cases} \tag{30}$$

and the Ricci scalar expression in the post-Newtonian approximation

$$R \simeq R^{(2)} + R^{(4)} \tag{31}$$

with

$$\begin{cases} R^{(2)} = \frac{2g^{(2)}_{rr} + r[2g^{(2)}_{tt,r} + 2g^{(2)}_{rr,r} + rg^{(2)}_{tt,rr}]}{r^2} \\ R^{(4)} = \frac{4g^{(2)2}_{rr} + 2rg^{(2)}_{rr}[2g^{(2)}_{tt,r} + 4g^{(2)}_{rr,r} + rg^{(2)}_{tt,rr}] + r\{-rg^{(2)2}_{tt,r} + 4g^{(4)}_{tt,r} + rg^{(2)}_{tt,r}g^{(2)}_{rr,r} - 2g^{(2)}_{tt}[2g^{(2)}_{tt,r} + rg^{(2)}_{tt,rr}] + 2rg^{(4)}_{tt,rr} + 2rg^{(2)}_{rr,tt}\}}{2r^2} \end{cases}. \tag{32}$$

In order to derive the post-Newtonian approximation for a generic function $f(R)$, one should specify the $f(R)$-Lagrangian into the field equations (5). This is a crucial point because once a certain Lagrangian is chosen, one will obtain a particular post-Newtonian approximation referred to such a choice. This means to lose any general prescription and to obtain corrections to the Newtonian potential which refer "univocally" to the considered $f(R)$ function. Alternatively, one can restrict to analytic $f(R)$ functions expandable with respect to a certain value $R = R_0$. In general, such theories are physically interesting and allow one to recover the GR results and the correct boundary and asymptotic conditions. Then we assume

$$f(R) = \sum_n \frac{f^n(R_0)}{n!}(R - R_0)^n \simeq f_0 + f_1R + f_2R^2 + f_3R^3 + \ldots, \tag{33}$$

where the $f(R)$ function is analytic at $R = 0$ (or, at least, its nonanalytic part, if it exists at all, goes to zero faster than R^3 at $R \to 0$) as the expansion implies. Furthermore, the coefficient f_1 must be positive in order to have a positive defined gravitational constant. On the other hand, it is possible to obtain the post-Newtonian approximation of $f(R)$ gravity considering such an expansion into the field equations (5) and expanding the system up to the orders O(0), O(2), and O(4). This approach provides general results and specific (analytic) Lagrangians are selected by the coefficients f_i in (33).

Let us now substitute the series (33) into the field equations (5). Developing the equations up to O(0), O(2), and O(4) orders in the case of vanishing matter, i.e., $T_{\mu\nu} = 0$, we have

$$\begin{cases} H^{(0)}_{\mu\nu} = 0, & H^{(0)} = 0 \\ H^{(2)}_{\mu\nu} = 0, & H^{(2)} = 0 \\ H^{(3)}_{\mu\nu} = 0, & H^{(3)} = 0 \\ H^{(4)}_{\mu\nu} = 0, & H^{(4)} = 0 \end{cases} \tag{34}$$

and, in particular, from the O(0)-order approximation, one obtains

$$f_0 = 0, \tag{35}$$

which trivially follows from the above assumption (15) that the spacetime is asymptotically Minkowskian. This result suggests a first consideration. If the Lagrangian is developable around a vanishing value of the Ricci scalar ($R_0 = 0$) the relation (35) will imply that the cosmological constant contribution has to be zero in vacuum whatever the $f(R)$-gravity theory. This result appears quite obvious but sometimes it is not considered in literature.

If we now consider the O(2)-order approximation, the equations system (34), in the vacuum case, results to be

$$\begin{cases} f_1 rR^{(2)} - 2f_1 g^{(2)}_{tt,r} + 8f_2 R^{(2)}_{,r} - f_1 r g^{(2)}_{tt,rr} + 4f_2 rR^{(2)} = 0 \\ f_1 rR^{(2)} - 2f_1 g^{(2)}_{rr,r} + 8f_2 R^{(2)}_{,r} - f_1 r g^{(2)}_{tt,rr} = 0 \\ 2f_1 g^{(2)}_{rr} - r[f_1 rR^{(2)} - f_1 g^{(2)}_{tt,r} - f_1 g^{(2)}_{rr,r} + 4f_2 R^{(2)}_{,r} + 4f_2 rR^{(2)}_{,rr}] = 0 \\ f_1 rR^{(2)} + 6f_2[2R^{(2)}_{,r} + rR^{(2)}_{,rr}] = 0 \\ 2g^{(2)}_{rr} + r[2g^{(2)}_{tt,r} - rR^{(2)} + 2g^{(2)}_{rr,r} + rg^{(2)}_{tt,rr}] = 0 \end{cases} . \tag{36}$$

