PHYSICAL REVIEW D **76,** 101701(R) (2007)

Calculation of the shear viscosity in SU(3) gluodynamics

Harvey B. Meyer*

Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 19 April 2007; published 5 November 2007)

We perform a lattice Monte Carlo calculation of the two-point functions of the energy-momentum tensor at finite temperature in the SU(3) gauge theory. Unprecedented precision is obtained thanks to a multilevel algorithm. The lattice operators are renormalized nonperturbatively and the classical discretization errors affecting the correlators are corrected for. A conservative upper bound for the shear viscosity to entropy density ratio is derived, $\eta/s < 1.0$, and our best estimate is $\eta/s = 0.134(33)$ at $T = 1.65T_c$ under the assumption of smoothness of the spectral function in the low-frequency region.

DOI: 10.1103/PhysRevD.76.101701 PACS numbers: 12.38.Gc, 12.38.Mh, 25.75.−q

I. INTRODUCTION

Models treating the system produced in heavy ion collisions at Relativistic Heavy Ion Collider (RHIC) as an ideal fluid have had significant success in describing the observed flow phenomena [1,2]. Subsequently the leading corrections due to a finite shear viscosity were computed [3], in particular, the flattening of the elliptic flow coefficient $v_2(p_T)$ above 1 GeV. It is therefore important to compute the QCD shear and bulk viscosities from first principles to establish this description more firmly. Small transport coefficients are a signature of strong interactions, which lead to efficient transmission of momentum in the system. Strong interactions in turn require nonperturbative computational techniques. Several attempts have been made to compute these observables on the lattice in the SU(3) gauge theory [4,5]. These calculations are based on the Kubo formulas, which relate each transport coefficient to a spectral function $\rho(\omega)$ at vanishing frequency. Even on current computers, these calculations are highly nontrivial, due to the falloff of the relevant correlators in Euclidean time (as x_0^{-5} at short distances), implying a poor signal-to-noise ratio in a standard Monte Carlo calculation. The second difficulty is to solve the ill-posed inverse problem for $\rho(\omega)$ given the Euclidean correlator at a finite set of points. Mathematically speaking, the uncertainty on a transport coefficient χ is infinite for any finite statistical accuracy, because adding $\epsilon\omega\delta(\omega)$ to $\rho(\omega)$ merely corresponds to adding a constant to the Euclidean correlator of order ϵ, while rendering χ infinite. Therefore, smoothness assumptions on $\rho(\omega)$ have to be made, which are reasonable far from the one-particle energy eigenstates, and can be proved in the hard-thermal-loop framework [6].

In this paper we present a new calculation which dramatically improves on the statistical accuracy of the Euclidean correlator relevant to the shear viscosity through the use of a two-level algorithm [7]. This allows us to derive a conservative upper bound on the viscosity and a useful estimate of the ratio η/s, which has acquired a special significance since its value $1/4\pi$ in a class of strongly coupled supersymmetric gauge theories [8] was conjectured to be an absolute lower bound for all substances [9].

II. METHODOLOGY

In the continuum, the energy-momentum tensor $\bar{T}_{\mu\nu}(x) = \frac{1}{4}\delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma} - F^a_{\mu\alpha}F^a_{\nu\alpha}$, being a set of Noether currents associated with translations in space and time, does not renormalize. With $L_0 = 1/T$ the inverse temperature, we consider the Euclidean two-point function $(0 < x_0 < L_0)$

$$C(x_0) = L_0^5 \int d^3\mathbf{x}\langle \bar{T}_{12}(0)\bar{T}_{12}(x_0, \mathbf{x})\rangle. \tag{1}$$

The tree-level expression is $C^{\text{t.l.}}(x_0) = \frac{32 d_A}{5\pi^2}(f(\tau) - \frac{\pi^4}{72})$, with $\tau = 1 - \frac{2x_0}{L_0}$, $d_A = 8$ the number of gluons, and $f(z) = \int_0^\infty ds s^4 \cosh^2(zs)/\sinh^2 s$. The correlator $C(x_0)$ is thus dimensionless and, in a conformal field theory, would be a function of Tx_0 only.

