Complete LQG propagator: Difficulties with the Barrett-Crane vertex

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Some components of the graviton two-point function have been recently computed in the context of loop quantum gravity, using the spinfoam Barrett-Crane vertex. We complete the calculation of the remaining components. We find that, under our assumptions, the Barrett-Crane vertex does *not* yield the correct long-distance limit. We argue that the problem is general and can be traced to the intertwiner independence of the Barrett-Crane vertex, and therefore to the well-known mismatch between the Barrett-Crane formalism and the standard canonical spin networks. In another paper we illustrate the asymptotic behavior of a vertex amplitude that can correct this difficulty.

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I. INTRODUCTION

A key problem in loop quantum gravity (LQG) [1–3] is to derive low-energy quantities from the full background-independent theory. A strategy for addressing this problem was presented in [4] and some components of the graviton propagator of linearized quantum general relativity

$$G^{abcd}(x, y) = \langle 0|h^{ab}(x)h^{cd}(y)|0\rangle \tag{1}$$

[$h^{ab}(x)$, $a, b = 1, \ldots 4$, is the linearized gravitational field] were computed in [5] (at first order) and [6] (to higher order) starting from the background-independent theory and using a suitable expansion. More precisely, the "diagonal" components $G^{aacc}(x, y)$ have been computed in the large-distance limit. This result has been extended to the three-dimensional theory in [7]; an improved form of the boundary states used in the calculation has been considered in [8]; and the exploration of some Planck-length corrections to the propagator of the linear theory has begun in [9]. See also [10].

Here we complete the calculation of the propagator. We compute the nondiagonal terms of $G^{abcd}(x, y)$, those where $a \neq b$ or $c \neq d$, and therefore derive the full tensorial structure of the propagator. The nondiagonal terms are important because they involve the *intertwiners* of the spin networks. Avoiding the complications given by the intertwiners' algebra was indeed the rationale behind the relative simplicity of the diagonal terms.

The dependence of the vertex from the intertwiners is a crucial aspect of the definition of the quantum dynamics. The particular version of the dynamics used in [5,6], indeed, is defined by the Barrett-Crane (BC) vertex [11], where the dependence on the intertwiners is trivial. This is an aspect of the BC dynamics that has long been seen as suspicious (see for instance [12]); and it is directly tested here.

We find that under our assumptions the BC vertex *fails* to give the correct tensorial structure of the propagator in the large-distance limit. We argue that this result is general, and cannot be easily corrected, say by a different boundary state. This result is of interest for a number of reasons. First, it indicates that the propagator calculations are nontrivial; in particular, they are not governed just by dimensional analysis, as one might have worried, and they do test the dynamics of the theory. Second, it reinforces the expectation that the BC model fails to yield classical general relativity (GR) in the long-distance limit. Finally, and more importantly, it opens the possibility of studying the conditions that an alternative vertex must satisfy, in order to yield the correct long-distance behavior. This analysis is presented in another paper [13].

The BC model exists in a number of variants [2,14]; the results presented here are valid for all of them. Alternative models have been considered, see for instance [15]. Recently, a vertex amplitude that modifies the BC amplitude, and which addresses precisely the problems that we find here, has been proposed [16,17], see also [18]. It would be of great interest to repeat the calculation presented here for the new vertex proposed in those papers.

This paper is organized as follows. In Sec. II we formulate the problem and we compute the action of the field operators on the intertwiner spaces. This calculation is a technical result with an interest in itself. Here we will use only part of this result, the rest will be relevant for a different paper. In Sec. III we discuss the form of the boundary state needed to describe a semiclassical geometry to the desired approximation. Section IV contains the main calculation. In Sec. V we discuss the interpretation of our result.

This paper is not self-contained: for full background, see [6]. For an introduction to the general ideas and the formalism, see the book [2]. However, we include here detailed appendixes, with all basic equations of the recoupling theory needed for the calculations. The appendix corrects

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some imprecisions in previous formularies and can be useful as a tool for further developments. We work entirely in the Euclidean theory.

II. THE PROPAGATOR IN LQG

We refer to [6] for the notation and the basic definitions. We want to compute

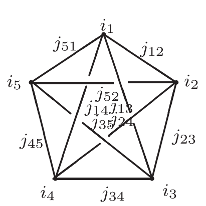
$$\mathbf{G}_{\mathbf{q}}^{abcd}(x,y) = \langle W|h^{ab}(x)h^{cd}(y)|\Psi_{\mathbf{q}}\rangle \tag{2}$$

to first order in λ. Here $\Psi_{\mathbf{q}}$ is a state peaked on $\mathbf{q}$, which is the (intrinsic and extrinsic) 3D geometry of the boundary of a spherical 4-ball of radius L in R^4, x and y are two points in this geometry, ab are tangent indices at x and cd tangent indices at y in this geometry. That is, $\mathbf{G}_{\mathbf{q}}^{abcd}(x,y)$ is a quantity that transforms covariantly under 3D diffeomorphisms acting *conjointly* on x, y, on the indices $abcd$, *and* on $\mathbf{q}$. $h^{ab}(x)$ is the fluctuation of the gravitational field over the Euclidean metric. W is the boundary functional, that defines the dynamics; it is assumed to be given here by a Barrett-Crane group field theory (GFT) [14,19] with coupling constant λ. We work here to first order in λ. Following [16], if we identify the terms generated by the GFT perturbative expansion with the ones obtained form a Regge-like lattice triangulation of GR, then we can interpret the expansion in λ as a cutoff in the degrees of freedom. More precisely, it corresponds to neglecting wavelengths much smaller than L. Degrees of freedom of wavelength larger than L do not matter, since we take a large L limit. We normalize here $\Psi_{\mathbf{q}}$ by $\langle W|\Psi_{\mathbf{q}}\rangle = 1$.

Consider the s-knot (abstract spin-network) basis $|s\rangle = |\Gamma, \mathbf{j}, \mathbf{i}\rangle$, where Γ is an abstract graph, $n, m\ldots$ label the nodes of Γ, $\mathbf{j} = \{j_{mn}\}$ are the spins and $\mathbf{i} = \{i_n\}$ the intertwiners of a spin network with graph Γ. Insert a resolution of the identity in (2)

$$\mathbf{G}_{\mathbf{q}}^{abcd}(x,y) = \sum_s \langle W|s\rangle\langle s|h^{ab}(x)h^{cd}(y)|\Psi_{\mathbf{q}}\rangle. \tag{3}$$

It is shown in [6] that at first order in λ, $\langle W|s\rangle = W[s] = W[\Gamma, \mathbf{j}, \mathbf{i}]$ where the dominant term in the large-spin limit contains only the contribution coming from Γ being the pentagonal graph, that is, for the s-knot

$s =$ 

. (4)

In this case, and from now on, we have five intertwiners $\mathbf{i} = \{i_n\}$, labeled by $n, m, \ldots = 1, \ldots, 5$ and ten spins $\mathbf{j} = \{j_{mn}\}$. We use equally the indices $i, j, k, \ldots = 1, \ldots, 5$ to indicate the nodes. Since the operators $h^{ab}(x)$ do not change the graph (they are operators acting on the spin and intertwiners variables $\mathbf{j}$, $\mathbf{i}$)

$$\mathbf{G}_{\mathbf{q}}^{abcd}(x,y) = \sum_{\mathbf{j},\mathbf{i}} W(\mathbf{j},\mathbf{i})h^{ab}(x)h^{cd}(y)\Psi(\mathbf{j},\mathbf{i}) \tag{5}$$

where $W(\mathbf{j},\mathbf{i}) = W[\Gamma_5, \mathbf{j}, \mathbf{i}]$ and $\Psi(\mathbf{j},\mathbf{i}) = \Psi_{\mathbf{q}}[\Gamma_5, \mathbf{j}, \mathbf{i}] = \langle\Gamma_5, \mathbf{j}, \mathbf{i}|\Psi_{\mathbf{q}}\rangle$, and the sum is over the 15 variables $(\mathbf{j},\mathbf{i}) = (j_{nm}, i_n)$. (We use the physicist's notation $h^{cd}(y)\Psi(\mathbf{j},\mathbf{i})$ for $[h^{cd}(y)\Psi](\mathbf{j},\mathbf{i})$.)

Following [6], we choose the form of $\Psi(\mathbf{j},\mathbf{i})$ by identifying Γ_5 with the dual skeleton of a regular triangulation of the three-sphere. Each node $n = 1, \ldots, 5$ corresponds to a tetrahedron $t_1 \ldots t_5$ and we choose the points x and y to be the centers x_n and x_m of the two tetrahedra t_n and t_m. We consider

$$\mathbf{G}_{\mathbf{q}n,m}^{ij,kl} := \mathbf{G}_{\mathbf{q}}^{abcd}(x_n, x_m)n_a^{(ni)}n_b^{(nj)}n_c^{(mk)}n_d^{(ml)}, \tag{6}$$

where $n_a^{(ij)}$ is the one-form normal to the triangle that bounds the tetrahedra t_i and t_j. From now on, we assume $n \neq m$. Since $h^{ab} = g^{ab} - \delta^{ab} = E^{ai}E_i^b - \delta^{ab}$, this is given by

$$\begin{aligned}\mathbf{G}_{\mathbf{q}n,m}^{ij,kl} &= \langle W|(E_n^{(ni)}\cdot E_n^{(nj)} - n^{(ni)}\cdot n^{(nj)})(E_m^{(mk)}\cdot E_m^{(ml)} \\ &\quad - n^{(mk)}\cdot n^{(ml)})|\Psi_{\mathbf{q}}\rangle \\ &= \sum_{\mathbf{j},\mathbf{i}} W(\mathbf{j},\mathbf{i})(E_n^{(ni)}\cdot E_n^{(nj)} - n^{(ni)}\cdot n^{(nj)})(E_m^{(mk)} \\ &\quad \cdot E_m^{(ml)} - n^{(mk)}\cdot n^{(ml)})\Psi(\mathbf{j},\mathbf{i}).\end{aligned} \tag{7}$$

where, $E_n^{(ml)} = E^a(\vec{x})n_a^{(ml)}$ is valued in the $su(2)$ algebra and, with abuse of notation, the scalar product between the triad fields indicates the product in the $su(2)$ algebra (in the internal space); while the scalar product among the one forms $n^{(ij)}$ is the one defined by the background metric δ^{ab}. In the rest of the paper, we compute the right hand side of (7).

A. Linearity conditions

Before proceeding to the actual computation of (7), let us pause to consider the following question. The four normal one-forms of a tetrahedron sum up to zero. Thus

$$\sum_{i\neq n} n_a^{(ni)} = 0. \tag{8}$$

This determines a set of linear conditions that must be satisfied by $\mathbf{G}_{\mathbf{q}n,m}^{ij,kl}$. In fact, from the last equation it follows immediately that

$$\sum_{i\neq n} \mathbf{G}_{\mathbf{q}n,m}^{ij,kl} = 0. \tag{9}$$

(The existence of conditions of this kind, of course, is necessary, since the four one forms $n_a^{(ni)}$ (for fixed n) span a three-dimensional space, namely, the space tangent to the boundary surface at x_n, and therefore the quantities

$\mathbf{G}^{ij,kl}_{\mathbf{q}n,m}$ are determined by the restriction of the bi-tensor $\mathbf{G}^{abcd}_{\mathbf{q}}$ to these tangent spaces.) How is it possible that the linear conditions (9) are satisfied by the expression (7)?

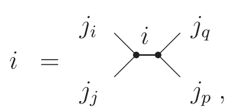
The answer is interesting. The operator $E^{(ni)}_n \cdot E^{(nj)}_n$ acts on the space of the intertwiners of the node n. This is the $SU(2)$ invariant part of the tensor product of the four $SU(2)$ irreducible representations determined by the four spins j_{ni}. In particular, $E^{(ni)}_n$ is the generator of $SU(2)$ rotations in the representation j_{ni}. Therefore

$$J = \sum_{i \neq n} E^{(ni)}_n \tag{10}$$

is the generator of $SU(2)$ rotations in the tensor product of these representations. But the intertwiners' space is precisely the $SU(2)$ invariant part of the tensor product. Therefore $J = 0$ on the intertwiner space. Inserting this in (7), Eq. (9) follows immediately. Therefore the linearity conditions between the projections of the propagator in the space tangent to the boundary surface are implemented by the $SU(2)$ invariance at the nodes.

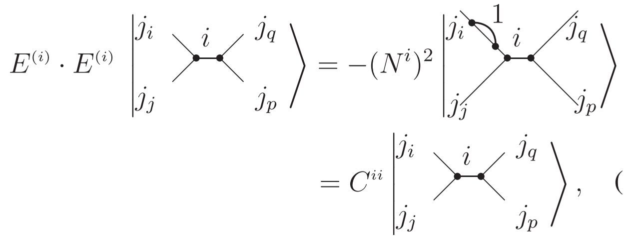
B. Operators

We begin by computing the action of the field operator $E^{(ni)}_n \cdot E^{(nj)}_n$ on the state. This operator acts on the intertwiner space at the node n. It acts as a "double grasping" [3] operator that inserts a virtual link (in the spin-one representation) at the node, connecting the links labeled ni and nj. The state of each node n ($n = 1, \ldots, 5$) is determined by five quantum numbers: the four spins j_{nj} ($n \neq j$, $j = 1, \ldots, 5$) that label the links adjacent to the node and a quantum number i_n of the virtual link that specifies the value of the intertwiner. In this section we study the action of this operator on a single node n; hence we drop for clarity the index n and write the intertwiner quantum number as i, the adjacent spins as j_i, j_j, j_p, j_q, and the operator as $E^{(i)} \cdot E^{(j)}$. We use the graphic notation of $SU(2)$ recoupling theory to compute the action of the operators on the spin-network states (see [2]). The basics of this notation are given in Appendix A and the details of the derivation of the action of the operator are given in Appendix C. Choose a given pairing at the node, say $(i, j)(p, q)$ (and fix the orientation, say clockwise, of each of the two trivalent vertices). We represent the node in the form



(11)

where we use the same notation i for the intertwiner and the spin of the virtual link that determines it. This basis diagonalizes the operator $E^{(i)} \cdot E^{(j)}$, but not the operators $E^{(i)} \cdot E^{(q)}$ and $E^{(i)} \cdot E^{(p)}$. We consider the action of these three "doublegrasping" operators on this basis. The simplest is the action of $E^{(i)} \cdot E^{(i)}$. Using the formulas in Appendix C we have easily

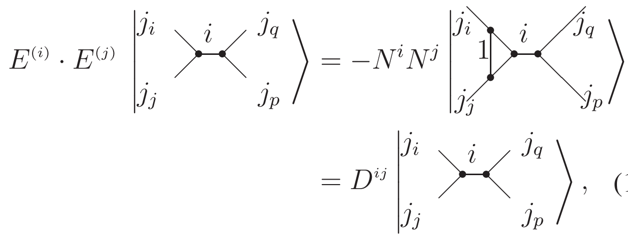


(12)

where

$$C^{ii} = C^2(j_i), \tag{13}$$

with $C^2(a) = a(a + 1)$ is the Casimir of the representation a. Just slightly more complicated is the action of $E^{(i)} \cdot E^{(j)}$

