Scalar-tensor cosmology at the general relativity limit: Jordan versus Einstein frame

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We consider the correspondence between the Jordan frame and the Einstein frame descriptions of scalar-tensor theory of gravitation. We argue that since the redefinition of the scalar field is not differentiable at the limit of general relativity the correspondence between the two frames is lost at this limit. To clarify the situation we analyze the dynamics of the scalar field in different frames for two distinct scalar-tensor cosmologies with specific coupling functions and demonstrate that the corresponding scalar field phase portraits are not equivalent for regions containing the general relativity limit. Therefore the answer to the question of whether general relativity is an attractor for the theory depends on the choice of the frame.

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I. INTRODUCTION

The generalization of Jordan-Fierz-Brans-Dicke theory of gravitation [1,2] known as the scalar-tensor theory [3–6], where the gravitational interaction is mediated by a scalar field together with the usual metric tensor, appears in various contexts of theoretical physics: as dilaton gravity in Kaluza-Klein, superstring, and supergravity theories, as the effective description of braneworld models [7], as an equivalent to modified $f(R)$ gravity [8], or in attempts to describe inflation [9,10] and dark energy [11].

The scalar-tensor theory (STT) can be formulated in the Jordan frame, where the scalar field Ψ is coupled nonminimally to the Ricci scalar R but not directly to the matter, whereas the scalar field kinetic term involves an arbitrary function $\omega(\Psi)$. It is possible to write the theory in the form reminiscent of the Einstein general relativity where the scalar field is minimally coupled to the Ricci scalar and its kinetic term is in the canonical form. In this case the field equations are mathematically less complicated, but at the price of making the matter couplings dependent on the scalar field. Going from the Jordan to the Einstein frame proceeds through two transformations:

(1) A conformal transformation of the Jordan frame metric $g_{\mu\nu}$ into the Einstein frame metric $\tilde{g}_{\mu\nu}$;

(2) A redefinition of the original scalar field Ψ into ϕ to give its kinetic term a canonical form.

The problem of physical interpretation and equivalence of these two frames has a long history, but discussions have mostly concerned only the role and properties of the conformal transformation (e.g., [6,12,13]). Much less attention has been paid to the redefinition of the scalar field used to put its kinetic term in the canonical form. The aim of our paper is to caution against the problems stemming from this transformation. The issue is relevant, e.g., in scalar-tensor cosmology where one is interested in whether the scalar field naturally evolves to an asymptotically constant value, in which case the solutions of STT for $g_{\mu\nu}$ can coincide with those of the Einstein general relativity. In earlier investigations, which were performed in the Jordan frame, the main tool was to estimate the late-time behavior of different types of solutions [10,14]. Damour and Nordtvedt [15] used the Einstein frame to derive a nonlinear equation for the scalar field decoupled from other variables and found that, e.g., in the case of a flat Friedmann-Lemaître-Robertson-Walker (FLRW) model and dust matter there exists an attractor mechanism taking the solutions of wide class of scalar-tensor theories to the limit of general relativity. Their approach was generalized to cases of curved FLRW models with nonvanishing self-interaction potentials with the result that in the flat model and dust matter the attractor mechanism is not rendered ineffective [16]. Yet, some authors [17,18] have argued under different assumptions, but still using the Einstein frame, that the attractor mechanism is not generic and may be replaced by repulsion. In the Jordan frame, the main tool of subsequent investigations has been the construction of viable cosmological models with present state very near to general relativity, leaving the question of generality somewhat aside [19–21].

In what follows, our aim is to indicate a possible source of these controversies. The plan of the paper is the following. In Sec. II we recall a few basic facts about the scalar-tensor theory and express some general considerations why the scalar field redefinition is problematic in the general relativity limit. In Sec. III we study two explicit examples, viz. $2\omega(\Psi) + 3 = \frac{3}{1-\Psi}$ and $2\omega(\Psi) + 3 = \frac{3}{|1-\Psi|}$, and by plotting the phase portraits for the Jordan frame Ψ and the Einstein frame ϕ demonstrate how the scalar field dynamics is qualitatively different in different frames. In Sec. IV we clarify why the previous studies of the attractor mechanism in the Einstein frame have yielded different results. We also make some comments on nonminimally coupled STT and the weak field [parametrized post-Newtonian (PPN)] limit. Finally in Sec. V we draw some

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conclusions, in particular, that if the Jordan frame formulation is taken to be definitive for a scalar-tensor theory then the conditions for the attractor mechanism towards general relativity should be reconsidered in the Jordan frame.

II. GENERAL CONSIDERATIONS

Our starting point is the action of a general scalar-tensor theory in the Jordan frame

$$S_{\rm J} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\Psi R(g) - \frac{\omega(\Psi)}{\Psi}\nabla^\rho\Psi\nabla_\rho\Psi\right] + S_m(g_{\mu\nu}, \chi_m), \tag{1}$$

where ∇_μ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$, $\omega(\Psi)$ is a coupling function, κ^2 is the bare gravitational constant, and S_m is the matter part of the action where χ_m includes all other fields. Different choices of the field dependent coupling function $\omega(\Psi)$ give us different scalar-tensor theories. We assume that $\Psi \in (0, \infty)$ or a subset of it and $\omega(\Psi) > -\frac{3}{2}$ to keep the effective Newtonian gravitational constant positive [4,21]. The corresponding field equations for the metric tensor $g_{\mu\nu}$ and the scalar field Ψ are given by

$$G_{\mu\nu}(g) = \frac{\kappa^2}{\Psi}T_{\mu\nu}(g) + \frac{1}{\Psi}(\nabla_\mu\nabla_\nu\Psi - g_{\mu\nu}\Box\Psi) + \frac{\omega(\Psi)}{\Psi^2}\left(\nabla_\mu\Psi\nabla_\nu\Psi - \frac{1}{2}g_{\mu\nu}\nabla^\rho\Psi\nabla_\rho\Psi\right), \tag{2}$$

$$\Box\Psi = \frac{\kappa^2}{(2\omega(\Psi)+3)}T(g) - \frac{1}{(2\omega(\Psi)+3)}\frac{d\omega}{d\Psi}\nabla^\mu\Psi\nabla_\mu\Psi. \tag{3}$$

