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State-relevant Maxwell's equation from Kaluza-Klein theory

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We study a five-dimensional perfect fluid coupled with Kaluza-Klein gravity. By dimensional reduction, a modified form of Maxwell's equation is obtained, which is relevant to the equation of state of the source. Since the relativistic magnetohydrodynamics and the three-dimensional formulation are widely used to study space matter, we derive the modified Maxwell's equations and relativistic magnetohydrodynamics in $3 + 1$ form. We then take an ideal Fermi gas as an example to study the modified effect, which can be visible under high-density or high-energy conditions, while the traditional Maxwell's equation can be regarded as a result in the low density and low temperature limit. We also indicate the possibility to test the state-relevant effect of Kaluza-Klein theory in a telluric laboratory.

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I. INTRODUCTION

A unified formulation of Einstein's theory of gravitation and Maxwell's theory of electromagnetism in four-dimensional (4D) spacetime was first proposed by Kaluza and Klein (KK) using a five-dimensional (5D) geometry [1,2]. A free test particle in 5D KK spacetime shows its electricity in the reduced 4D spacetime when it moves along the fifth dimension. Moreover, a 5D dust field coupled with KK gravity can curve the 5D spacetime in such a way that it provides exactly the source of the electromagnetic field in the 4D spacetime after the reduction [3]. In this paper we study the coupling of a 5D perfect fluid with KK gravity. It turns out that the 5D Einstein equation with a source gives a modification of Maxwell's equation (see Sec. II), which can show its state-relevant effect on high-density or high-temperature conditions (see Sec. IV). Thus this effect provides intriguing possibilities for the experimental test of the KK theory. Note that the KK theory which we are considering is purely classical. The KK theory has also been studied from the particle physics point of view (see e.g. [4]), which can show its effects in the high-energy scale of TeV, whereas the energy scale needed for testing our state-relevant effect is only around keV (see Sec. IV).

In order to reveal the physical implication of the modification more clearly, both the modified Maxwell's equations and the corresponding general relativistic magnetohydrodynamics (MHD) are reformulated in $3 + 1$ form (see Sec. III). The formalism is also useful for evolving numerically a relativistic MHD fluid in a spacetime characterized by a strong gravitational field. Taking an ideal Fermi gas as an example, the modification term is studied as a function of degeneracy and temperature parameters (see Sec. IV). Moreover, the modification terms for different components of the perfect fluid may be different. This would result in a net negative charge excess in high-temperature plasma. Recall that the electrical neutrality of atoms and of bulk matter has been examined precisely by a number of experiments [5]. However, in those experiments, either the objects considered are not ionized, or the ions in the objects cannot be regarded as a perfect fluid. Therefore, these experiments cannot provide definite opposite evidence to the classical KK theory since they do not satisfy our premise, but they do cast some doubts on it. Taking account of the state-relevant character, we suggest high-temperature plasma in a telluric laboratory or a dense-matter white dwarf in outer space as candidates to test the possible effects of the modified Maxwell's equation.

II. KALUZA-KLEIN GRAVITY COUPLED WITH 5D PERFECT FLUID

Using a 5D geometry, Kaluza and Klein proposed a unified formulation of gravity and electromagnetism in 4D spacetime [1,2]. The original KK theory assumed the so-called "*cylinder condition*," which means that there exists a spacelike killing vector field ξ^a on the 5D spacetime $(\hat{M}, \hat{g}_{ab})$ [6–8]. Note that the abstract index notation [9] is employed throughout the paper and the signature of the 5-metric is of the convention $(-, +, +, +, +)$. In addition, Kaluza also demanded that ξ^a be normalized, i.e.,

$$\phi \equiv \hat{g}_{ab}\xi^a\xi^b = 1. \qquad (1)$$

Later research shows that the Ansatz (1) may be dropped out and the ϕ may play a key role in the study of cosmology [10–13]. Being an extra dimension, the orbits of ξ^a are geometrically circles. The physical consideration that any displacement in the usual "physical" 4D spacetime (denoted as M) should be orthogonal to the extra dimension implies that the physical 4D metric should be defined as

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$$g_{ab} = \hat{g}_{ab} - \phi^{-1}\xi_a\xi_b, \tag{2}$$

and the projection operator onto M is

$$g^a{}_b = \hat{g}^a{}_b - \phi^{-1}\xi^a\xi_b. \tag{3}$$

For practical calculation, it is convenient to take a coordinate system $\{z^M = (x^\mu, y)|\mu = 0, 1, 2, 3\}$ with coordinate basis $(e_M)^a = \{(e_\mu)^a, (e_5)^a\}$ on $\hat{M}$ adapted to ξ^a, i.e., $(e_5)^a = (\frac{\partial}{\partial y})^a = \xi^a$. Then the 5-metric components $\hat{g}_{MN}$ take the form

$$\hat{g}_{MN} = \begin{pmatrix} g_{\mu\nu} + \phi B_\mu B_\nu & \phi B_\mu \\ \phi B_\nu & \phi \end{pmatrix}, \tag{4}$$

where $\hat{g}_{\mu 5} \equiv \phi B_\mu$. So, locally, the physical spacetime can be understood as a 4-manifold M with the coordinates $\{x^\mu\}$ endowed with the metric g_{ab}. The whole theory is governed by the 5D Einstein-Hilbert action

$$S_G = -\frac{1}{2\hat{k}}\int_{\hat{M}} d^4x dy\sqrt{-\hat{g}}\hat{R}. \tag{5}$$