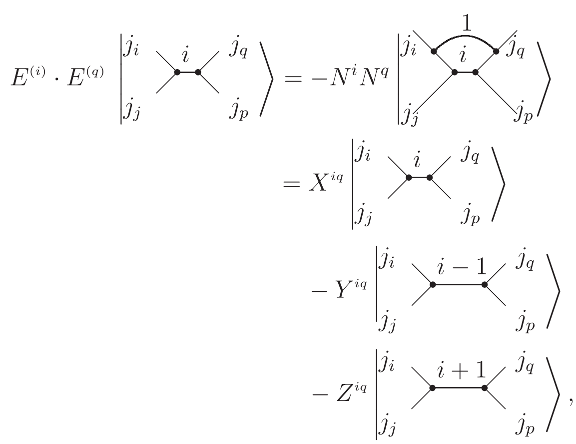


(14)

where

$$D^{ij} = \frac{C^2(i) - C^2(j_i) - C^2(j_j)}{2}. \tag{15}$$

In these two cases the action of the operator is diagonal. If, instead, the grasped links are *not* paired together, the action of the operator is not diagonal in this basis. In this case, the recoupling theory in the appendixes gives



(16)

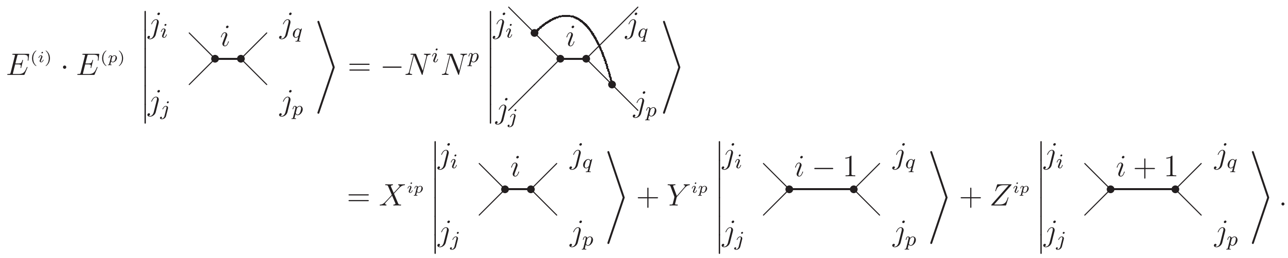
where

$$X^{iq} = -\frac{(C^2(i) + C^2(j_i) - C^2(j_j))(C^2(i) + C^2(j_q) - C^2(j_p))}{4C^2(i)}, \tag{17}$$

$$Y^{iq} = -\frac{1}{4i\,\dim(i)}\sqrt{(j^i + j^j + i + 1)(j^i - j^j + i)(-j^i + j^j + i)(j^i + j^j - i + 1)} \cdot \sqrt{(j^p + j^q + i + 1)(j^p - j^q + i)(-j^p + j^q + i)(j^p + j^q - i + 1)}, \tag{18}$$

$$Z^{iq} = -\frac{1}{4(i+1)\,\dim(i)}\sqrt{(j^i + j^j + i + 2)(j^i - j^j + i + 1)(-j^i + j^j + i + 1)(j^i + j^j - i)} \cdot \sqrt{(j^p + j^q + i + 2)(j^p - j^q + i + 1)(-j^p + j^q + i + 1)(j^p + j^q - i)}. \tag{19}$$

The last possibility is



(20)

Note that X^{ip} is exactly X^{iq} with p and q switched and $Y^{ip} = Y^{iq}$, $Z^{ip} = Z^{iq}$.

Finally, we have to take care of the orientation. As shown in the appendixes, the sign of the nondiagonal terms is influenced by the orientations: in the planar representation that we are using, there is a + sign if the added link intersects the virtual one and a -1 otherwise.

Summarizing, in a different notation and reinserting explicitly the index n of the node, we have the following action of the EE operators. If the grasped links are paired together we have the diagonal action

$$E^{(ni)} \cdot E^{(nj)}|\Gamma_5, \mathbf{j}, i_1, \ldots, i_n, \ldots, i_5\rangle = S_n^{ij}|\Gamma_5, \mathbf{j}, i_1, \ldots, i_n, \ldots, i_5\rangle, \tag{21}$$

where

$$S_n^{ij} = \begin{cases} C^{ii} = C^2(j_{ni}) & \text{if i = j,} \\ D^{ij} = \frac{C^2(i_n) - C^2(j_{ni}) - C^2(j_{nj})}{2} & \text{if } i \neq j. \end{cases} \tag{22}$$

If the grasped links are not paired together, we have the nondiagonal action

$$E^{(ni)} \cdot E^{(nq)}|\Gamma_5, \mathbf{j}, i_1, \ldots, i_n, \ldots, i_5\rangle = \begin{cases} X_n^{iq}|\Gamma_5, \mathbf{j}, i_1, \ldots, i_n, \ldots, i_5\rangle + Y_n^{iq}|\Gamma_5, \mathbf{j}, i_1, \ldots, i_n - 1, \ldots, i_5\rangle + Z_n^{iq}|\Gamma_5, \mathbf{j}, i_1, \ldots, i_n + 1, \ldots, i_5\rangle & \text{if } i \text{ opposite to } q, \\ X_n^{iq}|\Gamma_5, \mathbf{j}, i_1, \ldots, i, \ldots, i_5\rangle - Y_n^{iq}|\Gamma_5, \mathbf{j}, i_1, \ldots, i_n - 1, \ldots, i_5\rangle - Z_n^{iq}|\Gamma_5, \mathbf{j}, i_1, \ldots, i_n + 1, \ldots, i_5\rangle & \text{otherwise.} \end{cases} \tag{23}$$

This completes the calculation of the action of the gravitational field operators.

III. THE BOUNDARY STATE

The boundary state utilized in [6] was assumed to have a Gaussian dependence on the spins, and to be peaked on a particular intertwiner. This intertwiner was assumed to project trivially onto the BC intertwiner of the BC vertex. This was a simplifying assumption permitting us to avoid dealing with the intertwiners, motivated by the fact that intertwiners play no role for the diagonal terms. However, it was also pointed out in [6] that this procedure is not well defined, because of the mismatch between $SO(4)$ linearity and $SU(2)$ linearity (see the discussion in the appendixes of [6]). Here we face the problem squarely, and consider the intertwiner dependence of the boundary state explicitly.

A natural generalization of the Gaussian state used in [6], with a well-defined and nontrivial intertwiner dependence, is the state

$$\Phi(\mathbf{j}, \mathbf{i}) = C \exp\left\{-\frac{1}{2j_0}\sum_{(ij)(mr)} \alpha_{(ij)(mr)}(j_{ij} - j_0)(j_{mr} - j_0) + i\Phi\sum_{(ij)} j_{ij}\right\} \cdot \exp\left\{-\sum_n\left(\frac{(i_n - i_0)^2}{4\sigma} + \sum_{p\neq n}\phi(j_{np} - j_0)(i_n - i_0) + i\chi(i_n - i_0)\right)\right\}. \tag{24}$$

The first line of this equation is precisely the spin dependence of the state used in [6]. The second line contains a Gaussian dependence on the intertwiner variables. More precisely, it includes a diagonal Gaussian term, a nondiagonal Gaussian spin-intertwiner term, and a phase factor. We do not include nondiagonal intertwiner-intertwiner terms here. These will be considered in another paper.

Let us fix some of the constants appearing in (24), by requiring the state to be peaked on the expected geometry. The constant j_0 determines the background area A_0 of the faces, via $C(j_{nm}) = A_{nm}$. As in [6], we leave j_0 free to determine the overall scale. The constant Φ determines the background values of the angles between the normals to the tetrahedra. As in [6], we fix them to those of a regular four-simplex, namely $\cos\Phi = -1/4$.

The constant i_0 is the background value of the intertwiner variable. As shown in [6], the spin of the virtual link i_n is the quantum number of the *angle* between the normals of two triangles. More precisely, the Casimir $C(i_n)$ of the representation i_n is the operator corresponding to the classical quantity

$$C^2(i_n) = A_{ni} + A_{nj} + 2\vec{n}^{(ni)} \cdot \vec{n}^{(nj)}, \tag{25}$$

where i and j are the paired links at the node n and A_{ni} is the area of the triangle dual to the link (ni). The scalar product of the normals to the triangles can therefore be related to the Casimirs of spins and intertwiners:

$$n^{(ni)} \cdot n^{(nj)} = \frac{C(i_n) - C(j_{ni}) - C(j_{nj})}{2}. \tag{26}$$

For each node, the state must therefore be peaked on a value i_0 such that

$$i_0(i_0 + 1) = A_0 + A_0 + 2A_0A_0\cos\theta_{ij}, \tag{27}$$

where $\cos\theta_{ij}$ is the 3D dihedral angle between the faces of the tetrahedron. For the regular 4-symplex, in the large-distance limit we have $A_{ij} = j_0$, $\cos\theta_{ij} = -\frac{1}{3}$, which gives

$$i_0 = \frac{2}{\sqrt{3}} j_0. \tag{28}$$

This fixes i_0. Notice that in [6] our Eq. (25) refers to the Casimir of an $SO(4)$ simple representation and follows from the quantization of the Plebanski 2 form $B^{IJ} = e^I \wedge e^J$ associated with the discretized geometry. Exactly the same result follows from Eq. (14) directly from LQG.

Fixing i_0 in this manner determines only the mean value of the angle θ_{ij} between the two triangles that are paired together in the chosen pairing. What about the mean value of the angles between faces that are not paired together, such as θ_{iq}? It is shown in [20] that a state of the form $e^{(i-i_0)^2/\sigma}$ is peaked on $\theta_{iq} = 0$, which is not what we want; but the mean value of θ_{iq}, can be modified by adding a phase to the state. This is the analog of the fact that a phase changes the mean value of the momentum of the wave packet of a nonrelativistic particle, without affecting the mean value of the position. In particular, it was shown in [20] that by choosing the phase and the width of the Gaussian to be

$$\chi = \frac{\pi}{2}, \qquad \sigma = \frac{j_0}{3}, \tag{29}$$

we obtain a state whose mean value and variance for all angles is the same.

Let us therefore adopt here these values. Still, the present situation is more complicated than the case considered in [20], because the tetrahedron considered there had fixed *and equal* values of the external spins; while here the spins can take arbitrary values around the peak symmetric configuration $j_{nm} = j_0$. As a consequence, when repeating the calculation in [20], one finds additional spin-intertwiner Gaussian terms. These, however can be corrected by fixing the spin-intertwiner Gaussian terms in (24). A detailed calculation (see below), shows indeed that in the large j_0 limit, the state (24) transforms under change of pairing into a state with the same intertwiner mean value and the same variance σ, provided we also choose

$$\phi = -i\frac{3}{4j_0}, \tag{30}$$

which we assume from now on. With these values and introducing the difference variables $\delta i_n = i_n - i_0$ and $\delta j_{mr} = j_{mr} - j_0$ the wave functional, given in (24), reads

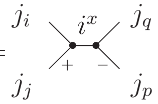
$$\Phi(\mathbf{j}, \mathbf{i}) = Ce^{-(1/2j_0)\sum \alpha_{(ij)(mr)}\delta j_{ij}\delta j_{mr} + i\Phi\sum_{ij}\delta j_{ij}} \times e^{-\sum_n((3(\delta i_n)^2/4j_0) - i(\sum_a(3/4j_0)\delta j_{an} - (\pi/2))\delta i_n)}. \tag{31}$$

This state, however, presents a problem, which we discuss in the next section.

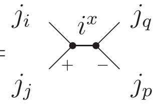
A. Pairing independence

It is natural to require that the state respects the symmetries of the problem. A moment of reflection shows that the state (31) does not. The reason is that the variables i_n are the spin of the virtual links *in one specific pairing*, and this breaks the symmetry of the four-simplex. The phases and variances chosen assure that the mean values are the desired ones, hence symmetric; but an explicit calculation confirms that the relative fluctuations of the angle variables determined by the state (31) depend on the pair chosen.

To correct the problem, recall that there are three natural bases in each intertwiner space, determined by the three possible pairings of these links. Denote them as follows.

$i^{x(+,-)} =$  (33)

where we conventionally denote $i^x \equiv i$ the basis in the pairing chosen as reference. These bases diagonalize the three noncommuting operators $E^{(i)} \cdot E^{(j)}$, $E^{(i)} \cdot E^{(q)}$ and $E^{(i)} \cdot E^{(p)}$, respectively. Furthermore a spin-network state is specified by the orientation of the three-valent nodes [21]; we fix this orientation by giving an ordering to the links. We write for instance

$i^{x(+,-)} =$  (33)

where the plus sign + (−) means anticlockwise (clockwise) ordering of the links in the two nodes. A complete basis in the space of the spin networks on Γ_5 is specified giving the pairing and the orientation at each node. In order to label the different bases, introduce at each node n a variable m_n that takes the values $m_n = x, y, z$, namely, that ranges over the three possible pairings at the node. Similarly, introduce a variable $o_n = \{(++), (+-), (-+), (--)\}$ that labels the possible orientations. To correct the pairing dependence of the state (24), let us first rewrite it in the notation

$$|\Phi_{\mathbf{q}}\rangle_{x++} = \sum_{\mathbf{j},\mathbf{i}^{x++}} \Phi[\mathbf{j}, \mathbf{i}^{x++}]|\mathbf{j}, \mathbf{i}^{x++}\rangle, \tag{34}$$

where the suffix $x++$ to the ket emphasizes the fact that the state has been defined with the chosen pairing and orientation at each node. We can now consider a new state obtained by summing (34) over all choices of pairings and orientations. That is, we change the definition of the boundary state to

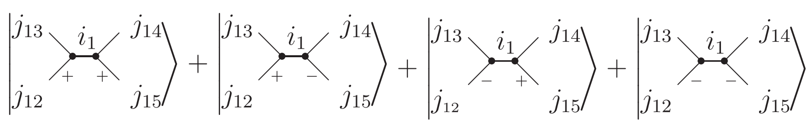
$$|\Psi_{\mathbf{q}}\rangle = \sum_{m_n, o_n} |\Phi_{\mathbf{q}}\rangle_{m_n o_n}, \tag{35}$$

where $\sum_{m_n,o_n} = \sum_{m_1\ldots m_5}\sum_{o_1\ldots o_5}$ and

$$|\Phi_{\mathbf{q}}\rangle_{m_n o_n} = \sum_{\mathbf{j},\mathbf{i}^{m_n o_n}} \Phi[\mathbf{j}, \mathbf{i}^{m_n o_n}]|\mathbf{j}, \mathbf{i}^{m_n o_n}\rangle, \tag{36}$$

namely $|\Phi_{\mathbf{q}}\rangle_{m_n o_n}$ is the same as the state $|\Phi_{\mathbf{q}}\rangle_{x++}$, but defined with a different choice of pairing at each node.