The spectral function is defined by

$$C(x_0) = L_0^5 \int_0^\infty \rho(\omega) \frac{\cosh\omega(\frac{1}{2}L_0 - x_0)}{\sinh\frac{\omega L_0}{2}} d\omega. \tag{2}$$

The shear viscosity is given by [4,10]

$$\eta(T) = \pi \frac{d\rho}{d\omega}\bigg|_{\omega=0}. \tag{3}$$

Important properties of ρ are its positivity, $\rho(\omega)/\omega \geq 0$, and parity, $\rho(-\omega) = -\rho(\omega)$. The spectral function that reproduces $C^{\text{t.l.}}(x_0)$ is

$$\rho^{\text{t.l.}}(\omega) = \frac{A_{\text{t.l.}}\omega^4}{\tanh\frac{1}{4}\omega L_0} + BL_0^{-4}\omega\delta(\omega), \tag{4}$$

$$A_{\text{t.l.}} = \frac{1}{10}\frac{d_A}{(4\pi)^2}, \qquad B = \left(\frac{2\pi}{15}\right)^2 d_A. \tag{5}$$

While the ω^4 term is expected to survive in the interacting theory with only logarithmic corrections, the δ function at

*meyerh@mit.edu

the origin corresponds to the fact that gluons are asymptotic states in the free theory and implies an infinite viscosity.

On the lattice, translations only form a discrete group, so that a finite renormalization is necessary, $\bar{T}_{\mu\nu}(g_0) = Z(g_0)\bar{T}^{(\mathrm{bare})}_{\mu\nu}$. We employ the Wilson action [11], $S_{\mathrm{g}} = \frac{1}{g_0^2}\sum_{x,\mu\neq\nu}\mathrm{Tr}\{1 - P_{\mu\nu}(x)\}$, on an $L_0 \cdot L^3$ hypertoroidal lattice, and the following discretized expression of the Euclidean energy:

$$\bar{T}^{(\mathrm{bare})}_{00}(x) \equiv \frac{2}{a^4 g_0^2}\Big[\sum_k \mathrm{Re}\,\mathrm{Tr}P_{0k}(x) - \sum_{k<1}\mathrm{Re}\,\mathrm{Tr}P_{kl}(x)\Big].$$

One of the lattice sum rules [12] can be interpreted as a nonperturbative renormalization condition for this particular discretization, from which we read off $Z(g_0) = 1 - \frac{1}{2}g_0^2(c_\sigma - c_\tau)$. The definition of the anisotropy coefficients $c_{\sigma,\tau}$ can be found in [13], where they are computed nonperturbatively. With a precision of about 1%, a Padé fit constrained by the one-loop result [14] yields

$$Z(g_0) = \frac{1 - 1.0225g_0^2 + 0.1305g_0^4}{1 - 0.8557g_0^2}, \qquad (6/g_0^2 \geq 5.7). \tag{6}$$

III. NUMERICAL RESULTS

We report results obtained on a $\beta \equiv 6/g_0^2 = 6.2$, 8×20^3 lattice and on a $\beta = 6.408$, 8×28^3 lattice. The first is thus at a temperature of $1.24T_c$, the second at $T = 1.65T_c$. We use the results for aT_c obtained in [15] and the nonperturbative lattice β function of [16] to determine this. We employ the two-level algorithm described in [7]. The computing time invested into the $1.65T_c$ simulation is about 860 PC days. Following [4], we discretize $\frac{1}{4}\langle(\bar{T}_{11} - \bar{T}_{22}) \times (\bar{T}_{11} - \bar{T}_{22})\rangle$ instead of $\langle\bar{T}_{12}\bar{T}_{12}\rangle$ (the two are equal in the continuum) to write $C(x_0) = \frac{L_0^5}{L^3}\langle O_\eta(0)O_\eta(x_0)\rangle + \mathrm{O}(a^2)$, where

$$O_\eta(x_0) \equiv \frac{1}{2}a^3\sum_{\mathbf{x}}\{\bar{T}_{11} - \bar{T}_{22}\}(g_0, x)$$
$$= \frac{2Z(g_0)}{ag_0^2}\sum_{\mathbf{x}}\mathrm{Re}\,\mathrm{Tr}\{P_{10} + P_{13} - P_{20} - P_{23}\}(x).$$