Although STT and general relativity are mathematically distinct theories, we may conventionally speak of "the general relativity limit of STT" in the sense of a regime of the solutions of STT where their observational predictions are identical with those of general relativity. In typical observational tests of gravitational theories the PPN formalism is used for slowly moving spherically symmetric systems in the weak field approximation. Nordtvedt [4] has demonstrated that the PPN parameters of a STT [with a distinct coupling function $\omega(\Psi)$] coincide with those of general relativity with the Newtonian gravitational constant $G_N = \kappa^2/\Psi_0$ if

$$\lim_{\Psi\to\Psi_0}\frac{1}{\omega(\Psi)} = 0, \qquad \lim_{\Psi\to\Psi_0}\frac{1}{\omega^3(\Psi)}\frac{d\omega}{d\Psi} = 0. \tag{4}$$

Let us denote the value $\Psi = \Psi_0 =$ const as "the general relativity limit of STT." This definition allows us to call a solution of STT as "approaching the general relativity limit" if the difference between these solutions is asymptotically vanishing.

Upon the conformal rescaling $\tilde{g}_{\mu\nu} = \Psi g_{\mu\nu}$ the action (1) transforms into

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-\tilde{g}}\left[R(\tilde{g}) - \frac{(2\omega+3)}{2\Psi^2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\Psi\tilde{\nabla}_\nu\Psi\right] + S_m(\Psi^{-1}\tilde{g}_{\mu\nu}, \chi_m), \tag{5}$$

where $\tilde{\nabla}_\mu$ denotes the covariant derivative with respect to the metric $\tilde{g}_{\mu\nu}$. The kinetic term of the scalar field obtains the canonical form by the means of a field redefinition

$$2(d\phi)^2 = \frac{(2\omega+3)}{2\Psi^2}(d\Psi)^2, \tag{6}$$

that determines a double-valued correspondence

$$\frac{d\Psi}{d\phi} = \mp\frac{2\Psi}{\sqrt{2\omega(\Psi)+3}}. \tag{7}$$

This double valuedness may be interpreted as defining two distinct Einstein frame theories which correspond to a Jordan frame theory, i.e., we may choose one of the two possible signs and keep it throughout all subsequent calculations. But in the literature one also meets another approach, where the scalar field is allowed to evolve from one branch (sign) to another. In order to fully clarify the issue we retain the possibility of both signs.

The resulting Einstein frame action is given by

$$S_{\rm E} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-\tilde{g}}[R(\tilde{g}) - 2\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi] + S_m(\Psi^{-1}(\phi)\tilde{g}_{\mu\nu}, \chi_m), \tag{8}$$

where the range of ϕ depends on the range of coupling function $\omega(\Psi)$ as given by Eq. (7) and can be determined only upon choosing a particular function $\omega(\Psi)$. The corresponding field equations are

$$G_{\mu\nu}(\tilde{g}) = \kappa^2 T_{\mu\nu}(\tilde{g}) + 2(\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\tilde{\nabla}^\rho\phi\tilde{\nabla}_\rho\phi), \tag{9}$$

$$\tilde{\Box}\phi = \frac{\kappa^2}{2}\alpha(\phi)T(\tilde{g}), \tag{10}$$

where

$$T_{\mu\nu}(\tilde{g}) = -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta S_m[\Psi^{-1}(\phi)\tilde{g}_{\mu\nu}, \chi_m]}{\delta\tilde{g}^{\mu\nu}}, \qquad \tilde{\nabla}^\mu T_{\mu\nu}(\tilde{g}) = -\alpha(\phi)T(\tilde{g})\tilde{\nabla}_\nu\phi, \tag{11}$$

and

$$\alpha(\phi) = \sqrt{\Psi}\frac{d(\sqrt{\Psi})^{-1}}{d\phi} = \pm\frac{1}{\sqrt{2\omega[\Psi(\phi)]+3}}. \tag{12}$$

"The limit of general relativity" corresponding to Eq. (4) is now given by $\phi = \phi_0$, satisfying $\alpha(\phi_0) = 0$.

The mathematical form of the scalar field redefinition (7) and of the ensuing Eq. (12) raise two concerns here.

(1) The property of double valuedness of $\phi(\Psi)$ is generally harmless, simply meaning that the original Jordan frame physics is represented by two equivalent copies in the Einstein frame description (related by $\phi \leftrightarrow -\phi$). However, these two copies meet at the point Ψ_0 corresponding to the limit of general relativity (4). Since $d\Psi/d\phi$ vanishes there, this point has to be a point of inflection or a local extremum of function $\Psi(\phi)$ (for an illustration see Fig. 1). The former case corresponds to picking the same sign in Eq. (7) on both sides $\Psi < \Psi_0$ and $\Psi > \Psi_0$, while in the latter case the derivative $d\Psi/d\phi$ changes sign, which occurs with changing the sign in Eq. (7). The second option remains the only possibility when the scalar field in the Jordan frame is assumed to have a restricted domain and Ψ_0 resides at its boundary. It turns out that the choice of the domain of Ψ and related issue of signs in Eq. (7) are significant and in Sec. IV we discuss how different assumptions yield qualitatively different results in the Einstein frame, namely, whether or not ϕ_0 is a generic attractor.