Since (by assumption) (24) does not depend on the orientation, the sum over the orientation of the node (say) 1, in (35) reduces to a term proportional to

$\sum_o |\mathbf{j}, i_1^o, i_2, i_3, i_4, i_5\rangle \sim$  . (37)

As shown in the appendixes, the change in orientation of a vertex produces the sign $(-1)^{a+b+c}$, where a, b, c are the three adjacent spins. Hence

$$\sum_o |\mathbf{j}, i_1^o, i_2, i_3, i_4, i_5\rangle \sim = (1 + (-1)^{j_{14}+j_{15}+i_1} + (-1)^{j_{12}+j_{13}+i_1} + (-1)^{j_{12}+j_{13}+j_{14}+j_{15}+2i_1})|\mathbf{j}, i_1^{++}, i_2, i_3, i_4, i_5\rangle$$
$$= \begin{cases} 4|\mathbf{j}, i_1^{++}, i_2, i_3, i_4, i_5, \rangle & \text{if } (j_{12}+j_{13}+i_1^{m_1} = 2n_1 \quad \text{and} \quad j_{14}+j_{15}+i_1^{m_1} = 2n_2), \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

We can therefore trade the sum over orientations in (35) with a condition on the spins summed over: at all trivalent vertices, the sum of the two external spins and the virtual spin, must be an even integer. (The factor 4 is absorbed in the normalization factor C.) With this understanding, we drop the sum over orientations in (35), which now reads

$$|\Psi_{\mathbf{q}}\rangle = \sum_{m_n} |\Phi_{\mathbf{q}}\rangle_{m_n}, \tag{39}$$

where all orientations are fixed. This state can of course also be expressed in terms of a single basis

$$|\Psi_{\mathbf{q}}\rangle = \sum_{\mathbf{j},\mathbf{i}} \Psi_{\mathbf{q}}(\mathbf{j}, \mathbf{i})|\mathbf{j}, \mathbf{i}\rangle, \tag{40}$$

where we have returned to the notation $i_n = i_n^{x,++}$. Its components are

$$\Psi(\mathbf{j}, \mathbf{i}) = \langle \mathbf{j}, \mathbf{i}|\Psi_{\mathbf{q}}\rangle = \sum_{m_n} \Phi(\mathbf{j}, \mathbf{i}^{m_n})\langle \mathbf{j}, \mathbf{i}|\mathbf{j}, \mathbf{i}^{m_n}\rangle. \tag{41}$$

The matrices of the change of basis $\langle \mathbf{j}, \mathbf{i}|\mathbf{j}, \mathbf{i}^{m_n}\rangle$ are (products of five) 6-j Wigner-symbols, as given by standard recoupling theory.

The state (39) is the boundary state we shall use. The complication of the sum over pairings is less serious than what could seem at first sight, due to a key technicality that we prove in the next section: the components of (39) become effectively orthogonal in the large-distance limit.

1. Orthogonality of the terms in different bases in the large j_0 limit

Suppose we want to compute the norm of the boundary state, in the limit of large j_0. From (39), this is given by

$$|\Psi|^2 = \sum_{m_n} \sum_{m'_n m_n} \langle \Phi_{\mathbf{q}} | \Phi_{\mathbf{q}} \rangle_{m'_n}. \tag{42}$$

We now show that in the large j_0 limit the nondiagonal terms of this sum (those with $m_n \neq m'_n$) vanish. Consider one of these terms, say

$$I \underset{m_n}{=} \langle \Phi_{\mathbf{q}} | \Phi_{\mathbf{q}} \rangle_{m'_n} = \sum_{\mathbf{j}\mathbf{i}^{m_n}} \sum_{\mathbf{j}'\mathbf{i}'^{m'_n}} \overline{\Phi(\mathbf{j}, \mathbf{i}^{m_n})}\, \Phi[\mathbf{j}', \mathbf{i}^{m'_n}] \langle \mathbf{j}\mathbf{i}^{m_n} | \mathbf{j}'\mathbf{i}'^{m'_n} \rangle \tag{43}$$

where, say, $m_n = (x, x, x, x, x)$ and $m'_n = (y, x, x, x, x)$. The scalar product is diagonal in the spins $\mathbf{j}$ and is given by 6-j symbol in the intertwiners quantum numbers. Hence

$$I = \sum_{\mathbf{j}} \sum_{\mathbf{i}^{m_n}} \sum_{\mathbf{i}^{m'_n}} \overline{\Phi(\mathbf{j}, \mathbf{i}^{m_n})}\, \Phi[\mathbf{j}, \mathbf{i}^{m'_n}] \langle i_1^x | i_1^y \rangle, \tag{44}$$

where (see Appendix D),

$$\langle i_1^x | i_1^y \rangle = (-1)^{j_{13}+j_{14}+i_1^x+i_1^y} \sqrt{d_{i_1^x} d_{i_1^y}} \begin{Bmatrix} j_{12} & j_{13} & i_1^x \\ j_{15} & j_{14} & i_1^y \end{Bmatrix}. \tag{45}$$

In the large j_0 limit, this sum can be approximated by an integral, as in [6]. Both the spin and the intertwiner sums become Gaussian integrals, peaked, respectively, on j_0 and i_0. The range of the sum over intertwiners is finite for finite j_0, because of the Clebsh Gordan conditions at the two trivalent node; but this range is much larger than the width of the Gaussian in the limit, and therefore the integral over the intertwiner variables too can be taken over the entire real line. In the limit, the 6-j symbol has the asymptotic value [22]

$$\begin{Bmatrix} j_{12} & j_{13} & i_1^x \\ j_{15} & j_{14} & i_1^y \end{Bmatrix} \approx \frac{e^{i(S_R+(\pi/4))} + e^{-i(S_R+(\pi/4))}}{\sqrt{12\pi V}}, \tag{46}$$

where S_R is the Regge action of a tetrahedron with side length determined by the spins of the $6j$ symbol, and V is its volume. Changing the sum into an integration and using this, we have

$$I = \int d\mathbf{j} \int d\mathbf{i} \int di_1^y \overline{\Phi(\mathbf{j}, \mathbf{i})} \Phi(\mathbf{j}, \mathbf{i}^{m'_n}) (-1)^{j_{13}+j_{14}+i_1^x+i_1^y} \times \frac{e^{i(S_R+(\pi/4))} + e^{-i(S_R+(\pi/4))}}{\sqrt{12\pi V}}. \tag{47}$$

Inserting the explicit form of the state (31) gives

$$I = \int d\mathbf{j} \int d\mathbf{i}\, e^{-(1/j_0)\sum \alpha_{(ij)(mr)}\delta j_{ij}\delta j_{mr} - \sum_{n\neq 1}(3(\delta i_n)^2/2j_0) - (3(\delta i_1)^2/4j_0) - i(\sum_a (3/4j_0)\delta j_{an} - (\pi/2))\delta i_1^x} \cdot \int di_1^y e^{-(3(\delta i_1^y)^2/4j_0)} e^{i(\sum_a (3/4j_0)\delta j_{an} - (\pi/2))\delta i_1^y} \frac{e^{i(S_R+\pi\delta i_1^y+(\pi/4))} + e^{-i(S_R-\pi\delta i_1^y+(\pi/4))}}{\sqrt{12\pi V}}. \tag{48}$$

In the limit, only the first terms in the expansion of the Regge action around the maximum of the peak of the Gaussian matter. We thus Taylor expand the Regge action in its six entries j_{1n}, i_1^x, i_1^y around the background values j_0 and i_0.

$$S_j[j_{na}] = \frac{\partial S_R}{\partial j_{1n}}\bigg|_{j_0,i_0} \delta j_{1n} + \frac{\partial S_R}{\partial i_1^x}\bigg|_{j_0,i_0} \delta i_1^x + \frac{\partial S_R}{\partial i_1^y}\bigg|_{j_0,i_0} \delta i_1^y + \text{higher order terms}. \tag{49}$$

The key point now is that the first of these terms is a rapidly oscillating phase factor in the j_{1n} variable. The Gaussian j_{1n} integration in (48) is suppressed by this phase factor. More precisely, the integral is like a Fourier transform in the j_{1n} variable, of a Gaussian centered around a large value of j_0 with variance proportional $\sqrt{j_0}$; this Fourier transform is then a Gaussian with variance $1/\sqrt{j_0}$, which goes to zero in the $j_0 \to \infty$ limit. QED.

2. Change of basis

For later convenience, let us also give here the expression of the state (31) under the transformation induced by the change of basis associated to a change of pairing. Say we change from the basis i^y to the basis i^x in the node $n = 1$. Then directly from (41) we have

$$\Phi'_{\mathbf{q}}[\mathbf{j}, i_1^x, i_2 \ldots i_5] = e^{-(1/2j_0)\sum \alpha_{(ij)(mr)}\delta j_{ij}\delta j_{mr} + i\sum \Phi\delta j_{ij}} e^{-\sum_{n\neq 1}((3(\delta i_n)^2/4j_0) - i(\sum_a (3/4j_0)\delta j_{an} - (\pi/2))\delta i_n)} \cdot \sum_{i_1^y} e^{-((3(\delta i_1^y)^2/4j_0) - i(\sum_a (3/4j_0)\delta j_{a1} - (\pi/2))\delta i_1^y)} (-1)^{j_{13}+j_{14}+i_1^x+i_1^y} \sqrt{d_{i_1^x} d_{i_1^y}} \begin{Bmatrix} j_{12} & j_{13} & i_1^x \\ j_{15} & j_{14} & i_1^y \end{Bmatrix} \tag{50}$$

where, we recall, the sum over intertwiners is under the condition (38) that gives $(-1)^{j_{13}+j_{14}+i_1^x} = 1$. We can evaluate the sum in the large j_0 limit by approximating it again with an integral. Inserting the asymptotic expansion of the $6j$ symbol, we have

$$\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, \ldots, i_5) = e^{-(1/2j_0)\sum \alpha_{(ij)(mr)}\delta j_{ij}\delta j_{mr} + i\sum \Phi\delta j_{ij}} e^{-\sum_{n\neq 1}((3(\delta i_n)^2/4j_0) - i(\sum_a (3/4j_0)\delta j_{an} - (\pi/2))\delta i_n)} e^{i\pi i_0} \cdot \int d\delta i_1^y e^{-(3(\delta i_1^y)^2/4j_0) - i(\sum_a (3/4j_0)\delta j_{a1} - (\pi/2))\delta i_1^y} \sqrt{d_{i_1^x} d_{i_1^y}} \frac{e^{i(S_R + \pi\delta i_1^y + (\pi/4))} + e^{-i(S_R - \pi\delta i_1^y + (\pi/4))}}{\sqrt{12\pi V}}. \quad (51)$$

This can be computed expanding the Regge action to second order around j_0 and i_0. As shown in Appendix F, the result is

$$\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, \ldots, i_5) = \Phi(\mathbf{j}, i_1^x, i_2, \ldots, i_5) N_1 \times e^{-iS[j_{1a}]} e^{-2i(\sum_a (3/4j_0)\delta j^{a1})\delta i_1^x}, \quad (52)$$

where N_1 is a normalization constant with $|N_1|^2 = 1$, and $S[j_{1a}]$ is the expansion of the Regge action linked to the tetrahedron associated with the $\{6j\}$ symbol (46) up to the second order *only in the link variables*, that is

$$S[j_{na}] = \frac{\partial S_R}{\partial j_{1n}}\bigg|_{j_0, i_0} \delta j_{1n} + \frac{\partial^2 S_R}{\partial j_{1n}\partial j_{1n'}}\bigg|_{j_0, i_0} \delta j_{1n}\delta j_{1n'} + \frac{1}{2}\frac{\partial^2 S_R}{\partial^2 j_{1n}}\bigg|_{j_0, i_0} (\delta j_{1n})^2. \quad (53)$$

This result follows from the choice (29) and (30) of the parameters in (24). In particular, the value $\chi_n = \frac{\pi}{2}$ makes the intertwiner phase equal, with opposite sign, to the term $\exp - i(\frac{\partial S_R}{\partial i_1^y}|_{j_0, i_0} \delta i_1^y - \pi\delta i_1^y)$, namely, the term in the expansion of the Regge action S_R linear in the variable δi_1^y. This selects one of the two exponentials in the asymptotic expansion (46), while the rapidly oscillating phase factor in the variables δi_1^y cancels the other.

The same calculation gives the $i^z \to i^x$ change of variable

$$\Phi''_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, \ldots, i_5) = \Phi(\mathbf{j}, i_1^x, i_2, \ldots, i_5) N_1 e^{-iS'[j_{1a}]} \times e^{-2i(\sum_a (3/4j_0)\delta j^{a1})\delta i_1^x}, \quad (54)$$

with the same constant N_1 as above. The only differences between (52) and (54) is that the arguments of the 6-j symbol enter with a different order, so that $S'(j_{12}, j_{13}, j_{14}, j_{15},) = S(j_{12}, j_{13}, j_{15}, j_{14})$.

Using these results, we can explicitly rewrite the state (39) in our preferred basis. We obtain easily

$$|\Psi_{\mathbf{q}}\rangle = 4^5 \sum_{\mathbf{j}, \mathbf{i}} \Phi(\mathbf{j}, \mathbf{i}) \prod_{n=1}^{5} G[\delta j_{na}, \delta i_n] |\mathbf{j}, \mathbf{i}\rangle, \quad (55)$$

where

$$G[\delta j_{na}, \delta i_n] = \left(1 + N_1 e^{-2i(\sum_a (3/4j_0)\delta j^{an})\delta i_n^x} (e^{-iS[j_{na}]} + e^{-iS'[j_{na}]})\right). \quad (56)$$

B. Mean values and variances

With these preliminaries completed, we can now check that mean values and relative fluctuations of areas and angles have the right behavior in the large scale limit. With the notation

$$\langle O \rangle := \frac{\langle \Psi_{\mathbf{q}} | O | \Psi_{\mathbf{q}} \rangle}{\langle \Psi_{\mathbf{q}} | \Psi_{\mathbf{q}} \rangle} \quad \text{and} \quad \Delta O = \sqrt{\langle O^2 \rangle - \langle O \rangle^2} \quad (57)$$

we demand

$$\langle j_{ni} \rangle = j_0 \quad \text{and} \quad \frac{\Delta j_{ni}}{\langle j_{ni} \rangle} \to 0 \quad \text{when } \mathrm{j}_0 \to \infty, \quad (58)$$

as in [6], as well as

$$\langle i_n^{m_n} \rangle = i_0 \quad \text{and} \quad \frac{\Delta i_n^{m_n}}{\langle i_n^{m_n} \rangle} \to 0 \quad \text{when } \mathrm{j}_0 \to \infty. \quad (59)$$

Notice that we demand this for all m_n, namely, for each node *in each pairing*.