Suppose the range of the fifth coordinate to be $0 \leq y \leq L$ and denote $k = \hat{k}/L$. Let $B_\mu = fA_\mu$, $f^2 = 2k$; then Eq. (5) becomes a coupling action on M as

$$\hat{S}_G = \int_M d^4x\sqrt{-g}\sqrt{\phi}\Big(-\frac{1}{2k}R + \frac{1}{4}\phi F_{ab}(A)F^{ab}(A)\Big), \tag{6}$$

where R is the curvature scalar of g_{ab} on M and $F_{ab}(A) \equiv 2\partial_{[a}A_{b]}$. Thus, it results in a 4D gravity g_{ab} coupled to an electromagnetic field A_a and a scalar field ϕ. It is clear that, under the Ansatz (1), 5D KK theory unifies the Einstein gravity and the source-free Maxwell's field in the standard formalism.

Now we consider a 5D perfect fluid

$$\hat{T}_{ab} = (\hat{p} + \hat{\mu})\hat{V}_a\hat{V}_b + \hat{p}\hat{g}_{ab}. \tag{7}$$

The 5-velocity $\hat{V}^a$ can be projected onto the physical spacetime (M, g_{ab}) as

$$u^a \equiv g^a{}_b\hat{V}^b = \hat{V}^\mu(e_\mu)^a - (B_\mu\hat{V}^\mu)(e_5)^a. \tag{8}$$

Note that we have $\hat{V}^a\hat{V}_a = -1$; hence it is easy to show that

$$\hat{V}^\mu\hat{V}_\mu \equiv \hat{g}_{ab}u^a u^b = g_{ab}\hat{V}^a\hat{V}^b = -1 - \frac{Q^2}{\phi}, \tag{9}$$

where $Q \equiv \hat{V}_5 = \hat{g}_{5a}\hat{V}^a$ [3]. The energy-momentum tensor can be projected on M as $\tilde{T}_{ab} \equiv g_a{}^c g_b{}^d\hat{T}_{cd}$. In order to obtain the observed 4D energy-momentum tensor T_{ab} on M, we have to integrate $\tilde{T}_{ab}$ along the extra dimension. In the light of (8) and (9) we obtain

$$T_{ab} = (\mu + p)v_a v_b + p g_{ab}, \tag{10}$$

where

$$p = \hat{p}\sqrt{\phi}L, \qquad \mu = \frac{\hat{\mu}L(Q^2 + \phi)}{\sqrt{\phi}} + \hat{p}L\frac{Q^2}{\sqrt{\phi}}, \qquad v_a = \frac{u_a}{\sqrt{-\hat{V}^\mu\hat{V}_\mu}}. \tag{11}$$

It is clear that T_{ab} is the energy-momentum tensor of a 4D perfect fluid in M, where μ and p are, respectively, the energy density and pressure density observed by a comoving observer in M.

We now consider the reduction of the 5D Einstein equation

$$\hat{R}_{ab} - \tfrac{1}{2}\hat{g}_{ab}\hat{R} = \hat{k}\hat{T}_{ab}, \tag{12}$$

which is equivalent to

$$\hat{R}_{ab} = \hat{k}(\hat{T}_{ab} - \tfrac{1}{3}\hat{T}^c_c\hat{g}_{ab}). \tag{13}$$

It is not difficult to show from Eq. (5) that the components of the 5D Ricci tensor $\hat{R}_{ab}$ can be expressed as [12]

$$\hat{R}_{55} = \frac{1}{2}k\phi^2F^{\sigma\rho}F_{\sigma\rho} - \frac{1}{2}\nabla^\mu\nabla_\mu\phi + \frac{1}{4\phi}(\nabla^\mu\phi)\nabla_\mu\phi, \tag{14}$$

$$\hat{R}_{\mu 5} = \frac{f}{2}\Big(\phi\nabla^\nu F_{\mu\nu} + \frac{3}{2}F_{\mu\nu}\nabla^\nu\phi\Big) + B_\mu\Big(\frac{1}{2}k\phi^2F^{\sigma\rho}F_{\sigma\rho} - \frac{1}{2}\nabla^\nu\nabla_\nu\phi + \frac{1}{4\phi}(\nabla^\nu\phi)\nabla_\nu\phi\Big), \tag{15}$$

$$\begin{aligned}\hat{R}_{\mu\nu} = {}& R_{\mu\nu} - k\phi F^\sigma{}_\mu F_{\sigma\nu} - \frac{1}{2\phi}\nabla_\mu\nabla_\nu\phi \\ &+ \frac{1}{4\phi^2}(\nabla_\mu\phi)\nabla_\nu\phi + B_\mu B_\nu\Big(\frac{1}{2}k\phi^2F^{\sigma\rho}F_{\sigma\rho} \\ &- \frac{1}{2}\nabla^\sigma\nabla_\sigma\phi + \frac{1}{4\phi}(\nabla^\sigma\phi)\nabla_\sigma\phi\Big) \\ &+ \frac{f}{2}B_\mu\Big(\phi\nabla^\sigma F_{\nu\sigma} + \frac{3}{2}F_{\nu\sigma}\nabla^\sigma\phi\Big) \\ &+ \frac{f}{2}B_\nu\Big(\phi\nabla^\sigma F_{\mu\sigma} + \frac{3}{2}F_{\mu\sigma}\nabla^\sigma\phi\Big),\end{aligned} \tag{16}$$