The trace equation (the fourth line in (36)), in particular, provides a differential equation with respect to the Ricci scalar which allows one to solve the system (36) at O(2) order:

$$\begin{cases} g^{(2)}_{tt} = \delta_0 - \frac{\delta_1(t)e^{-r\sqrt{-\xi}}}{3\xi r} + \frac{\delta_2(t)e^{r\sqrt{-\xi}}}{6(-\xi)^{3/2}r} \\ g^{(2)}_{rr} = \frac{\delta_1(t)[r\sqrt{-\xi}+1]e^{-r\sqrt{-\xi}}}{3\xi r} - \frac{\delta_2(t)[\xi r+\sqrt{-\xi}]e^{r\sqrt{-\xi}}}{6\xi^2 r} \\ R^{(2)} = \frac{\delta_1(t)e^{-r\sqrt{-\xi}}}{r} - \frac{\delta_2(t)\sqrt{-\xi}e^{r\sqrt{-\xi}}}{2\xi r} \end{cases} \tag{37}$$

where $\xi = \frac{f_1}{6f_2}$ and f_1 and f_2 are the expansion coefficients obtained by Taylor developing the analytic $f(R)$ Lagrangian. Let us notice that the integration constant δ_0 is correctly dimensionless, while the two arbitrary functions of time $\delta_1(t)$ and $\delta_2(t)$ have, respectively, the dimensions of lenght^{-1} and lenght^{-2}; ξ has the dimension lenght^{-2}. The functions $\delta_i(t)$ ($i = 1, 2$) are completely arbitrary since the differential equation system (36) contains only spatial derivatives. Besides, the integration constant δ_0 can be set to zero, as in the theory of the potential, since it represents an unessential additive quantity.

With these results in mind, the gravitational potential of a generic analytic $f(R)$ can be obtained. In fact, the first of (37) gives the second-order solution in terms of the metric expansion (see the definition (25)), but, as said above, this term coincides with the gravitational potential at the Newtonian order. In other words, we have $g_{tt} = 1 + 2\phi_{\rm grav} = 1 + g^{(2)}_{tt}$ and then the gravitational potential of a fourth-order gravity theory, analytic in the Ricci scalar R, is

$$\phi^{FOG}_{\rm grav} = \frac{K_1 e^{-r\sqrt{-\xi}}}{3\xi r} + \frac{K_2 e^{r\sqrt{-\xi}}}{6(-\xi)^{3/2}r}, \tag{38}$$

with $K_1 = \delta_1(t)$ and $K_2 = \delta_2(t)$.

As previously mentioned, one has to notice that the structure of the potential, for a given $f(R)$ theory, is determined by the parameter ξ, which depends on the first and the second derivative of the $f(R)$ function, once developed around a particular point R_0.

Furthermore, one has to consider that the potential (38) holds in the case of nonvanishing f_2 since we manipulated the equations in (36) dividing by such a quantity. As a matter of fact, the GR Newtonian limit cannot be achieved directly from the solution (38) but from the field equations (36) once the appropriate expressions in terms of the constants f_i are derived.

The solution (38) has to be discussed in relation to the sign of the term under the square root in the exponents. The first possibility is that the sign is positive, which means that f_1 and f_2 have an opposite signature. In this case, the solutions (37) and (38) can be rewritten introducing the scale parameter $l = |\xi|^{-1/2}$. In particular, considering $\delta_0 = 0$, the two $\delta_i(t)$ functions as constants, $k_1 = (\delta_1(t)/3)l$ and $k_2(t) = (\delta_2(t)/6)l^2$, and by introducing a radial coordinate $\tilde{r}$ in units of l, we have