It is easy to show that the state (39) satisfies (58). Because of the vanishing of the interference terms proven above, in the large j_0 limit the mean values reduce to the average of the mean values on each diagonal term.

$$\langle j_{ni} \rangle \approx \frac{\sum_{m_n}\sum_{\mathbf{j}}\sum_{\mathbf{i}_n^{m_n}} j_{ni} |\Phi[\mathbf{j}\mathbf{i}_n^{m_n}]|^2}{\sum_{m_n}\sum_{\mathbf{j}}\sum_{\mathbf{i}_n^{m_n}} |\Phi[\mathbf{j}\mathbf{i}_n^{m_n}]|^2} \approx \frac{\sum_{m_n} \int d\delta\mathbf{j} d\delta\mathbf{i}_n^{m_n} j_{ni} e^{-(1/j_0)\sum \alpha_{(ij)(mr)}\delta j_{ij}\delta j_{mr}} e^{-\sum_n (3(\delta i_n^{m_n})^2/2j_0)}}{\sum_{m_n} \int d\delta\mathbf{j} d\delta\mathbf{i}_n^{m_n} e^{-(1/j_0)\sum \alpha_{(ij)(mr)}\delta j^{ij}\delta j_{mr}} e^{-\sum_n (3(\delta i_n^{m_n})^2/2j_0)}} = j_0. \quad (60)$$

The calculation of the variance and mean value in the intertwiner variable is a bit more complicated. It is convenient to express the state in the pairing of the relevant variable using (52) and (54). With this, we have

$$\langle i_1^x \rangle \approx \frac{\sum_{m_n \neq m_1}\sum_{\mathbf{j}}\sum_{\mathbf{i}_{n\neq 1}^{m_n}}\sum_{i_1^x} i_1^x (|\Phi_{\mathbf{q}}|^2 + |\Phi'_{\mathbf{q}}|^2 + |\Phi''_{\mathbf{q}}|^2)}{\sum_{m_n}\sum_{\mathbf{j}}\sum_{\mathbf{i}_n^{m_n}} |\Phi_{\mathbf{q}}|^2} \approx 3\frac{\sum_{m_n \neq m_1}\sum_{\mathbf{j}}\sum_{\mathbf{i}_{n\neq 1}^{m_n}}\sum_{i_1^x} i_1^x |\Phi_{\mathbf{q}}|^2}{\sum_{m_n}\sum_{\mathbf{j}}\sum_{\mathbf{i}_n^{m_n}} |\Phi_{\mathbf{q}}|^2} = \frac{\sum_{m_n \neq m_1}\sum_{\mathbf{j}}\sum_{\mathbf{i}_{n\neq 1}^{m_n}}\sum_{i_1^x} i_1^x |\Phi_{\mathbf{q}}|^2}{\sum_{m_n \neq m_1}\sum_{\mathbf{j}}\sum_{\mathbf{i}_{n\neq 1}^{m_n}}\sum_{i_1^x} |\Phi_{\mathbf{q}}|^2} = i_0, \quad (61)$$

where we have used the (52) and (54) and the fact that the constant N_1 in these expression satisfies $|N_1|^2 = 1$. The same procedure can be used to compute the variance and check that (59) is satisfied.

IV. CALCULATION OF THE PROPAGATOR

We are now ready to compute all components of the propagator (7). Consider this quantity for a fixed value of m, n, i, j, k, l. Because of the sum in (56), the propagator can be written in the form:

$$\mathbf{G}^{ij,kl}_{\mathbf{q}n,m} = 4^5 \sum_{\mathbf{j}} \sum_{\mathbf{i}_n} \Phi(\mathbf{j}, \mathbf{i}) \prod_{n=1}^{5} G[\delta j_{na}, \delta i_n] \cdot \langle W|(E_n^{(ni)} \cdot E_n^{(nj)} - n^{(ni)} \cdot n^{(nj)})(E_m^{(mk)} \cdot E_m^{(ml)} - n^{(mk)} \cdot n^{(ml)})|\mathbf{j}, \mathbf{i}_n\rangle, \tag{62}$$

For a given value of m, n, i, j, k, l, we now can fix the reference choice of pairing so that (ij) (if different) are paired at the node n and (kl) (if different) are paired at the node m. With this choice of basis the action of the operators is diagonal, and we have

$$\mathbf{G}^{ij,kl}_{\mathbf{q}n,m} = 4^5 \sum_{\mathbf{j}} \sum_{\mathbf{i}_n} \Phi(\mathbf{j}, \mathbf{i}) \prod_{n=1}^{5} G[\delta j_{na}, \delta i_n](D_n^{ij} - n^{(ni)} \cdot n^{(nj)}) \times (D_m^{kl} - n^{(mk)} \cdot n^{(ml)})\langle W|\mathbf{j}, \mathbf{i}\rangle. \tag{63}$$

We use the same form of the Barret-Crane vertex as in [4,5]. This is given by

$$\langle W|\mathbf{j}, \mathbf{i}\rangle := W(\mathbf{j}, \mathbf{i}) = W(\mathbf{j}) \prod_n \langle i_{BC}|i_n\rangle = W(\mathbf{j}) \prod_n (2i_n + 1), \tag{64}$$

where $W(\mathbf{j})$ is the Barrett-Crane vertex, which is a function of the ten spins alone. In the large-distance limit, $\prod_n(2i_n + 1) = 2i_0^5$, hence

$$W(\mathbf{j}, \mathbf{i}) = 2i_0^5 W(\mathbf{j}). \tag{65}$$

Using this, (63) becomes

$$\mathbf{G}^{ij,kl}_{\mathbf{q}n,m} = \sum_{\mathbf{j}} W(\mathbf{j}) \sum_{\mathbf{i}_n^x} \Phi(\mathbf{j}, \mathbf{i}) \prod_{n=1}^{5} G[\delta j_{na}, \delta i_n] \times (D_n^{ij} - n^{(ni)} \cdot n^{(nj)})(D_m^{kl} - n^{(mk)} \cdot n^{(ml)}), \tag{66}$$

where we have absorbed numerical factors and i_0^5 in the normalization of the state. Each factor $G[\delta j_{na}, \delta i_n]$ in this expression has the form $(1 + Ne^{iS} + Ne^{iS'})$. The terms with the exponents contain rapidly oscillating phases in the spin variables, which again suppress the integral in the large j_0 limit. Therefore we can drop these factors.

The value of the eigenvalues D_n^{ij} is given in (22). The value of the product of normals is given in (26). Using these, we have

$$D_n^{ij} - n^{(ni)} \cdot n^{(nj)} = \frac{(C(i_n) - C(i_0)) - (C(j^{(ni)}) - C(j_0)) - (C(j^{(nj)}) - C(j_0))}{2}. \tag{67}$$

Expanding up to second order around the background values j_0 and i_0

$$C(j_j) - C(j_0) = (\delta j_j)^2 + 2\delta j_j j_0 + \delta j_j, \tag{68}$$

we obtain, in the large j_0 limit

$$D_n^{ij} - n^{(ni)} \cdot n^{(nj)} = \delta i_n i_0 - \delta j_j j_0 - \delta j_{nk} j_0. \tag{69}$$

Inserting this in (66) we have

$$\mathbf{G}^{ij,kl}_{\mathbf{q}n,m} = j_0^2 \sum_{\mathbf{j}} W(\mathbf{j}) \sum_{\mathbf{i}_n^x} \left(\frac{2}{\sqrt{3}}\delta i_n - \delta j_{ni} - \delta j_{nk}\right) \times \left(\frac{2}{\sqrt{3}}\delta i_m - \delta j_{mk} - \delta j_{ml}\right)\Phi(\mathbf{j}, \mathbf{i}). \tag{70}$$

In the case in which two of the indices of the propagator are parallel, say $i = j$, this reduces easily to

$$\mathbf{G}^{ii,kl}_{\mathbf{q}n,m} = 2j_0^2 \sum_{\mathbf{j}} W(\mathbf{j}) \sum_{\mathbf{i}_n^x} \delta j_{ni} \left(\frac{2}{\sqrt{3}}\delta i_m - \delta j_{mk} - \delta j_{ml}\right) \times \Phi(\mathbf{j}, \mathbf{i}). \tag{71}$$

While if $i = j$ and $k = l$ we recover the diagonal terms,

$$\mathbf{G}^{ii,kk}_{\mathbf{q}n,m} = 4j_0^2 \sum_{\mathbf{j}} W(\mathbf{j}) \sum_{\mathbf{i}_n^x} \delta j_{ni} \delta j_{mk} \Phi(\mathbf{j}, \mathbf{i}). \tag{72}$$

We can now evaluate (70). Inserting the explicit form of the state gives

$$\mathbf{G}^{ij,kl}_{\mathbf{q}n,m} = Cj_0^2 \sum_{\delta\mathbf{j}, \delta\mathbf{i}} W(\mathbf{j}) \left(\frac{2}{\sqrt{3}}\delta i_n - \delta j_{ni} - \delta j_{nk}\right) \times \left(\frac{2}{\sqrt{3}}\delta i_m - \delta j_{mk} - \delta j_{ml}\right) \cdot e^{-(1/2j_0)\sum \alpha_{(ij)(mr)}\delta j_{ij}\delta j_{mr} + i\sum \Phi \delta j_{ij}} \times e^{-\sum_n ((3(\delta i_n)^2/4j_0) - i(\sum_a (3/4j_0)\delta j_{an} + (\pi/2))\delta i_n)}. \tag{73}$$

Using the asymptotic expression for the BC vertex, we can proceed as in [4,5]. The rapidly oscillating phase term in the spins selects one of the factors of this expansion, giving

$$\mathbf{G}^{ij,kl}_{\mathbf{q}n,m} = \mathcal{N} j_0^2 \sum_{\delta j^{(ab)},\delta i_\alpha} \prod_{a<b} \dim(j^{(ab)}) \Big(\frac{2}{\sqrt{3}}\delta i_n - \delta j_{ni} - \delta j_{nk}\Big) \times \Big(\frac{2}{\sqrt{3}}\delta i_m - \delta j_{mk} - \delta j_{ml}\Big) \cdot e^{-(1/2j_0)(\alpha + iGj_0)_{(ij)(mn)}\delta j_{ij}\delta j_{mn}} \times e^{-\sum_n ((3(\delta i_n)^2/4j_0) - i(\sum_a (3/4j_0)\delta j^{an} + (\pi/2))\delta i_n)}, \quad (74)$$

where the phase factor $i\Phi\sum_{pq} j_{pq}$ in (73) has been absorbed by the corresponding phase factor in the asymptotic expansion of the $10j$ symbol $W(\mathbf{j})$ (see [22,23]), as in [5,6]. Here G is the matrix of the second derivatives of the Regge action (see [5,6]) and should not be confused with the G used in the appendixes. Finally,

$$\mathbf{G}^{ij,kl}_{\mathbf{q}n,m} = \mathcal{N}' j_0^2 \sum_{\delta j^{(ab)},\delta i_\alpha} \Big(\frac{2}{\sqrt{3}}\delta i_n - \delta j_{ni} - \delta j_{nk}\Big) \times \Big(\frac{2}{\sqrt{3}}\delta i_m - \delta j_{mk} - \delta j_{ml}\Big) \cdot e^{-(1/2j_0)(\alpha + iGj_0)_{(ij)(mn)}\delta j_{ij}\delta j_{mn}} \times e^{-\sum_n ((3(\delta i_n)^2/4j_0) - i(\sum_a (3/4j_0)\delta j_{an} + (\pi/2))\delta i_n)}. \quad (75)$$

We can rearrange this expression introducing the 15 components vector $\delta I^\alpha = (\delta j^{ab}, \delta i_n)$ and $\Theta^\alpha = (0, \chi_{i_n})$ and the 15×15 correlation matrix

$$M = \begin{pmatrix} A_{10\times 10} & C_{10\times 5} \\ C^T_{5\times 10} & S_{5\times 5} \end{pmatrix}, \quad (76)$$

where $A_{abcd} = \frac{1}{2}(\alpha + iGj_0)_{abcd}$ is a 10×10 matrix and $S_{nm} = I_{nm}\frac{3}{4}$ is a diagonal 5×5 matrix and C is a 10×5 matrix and C^T is its transpose, and evaluate it approximating the sum with an integral

$$\mathbf{G}^{ij,kl}_{\mathbf{q}n,m} = \mathcal{N}' j_0^2 \int d\delta I^\alpha \Big(\frac{2}{\sqrt{3}}\delta i_n - \delta j_{ni} - \delta j_j\Big) \times \Big(\frac{2}{\sqrt{3}}\delta i_m - \delta j_{mk} - \delta j_{ml}\Big) \times e^{-(M_{\alpha\beta}/j_0)\delta I^\alpha \delta I^\beta} e^{i\Theta_\alpha \delta I^\alpha}. \quad (77)$$

The matrix M is invertible and independent from j_0. Direct calculation using (G5) gives a sum of terms of the kind

$$\frac{e^{-j_0\Theta M^{-1}\Theta}}{\sqrt{\det M}} (j_0^3 M^{-1}_{\alpha\beta} - j_0^4 M^{-1}_{\alpha\gamma}\Theta^\gamma M^{-1}_{\beta\delta}\Theta^\delta). \quad (78)$$

These terms go to zero fast in the $j_0 \to \infty$ limit, and therefore do not match the expected large-distance behavior of the propagator.

One could hope to circumvent the problem behavior thanks to the normalization factor. Including this explicitly we have

$$\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}n,m} = \frac{\langle W|(E_n^{(ni)} \cdot E_n^{(nj)} - n^{(ni)} \cdot n^{(nj)})(E_m^{(mk)} \cdot E_m^{(ml)} - n^{(mk)} \cdot n^{(ml)})|\Psi_\mathbf{q}\rangle}{\langle W|\Psi_\mathbf{q}\rangle}. \quad (79)$$

The denominator gives

$$\langle W|\Psi_\mathbf{q}\rangle = \frac{e^{-j_0\Theta M^{-1}\Theta}}{\sqrt{\det M}}. \quad (80)$$

Terms of the kind (78) are still pathological, since they give

$$\Big(\frac{M^{-1}_{\alpha\beta}}{j_0} - M^{-1}_{\alpha\gamma}\Theta^\gamma M^{-1}_{\beta\delta}\Theta^\delta\Big) \quad (81)$$

in the limit. In conclusion, the calculation presented does not appear to give the correct low energy propagator.

V. CONCLUSIONS

The calculation presented above is based on a number of assumptions on the form of the boundary state. Could the negative result that we have obtained be simply the result of these assumptions being too strict, or otherwise wrong? Could, in particular, a different boundary state give the correct low energy behavior? Although we do not have any real proof, we do not think that this is the case. The original aim of the research program motivating this article was to find such a state; the negative result we report here has initially come as a disappointment, and we have fought against it at length. We have eventually reached the conclusion that the problem is more substantial, and is related to the BC vertex itself, at least as it is used in the present approach. There are several indications pointing to this conclusion.