(2) The property of $d\Psi/d\phi$ to vanish at Ψ_0 implies that as the field Ψ reaches the value Ψ_0 its dynamics as determined by the variational principle loses the correspondence with the dynamics of ϕ. Indeed, an infinitesimal variation of an action functional is invariant at a regular change of variables, so the variation of STT action functional can be given in two different forms:

$$\delta S = \frac{\delta S_J}{\delta\Psi}\delta\Psi + \frac{\delta S_J}{\delta g_{\mu\nu}}\delta g^{\mu\nu} = \frac{\delta S_E}{\delta\phi}\delta\phi + \frac{\delta S_E}{\delta\tilde{g}_{\mu\nu}}\delta\tilde{g}^{\mu\nu}. \tag{13}$$

But this relation may not hold if estimated at extremals $(\Psi_0, g_{\mu\nu})$, since $\delta\phi = \frac{d\phi}{d\Psi}\delta\Psi$ and $\frac{d\phi}{d\Psi}$ diverges there according to Eq. (7), i.e., the change of variables is not regular.

Here a remote analogy with coordinate patches in topologically nontrivial spaces suggests itself. For instance, if we describe particle's worldlines in terms of Schwarzschild coordinates we cannot go beyond the $r = 2m$ "boundary," however, if we use Kruskal coordinates we would be able to follow the particle's worldline beyond it. In the case of scalar-tensor theories, the choice of "field coordinates" could also entail similar effects. Yet, invariant description of STT in field space is still not well understood (e.g., [22]).

III. EXAMPLES

A. $2\omega(\Psi) + 3 = \frac{3}{(1-\Psi)}$

Let us consider a scalar-tensor cosmology with the coupling function

$$\omega(\Psi) = \frac{3}{2}\frac{\Psi}{(1-\Psi)}, \tag{14}$$

with a restricted domain $\Psi \in (0, 1]$, which arises as an effective description of Randall-Sundrum two-brane cosmology [23,24], and has also been considered before as an example of conformal coupling [10,20,25] or as a STT with vanishing scalar curvature [26]. The field equations for a flat Universe ($k = 0$) with the FLRW line element and perfect barotropic fluid matter, $p = (\Gamma - 1)\rho$, read

$$H^2 = -H\frac{\dot\Psi}{\Psi} + \frac{1}{4}\frac{\dot\Psi^2}{\Psi(1-\Psi)} + \frac{\kappa^2}{3}\frac{\rho}{\Psi}, \tag{15}$$

$$2\dot H + 3H^2 = -2H\frac{\dot\Psi}{\Psi} - \frac{3}{4}\frac{\dot\Psi^2}{\Psi(1-\Psi)} - \frac{\ddot\Psi}{\Psi} - \frac{\kappa^2}{\Psi}(\Gamma - 1)\rho, \tag{16}$$

$$\ddot\Psi = -3H\dot\Psi - \frac{1}{2}\frac{\dot\Psi^2}{(1-\Psi)} + \frac{\kappa^2}{3}(1-\Psi)(4-3\Gamma)\rho \tag{17}$$

($H \equiv \dot a/a$), while the conservation law is the usual

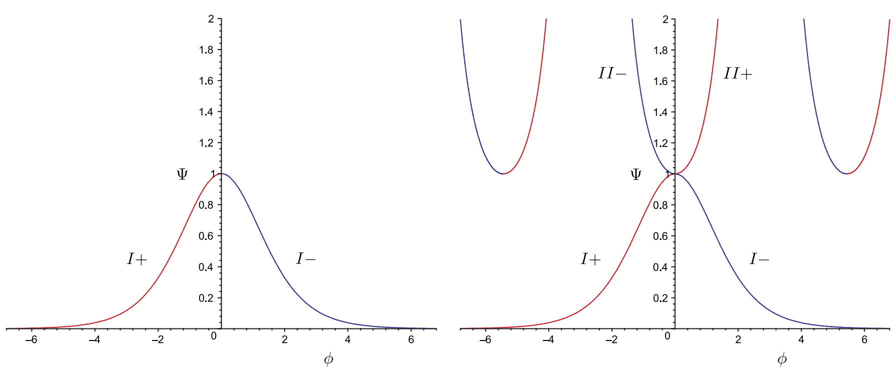

FIG. 1 (color online). Solution of the scalar field redefinition (7) in example A $2\omega(\Psi) + 3 = \frac{3}{1-\Psi}$ (left), and example B $2\omega(\Psi) + 3 = \frac{3}{|1-\Psi|}$ (right).

$$\dot{\rho} + 3H\Gamma\rho = 0. \tag{18}$$

The limit of general relativity (4) is reached at $\Psi \to 1$. Equations (15)–(17) are singular at this value, however, as we see soon, it corresponds to a fixed point in a dynamical system describing the scalar field.