The three electric-electric, magnetic-magnetic, and electric-magnetic contributions to $C(x_0)$ are computed separately and shown on Fig. 1. We apply the following technique to remove the tree-level discretization errors [17] separately to C_{BB}, C_{EE}, and C_{EB}. First, $\bar{x}_0$ is defined such that $C^{\mathrm{t.l.}}_{\mathrm{cont}}(\bar{x}_0) = C^{\mathrm{t.l.}}_{\mathrm{lat}}(x_0)$. The improved correlator is defined at a discrete set of points through $\bar{C}(\bar{x}_0) = C(x_0)$, and then augmented to a continuous function via $\bar{C}(\bar{x}_0^{(i)}) = \alpha + \beta C^{\mathrm{t.l.}}_{\mathrm{cont}}(\bar{x}_0^{(i)})$, $i = 1, 2$, where $\bar{x}_0^{(1)}$ and $\bar{x}_0^{(2)}$ correspond to two adjacent measurements.

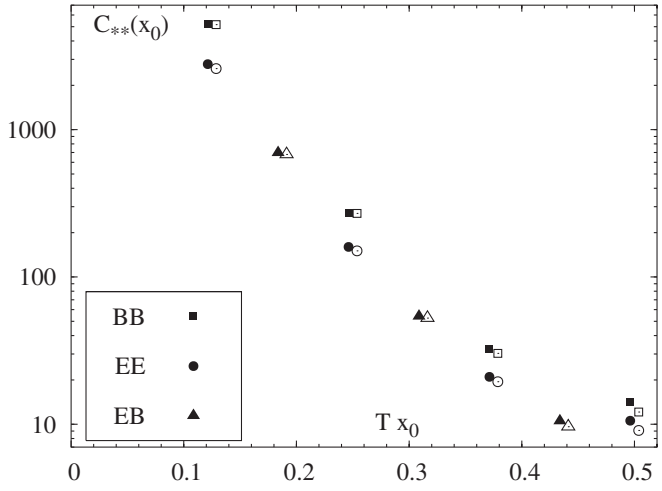

FIG. 1. The correlators that contribute to $C(x_0) = \frac{1}{4} \times (C_{BB} + C_{EE} + 2C_{EB})$. Filled symbols correspond to $T = 1.65T_c$, open symbols to $1.24T_c$. Error bars are smaller than the data symbols.

The resulting improved correlator, normalized by the continuum tree-level result, is shown on Fig. 2. One observes that the deviations from the tree-level result are surprisingly small, while deviations from conformality are visible. The latter is not unexpected at these temperatures, where p/T^4 is still strongly rising [18]. Finite-volume effects on the $T = 1.65T_c$ lattice are smaller than one part in 10^3 at tree level. Nonperturbatively, at the same temperature with resolution $L_0/a = 6$, increasing L/a from 20 to 30 reduces $\bar{C}(L_0/2)$ by a factor 0.922(73). While not statistically compelling as it stands, the effect deserves further investigation.