$$\begin{cases} g_{tt}^{(2)} = \delta_0 + \frac{\delta_1(t)l}{3}\frac{e^{-r/l}}{r/l} + \frac{\delta_2(t)l^2}{6}\frac{e^{r/l}}{r/l} = k_1\frac{e^{-\tilde{r}}}{\tilde{r}} + k_2\frac{e^{\tilde{r}}}{\tilde{r}} \\ g_{rr}^{(2)} = -\frac{\delta_1(t)l}{3}\frac{(r/l+1)e^{-r/l}}{r/l} + \frac{\delta_2(t)l^2}{6}\frac{(r/l-1)e^{r/l}}{r/l} = -k_1\frac{(\tilde{r}+1)e^{-\tilde{r}}}{\tilde{r}} + k_2\frac{(\tilde{r}-1)e^{\tilde{r}}}{\tilde{r}} \\ R^{(2)} = \frac{\delta_1(t)}{l}\frac{e^{-r/l}}{r/l} + \frac{\delta_2(t)}{2}\frac{e^{r/l}}{r/l} = \frac{3}{l^2}\left[k_1\frac{e^{-\tilde{r}}}{\tilde{r}} + k_2\frac{e^{\tilde{r}}}{\tilde{r}}\right] \end{cases} \quad (39)$$

by which we can recast the gravitational potential as

$$\phi_{\rm grav}^{FOG} = \frac{k_1 e^{-\tilde{r}}}{\tilde{r}} + \frac{k_2 e^{\tilde{r}}}{\tilde{r}}, \quad (40)$$

which is analogous to the result in [11], derived for the theory $R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}$ and coherent[3] with the results in Ref. [26], obtained for higher-order Lagrangians as $f(R, \Box R) = R + \sum_{k=0}^{p} a_k R\Box^k R$. In this last case, it was demonstrated that the number of Yukawa corrections to the gravitational potential was strictly related to the order of the theory. However, as discussed in [21], it is straightforward to show that the usual form Newton + Yukawa can be easily achieved by Eq. (40) through a coordinate change.

From (37) and (39), one can notice that the Newtonian limit of any analytic $f(R)$ theory is related only to the first and second term of the Taylor expansion of the given theory.

In other words, the gravitational potential is always characterized by the two Yukawa corrections, and only the first two terms of the Taylor expansion of a generical $f(R)$ Lagrangian turn out to be relevant. This is indeed a general result.

The diverging contribution, arising from the exponential growing mode, has to be carefully analyzed and, in particular, the physical relevance of this term must be evaluated in relation to the length-scale $(-\xi)^{-1/2}$. For very large r (i.e. $r >> (-\xi)^{-1/2}$) the weak field approximation turns out to be unphysical and (37) does not hold anymore. As a matter of fact, one can obtain a modified gravitational potential which can work as a standard Newtonian one, in the opportune limit, and provide interesting behaviors at larger scales, even in the presence of the growing mode, once the constants in (38) have been opportunely adjusted. Such a potential, once the growing exponential term is settled to zero, reproduces the Yukawa-like gravitational potential, phenomenologically introduced by Sanders [27] to explain the flat rotation curves of spiral galaxies without dark matter.

Besides, Yukawa-like corrections to the gravitational potential have been suggested in several approaches. For example, an interesting proposal is a model describing the gravitational interaction between dark matter and baryons. This points out that the interaction suppressed on small subgalactic scales can be described by means of a Yukawa contribution added to the standard Newtonian potential. Such a behavior is effectively suggested by the observations of the inner rotation curves of low-mass galaxies and provides a natural scenario in which to interpret the cuspy profile of dark matter halos observed in N-body simulations [28].

It is important to stress that the result we have obtained here is coherent with other calculations. In fact, since the Taylor expansion of an exponential potential is a power law series, it is not surprising to obtain a power law correction to the Newtonian potential [12] when a less rigorous approach is considered in order to calculate the weak field limit of a generic $f(R)$ theory. In particular, perturbative calculations will provide effective potentials which can be recovered by means of an appropriate approximation from the general case (40).

Let us now consider the opposite case in which the sign of ξ is negative and, as a consequence, the two Yukawa corrections in (39) are complex numbers.

Since the form of g_{tt}, the gravitational potential (40) turns out to be:

$$\phi_{\rm grav}^{FOG} = \frac{k_1 e^{-\imath\tilde{r}}}{\tilde{r}} + \frac{k_2 e^{\imath\tilde{r}}}{\tilde{r}}, \quad (41)$$

which can be recast as

$$\phi_{\rm grav}^{FOG} = \frac{1}{\tilde{r}}[(k_1 + k_2)\cos\tilde{r} + i(k_2 - k_1)\sin\tilde{r}]. \quad (42)$$

Such a gravitational potential, which could be discarded as nonphysically relevant, has the property to satisfy the Helmholtz equation, $\nabla^2\phi + k^2\phi = 4\pi G\rho$, where ϕ is the gravitational potential and ρ is a real function acting both as matter and the antimatter density. As discussed in [29], $\mathrm{Re}[\phi_{\rm grav}^{FOG}]$ can be addressed as a classically modified Newtonian potential corrected by a Yukawa factor while $\mathrm{Im}[\phi_{\rm grav}^{FOG}]$ could have significant implications for quantum mechanics. In particular, this term can provide an astrophysical, and in our case even theoretically well founded, origin for the puzzling decay $K_L \longrightarrow \pi^+\pi^-$ whose phase is related to an imaginary potential in the kaon mass matrix. Of course, these considerations, at this level, are only speculative, nevertheless it could be worth taking them into account for further investigations.