First, the trivial intertwiner dependence of the Barrett-Crane structure clashes with the intertwiner dependence of the boundary state that is needed to have a good semiclassical behavior. Since the variables associated to the angles between faces do not commute with one another, the boundary state cannot be sharp on a classical configuration. In order for a state peaked on a given angle to be also peaked on the other noncommuting angles, the state must have a phase dependence from intertwiners and spin variables. Following the general structure of quantum mechanics, one then expect the transition amplitude matching between coherent states to include a phase factor exactly balancing those phases. This is the case for instance for the free propagator of nonrelativistic quantum particles, as well as for the phases associated to the angles between tetrahedra in the calculation illustrated in [4,5]. However, no such phase factor appears in the BC vertex. In particular, the phase factor $i(\pi/2)\sum_p i_p$ present in the boundary state (necessary to have the complete symmetry of the state) is not matched by a corresponding factor in the vertex amplitude. This factor gives the rapidly oscillating term that suppresses the sum.

Second, as already mentioned, there is in fact a structural difficulty, already pointed out in [4,5], with the definition (64) of the amplitude, and we think that this difficulty is at the roots of the problem. Let us illustrate this difficulty in detail.

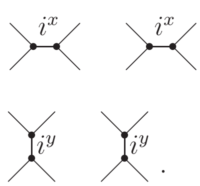
There are two possible interpretations of Eq. (64). The first is that this is true for one particular basis, namely, for $i_n = i_n^x$. Let us discard this possibility, which would imply that the BC the vertex itself would depend on a specific choice of pairing. The second is that it is (simultaneously) true in all possible bases, that is

$$\langle W|\mathbf{j}, \mathbf{i}^{m_n}\rangle = W(\mathbf{j})\prod_n (2i^{m_n} + 1) \tag{82}$$

for any choice of pairing, namely, for any choice of m_n. This is indeed the definition of the vertex that we have implicitly used. However, defined in this way, the vertex $\langle W|$ is not a *linear* functional on the state space. This is immediately evident by expressing, say $\langle i_1^y|$ on the $\langle i_1^x|$ basis.

We can say this in other words. The Barrett-Crane intertwiner is defined as a sum of simple $SO(4)$ intertwiners, that we can write as

$$\begin{aligned} i_{\rm BC} &= \sum_{i^x} (2i^x+1)|i^x, i^x\rangle \\ &= \sum_{i^y} (2i^y+1)|i^y, i^y\rangle \\ &= \sum_{i^x} (2i^x+1) \end{aligned}$$



$$= \sum_{i^y} (2i^y+1) \quad . \tag{83}$$

Hence

$$\langle i_{BC}|i^m, i^m\rangle = (2i^m + 1) \tag{84}$$

whatever is m. Since the simple $SO(4)$ intertwiner $|i^x, i^x\rangle$ diagonalizes the same geometrical quantity as the $SO(3)$ intertwiner $|i^x\rangle$, it is tempting to physically identify the two and write

$$\langle i_{BC}|i^m\rangle = (2i^m + 1). \tag{85}$$

But there is no state $\langle i_{BC}|$ in the $SO(3)$ intertwiner space that has this property. In other words, there is a mismatch between the linear structures of $SO(4)$ and $SO(3)$ in building up the theory that we have used.

In principle, the second difficulty could be circumvented by abandoning the standard canonical $SO(3)$ LQG structure, and its graviton operators, and replacing it with a purely $SO(4)$ one. We expect the first difficulty to still prevent this from working, but we have no definite result in this direction.

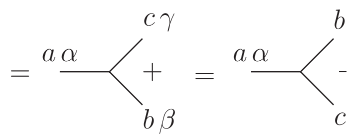
In another paper [13], we show that, perhaps surprisingly, a vertex with a suitable asymptotic behavior can overcame all these difficulties.

APPENDIX A: RECOUPLING THEORY

We give here the definitions at the basis of recoupling theory and the graphical notation that is used in the text. Our main reference source is [24].

(i) *Wigner* $3j$ *symbols.*—These are represented by a 3-valent node, the three lines stand for the angular momenta which are coupled by the $3j$ symbol. We denote the anticlockwise orientation with a + sign and the clockwise orientation with a sign −. in index notation $v^{\alpha\beta\gamma}$:

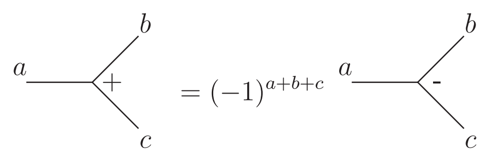
$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \tag{A1}$$



The symmetry relation $v^{\alpha\beta\gamma} = (-1)^{a+b+c} v^{\alpha\gamma\beta}$

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-1)^{a+b+c}\begin{pmatrix} a & c & b \\ \alpha & \gamma & \beta \end{pmatrix} \tag{A2}$$

implies



(A3)

(ii) *The Kroneker delta.*

$$\delta_{ab}\,\delta^{\alpha}_{\beta} = \quad a\alpha \underline{\qquad\qquad} b\beta. \tag{A4}$$

(iii) *Antisymmetric or "metric" tensor.* (1-j symbol.)—In vector notation: ${}^a\epsilon_{\alpha\beta}$

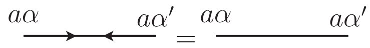
$$\begin{pmatrix} a \\ \alpha\beta \end{pmatrix} = (-1)^{a+\alpha}\delta_{\alpha-\beta} \tag{A5}$$

in graphical notation:

$$\delta_{ab}\begin{pmatrix} a \\ \alpha\beta \end{pmatrix} = \tag{A6}$$

the relations $\epsilon^{\alpha'\beta}\epsilon_{\alpha\beta} = \delta^{\alpha'}{}_{\alpha}$ and $\epsilon^{\alpha'\beta}\epsilon_{\beta\alpha} = -\delta^{\alpha'}{}_{\alpha}$, for the fundamental representation, read, for generic representations

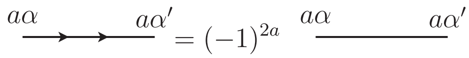
$$\sum_\beta \begin{pmatrix} a \\ \alpha'\beta \end{pmatrix}\begin{pmatrix} a \\ \alpha\beta \end{pmatrix} = \delta^{\alpha'}{}_\alpha \qquad \text{(A7)}$$



(A8)

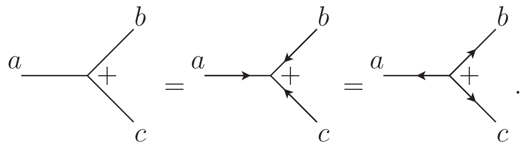
and

$$\sum_\beta \begin{pmatrix} a \\ \alpha'\beta \end{pmatrix}\begin{pmatrix} a \\ \beta\alpha \end{pmatrix} = (-1)^{2a}\delta^{\alpha'}{}_\alpha \qquad \text{(A9)}$$



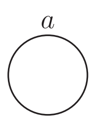
(A10)

From the properties of the $3j$ symbols it follows: in vector notation: $v^{\alpha\beta\gamma} = v_{\alpha\beta\gamma}$; in graphical notation:



(A11)

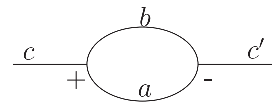
Trace of the identity

$^a\delta^\alpha{}_\alpha =$ 

$= 2a+1$ (A12)

(iv) *First orthogonality relation for* $3j$ *symbols.*

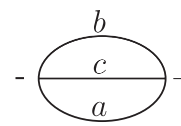
$$\sum_{\alpha,\beta}\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}\begin{pmatrix} a & b & c' \\ \alpha & \beta & \gamma' \end{pmatrix} = \frac{1}{2c+1}\delta_{cc'}\delta^\gamma{}_{\gamma'} \qquad \text{(A13)}$$



$= \frac{1}{2c+1}$ 

(A14)

This implies

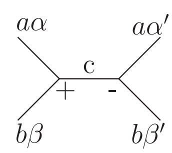


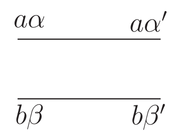
$= 1$ (A15)

(v) *Second orthogonality relation.*

$$\sum_{c\gamma}(2c+1)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}\begin{pmatrix} a & b & c \\ \alpha' & \beta' & \gamma \end{pmatrix} = \delta^\alpha{}_{\alpha'}\delta^\beta{}_{\beta'}. \qquad \text{(A16)}$$

Graphically

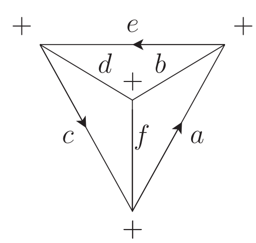
$\sum_c (2c+1)$ 

$=$ 

(A17)

(vi) *6j symbol.*

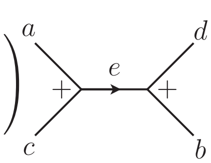
$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = \sum_{\alpha\epsilon\gamma}(-1)^{a+e+c-\alpha-\epsilon-\gamma}\begin{pmatrix} a & f & c \\ \alpha & \phi & -\gamma \end{pmatrix}\begin{pmatrix} c & d & e \\ \gamma & \delta & -\epsilon \end{pmatrix}\begin{pmatrix} e & b & a \\ \epsilon & \beta & -\alpha \end{pmatrix}\begin{pmatrix} b & d & f \\ \beta & \delta & \phi \end{pmatrix}$$



(A18)

(vii) *The* $4j$ *coefficient, or 4-valent node.*

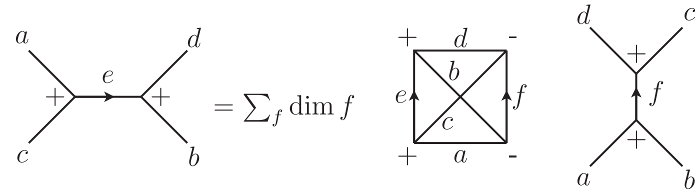
$$\begin{pmatrix} a & c & b & d \\ \alpha & \gamma & \beta & \delta \end{pmatrix} = \sum\nolimits_\epsilon (-1)^{e-\epsilon}\begin{pmatrix} e & a & c \\ \epsilon & \alpha & \gamma \end{pmatrix}\begin{pmatrix} e & b & d \\ -\epsilon & \beta & \delta \end{pmatrix}$$



(A19)

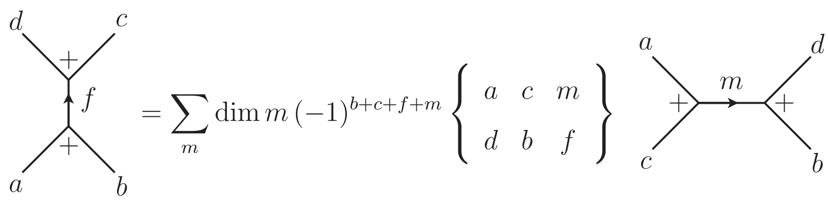
(viii) *Recoupling theorem.*

$$\begin{pmatrix} a & c & b & d \\ \alpha & \gamma & \beta & \delta \end{pmatrix} = \sum_f \dim f \, (-1)^{b+c+e+f} \begin{Bmatrix} a & b & f \\ d & c & e \end{Bmatrix} \begin{pmatrix} a & b & c & d \\ \alpha & \beta & \gamma & \delta \end{pmatrix}$$



(A20)

(ix) *Inverse transformation.*



(A21)

(x) *Orthogonality relation for the* $6j$ *symbols.*

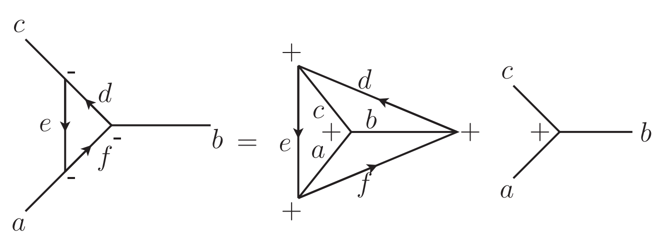
$$\sum_f \dim m \dim f \begin{Bmatrix} a & b & f \\ d & c & e \end{Bmatrix} \begin{Bmatrix} a & c & m \\ d & b & f \end{Bmatrix} = \delta_{em}. \tag{A22}$$

(xi) *Biedenharn-Elliot identity.*

$$\sum_x \dim x (-1)^{a+b+c+d+e+f+g+h+i+x} \begin{Bmatrix} e & f & x \\ b & a & i \end{Bmatrix} \begin{Bmatrix} a & b & x \\ c & d & h \end{Bmatrix} \begin{Bmatrix} d & c & x \\ f & e & g \end{Bmatrix} = \begin{Bmatrix} g & h & i \\ a & e & d \end{Bmatrix} \begin{Bmatrix} g & h & i \\ b & f & c \end{Bmatrix}. \tag{A23}$$

(xii) *The "basic rule."*

$$\sum_{\delta\epsilon\phi} (-1)^{d+e+f-\delta-\epsilon-\phi} \begin{pmatrix} d & e & c \\ -\delta & \epsilon & \gamma \end{pmatrix} \begin{pmatrix} e & f & a \\ -\epsilon & \phi & \alpha \end{pmatrix} \begin{pmatrix} f & d & b \\ -\phi & \delta & \beta \end{pmatrix} = \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}. \tag{A24}$$



(A25)

APPENDIX B: ANALYTIC EXPRESSIONS FOR $6j$ SYMBOLS

From [24].

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-1)^{a+b+c+d} \Delta(a,b,e)\Delta(a,c,f)\Delta(b,d,f)\Delta(c,d,e) \sum_z (-1)^z \frac{f(z)}{z!} \tag{B1}$$

where

$$\Delta(a,b,c) = \sqrt{\frac{(a+b-c)!(a+c-b)!(b+c-a)!}{(a+b+c+1)!}} \tag{B2}$$

and

$$f(z) = \frac{(a+b+c+d+1-z)!}{(e+f-a-d+z)!(e+f-b-c+z)!(a+b-e-z)!(c+d-e-z)!(a+c-f-z)!(b+d-e-f)!} \tag{B3}$$

The sum is extended to all the positive integers z, such that no factorial has negative argument.

The definition (B1) implies some restrictions on the arguments of the $6j$:

In particular the $\Delta(a,b,c)$ restricts the arguments to satisfy the triangle inequalities

$$(a+b-c)\geq 0 \qquad (a-b+c)\geq 0 \qquad (-a+b+c)\geq 0 \tag{B4}$$

and $a+b+c$ has to be an integer number.