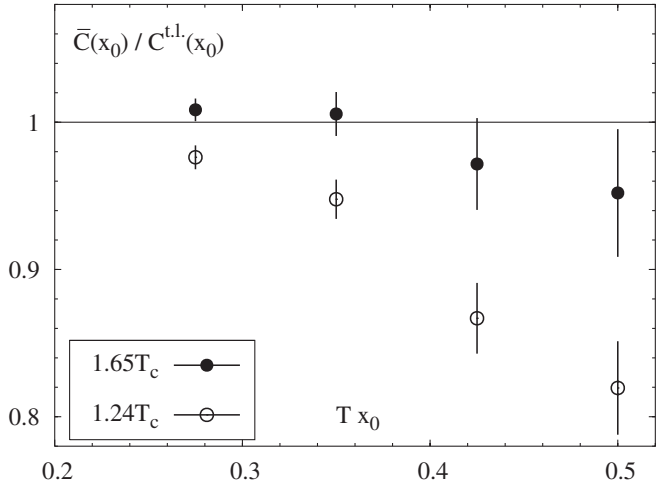

FIG. 2. The tree-level improved correlator $\bar{C}(x_0)$ normalized to the tree-level continuum infinite-volume prediction. The four points in each sequence are strongly correlated, but their covariance matrix is nonsingular.

The entropy density is obtained from the relation $s = (\epsilon + p)/T$ and the standard method to compute $\epsilon + p$ ([13], Eq. 1.14). We find $s/T^3 = 4.72(3)(5)$ and 5.70(2)(6), respectively, at $T/T_c = 1.24$ and 1.65 [the first error is statistical and the second is the uncertainty on $Z(g_0)$]. The Stefan-Boltzmann value is $32\pi^2/45$ in the continuum and 1.0867 times that value [13] at $L_0/a = 8$.

IV. UNSATISFACTORY ATTEMPTS TO EXTRACT THE VISCOSITY

In order to compare with previous studies [4,5], we fit $\bar{C}(x_0)$ with a Breit-Wigner ansatz

$$\rho(\omega)/\omega = \frac{F}{1 + b^2(\omega - \omega_0)^2} + \frac{F}{1 + b^2(\omega + \omega_0)^2}, \quad (7)$$

although it clearly ignores asymptotic freedom, which implies that $\rho(\omega) \sim \omega^4$ at $\omega \gg T$ [6]. The result of a correlated fit at $T = 1.65T_c$ using the points at $Tx_0 = 0.5$, 0.35, and 0.275 is $a^3F = 0.78(4)$, $(b/a)^2 = 240(30)$, and $a\omega_0 = 2.36(4)$, and hence $\eta/s|_{T=1.65T_c} = 0.33(3)$. A comparison of this to the result of Ref. [5], 0.4(4) around $1.5T_c$, illustrates the progress made in statistical accuracy.

An ansatz motivated by the hard-thermal-loop framework is [6]

$$\rho(\omega)/\omega = \frac{\eta/\pi}{1 + b^2\omega^2} + \theta(\omega - \omega_1)\frac{A\omega^3}{\tanh\omega/4T}. \quad (8)$$

It is capable of reproducing the tree-level prediction, Eq. (4), and it allows for a thermal broadening of the delta function at the origin. Fitting the $T = 1.65T_c$ points shown on Fig. 2, the χ^2 is minimized for $b = 0$ (effectively eliminating a free parameter), $A/A_{\text{t.l.}} = 0.996(8)$, $\omega_1/T = 7.5(2)$, and $\eta/s = 0.25(3)$, with $\chi^2_{\text{min}} = 4.0$. Thus, while the ansatz is hardly compatible with the data, which could be due in part to cutoff effects, the data favor A being close to its tree-level value.

V. A BOUND ON THE VISCOSITY

The positivity property of $\rho(\omega)$ allows us to derive an upper bound on the viscosity, based on the following assumptions:

(1) the contribution to the correlator from $\omega > \Lambda$ is correctly predicted by the tree-level formula,
(2) the width of any potential peak in the region $\omega < T$ is no less than T.