Let us now consider the system (34) up to the third-order contributions. The first important issue is that, at this order, one has to consider even the off-diagonal equation

$$f_1 g_{rr,t}^{(2)} + 2f_2 r R_{,tr}^{(2)} = 0, \quad (43)$$

[3] Let us remember that in the case of homogeneous and isotropic spacetime, higher-order curvature invariants as $R_{\mu\nu}R^{\mu\nu}$ and $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ reduce to R^2.

which relates the time derivative of the Ricci scalar to the time derivative of $g_{rr}^{(2)}$. From this relation, it is possible to draw a relevant consideration. One can deduce that if the Ricci scalar depends on time so it is for the metric components, and even the gravitational potential turns out to be influenced. This result agrees with the analysis provided in [20] where a complete description of the weak field limit of fourth-order gravity has been provided in terms of the dynamical evolution of the Ricci scalar. In that paper, it was demonstrated that if one supposes a time-independent Ricci scalar, static spherically symmetric solutions are allowed. Equation (43) confirms this result and provides the formal theoretical explanation of such a behavior. In particular, together with (39), it suggests that if one considers the problem at a lower level of approximation (i.e., the second order) the background spacetime metric can have static solutions according to the Birkhoff theorem; this is no more verified when the problem is faced with approximations of higher order. In other words, the debated issue to prove the validity of the Birkhoff theorem in the higher-order theories of gravity finds here its physical answer. In [20] and here, the validity of this theorem is demonstrated for $f(R)$ theories only when the Ricci scalar is time independent or, in addition, when the Newtonian limit solutions are investigated up to the second order of approximation in terms of a v/c expansion of the metric coefficients. Therefore, the Birkhoff theorem does not represent a general feature in the case of fourth-order gravity but, on the other hand, in the limit of small velocities and weak fields (which is enough to deal with the Solar System gravitational experiments), one can assume that the gravitational potential is effectively time independent according to (37) and (38).

The above results fix a fundamental difference between GR and fourth-order gravity theories. While in GR a spherically symmetric solution represents a stationary and static configuration difficult to be related to a cosmological background evolution, this is no more true in the case of higher-order gravity. In the latter case, a spherically symmetric background can have time-dependent evolution together with the radial dependence. In this sense, a relation between a spherical solution and the cosmological Hubble flow can be easily achieved.

The subsequent step concerns the analysis of the system (34) up to the O(4) order. Such an analysis provides the solutions, in terms of $g_{tt}^{(4)}$, the right order for the post-Newtonian parameters. Unfortunately, at this order of approximation, the system turns out to be too much involuted and a general solution is not possible.

From Eqs. (34), one can notice that the general solution is characterized only by the first three orders of the $f(R)$ expansion. Such a result is in agreement with the $f(R)$ reconstruction which can be induced by the post-Newtonian parameters adopting a scalar-tensor analogy (for details see [17,30]).

However, although we cannot achieve a complete description, an approximate estimation of the post-Newtonian parameter γ can be obtained recurring to the O(2) evaluation of the metric coefficients in the vacuum case.

It is important to notice that, since (37) suggests a modified gravitational potential (with respect to the standard Newtonian one) as a general solution of analytic $f(R)$ gravity models, there is no reason to ask for a post-Newtonian description for these theories. In fact, as previously said, the post-Newtonian analysis presupposes to evaluate deviations from the Newtonian potential at a higher-than-second-order approximation in terms of the quantity v/c. Thus, if the gravitational potential deduced from a given $f(R)$ theory of gravity is a general function of the radial coordinate, displaying a Newtonian behavior only in a certain regime (or in a given range of the radial coordinate), it could be meaningless to develop a general post-Newtonian formalism as in GR [25,31]. Of course, by a proper expansion of the gravitational potential for small values of the radial coordinate, and only in this limit, one can develop an analogous of the post-Newtonian limit for these theories with respect to the Newtonian behavior and estimate the deviations from it.