where ∇_a is the 4D covariant derivative operator associated with g_{ab}. Substituting Eq. (14) into Eq. (13), we obtain a coupling equation for the matter fields as

$$\frac{1}{2}k\phi^2F^{ab}F_{ab} = \sqrt{\phi}\nabla^a\nabla_a\sqrt{\phi} + k\sqrt{\phi}\mu\Big(1 - \frac{2\phi}{3(\phi + Q^2)}\Big) + k\sqrt{\phi}p\frac{Q^2 - \phi}{3(Q^2 + \phi)}. \tag{17}$$

Substituting Eq. (15) into Eq. (13) and using Eq. (17), we obtain an electromagnetic field equation with a source as

$$\phi\nabla^b F_{ab} + \frac{3}{2}F_{ab}\nabla^b\phi = \tilde{\gamma}\Big(1 + \frac{p}{\mu}\Big)J_a. \tag{18}$$

Here we have defined $\tilde{\gamma} \equiv \sqrt{(1+Q^2)/(\phi+Q^2)}$, $\rho \equiv \frac{f\mu Q}{\sqrt{\phi(1+Q^2)}}$, and $J^a \equiv \rho v^a$ [3]. Substituting Eq. (16) into Eq. (13) and using Eqs. (17) and (18), we obtain a 4D Einstein equation with a source as

$$G_{ab} = \frac{k}{\sqrt{\phi}}\bigg((\mu+p)v_a v_b + g_{ab}p + \phi^{3/2}\Big(F^c_a F_{bc} - \frac{1}{4}g_{ab}F^{cd}F_{cd}\Big) - \frac{1}{k}(g_{ab}\nabla^c\nabla_c\sqrt{\phi} - \nabla_a\nabla_b\sqrt{\phi})\bigg), \quad (19)$$

where G_{ab} is the Einstein tensor of g_{ab}. More generally, if the 5D perfect fluid consists of m components, $\hat{T}_{ab}$ then reads

$$\hat{T}_{ab} = \sum_{\eta=1}^{m}((\hat{p}_\eta + \hat{\mu}_\eta)\hat{V}_a(\eta)\hat{V}_b(\eta) + \hat{p}_\eta \hat{g}_{ab}). \quad (20)$$

By similar calculations, Eqs. (17)–(19) become, respectively,

$$\frac{1}{2}k\phi^2 F^{ab}F_{ab} = \sqrt{\phi}\nabla^a\nabla_a\sqrt{\phi} + k\sqrt{\phi}\sum_{\eta=1}^{m}\mu(\eta) \times \bigg(1 - \frac{2\phi}{3(\phi+Q(\eta)^2)}\bigg) + k\sqrt{\phi}\sum_{\eta=1}^{m}p(\eta)\frac{Q(\eta)^2-\phi}{3(Q(\eta)^2+\phi)}, \quad (21)$$

$$\phi\nabla^b F_{ab} + \frac{3}{2}F_{ab}\nabla^b\phi = \sum_{\eta=1}^{m}\tilde{\gamma}(\eta)\bigg(1+\frac{p(\eta)}{\mu(\eta)}\bigg)J_a(\eta), \quad (22)$$

$$G_{ab} = \frac{k}{\sqrt{\phi}}\bigg(\sum_{\eta=1}^{m}((\mu(\eta)+p(\eta))v_a(\eta)v_b(\eta) + g_{ab}p(\eta)) + \phi^{3/2}\Big(F^c_a F_{bc} - \frac{1}{4}g_{ab}F^{cd}F_{cd} - \frac{1}{k}(g_{ab}\nabla^c\nabla_c\sqrt{\phi} - \nabla_a\nabla_b\sqrt{\phi})\Big)\bigg). \quad (23)$$

It is interesting to see the results when $\phi \equiv 1$. Equations (17)–(19) become, respectively,

$$\frac{1}{2}kF^{ab}F_{ab} = k\mu\bigg(1 - \frac{2}{3(1+Q^2)}\bigg) + kp\frac{Q^2-1}{3(Q^2+1)}, \quad (24)$$

$$\nabla^b F_{ab} = \bigg(1+\frac{p}{\mu}\bigg)J_a, \quad (25)$$

$$G_{ab} = k(T^{(\mathrm{fluid})}_{ab} + T^{(\mathrm{em})}_{ab}), \quad (26)$$

where $T^{(\mathrm{fluid})}_{ab} \equiv (\mu+p)v_a v_b + pg_{ab}$ and $T^{(\mathrm{em})}_{ab} = F^c_a F_{bc} - \frac{1}{4}g_{ab}F^{cd}F_{cd}$ are, respectively, the usual energy-momentum tensors of the 4D perfect fluid and electromagnetic field. Equation (26) is the standard 4D Einstein equation, while Eq. (25) is not the same as the standard 4D Maxwell's equation $\nabla^b F_{ab} = J_a$. The new term $(1+\frac{p}{\mu})$ brings an effective charge which can be considered as a state-relevant effect, as will be discussed later. We thus call Eq. (25) the state-relevant Maxwell's equation.