The Einstein frame description is obtained by conformally rescaling the metric, $\tilde{g}_{\mu\nu} = \Psi g_{\mu\nu}$, followed by a coordinate transformation to keep the FLRW form of the line element,

$$\tilde{a} = \sqrt{\Psi}a, \qquad d\tilde{t} = \sqrt{\Psi}dt, \qquad \tilde{\rho} = \Psi^{-2}\rho. \tag{19}$$

The redefinition (7) of the scalar field which gives its kinetic term the usual canonical form,

$$\frac{d\phi}{d\Psi} = \mp\sqrt{\frac{3}{4}\frac{1}{\Psi^2(1-\Psi)}}, \tag{20}$$

is solved by

$$\pm\phi = \sqrt{3}\mathrm{arctanh}(\sqrt{1-\Psi}), \qquad \pm\sqrt{1-\Psi} = \tanh\left(\frac{\phi}{\sqrt{3}}\right). \tag{21}$$

The solution is plotted on Fig. 1 left. There are two branches $I+$ and $I-$ corresponding to the positive and negative signs in Eq. (20) respectively. The map $\Psi \to \phi$ is double valued, the two branches $\phi \in (-\infty, 0]$ and $\phi \in (\infty, 0]$ define two Einstein frame copies of the Jordan frame physics of $\Psi \in (0, 1]$. The two branches meet at the point $\phi_0 = 0$, which corresponds to the limit of general relativity, $\Psi_0 = 1$. For this point there is a choice to be made with two options: either we allow ϕ to pass from one branch to another, or not. The first option would mean that ϕ can jump from one copy of the Einstein frame description to another equivalent copy. In the Jordan frame description this corresponds to Ψ bouncing back from Ψ_0. The second option would mean that the evolution of ϕ has to end at ϕ_0 even when it reaches this point with nonvanishing speed. Of course, there would be no problem, if the equations for ϕ were already "aware" of this and never allowed ϕ to reach ϕ_0 with nonvanishing speed. Unfortunately this is not so, as we will see in the following.

The Einstein frame equations read

$$\tilde{H}^2 = \frac{1}{3}\dot{\phi}^2 + \frac{\kappa^2}{3}\tilde{\rho} \tag{22}$$

$$2\dot{\tilde{H}} + 3\tilde{H}^2 = -\dot{\phi}^2 - \kappa^2(\Gamma - 1)\tilde{\rho}, \tag{23}$$

$$\ddot{\phi} + 3\tilde{H}\dot{\phi} = -\frac{1}{2}\kappa^2\alpha(\phi)(4 - 3\Gamma)\tilde{\rho}, \tag{24}$$

$$\dot{\tilde{\rho}} + 3\tilde{H}\Gamma\tilde{\rho} = \alpha(\phi)(4 - 3\Gamma)\tilde{\rho}\,\dot{\phi}. \tag{25}$$

Here

$$\alpha(\phi) = \frac{1}{\sqrt{3}}\tanh\left(\frac{\phi}{\sqrt{3}}\right) \tag{26}$$

acts as a coupling function in the wave equation for the scalar field (24) and also occurs in the matter conservation law (25). The limit of general relativity, $\alpha(\phi_0) = 0$, is at $\phi_0 = 0$.

In the following let us consider the case of dust matter ($\Gamma = 1$). Equations (15)–(18) and (22)–(25) can be numerically integrated (Fig. 2). The result explicitly supports the concern that the dynamics of the scalar field can be different in different frames when the limit of general relativity is reached: while the Jordan frame solution converges to the limit of general relativity ($\Psi_0 = 1$), the Einstein frame solution of the same initial conditions (properly transformed from the Jordan frame) evolves through the corresponding point ($\phi_0 = 0$). Here we allowed ϕ to jump from the branch $I-$ to the branch $I+$, since otherwise it must have been stopped abruptly at $\phi_0 = 0$, which is not in accordance with Eqs. (22)–(24). To confirm that this difference in the behavior of the Jordan and the Einstein frame descriptions is not due to numerical effects, but is truly encoded in the dynamics, we have to look at the phase portraits [27].

By a change of variables introduced by Damour and Nordtvedt [15] it is possible to combine the field equations to get a dynamical equation for the scalar field which does

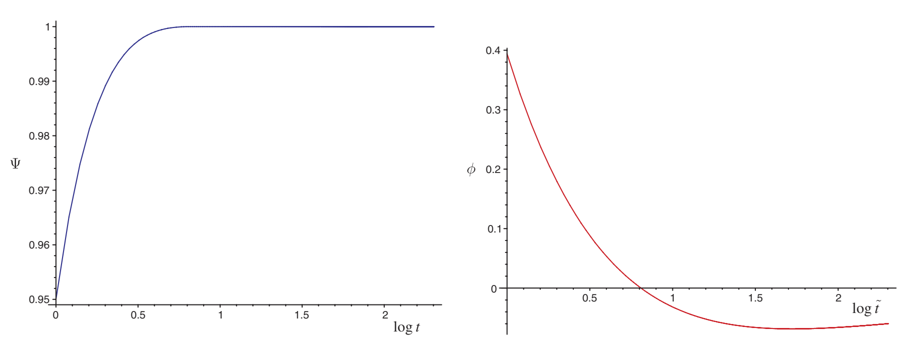

FIG. 2 (color online). Numerical solution of example A with the initial condition $\Psi(0) = 0.95$, $\dot{\Psi}(0) = 0.095$, $\rho(0) = 1$, $a(0) = 1$ in the Jordan frame (left) and Einstein frame (right). Note that since $\Psi \approx 1$ the respective time variables t and $\tilde{t}$ differ only slightly.