In order to have an effective estimation of the post-Newtonian parameter γ, we can proceed in the following way. Expanding g_{tt} and g_{rr}, obtained at the second order in (39) with respect to the dimensionless coordinate $\tilde{r}$, one has[4]

$$\begin{aligned} g_{tt}^{(2)} &= (k_2 - k_1) + \frac{k_1 + k_2}{\tilde{r}} + \frac{k_1 + k_2}{2}\tilde{r} + \mathrm{O}[2], \\ g_{rr}^{(2)} &= -\frac{k_1 + k_2}{\tilde{r}} + \frac{k_1 + k_2}{2}\tilde{r} + \mathrm{O}[2], \end{aligned} \tag{44}$$

where, clearly, $k_1 + k_2 = GM$ and $k_1 = k_2$ in the standard case. When $\tilde{r} \to 0$ (i.e., when the coordinate $r << \sqrt{-\xi}$) the linear and the higher-than-first-order terms are vanishingly small and only the first Newtonian term survives. Since the post-Newtonian parameter γ is strictly related to the coefficients of the $1/r$ term into the expressions of g_{tt} and g_{rr}, one can actually obtain an effective estimation of this quantity confronting the coefficients of the Newtonian terms relative to both of the expressions in (44). Being $\gamma = 1$ in GR, the difference between these two coefficients gives the effective deviation from the GR expectation value.

It is easy to derive that a generic fourth-order gravity theory provides a post-Newtonian parameter γ which is consistent with the GR prescription ($\gamma = 1$) if $k_1 = k_2$. Conversely, deviations from such a behavior can be accommodated by tuning the relation between the two integration constants k_1 and k_2. This is equivalent to adjusting the form of the $f(R)$ theory in such a way to obtain the right

[4]In this case the symbol O[2] is referred to higher-than-first-order contributions the dimensionless coordinate $\tilde{r}$.

GR limit, and then the Newtonian potential. This result agrees with the viewpoint that asks for the recovering of GR behavior from generic $f(R)$ theories in the post-Newtonian limit [32,33]. This is particularly true when the $f(R)$ Lagrangian behaves, in the weak field and small velocities regime, as the Hilbert-Einstein Lagrangian.

On the other side, if deviations from these regime are observed, a $f(R)$ Lagrangian, built up with a third-order polynomial in the Ricci scalar, can be suitable to interpret such a behavior (see [30]).

Actually, the degeneracy regarding the integration constants can be partially broken once a complete post-Newtonian parametrization is developed in the presence of matter. In such a case, the integration constants remain constrained by the Boltzmann-Vlasov equation which describes the conservation of matter at these scales [34].

Up to now, the discussion has been developed without any gauge choice. In order to overcome the difficulties related to the nonlinearities of calculations, we can consider some gauge choice obtaining less general solutions for the metric entries. A natural choice is represented by the conditions (20) which coincide with the standard post-Newtonian gauge

$$h_{jk,}{}^{k} - \frac{1}{2}h_{,j} = \mathrm{O}(4), \qquad h_{0k,}{}^{k} - \frac{1}{2}h^{k}{}_{k,0} = \mathrm{O}(5), \tag{45}$$

where $h_{\mu\nu}$ accounts for deviations from the Minkowski metric ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$). In this case the Ricci curvature tensor becomes[5]

$$\begin{cases} R_{tt|_{hg}} \simeq R^{(2)}_{tt|_{hg}} + R^{(4)}_{tt|_{hg}} \\ R_{rr|_{hg}} \simeq R^{(2)}_{rr|_{hg}} \end{cases} \tag{46}$$

where

$$\begin{cases} R^{(2)}_{tt|_{hg}} &= \frac{rg^{(2)}_{tt,rr}+2g^{(2)}_{tt,r}}{2r} \\ R^{(4)}_{tt|_{hg}} &= \frac{rg^{(4)}_{tt,rr}+2g^{(4)}_{tt,r}+r[g^{(2)}_{rr}g^{(2)}_{tt,rr}-g^{(2)}_{tt,tt}-g^{(2)}_{tt,rr}]}{2r} \\ R^{(2)}_{rr|_{hg}} &= \frac{rg^{(2)}_{rr,rr}+2g^{(2)}_{rr,r}}{2r} \\ R^{(2)}_{\theta\theta|_{hg}} &= R^{(2)}_{\phi\phi|_{hg}} = 0 \end{cases} \tag{47}$$