III. MAXWELL'S EQUATIONS AND RELATIVISTIC MHD IN 3 + 1 FORM

Since relativistic MHD is widely used to study space matter and 3D formulation is frequently applied to dealing with specific issues [14], we now derive the modified results of Maxwell's equations and relativistic MHD in 3 + 1 form. The 4D spacetime M is foliated into a family of nonintersecting spacelike 3-surfaces Σ, which arise, at least locally, as level surfaces of a scalar time function t. The spatial metric γ_{ab} on the three-dimensional hypersurfaces Σ is induced by the spacetime metric g_{ab} according to

$$\gamma_{ab} = g_{ab} + n_a n_b, \quad (27)$$

where n^a is the unit normal vector to the slices and thus $n_a = -\alpha\nabla_a t$. Here the normalization factor α is called the lapse function. The time vector t^a is dual to the foliation 1-form $\nabla_a t$ and can be decomposed as

$$t^a = \alpha n^a + \beta^a, \quad (28)$$

where the shift vector β is spatial, i.e., $n_a\beta^a = 0$. Since the extrinsic curvature K_{ab} of Σ can be written as

$$K_{ab} = -\nabla_a n_b - n_a a_b, \quad (29)$$

where $a_a \equiv n^b\nabla_b n_a$, the divergence of n^a satisfies

$$\nabla_a n^a = -K. \quad (30)$$

Here K is the trace of K_{ab}.

First we write the modified Maxwell's equations in 3 + 1 form. The Faraday tensor F^{ab} can be decomposed as

$$F^{ab} = n^a E^b - n^b E^a + \epsilon^{abc}B_c, \quad (31)$$

where E^a and B^a are the electric and magnetic fields observed by a normal observer n^a. Both fields are purely spatial, whereby

$$E^a n_a = 0 \quad \text{and} \quad B^a n_a = 0, \quad (32)$$

and the three-dimensional Levi-Civita symbol ϵ_{abc} is defined by

$$\epsilon_{abc} = n^d\epsilon_{dabc} \quad \text{or} \quad \epsilon^{abc} = n_d\epsilon^{dabc}. \quad (33)$$

The electromagnetic current 4-vector J^a is decomposed as

$$J^a = n^a\rho_e + j^a, \quad (34)$$

where ρ_e and j^a are the charge density and 3-current as observed by a normal observer n^a. Note that j^a is purely spatial, i.e., $j^a n_a = 0$. With these definitions, the modified Maxwell's equations (17) and (18) and $\nabla_{[a}F_{bc]} = 0$ can be cast into 3 + 1 form as

$$k\phi^2(B^2 - E^2) = \sqrt{\phi}(D^a D_a\sqrt{\phi} - (\alpha^{-1}(\partial_t - \mathcal{L}_\beta))^2\sqrt{\phi} + K\alpha^{-1}(D_a\sqrt{\phi})(D^a \ln\alpha)(\partial_t - \mathcal{L}_\beta)\sqrt{\phi}) + k\sqrt{\phi}\mu\left(1 - \frac{2\phi}{3(\phi + Q^2)}\right) + k\sqrt{\phi}p\frac{Q^2 - \phi}{3(\phi + Q^2)}, \tag{35}$$

$$D_a E^a = \phi^{-1}\left(\tilde{\gamma}\left(1 + \frac{p}{\mu}\right)\rho_e - \frac{3}{2}E^a D_a\phi\right), \tag{36}$$

$$\phi\mathcal{L}_t E^a = \phi(\alpha K E^a + \mathcal{L}_\beta E^a + \epsilon^{abc}D_b(\alpha B_c)) - \alpha\tilde{\gamma}\left(1 + \frac{p}{\mu}\right)j^a + \frac{3}{2}\alpha(\epsilon^{abc}B_b D_c\phi - E^a\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\phi), \tag{37}$$

$$D_a B^a = 0, \tag{38}$$

$$\mathcal{L}_t B^a = -\epsilon^{abc}D_b(\alpha E_c) + \alpha K E^a + \mathcal{L}_\beta B^a. \tag{39}$$

Here $\mathcal{L}_\beta$ denotes the Lie derivative along β^a, and D_a is the covariant derivative operator associated to γ_{ab}. Note that the Lie derivative of a spacelike tensor $A^{a\cdots b}{}_{c\cdots d}$ along s^a is defined conventionally as $\tilde{\mathcal{L}}_s A^{a\cdots b}{}_{c\cdots d} \equiv \gamma^a_e \cdots \gamma^b_f \gamma^g_c \cdots \gamma^h_d \mathcal{L}_s A^{e\cdots f}{}_{g\cdots h}$, and we write $\tilde{\mathcal{L}}_s$ as $\mathcal{L}_s$ for short. Note also that the formula $n^b\nabla_b n_a = a_a = D_a \ln\alpha$ is used in the above calculation [15]. If one considered a 5D perfect fluid consisting of m components, the terms $\tilde{\gamma}(1 + \frac{p}{\mu})\rho_e$ and $\tilde{\gamma}(1 + \frac{p}{\mu})j^a$ in Eqs. (36) and (37) would be replaced by $\sum_{\eta=1}^m \tilde{\gamma}(\eta)(1 + \frac{p(\eta)}{\mu(\eta)})\rho_e(\eta)$ and $\sum_{\eta=1}^m \tilde{\gamma}(\eta)(1 + \frac{p(\eta)}{\mu(\eta)})j^a$. When $\phi \equiv 1$, one can see from Eq. (36) that $\tilde{\rho}_e \equiv (1 + p/\mu)\rho_e$ is the effective charge density serving as the source of the electric field. This effective charge density is state relevant; i.e., it is dependent on p/μ. Its significance will be discussed later.