The standard QCD sum rule practice is to use perturbation theory from the energy lying midway between the lightest state and the first excitation. With this in mind we choose $\Lambda = \max(\frac{1}{2}[M_2 + M_{2^*}] \approx 2.6\ \text{GeV}, 5T)$, where $M_{2^{(*)}}$ are the masses of the two lightest tensor glueballs. To derive the upper bound we conservatively assume that, for $\omega < \sqrt{2}T$, $\rho(\omega)/\omega$ is a Breit-Wigner of width $\Gamma = T$ centered at the origin (a Gaussian would not make any significant difference). At $T < 2T_c$, this is numerically consistent with resummed perturbation theory, which predicts a Breit-Wigner centered at the origin of width $\Gamma = 2\gamma$ [6], where $\gamma \approx \alpha_s NT$ is the gluon damping rate. The tree-level spectral function instead badly violates assumption (2). From $C(\frac{1}{2}L_0) \geq L_0^5[\int_0^{\sqrt{2}T} \rho_{\text{BW}}(\omega) + \int_\Lambda^\infty \rho_{\text{t.l.}}(\omega)] \times \frac{d\omega}{\sinh\omega L_0/2}$ we obtain (with 90% statistical confidence level)

$$\eta/s < \begin{cases} 0.96 & (T = 1.65T_c), \\ 1.08 & (T = 1.24T_c). \end{cases} \quad (9)$$

VI. THE SPECTRAL FUNCTION

As illustrated above, it is rather difficult to find a functional form for $\rho(\omega)$ that is both physically motivated and fits the data. In a more model-independent approach, $\rho(\omega)$ is expanded in an orthogonal set of functions, which grows as the lattice resolution on the correlator increases, and becomes complete in the limit of $L_0/a \to \infty$. We proceed to determine the function $\bar{\rho}(\omega) \equiv \rho(\omega)/\tanh(\frac{1}{2}\omega L_0)$ by making the ansatz

$$\bar{\rho}(\omega) = m(\omega)[1 + a(\omega)], \quad (10)$$

where $m(\omega) > 0$ has the high-frequency behavior of Eq. (4), and correspondingly define $\bar{K}(x_0, \omega) = \cosh\omega(x_0 - \frac{1}{2}L_0)/\cosh\frac{1}{2}\omega L_0$. Suppose that $m(\omega)$ already is a smooth approximate solution to $\bar{\rho}(\omega)$; inserting (10) into Eq. (2), one requires that $a(\omega) = \sum_\ell c_\ell a_\ell(\omega)$, with $\{a_\ell\}$ a basis of functions which is as sensitive as possible to the discrepancy between the lattice correlator and the correlator generated by $m(\omega)$. These are the eigenfunctions of largest eigenvalue of the symmetric kernel $G(\omega, \omega') \equiv \int_0^{L_0} \frac{dx_0}{L_0} M(x_0, \omega)M(x_0, \omega')$, where $M(x_0, \omega) \equiv \bar{K}(x_0, \omega)m(\omega)$. These functions satisfy $\int_0^\infty d\omega a_\ell(\omega)a_{\ell'}(\omega) = \delta_{\ell\ell'}$ and have an increasing number of nodes as their eigenvalue decreases. Thus the more data points available, the larger the basis and the finer details of the spectral function one is able to determine.

To determine the spectral function from N points of the correlator, we proceed by first discretizing the ω variable into an N_ω-vector. The final spectral function is given by the last member $\rho^{(N)}$ of a sequence whose first member is $\rho^{(0)} = m$ and whose general member $\rho^{(n)}$ reproduces n points (or linear combinations) of the lattice correlator. For $n \geq 1$, $\rho^{(n)} = \rho^{(n-1)}[1 + \sum_{\ell=1}^n c_\ell^{(n)} a_\ell^{(n)}]$ and the functions $a_\ell^{(n)}(\omega)$ are found by the singular-value decomposition [19] of the $N_\omega \times n$ matrix $M^{(n)t}$, where $M_{ij}^{(n)} \equiv \bar{K}(x_0^{(i)}, \omega_j) \times \bar{\rho}^{(n-1)}(\omega_j)$. The "model" $m(\omega)$ is thus updated and agrees with $\rho(\omega)$ at the end of the procedure. We first performed this procedure on coarser lattices with $L_0/a = 6$ at the same temperatures, starting from $m(\omega) = A_{\text{t.l.}}\omega^4/(\tanh(\frac{1}{4}\omega L_0)\tanh(\frac{1}{2}\omega L_0)\tanh^2(c\omega L_0))$ with $\frac{1}{4} \leq c \leq \frac{1}{2}$, and then recycled the output as seed for the $L_0/a = 8$ lattices. On the latter we used the $N = 4$ points shown on Fig. 2.