while the Ricci scalar expressions at the O(2) and O(4) orders read

$$\begin{cases} R^{(2)}_{|hg} &= \frac{rg^{(2)}_{tt,rr}+2g^{(2)}_{tt,r}-rg^{(2)}_{rr,rr}-2g^{(2)}_{rr,r}}{2r} \\ R^{(4)}_{|hg} &= \frac{rg^{(4)}_{tt,rr}+2g^{(4)}_{tt,r}+r[g^{(2)}_{rr}g^{(2)}_{tt,rr}-g^{(2)}_{tt,tt}-g^{(2)}_{tt,rr}]-g^{(2)}_{tt}[rg^{(2)}_{tt,rr}+2g^{(2)}_{tt,r}]-g^{(2)}_{rr}[rg^{(2)}_{rr,rr}+2g^{(2)}_{rr,r}]}{2r} \end{cases}. \tag{48}$$

The gauge choice does not affect the Christoffel. Thus, by solving the system (34), with the simplification induced by the gauge, one obtains

$$\begin{cases} g_{tt|_{hg}}(t,r) = 1 + \frac{k_1}{r} + \frac{k_2}{r^2} + \frac{k_3\log r}{r} \\ g_{rr|_{hg}}(t,r) = 1 + \frac{k_4}{r} \end{cases} \tag{49}$$

where the constants k_1, k_4 are relative to the O(2) order of approximation, while k_2 and k_3 are related to the O(4) order. The Ricci scalar is zero both at O(2) and at O(4) approximation orders.

Equations (49) suggest some interesting remarks. It is easy to check that the GR prescriptions are immediately recovered for $k_1 = k_4$ and $k_2 = k_3 = 0$. The g_{rr} component displays only the second-order term, as required by a GR-like behavior, while the g_{tt} component shows also the fourth-order corrections which determine the second post-Newtonian parameter β [25]. It has to be stressed here that a full post-Newtonian formalism requires that matter in the system (34) be taken into account: the presence of matter links the second- and fourth-order contributions in the metric coefficients [25].

V. THE POST-MINKOWSKIAN APPROXIMATION

In the previous section we have developed a general analytic procedure to deduce the Newtonian and the post-Newtonian limit of $f(R)$ gravity in the absence of matter or far from matter sources. Here we want to discuss a different limit of these theories, pursued when the small velocity assumption is relaxed and only the weak field approximation is retained. This situation is related to the Minkowski limit of the underlying gravity theory as well, as the discussion of the previous section was related to the Newtonian one. In order to develop such an analysis, we can reasonably resort to the metric (4), considering the gravitational potentials $A(t, r)$ and $B(t, r)$ in the suitable form

$$\begin{cases} A(t,r) = 1 + a(t,r) \\ B(t,r) = 1 + b(t,r) \end{cases} \tag{50}$$

with $a(t, r), b(t, r) \ll 1$. Let us now perturb the field equations (5) considering, again, the Taylor expansion (33) for a generic $f(R)$ theory. For the vacuum case ($T_{\mu\nu} = 0$), at the first order with respect to a and b, it is

$$\begin{cases} f_0 = 0 \\ f_1\{R^{(1)}_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}R^{(1)}\} + \mathcal{H}^{(1)}_{\mu\nu} = 0 \end{cases} \tag{51}$$

where

[5] We have indicated with the subscript hg the harmonic gauge variables.

$$\mathcal{H}^{(1)}_{\mu\nu} = -f_2\{R^{(1)}_{,\mu\nu} - \Gamma^{(0)\rho}{}_{\mu\nu}R^{(1)}_{,\rho} - g^{(0)}_{\mu\nu}[g^{(0)\rho\sigma}{}_{,\rho}R^{(1)}_{,\sigma} + g^{(0)\rho\sigma}R^{(1)}_{,\rho\sigma} + g^{(0)\rho\sigma}\ln\sqrt{-g}^{(0)}_{,\rho}R^{(1)}_{,\sigma}]\}. \quad (52)$$

In this approximation, the Ricci scalar turns out to be zero while the derivatives, in the previous relations, are calculated at $R = 0$.