Second we rewrite the modified relativistic MHD in 3 + 1 form. Note that the total energy-momentum tensor in M can be read off from the right-hand side of Eq. (19) as

$$T^{ab} \equiv T^{ab}_{(\text{fluid})} + \tilde{T}^{ab}_{(\text{em})} + T^{ab}_{(\phi)}, \tag{40}$$

where

$$\tilde{T}^{ab}_{(\text{em})} \equiv \phi^{3/2}T^{ab}_{(\text{em})} = \phi^{3/2}(F^{ac}F^b{}_c - \tfrac{1}{4}g^{ab}F^{cd}F_{cd}), \tag{41}$$

$$T^{ab}_{(\phi)} \equiv -\frac{1}{k}(g^{ab}\nabla^c\nabla_c\sqrt{\phi} - \nabla^a\nabla^b\sqrt{\phi}). \tag{42}$$

It is straightforward to see that [16]

$$\nabla_b T^{ab}_{(\text{em})} = F^{ac}\nabla^b F_{bc}. \tag{43}$$

In the light of Eqs. (27)–(33), we obtain the 3 + 1 form

$$\nabla_b T^{ab}_{(\text{em})} = n^a(-KE^2 + \alpha^{-1}E_b(\mathcal{L}_t - \mathcal{L}_\beta)E^b - \alpha^{-1}\epsilon^{bcd}E_b D_c(\alpha B_d)) - E^a D_b E^b + \epsilon^{abc}B_c\alpha^{-1}(\mathcal{L}_t - \mathcal{L}_\beta - \alpha K)E_b + \alpha^{-1}B_b D^a(\alpha B^b) - \alpha^{-1}B_b D^b(\alpha B^a), \tag{44}$$

$$T^{ab}_{(\text{em})}\nabla_b\phi^{3/2} = \frac{3\sqrt{\phi}}{2}\Big(\frac{1}{2}(B^2 + E^2)(D^a\phi + n^a\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\phi) - (E^aE^b + B^aB^b)D_b\phi + E_cB_d(n^a\epsilon^{bcd}D_b\phi + \epsilon^{acd}\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\phi)\Big). \tag{45}$$

Here $E^2 \equiv E_aE^a$ and $B^2 \equiv B_aB^a$. Recall that the relation between the three-dimensional Riemann tensor ${}^3R^d_{abc}$ and the 4D one R^d_{abc} reads

$${}^3R^d_{abc} = \gamma^e_a\gamma^f_b\gamma^l_c\gamma^d_m R^m_{efl} - 2K_{c[a}K_{b]}{}^d. \tag{46}$$

Using

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\nabla^c\sqrt{\phi} = -R^c_{abd}\nabla^d\sqrt{\phi}, \tag{47}$$

a lengthy but straightforward calculation gives

$$\nabla_b T^{ab}_{(\phi)} = \frac{1}{2k\sqrt{\phi}}({}^3R^{ab}D_b\phi + \alpha^{-1}(D_bK^{ab} - D^aK)(\partial_t - \mathcal{L}_\beta)\phi - (D^b\phi)(\alpha^{-1}(\mathcal{L}_t - \mathcal{L}_\beta)K^a_b + K^{ac}K_{bc} + D^aD_b\ln\alpha + (D^a\ln\alpha)D_b\ln\alpha) + n^a(\alpha^{-1}((\partial_t - \mathcal{L}_\beta)\phi)(-K^{bc}K_{bc} + D^bD_b\ln\alpha + a^2 + \alpha^{-1}(\partial_t - \mathcal{L}_\beta)K) + (D_bK^{bc})D_c\phi - (D^bK)D_b\phi - K_{bc}(D^b\phi)D^c\ln\alpha)), \tag{48}$$

where ${}^3R^{ab}$ is the three-dimensional Ricci tensor and $a^2 \equiv a^a a_a$. For a perfect fluid, the energy-momentum tensor $T^{ab}_{(\mathrm{fluid})}$ can also be written as

$$T^{ab}_{(\mathrm{fluid})} = \rho h v^a v^b + p g^{ab}, \tag{49}$$

where ρ is the rest-mass density as seen by an observer comoving with the fluid v^a, p is the pressure, and h the specific enthalpy

$$h = 1 + \epsilon + p/\rho. \tag{50}$$

Hence one has $\mu = (1 + \epsilon)\rho$. The local conservation of the 4D Einstein tensor G^{ab} leads to

$$\nabla_b(T^{ab}/\sqrt{\phi}) = 0. \tag{51}$$

We assume that $T^{ab}_{(\phi)}$ does not contribute to the number of baryons. Thus we have the conservation of baryons as

$$\nabla_a(\rho v^a) = 0, \tag{52}$$

which is decomposed into 3 + 1 form as

$$D_a(\rho\tilde{v}^a) + \alpha^{-1}(\partial_t - \mathcal{L}_\beta)(\rho W) - \rho W K + \rho\tilde{v}^a D_a\ln\alpha = 0, \tag{53}$$