not manifestly contain the scale factor or matter density. In the Jordan frame this amounts to defining a new time variable [18]

$$dp = h_c dt \equiv \left(H + \frac{\dot{\Psi}}{2\Psi}\right)dt. \tag{27}$$

Then from Eqs. (15)–(17) the following "master" equation for the scalar field can be derived [18,24]:

$$8(1-\Psi)\frac{\Psi''}{\Psi} - 3\left(\frac{\Psi'}{\Psi}\right)^3 - 2(3-5\Psi)\left(\frac{\Psi'}{\Psi}\right)^2 + 12(1-\Psi)\frac{\Psi'}{\Psi} - 8(1-\Psi)^2 = 0, \tag{28}$$

where primes denote the derivatives with respect to p. The Friedmann constraint (15) in terms of the new time variable p can be written as

$$h_c^2 = \frac{\kappa^2 \rho}{3\Psi(1 - \frac{\Psi'^2}{4\Psi^2(1-\Psi)})}. \tag{29}$$

Assuming that ρ is positive definite, the constraint restricts the dynamics to explore only the region

$$|\Psi'| \leq |2\Psi\sqrt{1-\Psi}|. \tag{30}$$

Notice, Eq. (29) assures that the time reparametrization (27) works fine, as within the borders of the allowed phase space p time and t time always run in the same direction. Also, from $\dot{\Psi} = h_c \Psi'$ it is easy to see that $\dot{\Psi} = 0$ corresponds to $\Psi' = 0$, while $\dot{\Psi} = \pm\infty$ corresponds to the boundary $\Psi' = \pm 2\Psi\sqrt{1-\Psi}$.

Let us introduce variables $x \equiv \Psi$, $y \equiv \Psi'$ and write Eq. (28) as a dynamical system

$$\begin{cases} x' = y \\ y' = \frac{3y^3}{8x^2(1-x)} + \frac{(3-5x)y^2}{4x(1-x)} - \frac{3y}{2} + x(1-x). \end{cases} \tag{31}$$

There are two fixed points:

(i) A saddle point at ($x = 0$, $y = 0$), with repulsive and attractive eigenvectors tangential to the upper and lower boundaries $y = \pm 2x\sqrt{1-x}$, respectively,

(ii) A spiralling attractor at ($x = 1$, $y = 0$), but notice here the trajectories also need to respect the boundaries of the allowed region.

As can be seen from the phase portrait (Fig. 3 left) all trajectories begin in the infinitesimal vicinity of one of the two fixed points. Also all trajectories are collected by the attractor, except for the marginal trajectory along the boundary $y = -2x\sqrt{1-x}$, which runs into the saddle point. Translating back to the original time t it turns out that the attractor corresponds to the limit of general relativity ($\Psi \to 1$, $\dot{\Psi} \to 0$) for all trajectories within the allowed phase space.

In the Einstein frame the new time variable is given by [15,18]

$$dp = \tilde{H} d\tilde{t}, \tag{32}$$

and from Eqs. (22)–(24) follows an analogous master equation

$$\frac{2}{3 - \phi'^2}\phi'' + \phi' = -\alpha(\phi), \tag{33}$$

where primes denote the derivatives with respect to p and $\alpha(\phi)$ is given by Eq. (26). Now the allowed phase space is constrained by

$$\phi' \leq \pm\sqrt{3}, \tag{34}$$

$\dot{\phi} = 0$ corresponds to $\phi' = 0$, while $\dot{\phi} = \pm\infty$ corresponds to the boundary $\phi' = \pm\sqrt{3}$. In the variables $x \equiv \phi$, $y \equiv \phi'$ Eq. (33) reads

$$\begin{cases} x' = y, \\ y' = -y(3 - y^2) - \frac{(3-y^2)}{\sqrt{3}} \tanh(\frac{x}{\sqrt{3}}). \end{cases} \tag{35}$$

There is one fixed point:

(i) An attractor at ($x = 0$, $y = 0$).

As can be observed from the phase portrait (Fig. 3 right)

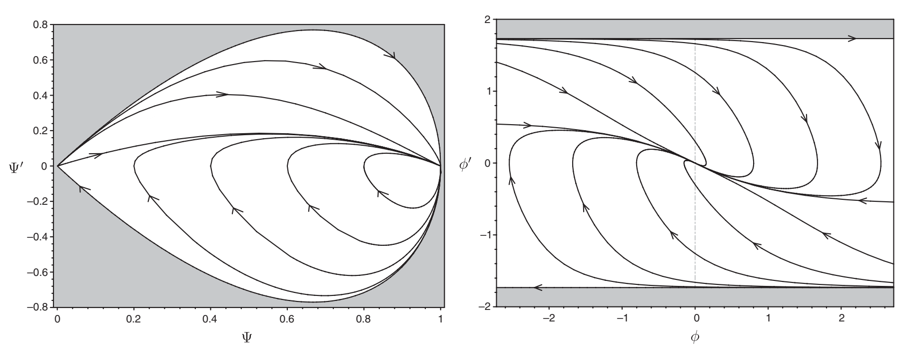

FIG. 3. Example A phase portraits of the scalar field master Eq. (28) in the Jordan frame (left) and its analogue (26) and (33) in the Einstein frame (right).

the attractor collects all the trajectories, except the marginal ones which run along the boundaries.