Let us now consider the limit for large r, i.e., we study the problem far from the source of the gravitational field. In such a case Eqs. (51) become

$$\begin{cases} \frac{\partial^2 a(t,r)}{\partial r^2} - \frac{\partial^2 b(t,r)}{\partial t^2} = 0 \\ f_1[a(t,r) - b(t,r)] - 8f_2[\frac{\partial^2 b(t,r)}{\partial r^2} + \frac{\partial^2 a(t,r)}{\partial t^2} - 2\frac{\partial^2 b(t,r)}{\partial t^2}] = \Psi(t) \end{cases} \quad (53)$$

where $\Psi(t)$ is a generic time-dependent function. Equations (53) are two coupled wave equations in terms of the two functions $a(t, r)$ and $b(t, r)$. Therefore, we can ask for wavelike solutions for the gravitational potentials $a(t, r)$ and $b(t, r)$

$$\begin{cases} a(t, r) = \int \frac{d\omega dk}{2\pi}\tilde{a}(\omega, k)e^{i(\omega t - kr)} \\ b(t, r) = \int \frac{d\omega dk}{2\pi}\tilde{b}(\omega, k)e^{i(\omega t - kr)} \end{cases} \quad (54)$$

and substitute these into (53). In order to simplify the calculations, we can set $\Psi(t) = 0$ since, as said, this is an arbitrary time function. Equations (53) are satisfied if

$$\begin{cases} \tilde{a}(\omega, k) = \tilde{b}(\omega, k), & \omega = \pm k \\ \tilde{a}(\omega, k) = [1 - \frac{3\xi}{4k^2}]\tilde{b}(\omega, k), & \omega = \pm\sqrt{k^2 - \frac{3\xi}{4}} \end{cases} \quad (55)$$

where, as before, $\xi = \frac{f_1}{6f_2}$. In particular, for $f_1 = 0$ or $f_2 = 0$ one obtains solutions with a dispersion relation $\omega = \pm k$. In other words, for $f_i \neq 0$ ($i = 1, 2$), that is in the case of nonlinear $f(R)$, the above dispersion relation suggests that massive modes are in order. In particular, for $\xi < 0$, the mass of the graviton is $m_{\rm grav} = -\frac{3\xi}{4}$ and, coherently, it is obtained for a modified real gravitational potential. As a matter of fact, a gravitational potential deviating from the Newtonian regime induces a massive degree of freedom into the particle spectrum of the gravity sector with an interesting perspective for the detection and the production of gravitational waves [35]. It has to be noted that the presence of massive gravitons in the wave spectrum of higher-order gravity is a well-known result since the paper of [11]. Nevertheless it is our opinion that this issue has been always considered under a negative perspective and has been not sufficiently investigated. Furthermore, if $\xi > 0$, even the solution

$$\begin{cases} a(\tilde{t}, \tilde{r}) = (a_0 + a_1\tilde{r})e^{\pm\frac{\sqrt{3}}{2}\tilde{t}} \\ b(\tilde{t}, \tilde{r}) = (b_0 + b_1\tilde{t})\cos[\frac{\sqrt{3}}{2}\tilde{r}] + (b'_0 + b'_1\tilde{t})\sin[\frac{\sqrt{3}}{2}\tilde{r}] + b''_0 + b''_1\tilde{t} \end{cases} \quad (56)$$

with $a_0, a_1, b_0, b_1, b'_0, b'_1, b''_0, b''_1$ constants is admitted. The variables $\tilde{r}$ and $\tilde{t}$ are expressed in units of $\xi^{-1/2}$. In the post-Minkowskian approximation, as expected, the gravitational field propagates by means of wavelike solutions. This result suggests that investigating the gravitational waves behavior of fourth-order gravity can represent an interesting issue where a new phenomenology (massive gravitons) has to be seriously taken into account. Besides, such massive degrees of freedom could be a realistic and testable candidate for cold dark matter, as discussed in [36].

VI. CONCLUSIONS

In this paper, we have developed a general analytic approach to deal with the weak field and small velocity limit (Newtonian limit) of a generic $f(R)$ gravity theory. The scheme can be adopted also to correctly calculate the post-Newtonian parameters of such theories without any redefinition of the degrees of freedom by some scalar field leading to the so called O'Hanlon Lagrangian [37]. In fact, considering this latter approach, we get a Brans-Dicke-like theory with a vanishing kinetic term and then the post-Newtonian parameter γ results $\gamma = 1/2$ and not $\gamma \sim 1$ as observed. This result is misleading in the weak field limit. In the approach presented here, we do not need any change from the Jordan to the Einstein frame [30,38]. Apart from the possible shortcomings related to noncorrect changes of variables, any $f(R)$ theory can be rewritten as a scalar-tensor one or an ideal fluid, as shown in [39–41]. In those papers, it has been demonstrated that such different representations give rise to physically nonequivalent theories, and then also the Newtonian and post-Newtonian approximations have to be handled very carefully because the results could not be equivalent. In fact, the further geometric degrees of freedom of $f(R)$ gravity (with respect to GR), the scalar field and the ideal fluid have weak field behaviors strictly depending on the adopted gauge which could not be equivalent or difficult to compare. In order to circumvent these possible sources of shortcomings, one should state the frame (Jordan or Einstein) at the very beginning and then remain in such a frame along all the calculations up to the final results. Adopting this procedure, arbitrary limits and noncompatible results should be avoided.