where $\tilde{v}^a \equiv v^a - Wn^a$. The equation for the conservation of energy is obtained by contracting Eq. (51) with n_b as

$$H = \alpha^{-1}(\partial_t - \mathcal{L}_\beta)p - W\rho(\tilde{v}^aD_ah + W\alpha^{-1}(\partial_t - \mathcal{L}_\beta)h) - \rho h(W\alpha^{-1}(\partial_t - \mathcal{L}_\beta)W + \tilde{v}^aD_aW + W\tilde{v}^aD_a\ln\alpha - K_{ab}\tilde{v}^a\tilde{v}^b), \tag{54}$$

and the Euler equation is obtained by projecting Eq. (51) onto Σ as

$$\rho h\tilde{v}^aD_b\tilde{v}^a = \rho h(2WK^{ab}\tilde{v}_b - W^2D^a\ln\alpha) - \rho hW\alpha^{-1}(\mathcal{L}_t - \mathcal{L}_\beta)\tilde{v}^a - \rho\tilde{v}^a(W\alpha^{-1}(\partial_t - \mathcal{L}_\beta)h + \tilde{v}^bD_bh) - D^ap - M^a, \tag{55}$$

where

$$H \equiv \phi^{3/2}(-KE^2 + \alpha^{-1}E_a(\mathcal{L}_t - \mathcal{L}_\beta)E^a - \alpha^{-1}\epsilon^{abc}E_aD_b(\alpha B_c)) + \frac{\sqrt{\phi}}{2}(B^2 + E^2)\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\phi + \sqrt{\phi}\epsilon^{abc}E_aB_bD_c\phi + \frac{1}{2k\sqrt{\phi}}(\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\phi(-K^{ab}K_{ab} + D^aD_a\ln\alpha + a^2 + \alpha^{-1}(\partial_t - \mathcal{L}_\beta)K) + (D_aK^{ab})D_b\phi - (D^aK)D_a\phi) - \frac{W}{2\phi}(\mu + p)\tilde{v}^bD_b\phi + \frac{1}{2k\phi}(D_a\phi)(\alpha^{-1}(\mathcal{L}_t - \mathcal{L}_\beta)D^a\sqrt{\phi} - K_{ab}D^b\sqrt{\phi}) + \frac{1}{2\phi}\alpha^{-1}\Big(-W^2(\mu + p) + p - \frac{1}{k}(D^aD_a\sqrt{\phi} + K\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\sqrt{\phi} + (D_a\sqrt{\phi})D^a\ln\alpha)\Big)(\partial_t - \mathcal{L}_\beta)\phi \tag{56}$$

and

$$M^a \equiv +\sqrt{\phi}\Big(\frac{1}{2}(B^2 + E^2)D^a\phi - (E^aE^b + B^aB^b)D_b\phi + \epsilon^{abc}E_bB_c\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\phi\Big) - \phi^{3/2}E^aD_bE^b + \phi^{3/2}\alpha^{-1}(\epsilon^{abc}B_c(\mathcal{L}_t - \mathcal{L}_\beta - \alpha K)E_b + B_bD^a(\alpha B^b) - B_bD^b(\alpha B^a)) + \frac{1}{2k\sqrt{\phi}}({}^3R^{ab}D_b\phi + \alpha^{-1}(D_bK^{ab} - D^aK)(\partial_t - \mathcal{L}_\beta)\phi) - (D^b\phi)(\alpha^{-1}(\mathcal{L}_t - \mathcal{L}_\beta)K^a{}_b + K^{ac}K_{bc} + D^aD_b\ln\alpha + (D^a\ln\alpha)D_b\ln\alpha) - \frac{1}{2\phi}(\mu + p)(\tilde{v}^bD_b\phi + W\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\phi)\tilde{v}^a - \frac{1}{2\phi}pD^a\phi + \frac{1}{2k\phi}(D^a\phi)(D^bD_b\sqrt{\phi} - (\alpha^{-1}(\partial_t - \mathcal{L}_\beta))^2\sqrt{\phi} + K\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\sqrt{\phi} + (D_b\sqrt{\phi})D^b\ln\alpha) - \frac{1}{2k\phi}((D_b\phi)D^aD^b\sqrt{\phi} + (\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\sqrt{\phi})K^{ab}D_b\phi - (\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\phi)D^a(\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\sqrt{\phi})). \tag{57}$$

Here $(\alpha^{-1}(\partial_t - \mathcal{L}_\beta))^2\sqrt{\phi}$ denotes $\alpha^{-1}(\partial_t - \mathcal{L}_\beta) \times (\alpha^{-1}(\partial_t - \mathcal{L}_\beta)\sqrt{\phi})$. Note that Eqs. (53)–(55) comprise the basic formulas for the modified relativistic MHD in three-dimensional form. In the special case $\phi \equiv 1$, we have

$$H = -KE^2 + \alpha^{-1}E_a(\mathcal{L}_t - \mathcal{L}_\beta)E^a - \alpha^{-1}\epsilon^{abc}E_aD_b(\alpha B_c), \quad (58)$$

$$M^a = -E^aD_bE^b + \alpha^{-1}\epsilon^{abc}B_c(\mathcal{L}_t - \mathcal{L}_\beta - \alpha K)E_b + \alpha^{-1}(B_bD^a(\alpha B^b) - B_bD^b(\alpha B^a)), \quad (59)$$

which accord with the conventional form [14].