The expression (B1) reduces to the following simple expressions used in the calculation

$$\begin{Bmatrix} a & a & 1 \\ b & b & e \end{Bmatrix} = \frac{(-1)^{a+b+e+1}}{2} \frac{C^2(a)+C^2(b)-C^2(e)}{\sqrt{C^2(a)\dim(a)C^2(b)\dim(b)}} \tag{B5}$$

$$\begin{Bmatrix} e & e-1 & 1 \\ a & a & b \end{Bmatrix} = \frac{(-1)^{a+b+e}}{2} \sqrt{\frac{(a+b+e+1)(a-b+e)(-a+b+e)(a+b-e+1)}{C^2(a)\dim(a)e\dim(e)\dim(e-1)}} \tag{B6}$$

$$\begin{Bmatrix} e & e+1 & 1 \\ a & a & b \end{Bmatrix} = \frac{(-1)^{a+b+e+1}}{2} \sqrt{\frac{(a+b+e+2)(a-b+e+1)(-a+b+e+1)(a+b-e)}{C^2(a)\dim(a)(e+1)\dim(e)\dim(e+1)}}. \tag{B7}$$

The $6j$ symbol is invariant for interchange of any two columns, and also for interchange of the upper and lower arguments in each of any two columns:

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = \begin{Bmatrix} a & e & b \\ d & f & c \end{Bmatrix} = \begin{Bmatrix} e & a & b \\ f & d & c \end{Bmatrix} = \begin{Bmatrix} a & c & f \\ d & b & e \end{Bmatrix} = \begin{Bmatrix} d & c & e \\ a & b & f \end{Bmatrix}, \text{ etc.} \tag{B8}$$

We have also used the trivial facts

$$(-1)^a = (-1)^{-a} \qquad \forall\ a\in\mathbb{Z}, \qquad (-1)^{2a} = 1$$
$$\forall\ a\in\mathbb{Z}, \qquad (-1)^{3s} = (-1)^{-s} \qquad \forall\ s\in\frac{\mathbb{Z}}{2}$$

in the calculations involving the $6j$ symbols

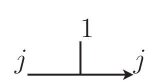
APPENDIX C: GRASPING OPERATORS

The operator $E^a(\vec{x})n_a^{(ni)}$ is the "grasping operator" that acts on the spin network's link dual to the triangle with normal $n_a^{(ni)}$. Let say that this link is in the j representation; $E^a(\vec{x})n_a^{ni}$ will acts inserting an $SU(2)$ generator in the same representation [2] or equivalently, by inserting an intertwiner between the (j) rep and the rep 1, namely, a $3j$ symbol not normalized:

$$E^{(ni)}(\vec{x})^{i\alpha}{}_{\beta} = i^{(j)}J^{i\alpha}{}_{\beta} = iN^j v^{i\alpha}{}_{\beta} \tag{C1}$$

where ${}^{(j)}J^{i\alpha}{}_{\beta}$ is the $SU(2)$ generator in the j representation $(i=-1,0,1)$, $(\alpha,\beta=-j,\ldots,j)$, N^j is a normalization factor and $v^{i\alpha\beta}$ is the normalized $3j$ symbol. The action of the operator $E^{(ni)}$ is then determined by the representation of the links on which it acts; in the following we will call $E^{(j)}$ an operator acting on the link with rep j.

Graphically, with our conventions

$E_n^{(j)} = i\,N^{(j)}$  (C2)

(Note the arrow that reflect the lowered magnetic index.)

To fix the normalization factor N^j is enough to square the expression (C1), use (A14)

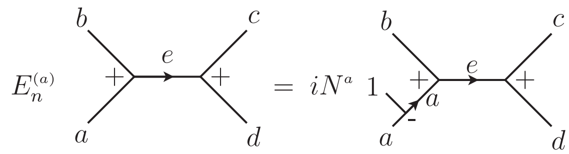
$${}^{j}J^{2\alpha}{}_{\beta} = C^2(j)^{j}I^{\alpha}{}_{\beta} = \frac{(N^j)^2}{\dim j} I^{\alpha}{}_{\beta}j \tag{C3}$$

and take the trace of the previous equation (where ${}^{j}I^{\alpha}{}_{\beta}$ is the identity in the rep j), obtaining

$$N^j = \sqrt{j(j+1)\dim j}. \quad \text{(C4)}$$

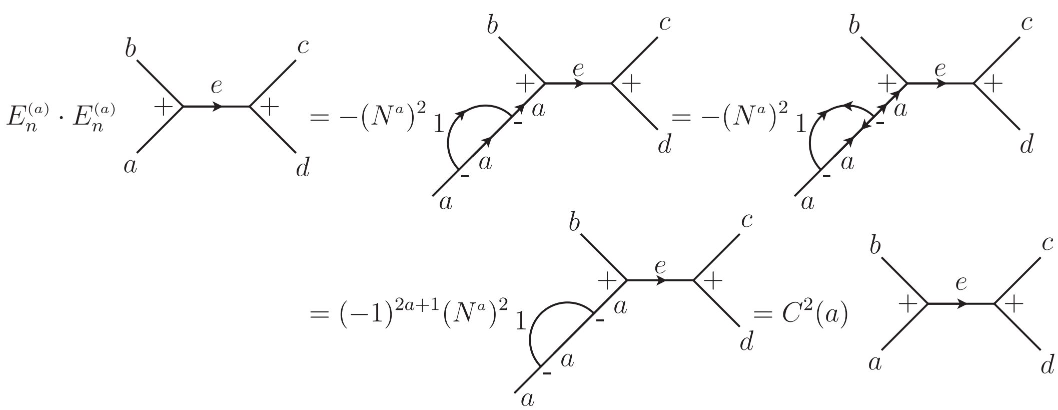
Our triangulated manifold consist of a 4 simplex made of 5 tetrahedron t_n, bounded by triangles t_{nm}. In the dual picture the 4 simplex is represented by the pentagonal net where the tetrahedra are the 4-valent nodes n, labeled by the intertwiners i_n in a given pairing, and the triangles are the links nm labeled by the spin numbers j^{nm}.

In our calculation we act with the operator $E^a(\vec{x})n_a^{(nl)}$ on the tetrahedron t_n in the direction $n_a^{(nl)}$ orthogonal to the triangle t_{nl}; in the dual picture we are then acting on the 4-valent nodes n and precisely on the link j^{ni}. To enlighten the notation, fixed a node n, we will call the four possible colorings corresponding to the 4 directions ni with a, b, c, d where the letter indicates the representation of the links. Graphically the action of a single grasping operator operating on the link a for example is



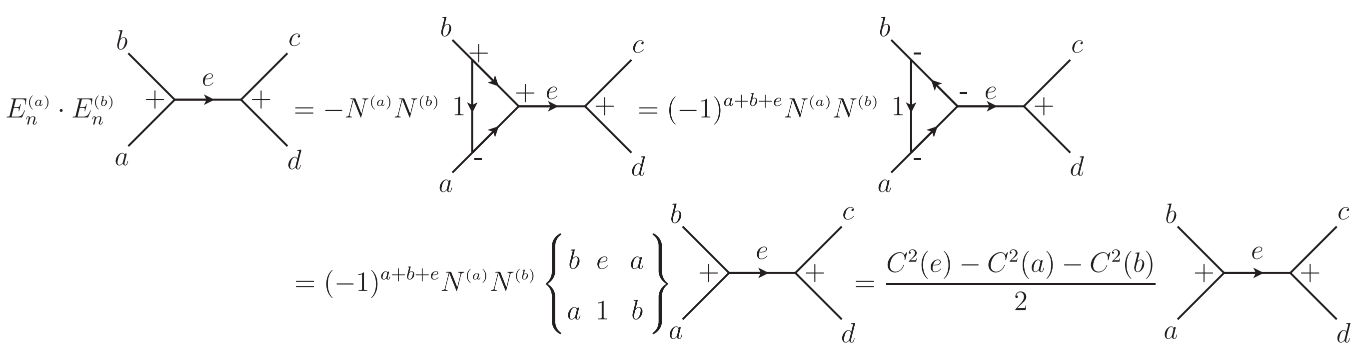
(C5)

The action of our operators $E_n^{(ni)} \cdot E_n^{(nj)}$ on a node in a fixed pairing can then produce four different results depending on the two directions n^{ni},n^{nj}



(C6)

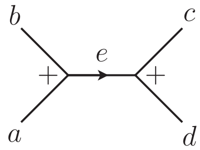
where in the last equalities we have used the relation (A8), (A10), and (A11) to eliminate the arrows and the (A3) to solve the loop using (A14). The other possible case is

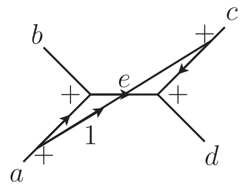


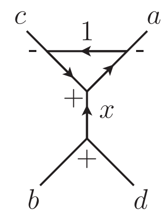
(C7)

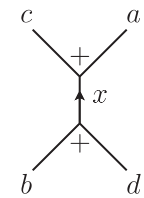
where we have changed the orientations of the 3-valent nodes to simplify the loop, using the basic identity (A25), and used the symmetry properties of $6j$ symbols and its explicit expression (B5).

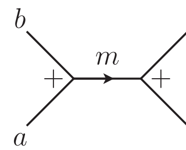
The other possible action is

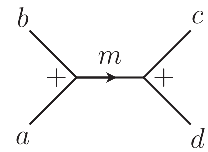
$$E_n^{(a)} \cdot E_n^{(c)}$$ 

$$= -N^{(a)}N^{(c)}$$ 

$$= N^{(a)}N^{(c)} \sum_x (-1)^{a+d+e+x} \dim x \begin{Bmatrix} b & d & x \\ c & a & e \end{Bmatrix}$$ 

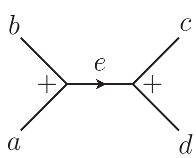
$$= N^{(a)}N^{(c)} \sum_x (-1)^{a+c+x}(-1)^{a+d+e+x} \dim x \begin{Bmatrix} b & d & x \\ c & a & e \end{Bmatrix}\begin{Bmatrix} c & a & x \\ a & c & 1 \end{Bmatrix}$$ 

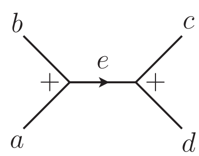
$$= N^{(a)}N^{(c)} \sum_x (-1)^{a+c+x}(-1)^{a+d+e+x} \dim x \sum_m \dim m (-1)^{a+d+m+x} \cdot$$
$$\cdot \begin{Bmatrix} b & d & x \\ c & a & e \end{Bmatrix}\begin{Bmatrix} c & a & x \\ a & c & 1 \end{Bmatrix}\begin{Bmatrix} b & a & m \\ c & d & x \end{Bmatrix}$$ 

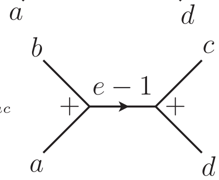
$$= -N^{(a)}N^{(c)}(-1)^{3d+a+b-c} \sum_m \dim m \begin{Bmatrix} e & m & 1 \\ a & a & b \end{Bmatrix}\begin{Bmatrix} e & m & 1 \\ c & c & d \end{Bmatrix}$$ 

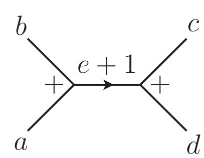
(C8)

In the derivation of the result we have used, in order, the recoupling theorem (A21) to change the pairing of the node, the basic rule (A25) to solve the loop, the inverse transformation (A21) to put the graph on the starting pairing and the Biedenharn-Elliot identity (A23), having adjusted the sign factors, using the triangles inequalities of the $3j$ symbols defining the $6j$. To analyze the result we have to look at the existence conditions of the $\{6j\}$ (Appendix B) concluding that m can only take the values $e-1$, e, $e+1$, the final result is then

$$E_n^{(a)} \cdot E_n^{(c)}$$ 

$$= X_e^{ac}$$ 

$$+ Y_e^{ac}$$ 

$$+ Z_e^{ac}$$ 

(C9)

The form of the coefficient form is easily calculated inserting the explicit expression of the $\{6j\}$ symbols given in Appendix B

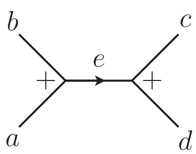
$$X_e^{ac} = -N^{(a)}N^{(c)}(-1)^{3d+a+b-c} \dim(e)\begin{Bmatrix} e & e & 1 \\ a & a & b \end{Bmatrix}\begin{Bmatrix} e & e & 1 \\ c & c & d \end{Bmatrix}$$
$$= -\frac{(-1)^{2(a+b+e)}}{4} \frac{(C^2(b) - C^2(a) - C^2(e))(C^2(d) - C^2(c) - C^2(e))}{C^2(e)} \qquad \text{(C10)}$$

$$
\begin{aligned}
Y_e^{ac} &= -N^{(a)}N^{(c)}(-1)^{3d+a+b-c}\dim(e-1)\begin{Bmatrix} e & e-1 & 1 \\ a & a & b \end{Bmatrix}\begin{Bmatrix} e & e-1 & 1 \\ c & c & d \end{Bmatrix} \\
&= -\frac{(-1)^{2(a+b+e)}}{4e\dim(e)}\sqrt{(a+b+e+1)(a-b+e)(-a+b+e)(a+b-e+1)} \\
&\quad \cdot \sqrt{(c+d+e+1)(-c+d+e)(c-d+e)(c+d-e+1)}
\end{aligned} \tag{C11}
$$

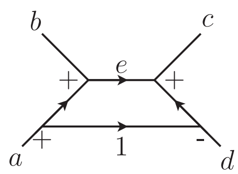
$$
\begin{aligned}
Z_e^{ac} &= -N^{(a)}N^{(c)}(-1)^{3d+a+b-c}\dim(e+1)\begin{Bmatrix} e & e+1 & 1 \\ a & a & b \end{Bmatrix}\begin{Bmatrix} e & e+1 & 1 \\ c & c & d \end{Bmatrix} \\
&= -\frac{(-1)^{2(a+b+e+1)}}{4(e+1)\dim(e)}\sqrt{(a+b+e+2)(a-b+e+1)(-a+b+e+1)(a+b-e)} \\
&\quad \cdot \sqrt{(c+d+e+2)(-c+d+e+1)(c-d+e+1)(c+d-e)}.
\end{aligned} \tag{C12}
$$

Note that by definition $(a+b+e)$ is an integer, so there are not sign factors appearing in these expressions. The last term is

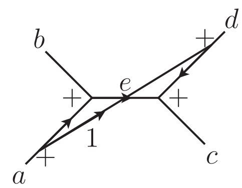
$E_n^{(a)} \cdot E_n^{(d)}$



$= -N^{(a)}N^{(d)}$

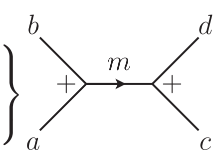


$= -N^{(a)}N^{(d)}(-1)^{c+d+e}$



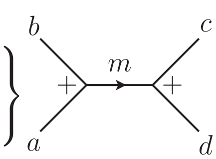
$$= -N^{(a)}N^{(d)}(-1)^{3c+a+b-d}(-1)^{c+d+e}$$

$$\times \sum_m \dim m \begin{Bmatrix} e & m & 1 \\ a & a & b \end{Bmatrix}\begin{Bmatrix} e & m & 1 \\ d & d & c \end{Bmatrix}$$