In this paper, we have considered the Taylor expansion of a generic $f(R)$ theory, obtaining general solutions in terms of the metric coefficients up to the third order of approximation when matter is neglected. In particular, the solution relative to the g_{tt} metric component gives the gravitational potential which is corrected with respect to the Newtonian one of $f(R) = R$. The general gravitational potential is given by a couple of Yukawa-like terms, combined with the Newtonian potential, which is effectively achieved at small distances. In relation to the sign of the characteristic coefficients entering the g_{tt} component, one can obtain real or complex solutions. In both cases, the resulting gravitational potential has physical meanings. This degeneracy could be removed once standard matter is introduced into dynamics.

The complete analysis allows one to obtain direct information on the post-Newtonian formalism: the post-Newtonian parameters can be fully characterized considering the integration constants in the gravitational potential. Nevertheless this study is beyond the aim of this paper and will be developed in a forthcoming research project.

Furthermore, it has been shown that the Birkhoff theorem is not a general result for $f(R)$ gravity. This is a fundamental difference between GR and fourth-order gravity. While in GR a spherically symmetric solution is, in any case, stationary and static, here time-dependent evolution can be achieved depending on the order of perturbations.

Finally, we have discussed the differences between the post-Newtonian and the post-Minkoskian limit in $f(R)$ gravity. The main result of such an investigation is the presence of massive degrees of freedom in the spectrum of gravitational waves which are strictly related to the modifications occurring in the gravitational potential. This occurrence could constitute an interesting opportunity for the detection and investigation of gravitational waves.

APPENDIX

In this appendix, we show that the harmonic gauge can be suitably reduced to the form (20). Such a gauge is usually characterized by the condition $g^{\sigma\tau}\Gamma^{\mu}_{\sigma\tau} = 0$. For $\mu = 0$ one has

$$2g^{\sigma\tau}\Gamma^{0}_{\sigma\tau} \approx g^{(2)0,0}{}_{0} - 2g^{(3)0,m}{}_{m} + g^{(2)m,0}{}_{m} = 0, \quad \text{(A1)}$$

and for $\mu = i$

$$2g^{\sigma\tau}\Gamma^{i}_{\sigma\tau} \approx g^{(2)0,i}{}_{0} + 2g^{(2)mi}{}_{,m} - g^{(2)m,i}{}_{m} = 0. \quad \text{(A2)}$$

Differentiating Eq. (A1) with respect to x^0, x^j, and (A2) and with respect to x^0, one obtains

$$g^{(2)0}{}_{0,00} - 2g^{(3)m}{}_{0,0m} + g^{(2)m}{}_{m,00} = 0, \quad \text{(A3)}$$

$$g^{(2)0}{}_{0,0j} - 2g^{(3)m}{}_{0,jm} + g^{(2)m}{}_{m,0j} = 0, \quad \text{(A4)}$$

$$g^{(2)0}{}_{0,0i} + 2g^{(2)m}{}_{i,0m} - g^{(2)m}{}_{m,0i} = 0. \quad \text{(A5)}$$

On the other side, combining Eq. (A4) and (A5) we get

$$g^{(2)m}{}_{m,0i} - g^{(2)m}{}_{i,0m} - g^{(3)m}{}_{0,mi} = 0. \quad \text{(A6)}$$

Finally, differentiating Eq. (A2) with respect to x^j, one has

$$g^{(2)0}{}_{0,ij} + 2g^{(2)m}{}_{i,jm} - g^{(2)m}{}_{m,ij} = 0 \quad \text{(A7)}$$

and redefining indexes as $j \longrightarrow i$, $i \longrightarrow j$ since these are mute indexes, we get

$$g^{(2)0}{}_{0,ij} + 2g^{(2)m}{}_{j,im} - g^{(2)m}{}_{m,ij} = 0. \quad \text{(A8)}$$

Combining Eq. (A7) and (A8) we obtain

$$g^{(2)0}{}_{0,ij} + g^{(2)m}{}_{i,jm} + g^{(2)m}{}_{j,im} - g^{(2)m}{}_{m,ij} = 0. \quad \text{(A9)}$$

The relations (A3), (A6), and (A9) guarantee the viability of (20).

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