IV. DISCUSSION ON THE EFFECTIVE CHARGE

From the state-relevant Maxwell's equation (25) and its 3 + 1 form Eq. (36), we can see that $\tilde{\rho}_e \equiv (1 + p/\mu)\rho_e$ is the effective charge density of a perfect fluid. Such an effective charge density is relevant to the equation of state of the fluid, and its rationality should be carefully checked. In this section, we adopt an ideal Fermi gas as an example to see under what condition the modified term p/μ can show a visible effect. We employ in terms of the dimensionless degeneracy and temperature parameters

$$\eta = \frac{\tilde{\mu}}{k_BT}, \qquad \beta = \frac{k_BT}{mc^2}, \quad (60)$$

where $\tilde{\mu}$ is the chemical potential, m is the mass of the fermion, and k_B is the Boltzmann constant. The gas is degenerate for $\eta \gg 0$ and nondegenerate for $\eta \ll 0$. On the other hand, the gas is extremely relativistic for $\beta \gg 1$ and nonrelativistic for $\beta \ll 1$ [17]. The zero of energy for the particles is chosen so that the thermodynamic potential reads

$$\Omega = -Vk_BT\int\frac{gd^3\tilde{p}}{h^3}\ln\left[1 + \exp\frac{\tilde{\mu} - \varepsilon}{k_BT}\right], \quad (61)$$

where $\tilde{p}$ is the momentum, g is the statistical weight, and

$$\varepsilon = \sqrt{(\tilde{p}c)^2 + (mc^2)^2} - mc^2 \quad (62)$$

is the kinetic energy. The number density n, pressure p, and internal energy density E (per volume) of an ideal Fermi gas are, respectively,

$$n = K\beta^{3/2}[F_{1/2}(\eta, \beta) + \beta F_{3/2}(\eta, \beta)], \quad (63)$$

$$p = mc^2K\beta^{5/2}[\tfrac{2}{3}F_{3/2}(\eta, \beta) + \tfrac{1}{3}\beta F_{5/2}(\eta, \beta)], \quad (64)$$

$$E = mc^2K\beta^{5/2}[F_{3/2}(\eta, \beta) + \beta F_{5/2}(\eta, \beta)], \quad (65)$$

where $K = 4\sqrt{2}\pi g(mc/h)^3$, and the Fermi integral is

$$F_k(\eta, \beta) \equiv \int_0^{+\infty}\frac{z^k(1 + \frac{1}{2}\beta z)^{1/2}dz}{e^{z-\eta} + 1} \quad (k > -1). \quad (66)$$

Then we have

$$\frac{p}{\mu} = \frac{p}{E + nmc^2} = \frac{1}{3 + 3F_{1/2}(\eta, \beta)/(2\beta F_{3/2}(\eta, \beta) + \beta^2F_{5/2}(\eta, \beta))}. \quad (67)$$

The relation between p/μ and (η, β) is shown in Fig. 1. One can see clearly that both degenerate and relativistic conditions can lead to the value of p/μ comparable to $1/3$ (which is the value of p/μ for radiation).

Now we study the two kinds of conditions, respectively. For a nondegenerate ideal Fermi gas (for example, $\eta = -30$), the value of p/μ is drawn from the nonrelativistic ($\beta = 0$) to the relativistic ($\beta = 2$) regime in Fig. 2. It is obvious that we need not go to an extremely relativistic condition since p/μ is already close to $1/3$ when $\beta = 2$. In the specific calculation for an electron gas, we set $p/\mu = 0.002$ when $k_BT = 1$ keV. This result indicates the possibility of testing the theory in a telluric laboratory. For a nondegenerate electron gas at $T = 273$ K, one may estimate the modification term as $p/\mu \sim k_BT/m_ec^2 \sim 10^{-8}$. On the other hand, in the experiments on the equality of the electric charges of protons and electrons, these charges in a conductor are found to be equal within 10^{-19} or better (see e.g. [18]). However, the proton system

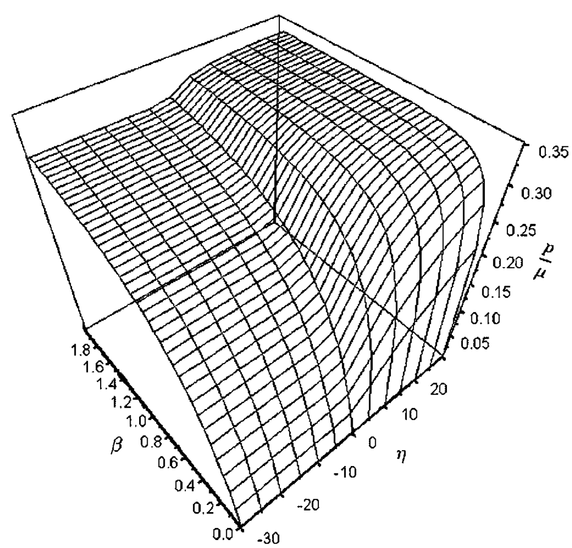

FIG. 1. Modified term p/μ as a function of the degeneracy parameter η and relativistic parameter β.