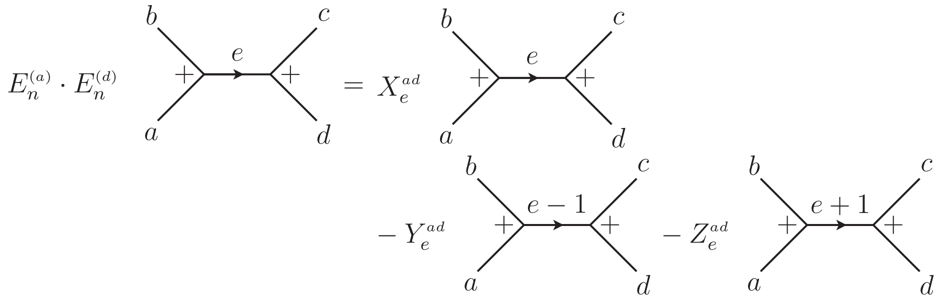
$$= -N^{(a)}N^{(d)}(-1)^{a+b+e}$$

$$\times \sum_m (-1)^{c+d+m} \dim m \begin{Bmatrix} e & m & 1 \\ a & a & b \end{Bmatrix}\begin{Bmatrix} e & m & 1 \\ d & d & c \end{Bmatrix} \tag{C13}$$



,

The result is obtained flipping the two link's c and d to recast the graph in the form (C8), using the previous result and flipping back the graph in the summation. Keeping in mind that the product of $\{6j\}$ appearing in the nondiagonal terms is left unchanged by the change $c \to d$, the final result is then the same as (C9) apart from the sign of the nondiagonal terms and the change $c \to d$ in the diagonal one



(C14)

where

$$X_e^{ad} = -N^{(a)}N^{(d)}(-1)^{a+b+c+d+2e}\dim(e)\begin{Bmatrix} e & e & 1 \\ a & a & b \end{Bmatrix}\begin{Bmatrix} e & e & 1 \\ d & d & c \end{Bmatrix} = -\frac{1}{4}\frac{(C^2(b)-C^2(a)-C^2(e))(C^2(c)-C^2(d)-C^2(e))}{C^2(e)}. \tag{C15}$$

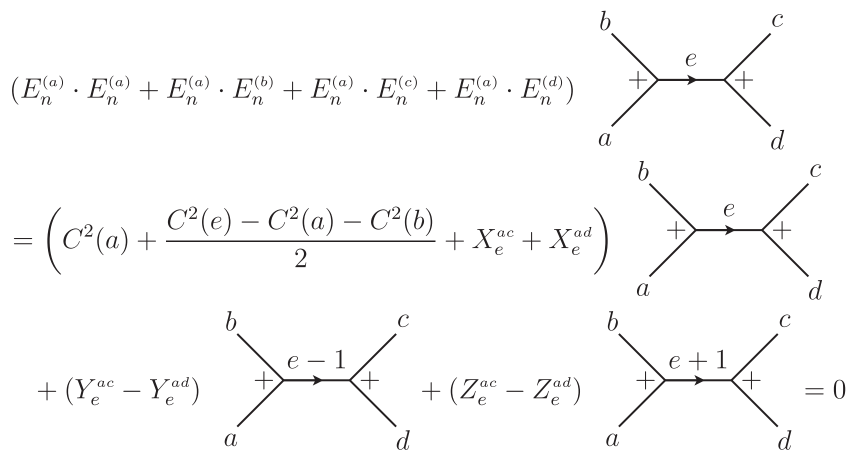
Note that by definition

$$Y_e^{ac} = Y_e^{ad} \qquad Z_e^{ac} = Z_e^{ad}. \tag{C16}$$

The operators that we have calculated have to satisfy

$$E_n^{(a)} \cdot E_n^{(a)} + E_n^{(a)} \cdot E_n^{(b)} + E_n^{(a)} \cdot E_n^{(c)} + E_n^{(a)} \cdot E_n^{(d)} = 0 \tag{C17}$$

as a direct consequence of (8) which, at quantum level, implies that a four-valent node (by definition an intertwiner) is invariant under the action of the group. A direct calculation on our four-valent node shows that this is indeed the case



(C18)

being 0 the coefficient of all the states.

APPENDIX D: NORMALIZATION OF THE SPIN-NETWORK STATES

Following [2], we define a spin network $S = (\Gamma, j_l, i_n)$ as given by a graph Γ with a given orientation (or ordering of the links) with L links and N nodes, and by a representation j_l associated to each to each link and an intertwiner i_n to each node. As a functional of the connection, a spin-network state is given by

$$\Psi_S[A] = \langle A|S\rangle \equiv (\otimes_l R^{j_l}(H[A,\gamma_l])) \cdot (\otimes_n i_n) \tag{D1}$$

where the notation $\cdot$ indicates the contraction between dual spaces and $R^{j_l}(H[A,\gamma_l])$ is the j_l representation of the holonomy group element $H[A,\gamma_l]$ along the curve γ_l of the gravitation field connection A. In the paper we have used states normalized in such a way that

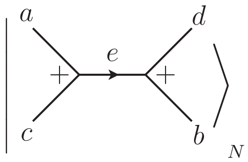
$$\langle S|S'\rangle = \delta_{S,S'}. \tag{D2}$$

Following [25,26] we can see that the scalar product reduces to the evaluation of the spin network and that the definition of the spin-network state has to be properly normalized in order for (D2) to be satisfied. Here we have used three-valent intertwiners [$3j$ Wigner symbols

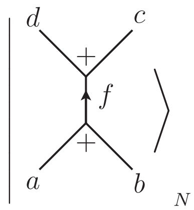
(A1)] normalized to 1, so that the evaluation of the theta graph gives 1: see (A15). This means that the formula (8.7) of [25] defining a normalized spin-network state in our case reads

$$|S\rangle_N = \sqrt{\prod_{e\in\mathcal{E}} \dim j_e}\,|S\rangle, \tag{D3}$$

where $\mathcal{E}$ is the set of real and virtual edges (intertwiner links of the decomposition of multivalent nodes). We can then see that the recoupling theorem (A21) when applied to the spin-network normalized state becomes



$$= \sum_f \sqrt{\dim e}\sqrt{\dim f}(-1)^{b+c+e+f} \times \begin{Bmatrix} a & b & f \\ d & c & e \end{Bmatrix}$$



(D4)

APPENDIX E: REGGE ACTION AND ITS DERIVATIVES

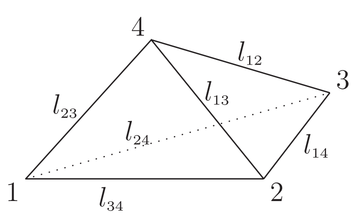
Following [27], we can write the asymptotic formula of a $6j$ symbol as

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \approx \frac{1}{\sqrt{12\pi V}} \cos\left(S_R + \frac{\pi}{4}\right) \tag{E1}$$

where

$$S_R = \sum_{i,j=1}^{4} l_{ij}\phi_{ij} \tag{E2}$$

where S_R is the Regge action of the tetrahedron



(E3)

associated to the $6j$ symbol, and $\phi_{ij} = \phi_{ji}$ $(i \neq j)$ are the dihedral angle at the edge l_{ij}. The edge lengths in terms of the $6j$ entries are $l_{12} = a + \frac{1}{2}$, $l_{13} = b + \frac{1}{2}$, $l_{14} = c + \frac{1}{2}$, $l_{34} = d + \frac{1}{2}$, $l_{23} = b + \frac{1}{2}$ and $l_{hh} = 0$, $l_{hk} = l_{kh}$.

The dihedral angles can be expressed in terms of the volume and the areas of the tetrahedron

$$A_i A_j \sin\phi_{ij} = \frac{3}{2} l_{ij} V \tag{E4}$$

where A_i is the area of the triangle opposite to the vertex i (A_i, A_j are the areas of the triangles that share the edge l_{ij}). We are interested in the expansion of the Regge action in the variables l_{ij}; we can express everything in terms of the edge length expressing the volume and the areas using the formula

$$V_d^2 = \frac{(-1)^{d+1}}{2^d (d!)^2} \det C_d \tag{E5}$$

where V_d is the volume of a simplex of dimension d and C_d is the Cayley matrix of dimension d; in particular, given 6 edges for the tetrahedron or 3 for the triangle, with the following Cayley matrix we can calculate all the quantities appearing in (E4)

$$C_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & l_1^2 & l_2^2 & l_3^2 \\ 1 & l_1^2 & 0 & l_4^2 & l_5^2 \\ 1 & l_2^2 & l_4^2 & 0 & l_6^2 \\ 1 & l_3^2 & l_5^2 & l_6^2 & 0 \end{pmatrix}$$
$$C_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & l_1^2 & l_2^2 \\ 1 & l_1^2 & 0 & l_3^2 \\ 1 & l_2^2 & l_3^2 & 0 \end{pmatrix}. \tag{E6}$$

We are interested in the asymptotic expansion of the $6j$ symbol that realizes the change of pairing at a given node; in the node 1 for example

$$\begin{Bmatrix} j_{12} & j_{13} & i_1^x \\ j_{15} & j_{14} & i_1^y \end{Bmatrix} \tag{E7}$$

with link variables j_{1n} centered around j^0 and intertwiners variables $i_1^{m_n}$ centered around $i^0 = \frac{2}{\sqrt{3}} j^0$. Using the previous formula we can calculate the coefficients of the Regge action expansion linked to this symbol. The relevant derivatives for our calculation are (see also the appendix of [9])

$$\frac{\partial S_R^A}{\partial i_1^x}\bigg|_{j^0,i^0} = \frac{\partial S_R^A}{\partial i_1^y}\bigg|_{j^0,i^0} = \frac{\pi}{2}, \tag{E8}$$

$$\frac{\partial^2 S_R^A}{\partial j_{1n}\partial i_1^x}\bigg|_{j^0,i^0} = \frac{\partial^2 S_R^A}{\partial j_{1n}\partial i_1^y}\bigg|_{j^0,i^0} = \frac{3}{4j^0}, \tag{E9}$$

$$\frac{\partial^2 S_R^A}{\partial i_1^x \partial i_1^y}\bigg|_{j^0,i^0} = -\frac{\sqrt{3}}{j^0}, \tag{E10}$$

$$\frac{\partial^2 S_R^A}{\partial^2 i_1^x}\bigg|_{j^0,i^0} = \frac{\partial^2 S_R^A}{\partial^2 i_1^y}\bigg|_{j^0,i^0} = -\frac{\sqrt{3}}{2j^0}. \tag{E11}$$

APPENDIX F: CHANGE OF PAIRING ON THE BOUNDARY STATE

Here we show how one of the coefficients defined by (36) transforms under the change of basis determined by a different pairing. In particular, we show that with the choice of parameters in (24), Eq. (50) becomes (52). Under the change of basis,

$$\Phi'_{\mathbf{q}}[\mathbf{j}, i_1^x, i_2 \dots i_5] = \sum_{i_1^y} \Phi_{\mathbf{q}}[\mathbf{j}, i_1^y, i_2 \dots i_5](-1)^{j_{13}+j_{14}+i_1^x+i_1^y} \times \sqrt{d_{i_1^x} d_{i_1^y}} \begin{Bmatrix} j_{12} & j_{13} & i_1^x \\ j_{15} & j_{14} & i_1^y \end{Bmatrix}. \tag{F1}$$

With the choice of the boundary state defined by (24), this reads

$$\Phi'_{\mathbf{q}}[\mathbf{j}, i_1^x, i_2 \dots i_5] = e^{-(1/2j_0)\sum \alpha_{(ij)(mr)}\delta j_{ij}\delta j_{mr} + i\sum \Phi \delta j_{ij}} e^{-\sum_{n\neq 1}(((\delta i_n^{mn})^2/4\sigma_{i^{mn}}) + \sum_a \phi_{j_{na} i_n^{mn}} \delta j^{an} \delta i_n^{mn} + i\chi_{i_n^{mn}} \delta i_n^{mn})} \cdot \sum_{i_1^y} e^{-(((\delta i_1^y)^2/4\sigma_{i_1^y}) + \sum_a \phi_{j_{a1} i_1^y} \delta j^{a1} \delta i_1^y + i\chi_{i_1^y} \delta i_1^y)} (-1)^{j_{13}+j_{14}+i_1^x+i_1^y} \sqrt{d_{i_1^x} d_{i_1^y}} \begin{Bmatrix} j_{12} & j_{13} & i_1^x \\ j_{15} & j_{14} & i_1^y \end{Bmatrix}. \tag{F2}$$

Expanding the $6j$ symbol in the large-j limit, and applying the relation (38) we get

$$\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, \dots, i_5) = e^{-(1/2j^0)\sum \alpha_{(ij)(mr)}\delta j^{ij}\delta j^{mr} + i\sum \Phi \delta j^{ij}} e^{-\sum_{n\neq 1}(((\delta i_n^{mn})^2/4\sigma_{i^{mn}}) + \sum_a \phi_{j_{na} i_n^{mn}} \delta j^{an} \delta i_n^{mn} + i\chi_{i_n^{mn}} \delta i_n^{mn})} \cdot \frac{e^{i\pi i_0}}{2} \times \int d\delta i_1^y e^{-(((\delta i_1^y)^2/4\sigma_{i_1^y}) + \sum_a \phi_{j_{a1} i_1^y} \delta j^{a1} \delta i_1^y + i\chi_{i_1^y} \delta i_1^y)} \sqrt{d_{i_1^x} d_{i_1^y}} \frac{e^{i(S_R + \pi\delta i_1^y + (\pi/4))} + e^{-i(S_R - \pi\delta i_1^y + (\pi/4))}}{\sqrt{12\pi V}}. \tag{F3}$$

We expand the Regge action up to second order in all its 6 entries; the external link around j^0 and the intertwiners around i^0

$$S_R[j_{1n}, i_1^y, i_1^x] = S_R[j^0, i^0] + \frac{\partial S_R}{\partial j_{1n}}\bigg|_{j^0,i^0} \delta j_{1n} + \frac{\partial S_R}{\partial i_1^x}\bigg|_{j^0,i^0} \delta i_1^x + \frac{\partial S_R}{\partial i_1^y}\bigg|_{j^0,i^0} \delta i_1^y + \frac{\partial^2 S_R}{\partial j_{1n}\partial j_{1n'}}\bigg|_{j^0,i^0} \delta j_{1n}\delta j_{1n'} + \frac{\partial^2 S_R}{\partial j_{1n}\partial i_1^x}\bigg|_{j^0,i^0} \delta j_{1n}\delta i_1^x + \frac{\partial^2 S_R}{\partial j_{1n}\partial i_1^y}\bigg|_{j^0,i^0} \delta j_{1n}\delta i_1^y + \frac{\partial^2 S_R}{\partial i_1^x \partial i_1^y}\bigg|_{j^0,i^0} \delta i_1^x \delta i_1^y + \frac{1}{2}\frac{\partial^2 S_R}{\partial^2 j_{1n}}\bigg|_{j^0,i^0} (\delta j_{1n})^2 + \frac{1}{2}\frac{\partial^2 S_R}{\partial^2 i_1^x}\bigg|_{j^0,i^0} (\delta i_1^x)^2 + \frac{1}{2}\frac{\partial^2 S_R}{\partial^2 i_1^y}\bigg|_{j^0,i^0} (\delta i_1^y)^2 + \dots. \tag{F4}$$