Despite both cases exhibiting an attractor behavior, the Jordan and Einstein frame phase portraits are not equivalent. The Einstein frame portrait is symmetric under $x \leftrightarrow -x$, $y \leftrightarrow -y$, related to the two branches (two copies) discussed above. The transition form one branch to another is smooth and there is no constraint on the Einstein frame dynamics to prevent the trajectories from passing through $\phi = 0$. In fact, all the Einstein frame trajectories do cross once from one branch to another, except for the two trajectories which flow directly from $\phi = \pm\infty$ to the fixed point. This general behavior confirms that the Einstein frame solution on Fig. 2 right does indeed evolve through $\phi = 0$ and the crossing is not an artifact of numerical errors in a sensitive region. However, the passing of ϕ from one branch to another would in the Jordan frame description correspond to Ψ evolving to $\Psi = 1$ and then bouncing back to $\Psi < 1$. This does not happen, as is illustrated by the solution on Fig. 2 left, which monotonously converges to $\Psi = 1$. The analysis of the Jordan frame phase portrait makes it completely clear. No trajectory does change from $\Psi' > 0$ to $\Psi' < 0$, all trajectories with $\Psi' > 0$ necessarily flow towards $\Psi = 1$, and $\Psi = 1$ is a fixed point, i.e., there is no way back.

An alternative option would be to cut the Einstein frame phase portrait along $\phi = 0$ into two copies and maintain both separately. Then there would be no problematic crossing from one branch to another, however, in this case there is a mismatch between the extent of the past or future of the solutions in different frames. All generic Einstein frame solutions either terminate at finite time (run to $\phi = 0$ with $\phi' \neq 0$) or begin at finite time (emerge at $\phi = 0$ with $\phi' \neq 0$). Yet, all Jordan frame solutions have infinite past and infinite future (they begin near a fixed point and run into a fixed point). On Fig. 2 this would correspond to terminating the Einstein frame solution at $\phi = 0$ at a finite time, while its Jordan frame counterpart can enjoy an infinite time in approaching $\Psi = 1$.

The reason for the incompatibility of the Jordan and Einstein frame pictures lies, of course, in the singular behavior of the transformation (20) at $\phi = 0$, which maps the point ($\Psi = 1$, $\Psi' = 0$) in the Jordan frame to the whole line ($\phi = 0$, $|\phi'| < \sqrt{3}$) on the Einstein frame phase diagram. The Jordan frame solutions which approach $\Psi \to 1$ with $\Psi' \to 0$ get mapped to the Einstein frame solutions $\phi \to 0$ with arbitrary ϕ' which therefore do not necessarily stop at $\phi = 0$, but can evolve through. This is a manifestation of our general observation that at the limit of general relativity the dynamics of the Einstein frame ϕ loses any correspondence with the dynamics of the Jordan frame Ψ. The fact that the Einstein frame description involves two copies of the Jordan frame physics and the problem whether or not to glue these copies together really becomes an issue since the ϕ trajectories lose correspondence with the Ψ trajectories at this point. None of the two options on how to deal with the two branches yields an acceptable result.

B. $2\omega(\Psi) + 3 = \frac{3}{|1-\Psi|}$

As a second example, let us consider a scalar-tensor cosmology with the coupling function

$$\omega(\Psi) = \frac{3}{2}\frac{1-|1-\Psi|}{|1-\Psi|}, \qquad \Psi \in (0, +\infty), \tag{36}$$

which belongs to subclasses (a) and (c) in the classification proposed by Barrow and Parsons [21] and was studied before by Serna *et al.* [18]. The field equations for a flat Universe ($k = 0$) with the FLRW line element and perfect fluid matter now read

$$H^2 = -H\frac{\dot{\Psi}}{\Psi} + \frac{1}{4}\frac{1-|1-\Psi|}{|1-\Psi|}\left(\frac{\dot{\Psi}}{\Psi}\right)^2 + \frac{\kappa^2}{3}\frac{\rho}{\Psi}, \tag{37}$$

$$2\dot{H} + 3H^2 = -2H\frac{\dot{\Psi}}{\Psi} - \frac{3}{4}\frac{1-|1-\Psi|}{|1-\Psi|}\left(\frac{\dot{\Psi}}{\Psi}\right)^2 - \frac{\ddot{\Psi}}{\Psi} - \frac{\kappa^2}{\Psi}(\Gamma - 1)\rho, \tag{38}$$

$$\ddot{\Psi} = -3H\dot{\Psi} - \frac{1}{2}\frac{\dot{\Psi}^2}{(1-\Psi)} + \frac{\kappa^2}{3}|1-\Psi|(4 - 3\Gamma)\rho. \tag{39}$$

In the case of dust ($\Gamma = 1$) an analogue of the master Eq. (28) is given by

$$8|1-\Psi|\frac{\Psi''}{\Psi} - 3\left(\frac{\Psi'}{\Psi}\right)^3 - 2\frac{|1-\Psi|}{(1-\Psi)}(3 - 5\Psi)\left(\frac{\Psi'}{\Psi}\right)^2 + 12|1-\Psi|\frac{\Psi'}{\Psi} - 8(1-\Psi)^2 = 0, \tag{40}$$

while the Friedmann equation constrains the dynamics to explore the region

$$|\Psi'| \leq |2\Psi\sqrt{|1-\Psi|}| \tag{41}$$

only. We can write Eq. (40) as a dynamical system and study the respective phase portrait as before, see Fig. 4 left. The phase portrait in the region $\Psi \leq 1$ is identical with the previous case (Fig. 3 left), while the region $\Psi \geq 1$ is now a new feature. These two regions meet at the point ($\Psi = 1$, $\Psi' = 0$), which is also a fixed point. It turns out that this fixed point has different properties for the regions $\Psi \leq 1$ and $\Psi \geq 1$. For the trajectories in the region $\Psi \leq 1$ it functions as a spiralling attractor as we learned before. For the trajectories in the $\Psi \geq 1$ region, however, it is a saddle point with attractive and repulsive eigenvectors tangential to the lower and upper boundaries (41), respectively. Therefore all generic trajectories in the $\Psi \geq 1$ region start at $\Psi = \infty$, come arbitrarily close to $\Psi = 1$ but get turned around and run back to $\Psi = \infty$. It is not