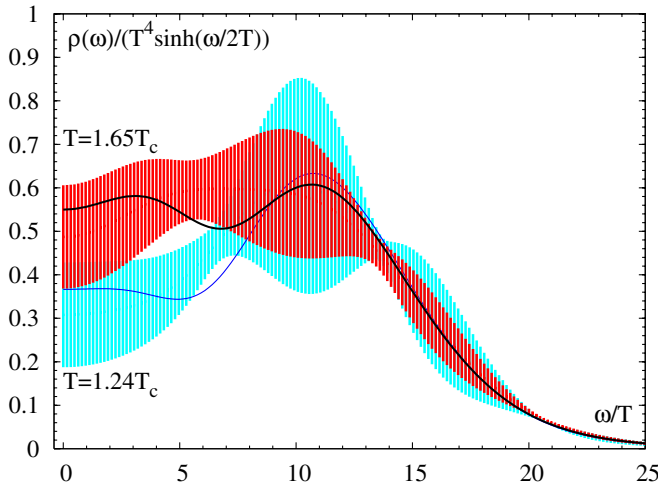

FIG. 3 (color online). The result for $\rho(\omega)$. The meaning of the error bands and the curves is described in the text. The area under them equals $\bar{C}(L_0/2) = 8.05(31)$ and $9.35(42)$ for $1.24T_c$ and $1.65T_c$, respectively, and $\eta/T^3 = (\pi/2) \times$ intercept.

The next question to address is the uncertainty on $\rho(\omega)$. It is important to realize that, even in the absence of statistical errors, a systematic uncertainty subsists due to the finite number of basis functions we can afford to describe $\rho(\omega)$ with. A reasonable measure of this uncertainty is by how much $\rho(\omega)$ varies if one doubles the resolution on $C(x_0)$. This can be estimated by "generating" new points by using the computed $\rho^{(N)}(\omega)$. On the other hand, we perform a two-point interpolation in x_0 space [we chose the form $(\alpha + \beta(x_0 - \frac{1}{2}L_0)^2)/\sin^5(\pi x_0/L_0)$], and take the difference between these and the generated ones as their systematic uncertainty. In practice, this difference is added in quadrature with the statistical uncertainty. Next we repeat the procedure to find ρ described above with $N \to 2N$: if we use as seed $\rho^{(N)}$, then by construction it is left invariant by the iterative procedure, but the derivatives of $\rho^{(2N)}$ with respect to the $2N$ points of the correlator can be evaluated. The error on $\rho(\omega)$ is then obtained from a formula of the type $(\delta\rho)^2 = \sum_{i=1}^{2N} (\frac{\partial\rho}{\partial C_i})^2 (\delta C_i)^2$, which, however, keeps track of correlations in x_0 and Monte Carlo time. This is the error band shown on Fig. 3, and the corresponding shear viscosity values are

$$\eta/s = \begin{cases} 0.134(33) & (T = 1.65T_c), \\ 0.102(56) & (T = 1.24T_c). \end{cases} \qquad (11)$$

These results are still subject to the assumption of smoothness of the spectral function on the scale T for $\omega < T$, which can be weakened (but not removed entirely) by simulating on a finer lattice. Finally the positivity of $\rho(\omega)$ provides a consistency check in our method.

It is also interesting to check for the stability of the solution under the use of a larger basis of functions. If instead of starting from $\rho^{(N)}(\omega)$ we restart from $\rho^{(0)}$ (the output of the $L_0/a = 6$ lattice) and fit the $2N$ (dependent) points using $2N$ basis functions $\{a_\ell\}$, we obtain the curves drawn on Fig. 3. As one would hope, the oscillations of $\rho^{(2N)}(\omega)$ are covered by the error band.