In the background in which we are interested, $i^0 = \frac{2}{\sqrt{3}} j^0$ and $\frac{\partial S_R}{\partial i_1^y}|_{j^0,i^0} = \frac{\partial S_R}{\partial i_1^x}|_{j^0,i^0} = \frac{\pi}{2}$. The value $\chi = \frac{\partial S_R}{\partial i_1^y}|_{j^0,i^0} = \frac{\pi}{2}$, yields a phase in the intertwiner variable $e^{-i(\pi/2)\delta i_1^y}$ that cancels one of the two rapidly oscillating phase factor due to the linear term of the expansion of the Regge action. In particular the linear part in the intertwiner variable of the first exponential $e^{i((\partial S_R/\partial i_1^y)|_{j^0,i^0} + \pi)\delta i_1^y} = e^{i(3\pi/2)\delta i_1^y}$ combines with the boundary phase factor but the linear part of the second one $e^{-i((\partial S_R/\partial i_1^y)|_{j^0,i^0} - \pi)\delta i_1^y} = e^{i(\pi/2)\delta i_1^y}$ is canceled: for the same mechanism described in [6] only the second term in the summation (F3) survives. Denoting $\tilde{S_R} = S_R - i(\pi/2)\delta i_1^y$, we have that (F3) reduces to

$$\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, \dots, i_5) = e^{-(1/2j^0)\sum \alpha_{(ij)(mr)}\delta j^{ij}\delta j^{mr} + i\sum \Phi \delta j^{ij}} e^{-\sum_{n\neq 1}(((\delta i_n^{mn})^2/4\sigma_{i^{mn}}) + \sum_a \phi_{j_{na} i_n^{mn}} \delta j^{an} \delta i_n^{mn} + i(\pi/2)\delta i_n^{mn})} \cdot \frac{e^{i\pi i_0}}{2} \times \int d\delta i_1^y e^{-(((\delta i_1^y)^2/4\sigma_{i_1^y}) + \sum_a \phi_{j_{1a} i_1^y} \delta j^{a1} \delta i_1^y + i(\pi/2)\delta i_1^y)} \sqrt{d_{i_1^x} d_{i_1^y}} \frac{e^{-i(\tilde{S_R} + (\pi/4))}}{\sqrt{12\pi V}}. \tag{F5}$$

From [9], we have that denoting $\mu = \sqrt{\frac{d_{i_1^x} d_{i_1^y}}{12\pi V}}$, the dominant term is $\mu[j^0]$. We take $\mu[j^0]$ out of the integration and evaluate the integral following [20]. To simplify the notation, rename the second derivative of the Regge action $G_{j^{na}, i_n^{mn}} = \frac{\partial^2 S_R}{\partial j_{na} \partial i_n^{mn}}|_{j^0,i^0}$, $G_{i_n^{m}{}_{n'}, i_n^{mn}} = \frac{\partial^2 S_R}{\partial i_n^{m}{}_{n'} \partial i_n^{mn}}|_{j^0,i^0}$ and indicate with $S[j_{na}]$ (53) the part of the Regge action that depends only on the boundary links involved in the $6j$ symbol considered and with no dependence from the intertwiners. Substituting we get

$$\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, \dots, i_5) = e^{-(1/2j^0)\sum \alpha_{(ij)(mr)}\delta j^{ij}\delta j^{mr} + i\sum \Phi \delta j^{ij}} e^{-\sum_{n\neq 1}(((\delta i_n^{mn})^2/4\sigma_{i^{mn}}) + \sum_a \phi_{j_{na} i_n^{mn}} \delta j^{an} \delta i_n^{mn} + i(\pi/2)\delta i_n^{mn})} \cdot \frac{e^{i\pi i_0}}{2} e^{-i(\pi/4)} \mu[j^0] e^{-iS_R[j^0,i^0]} \cdot e^{-iS_j[j_{1a}]} e^{-i(\pi/2)\delta i_1^x} e^{-i(\sum_a G_{j_{1a} i_1^x} \delta j^{a1})\delta i_1^x} e^{-(i/2)G_{i_1^x i_1^x}(\delta i_1^x)^2} \cdot \int d\delta i_1^y e^{-(1/2)((1/2\sigma_{i_1^y}) + iG_{i_1^y i_1^y})(\delta i_1^y)^2} e^{-iG_{i_1^x i_1^y}\delta i_1^x \delta i_1^y} e^{-(\sum_a (\phi_{j_{1a} i_1^y} + iG_{j_{1a} i_1^y})\delta j^{a1})\delta i_1^y} \tag{F6}$$

The choice $\phi = -iG_{j_{1a}i_1^y} = -i\frac{3}{4j^0}$ eliminates the argument of the last exponential, so that we fall into the same as calculation [20], and we can transform the Gaussian in another Gaussian with the same variance. Evaluating the integral we get

$$\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, \ldots, i_5) = e^{-(1/2j^0)\sum \alpha_{(ij)(mr)}\delta j^{ij}\delta j^{mr} + i\sum \Phi\delta j^{ij}} e^{-\sum_{n\neq 1}(((\delta i_n^{m_n})^2/4\sigma_{i^{m_n}}) - i(\sum_a (3/4j^0)\delta j^{an} - (\pi/2))\delta i_n^{m_n})} \cdot \sqrt{\frac{\pi}{2(\frac{1}{2\sigma_{i_1^y}} + iG^A_{i_1^y i_1^y})}} \cdot e^{i\pi i_0} e^{-i(\pi/4)} \mu[j^0] e^{-iS^A[j^0, i^0]} e^{-iS_j^A[j_{1a}]} e^{-i(\pi/2)\delta i_1^x} e^{-i(\sum_a G_{j_{1a}i_1^x}\delta j^{a1})\delta i_1^x} e^{-(1/2)((G^2_{i_1^x i_1^y}/((1/2\sigma_{i_1^y}) + iG^A_{i_1^y i_1^y})) + iG^A_{i_1^x i_1^x})(\delta i_1^x)^2}. \tag{F7}$$

The Gaussian in the last equation has variance

$$\sigma_{i_1^x} = \frac{1}{2}\left(\frac{G^2_{i_1^x i_1^y}}{(\frac{1}{2\sigma_{i_1^y}} + iG_{i_1^y i_1^y})} + iG_{i_1^x i_1^x}\right)^{-1} \tag{F8}$$

as in [20]. Proceeding in the same way, we fix σ so that both $\sigma_{i_1^y}$ and $\sigma_{i_1^x}$ are real quantities. Remarkably the auxiliary tetrahedron described by S_R is isosceles and in this case $\sigma_{i_1^x} = \sigma_{i_1^y} = j^0/3$.

The final form of the coefficient is then

$$\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, \ldots, i_5) = e^{-(1/2j^0)\sum \alpha_{(ij)(mr)}\delta j^{ij}\delta j^{mr} + i\sum \Phi\delta j^{ij}} e^{-\sum_{n\neq 1}(((\delta i_n^{m_n})^2/4\sigma_{i^{m_n}}) - i(\sum_a (3/4j^0)\delta j^{an} - (\pi/2))\delta i_n^{m_n})} \cdot N_1 e^{-iS_j[j_{1a}]} e^{-(1/4)(1/\sigma_{i_1^x})(\delta i_1^x)^2} e^{-i(\sum_a G_{j_{1a}i_1^x}\delta j^{a1} + (\pi/2))\delta i_1^x} \tag{F9}$$

where

$$N_1 = \sqrt{\frac{\pi}{2(\frac{1}{2\sigma_{i_1^y}} + iG_{i_1^y i_1^y})}} e^{i\pi i_0} e^{-i(\pi/4)} \mu[j^0] e^{-iS_R[j^0, i^0]} \tag{F10}$$

and we have the result (52).

Summarizing, the parameters (29) and (30) are determined by the requirement that the Gaussian has the same shape in all bases.

APPENDIX G: SIMPLE GAUSSIAN INTEGRALS USED IN THE CALCULATION

$$\int_{-\infty}^{+\infty} dx^D \exp -\frac{1}{2}x^a A_{ab} x^b = \frac{(2\pi)^{(D/2)}}{\sqrt{\det A}}, \tag{G1}$$

$$\int_{-\infty}^{+\infty} dx^D x^i x^j \exp -\frac{1}{2}x^a A_{ab} x^b = \frac{(2\pi)^{(D/2)}}{\sqrt{\det A}} A^{-1}_{ij}, \tag{G2}$$

$$\int_{-\infty}^{+\infty} dx^D \exp -\frac{1}{2}x^a A_{ab} x^b + i\theta_a x^a = \frac{(2\pi)^{(D/2)}}{\sqrt{\det A}} \exp -\frac{1}{2}\theta^a A^{-1}_{ab}\theta^b, \tag{G3}$$

$$\int_{-\infty}^{+\infty} dx^D x^i \exp -\frac{1}{2}x^a A_{ab} x^b + i\theta_a x^a = \frac{(2\pi)^{(D/2)}}{\sqrt{\det A}} iA^{-1}_{ia}\theta^a \exp -\frac{1}{2}\theta^a A^{-1}_{ab}\theta^b, \tag{G4}$$

$$\int_{-\infty}^{+\infty} dx^D x^i x^j \exp -\frac{1}{2}x^a A_{ab} x^b + i\theta_a x^a = \frac{(2\pi)^{(D/2)}}{\sqrt{\det A}} (A^{-1}_{ia}\theta^a A^{-1}_{jb}\theta^b - A^{-1}_{ij}) \exp -\frac{1}{2}\theta^a A^{-1}_{ab}\theta^b, \tag{G5}$$

$$\int_{-\infty}^{+\infty} dx x^m \exp -\frac{1}{2}ax^2 + i\theta x = \sqrt{\frac{2\pi}{a}} (-i)^m \frac{\partial^m}{\partial\theta^m} \exp -\frac{1}{2a}\theta^2. \tag{G6}$$

[1] A. Ashtekar, arXiv:gr-qc/0702030; A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **21**, R53 (2004); T. Thiemann, *Introduction to Modern Canonical Quantum General Relativity* (Cambridge University Press,

Cambridge, U.K., 2007).

[2] C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, U.K., 2004).

[3] C. Rovelli and L. Smolin, Phys. Rev. Lett. **61**, 1155 (1988); Nucl. Phys. **B331**, 80 (1990); A. Ashtekar, C. Rovelli, and L. Smolin, Phys. Rev. Lett. **69**, 237 (1992); C. Rovelli and L. Smolin, Nucl. Phys. **B442**, 593 (1995); Nucl. Phys. **B456**, 734 (1995); A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **14**, A55 (1997); Adv. Theor. Math. Phys. **1**, 388 (1997).

[4] L. Modesto and C. Rovelli, Phys. Rev. Lett. **95**, 191301 (2005).

[5] C. Rovelli, Phys. Rev. Lett. **97**, 151301 (2006).

[6] E. Bianchi, L. Modesto, C. Rovelli, and S. Speziale, Classical Quantum Gravity **23**, 6989 (2006); arXiv:gr-qc/0604044.

[7] S. Speziale, J. High Energy Phys. 05 (2006) 039.

[8] E. R. Livine and S. Speziale, J. High Energy Phys. 11 (2006) 092.

[9] E. Livine, S. Speziale, and J. Willis, Phys. Rev. D **75**, 024038 (2007).

[10] B. Dittrich, L. Freidel, and S. Speziale, arXiv:0707.4513 [Phys. Rev. D (to be published)].

[11] J. W. Barrett and L. Crane, J. Math. Phys. (N.Y.) **39**, 3296 (1998).

[12] J. C. Baez, J. D. Christensen, T. R. Halford, and D. C. Tsang, Classical Quantum Gravity **19**, 4627 (2002).

[13] E. Alesci and C. Rovelli ‘‘Complete LQG Propagator: II. Asymptotics of the Vertex’’ (unpublished).

[14] R. De Pietri, L. Freidel, K. Krasnov, and C. Rovelli, Nucl. Phys. **B574**, 785 (2000); D. Oriti and R. M. Williams, Phys. Rev. D **63**, 024022 (2000); A. Perez and C. Rovelli, in *Proceedings of the Ninth Marcel Grossmann Meeting on General Relativity*, edited by R. T. Jantzen, R. Ruffini, and V. G. Gurzadyan (World Scientific, Singapore, 2002), pp. 1315–1317; A. Perez and C. Rovelli, Nucl. Phys. **B599**, 255 (2001).

[15] M. P. Reisenberger, arXiv:gr-qc/9711052.

[16] J. Engle, R. Pereira, and C. Rovelli, Phys. Rev. Lett. **99**, 161301 (2007).

[17] E. R. Livine and S. Speziale, Phys. Rev. D **76**, 084028 (2007).

[18] S. Alexandrov, arXiv:0705.3892.

[19] D. Oriti, arXiv:gr-qc/0607032; L. Freidel, Int. J. Theor. Phys. **44**, 1769 (2005).

[20] S. Speziale and C. Rovelli, Classical Quantum Gravity **23**, 5861 (2006).

[21] C. Rovelli and L. Smolin, Phys. Rev. D **52**, 5743 (1995).

[22] L. Freidel and D. Louapre, Classical Quantum Gravity **20**, 1267 (2003).

[23] J. W. Barrett and R. M. Williams, Adv. Theor. Math. Phys. **3**, 209 (1999); J. C. Baez, J. D. Christensen, and G. Egan, Classical Quantum Gravity **19**, 6489 (2002); J. W. Barrett and C. M. Steele, Classical Quantum Gravity **20**, 1341 (2003); J. D. Christensen and G. Egan, Classical Quantum Gravity **19**, 1185 (2002).

[24] D. M. Brink and G. R. Satchler, *Angular Momentum* (Clarendon Press, Oxford, 1968).

[25] R. De Pietri and C. Rovelli, Phys. Rev. D **54**, 2664 (1996).

[26] R. De Pietri, Classical Quantum Gravity **14**, 53 (1997).

[27] G. Ponzano and T. Regge, in *Spectroscopic and Group Theoretical Methods in Physics*, edited by F. Bloch (North-Holland, Amsterdam, 1968).