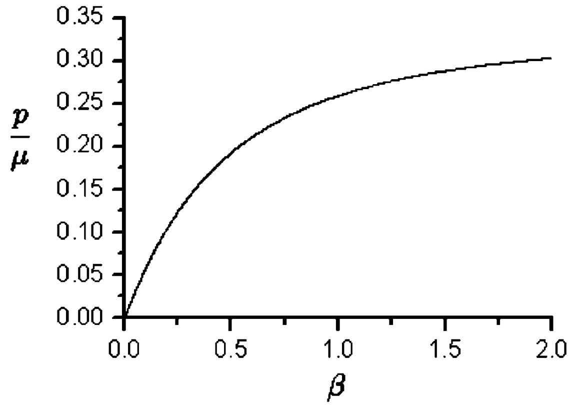

FIG. 2. Modified term p/μ as a function of the relativistic parameter β in the nondegenerate condition $\eta = -30$.

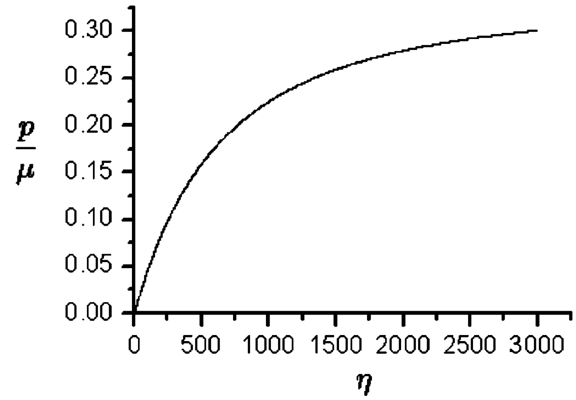

FIG. 3. Modified term p/μ as a function of the degeneracy parameter η in the nonrelativistic condition $\beta = 10^{-9}$.

in a conductor cannot be seen as a perfect fluid and hence does not satisfy our premise. Hence the effective charges of protons in a conductor cannot be directly obtained by our modified equations. So, those experiments are not in severe contradiction with the KK theory. For similar reasons, the experiments reported in Ref. [5] cannot provide definite opposite evidence to the KK theory either. But these kind of experiments do cast some doubts on the classical KK theory. Note that both the electron system and the ion system could be regarded as a perfect fluid in high-temperature plasma. In a thermal equilibrium state the electron and ion in a plasma have the same temperature. Hence they would have different values of p/μ. Actually the value of p/μ for the ion is much smaller than the one for the electron when $k_B T$ takes a value from keV to MeV. It turns out that the two important physical parameters for the description of plasma-Debye length and plasma frequency [19] have to be modified in our 5D theory as

$$\lambda_D = \left[\frac{\epsilon_0 k_B T}{n_e e^2 (1 + p_e/\mu_e)}\right]^{1/2} \tag{68}$$

and

$$\omega_p = \left[\frac{n_e e^2 (1 + p_e/\mu_e)}{m_e \epsilon_0}\right]^{1/2}, \tag{69}$$

where n_e is the number density of the electron and ϵ_0 the permittivity of the vacuum. Since the electromagnetic wave whose frequency is lower than ω_p will be reflected while others can transmit through the plasma, the plasma frequency can be measured accurately [20]. Therefore it is possible to test the prediction from the 5D KK theory in a telluric laboratory. For a degenerate idea Fermi gas, the relation between p/μ and η is demonstrated in Fig. 3. Recall that the white dwarf is known to resist the gravity by an electronic degenerate pressure. It is also possible to test the 5D theory by certain relevant phenomena in outerspace.

Note that the vacuum polarization in quantum electrodynamics (QED) also leads to an effective charge of a pointlike particle [21,22], so the effective charge viewpoint does not merely come from the KK theory. For the Fermi gas in KK theory, the larger the density and the temperature, the larger the effective charge factor $\tilde{\rho}_e/\rho_e$, which approaches $4/3$ as a limit, whereas for QED the higher the energy scale (or shorter the distance), the larger the effective charge $e_{\rm eff}/e$, which approaches infinity as a limit. Therefore the state-relevant Maxwell's equation and QED give similar results of larger effective charges. However, the state-relevant effect in KK theory is a pure classical effect due to the extra dimension of spacetime, whereas the QED effect is a quantum effect irrespective of any extra dimension. Hence, one does not expect them to be the same. It is easy to distinguish the two effects by comparing their character.

In summary, the coupling of 5D perfect fluid to KK gravity is fully studied. The 4D effective equations of this 5D coupling system are derived. In particular, the modified Maxwell's equation which is relevant to the equation of state of the source is obtained. To facilitate applications, we also derive the $3 + 1$ form of the modified Maxwell's equations and the relativistic MHD. It turns out that the effective charge density in the KK theory can be written as $\tilde{\rho}_e \equiv (1 + p/\mu)\rho_e$. Moreover, using an ideal Fermi gas model, we study the modification term p/μ as a function of the degeneracy parameter η and the relativity parameter β. It reveals that the traditional Maxwell's equation is the low density and low temperature limit of the state-relevant Maxwell's equation. We thus indicate the possibility to test the state-relevant effect both in a telluric laboratory and in astrophysical phenomena.

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