VII. CONCLUSION

Using state-of-the-art lattice techniques, we have computed the correlation functions of the energy-momentum tensor to high accuracy in the SU(3) pure gauge theory. We have calculated the leading high-temperature cutoff effects and removed them from the correlator relevant to the shear viscosity, and we normalized it nonperturbatively, exploiting existing results. We obtained the entropy density with an accuracy of 1%. The most conservative result obtained on the shear viscosity is the upper bound, Eq. (9), which comes from lumping the area under the curve on Fig. 3 in the interval $[0, 6T]$ into a peak of width $\Gamma = T$ centered at the origin. Second, our best estimate of the shear viscosity is given by Eq. (11), subject to the assumption formulated below the equation, using a new method of extraction of the spectral function. The errors contain an estimate of the systematic uncertainty associated with the limited resolution in Euclidean time. We are extending the calculation to finer lattice spacings and larger volumes to further consolidate our findings.

The values (11) are intriguingly close to saturating the KSS bound [9] $\eta/s \geq 1/4\pi$. We note that in perturbation theory, $(\eta/s)_{N_t=3}/(\eta/s)_{N_t=0}$ is close to 1 [20]. Assuming perturbation theory predicts this double ratio more accurately than η/s itself, our results corroborate the picture of a near-perfect fluid that has emerged from the RHIC experiments, with the magnitude of the anisotropic flow incompatible with $\eta/s \gtrsim 0.2$ [3].

ACKNOWLEDGMENTS

I thank Krishna Rajagopal and Philippe de Forcrand for their encouragement and many useful discussions. This work was supported in part by funds provided by the U.S. Department of Energy under Cooperative Research Agreement No. DE-FC02-94ER40818.

[1] P. F. Kolb, P. Huovinen, U. W. Heinz, and H. Heiselberg, Phys. Lett. B **500**, 232 (2001); P. Huovinen, P. F. Kolb, U. W. Heinz, P. V. Ruuskanen, and S. A. Voloshin, Phys. Lett. B **503**, 58 (2001).

[2] D. Teaney, J. Lauret, and E. V. Shuryak, Phys. Rev. Lett. **86**, 4783 (2001).
[3] D. Teaney, Phys. Rev. C **68**, 034913 (2003).
[4] F. Karsch and H. W. Wyld, Phys. Rev. D **35**, 2518 (1987).
[5] A. Nakamura and S. Sakai, Phys. Rev. Lett. **94**, 072305 (2005).
[6] G. Aarts and J. M. Martinez Resco, J. High Energy Phys. 04 (2002) 053.
[7] H. B. Meyer, J. High Energy Phys. 01 (2004) 030.
[8] G. Policastro, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001).
[9] P. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005).
[10] A. Hosoya, M. A. Sakagami, and M. Takao, Ann. Phys. (N.Y.) **154**, 229 (1984).
[11] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[12] C. Michael, Phys. Rev. D **53**, 4102 (1996).
[13] J. Engels, F. Karsch, and T. Scheideler, Nucl. Phys. **B564**, 303 (2000).
[14] F. Karsch, Nucl. Phys. **B205**, 285 (1982).
[15] B. Lucini, M. Teper, and U. Wenger, J. High Energy Phys. 01 (2004) 061.
[16] S. Necco and R. Sommer, Nucl. Phys. **B622**, 328 (2002).
[17] R. Sommer, Nucl. Phys. **B411**, 839 (1994).
[18] G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lutgemeier, and B. Petersson, Nucl. Phys. **B469**, 419 (1996).
[19] R. K. Bryan, Eur. Biophys. J. **18**, 165 (1990).
[20] P. Arnold, G. D. Moore, and L. G. Yaffe, J. High Energy Phys. 05 (2003) 051.