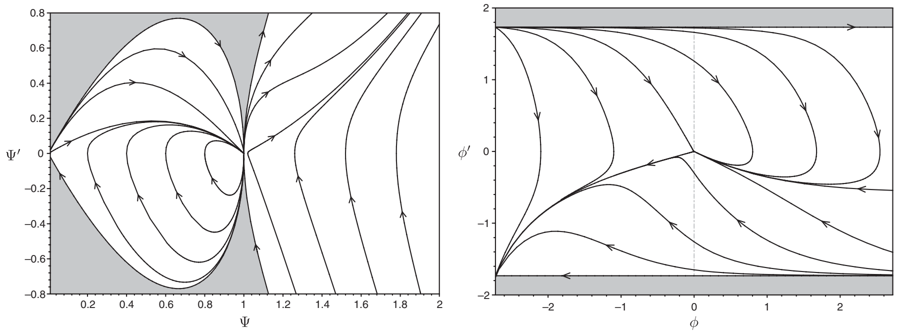

FIG. 4. Example B phase portraits of the scalar field master Eq. (40) in the Jordan frame (left) and its analogue (33) and (43) in the Einstein frame (right).

possible for the trajectories to pass from the region $\Psi \leq 1$ to the region $\Psi \geq 1$, or vice versa.

The Einstein frame view with the canonical scalar field kinetic term is obtained from Eq. (7), the solution is given by

$$\pm \phi = \begin{cases} \sqrt{3}\text{arctanh}(\sqrt{1-\Psi}), & \Psi \leq 1, \\ -\sqrt{3}\arctan(\sqrt{\Psi - 1}), & \Psi \geq 1, \end{cases} \tag{42}$$

see Fig. 1 right. As in the previous case, the solution has two branches $(I+, II+)$ and $(I-, II-)$ related to the $\mp$ sign in Eq. (7) and to be interpreted as two equivalent Einstein frame copies of the Jordan frame dynamics. [Actually the transformation (42) is infinitely many-valued in the domain $\Psi \geq 1$, since for each Ψ we have $-\sqrt{3}\arctan(\sqrt{\Psi - 1}) = \sqrt{3}(\phi_c + n\pi)$, $\phi_c \in [-\frac{\pi}{2}, 0]$, but in what follows we ignore this extra complication and assume $n = 0$.]

Let us focus on one of these branches by taking the + sign in Eq. (42). Then $\Psi \in (0, 1]$ gets mapped onto $\phi \in (\infty, 0]$ and $\Psi \in [1, \infty)$ gets mapped onto $\phi \in [0, -\frac{\pi}{2}\sqrt{3})$. The Einstein frame field equations have the same form as in the example considered previously (22)–(24), but with the coupling function $\alpha(\phi)$ given by

$$\alpha(\phi) = \begin{cases} \frac{1}{\sqrt{3}}\tanh(\frac{\phi}{\sqrt{3}}) & \phi \geq 0, \\ -\frac{1}{\sqrt{3}}\tan(\frac{\phi}{\sqrt{3}}) & \phi \leq 0. \end{cases} \tag{43}$$

The limit to general relativity corresponds to the value $\phi = 0$ as before.

The master equation for ϕ retains its form (33) as well, but with the coupling function (43). The corresponding Einstein frame phase portrait on Fig. 4 right exhibits no symmetry reflecting the fact that we have chosen only one branch of $\phi(\Psi)$. (The other branch would have given a mirror portrait with $\phi \to -\phi$.) The point ($\phi = 0$, $\phi' = 0$) is still a fixed point, but characterized by different properties with respect to the regions $\phi \geq 0$ and $\phi \leq 0$. For $\phi \geq 0$ it is an attractor, but for $\phi \leq 0$ it is a saddle point.

Despite the properties of this fixed point being the same in the respective regions of the Einstein and Jordan frame, the phase portraits are clearly not equivalent in the two frames. While the Jordan frame trajectories are unable to cross the general relativity limit $\Psi = 1$, the generic Einstein frame trajectories do it once. In particular, all the Jordan frame trajectories with $\Psi < 1$ converge to the general relativity fixed point, but only some of the corresponding Einstein frame trajectories with $\phi > 0$ are collected by the fixed point while others pass through $\phi = 0$ and get repelled from general relativity. Similarly, all the generic Jordan frame trajectories with $\Psi > 1$ can only get arbitrarily close to general relativity, but in the Einstein frame only some of the corresponding trajectories with $\phi < 0$ are repelled while some can pass through $\phi = 0$ and end up at the fixed point. Therefore, although the issue of the Einstein frame trajectories jumping from one branch to another does not arise in this case, the problem of the losing the correspondence between the Jordan and Einstein frame dynamics at the general relativity limit is still manifest.

IV. DISCUSSION

A. General relativity as a late time attractor for generic scalar-tensor theories

Studies of this question have usually relied on the Einstein frame where the equations are mathematically less complicated. Damour and Nordtvedt [15] investigated Eq. (33) in the linear approximation of an arbitrary coupling function at the point of general relativity ($\phi = 0$), assuming $\alpha(\phi) \sim \phi$ which corresponds to a quadratic "potential" $P(\phi) \sim \phi^2$, introduced as $\alpha \equiv dP/d\phi$. In the case of dust matter they found an oscillatory behavior of the scalar field with late-time relaxation to general relativity. In comparison, Serna *et al.* [18] obtain $\alpha(\phi) \sim |\phi|$ for small values of ϕ from the examples of Barrow and Parsons [21] in the Jordan frame. Now the corresponding potential has no minimum, $P \sim \text{sign}(\phi)\phi^2$, and general relativity ($\phi = 0$) is a point of inflection making possible also repulsion from general relativity.

Both of these two cases are contained in our examples as a linear approximation near $\phi = 0$: Eq. (26) implies $\alpha(\phi) \sim \phi$ and Eq. (43) implies $\alpha(\phi) \sim |\phi|$. The respective qualitative behavior can be inferred from the phase portraits (Figs. 3 and 4 right) in the neighborhood of the fixed point ($\phi = 0$, $\phi' = 0$). Also recall that the first case involved allowing ϕ to pass from one sign in Eq. (21) to another, while in the second case ϕ was evolving according to Eq. (42) with a fixed sign.

In fact, using our phase portraits it is also possible to combine portraits for the cases of $\alpha(\phi) \sim -\phi$ and $\alpha(\phi) \sim -|\phi|$. Gluing together the left half of Fig. 4 right ($\phi \leq 0$) with its image under the transformation $\phi \rightarrow -\phi$, $\phi' \rightarrow -\phi'$ gives the phase portrait for $\alpha(\phi) \sim -\phi$, generically characterized by repulsion from general relativity. Reflection $\phi \rightarrow -\phi$ of the full Fig. 4 right yields the portrait for $\alpha(\phi) \sim -|\phi|$ with properties similar to the $\alpha(\phi) \sim |\phi|$ case.

It is clear that the possibility of general relativity being an Einstein frame attractor crucially depends on the form of the coupling function $\alpha(\phi)$ and without knowing it at least in the neighborhood of general relativity no conclusions can be drawn. This is in accord with the results of Gérard and Mahara [17] who tried to determine a generic behavior around the general relativity in the Einstein frame without specifying the coupling function and concluded that the potential P can but need not be bounded from below.

However, if we want to translate the results into the Jordan frame description the Einstein frame analysis is not reliable, as conjectured by the general remarks in Sec. II and explicitly demonstrated by the two examples in Sec. III. For the Jordan frame conclusions about the STT convergence to general relativity the analysis must be carried out in the Jordan frame.

B. Nonminimally coupled STT

Sometimes a different action of scalar-tensor theory is considered [6,31]

$$S_\xi = \frac{1}{2\kappa^2} \int d^4x\sqrt{-g}[(1 - \xi\kappa^2\phi^2)R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi] + S_{\text{matter}}. \tag{44}$$

It is equivalent to the action (1) of the scalar-tensor theory in the Jordan frame with a specific coupling function ω, if a redefinition of the scalar field is performed,

$$\frac{d\Psi}{d\phi} = \mp\sqrt{\frac{\Psi}{\omega(\Psi)}}. \tag{45}$$

However, analogously to the redefinition (7) it (i) contains a sign ambiguity and (ii) is singular at the limit to general relativity, $\omega \rightarrow \infty$. It seems that the actions S_ξ and S_J are not equivalent at the limit to general relativity since S_J is obtained from S_ξ through a singular transformation (45).

Note that Faraoni [32] has also recently pointed out that the correspondence between modified $f(R)$ theories and scalar-tensor theories of gravity breaks down in the limit to general relativity. This indicates that general relativity may be a rather special theory for its different modifications.

C. PPN

We have demonstrated that there are essential differences at the limiting process to general relativity between the scalar field Ψ in the Jordan frame and the canonical scalar field ϕ in the Einstein frame. In principle, the differences may be reflected in present day observations, but only indirectly, through possible differences in the form of the solutions for the scalar fields. The Eddington parameters which determine direct observational consequences and are given in terms of the coupling function $\omega(\Psi)$ in the Jordan frame [4,5] depend only on the quantities without sign ambiguity in the Einstein frame [15],

$$\begin{aligned} \alpha^2(\phi) &= \frac{1}{2\omega[\Psi(\phi)] + 3}, \\ \frac{d\alpha}{d\phi} &= \frac{2}{G(t_0)} \frac{(2\omega[\Psi(\phi)] + 4)}{(2\omega[\Psi(\phi)] + 3)^3} \frac{d\omega}{d\Psi}, \end{aligned} \tag{46}$$

where $G(t_0)$ is the present day measured gravitational constant.

V. CONCLUSION

The action functionals S_J and S_E of the Jordan and the Einstein frame description are equivalent in the sense that they are connected by conformal transformation of the metric and redefinition of the scalar field. However, at the limit of general relativity the redefinition of the scalar field is singular and the correspondence between the different frames is lost. This results in a different behavior of solutions of the field equations at this limit, e.g., in our examples of FLRW cosmology, the scalar field Ψ in the Jordan frame never crosses its general relativistic value $\Psi_0 = 1$, but scalar field ϕ in the Einstein frame may oscillate around its general relativistic value $\phi_0 = 0$. We argue that these solutions cannot be properly set into correspondence using the redefinition of the scalar field (7). In order to investigate the scalar field as it approaches to the limit of general relativity, we must choose the frame from the very beginning by using some additional assumptions. If our choice is that the Jordan frame is basic, then the attractor mechanism towards general relativity must be reconsidered in the Jordan frame [33].

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