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# Possible ferromagnetism in the large $N_c$ and $N_f$ limit of quark matter

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We consider high density quark matter in the large $N_c$ and $N_f$ limit with $N_f/N_c$ fixed. In this limit, the color superconductivity disappears. We discuss that the chiral density wave state is also absent in the limit, if we assume the existence of the nonperturbative magnetic screening effect as indicated by recent lattice study. We argue that ferromagnetism can become a candidate for the ground state if quarks are massive.

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There have been extensive studies of high density quark matter [1], which may be realized in the core of neutron stars and may be accessible in the relativistic heavy ion collision experiments. Because of the asymptotic freedom of QCD, the interaction between quarks at high density is dominated by a one-gluon-exchange (OGE) process. In the OGE interaction we have an attractive channel (the color antisymmetric $\underline{3}$ channel), which inevitably induces the Cooper instability near the Fermi surface. Thus the ground state of quark matter at high density region would be in a BCS state, i.e., the color superconductivity [2,3].

In the high density quark matter, three light flavors are relevant, i.e., up, down, and strange quarks, which have small but nonzero masses [4]. In the SU(3) chiral limit, the ground state is supposed to be the color-flavor-locked (CFL) phase [5]. Indeed, because all the three colors can participate in the Cooper pairing, the CFL phase is favored compared to the 2-flavor color superconductivity (2SC) state [6,7], where only two colors are in the superconducting state. In the CFL state, the three flavors are locked to the three colors and consequently the chiral symmetry is spontaneously broken according to $SU_{color}(3) \otimes SU_L(3) \otimes SU_R(3) \to SU_{color+L+R}(3)$. Thus the superconducting state at the high enough density, where the quark masses of the three flavors can be neglected compared with the chemical potential, is believed to be in the CFL phase.

Although we have three colors in this world, it would be fruitful to consider an SU($N_c$) gauge theory treating the number of color, $N_c$, as a free parameter [8–10]. In fact, a meaningful large $N_c$ limit is obtained by making the gauge coupling $g$ scale as $g \sim 1/\sqrt{N_c}$ and by carrying out an expansion in terms of $1/N_c$ around the large $N_c$ limit.

In the large $N_c$ expansion, the leading term is given by planar diagrams, while nonplanar diagrams contribute to the next-to-next leading term of $\mathcal{O}(1/N_c^2)$. The diagrams with one quark loop are suppressed by a $1/N_c$ factor, giving the next-to-leading contribution.

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It would be of natural interest to ask what happens to high density quark matter if we take the large $N_c$ limit. The color superconductivity vanishes in the large $N_c$ limit because the Cooper pair is not a color singlet [11–16]. Actually, the superconducting gap $\Delta$ in the perturbative regime is estimated to be [17] $\Delta \sim \mu \exp(-\sqrt{6N_c/(N_c+1)}\pi^2/g)$, which goes to zero exponentially as $N_c \to \infty$. This means that the color superconductivity is a phenomenon that can never be seen in the $1/N_c$ expansion, that is, a nonperturbative phenomenon with respect to the $1/N_c$ expansion, like the pair creation process of baryon-antibaryon through a virtual photon [9].

Instead of the color superconductor, the ground state of quark matter in the large $N_c$ limit is believed to be replaced by the chiral density wave state [11–14], in which the condensate of particle-hole pairs is deformed with a finite spatial wave number.

We have another free parameter in QCD, i.e., the number of flavor, $N_f$. It is meaningful to take the large $N_c$ and $N_f$ limit simultaneously with $N_f/N_c$ fixed [18–20]. In this limit, the asymptotic freedom of QCD is preserved [10]. One of the features of the large $N_c$ and $N_f$ limit is that quark loops are not suppressed. This is because the factor $N_f$ arising from the quark loop compensates the suppression factor $1/N_c$. Thus all the planar diagrams including quark loops constitute the leading contribution in the expansion. We also note that the $U_A(1)$ anomaly is not suppressed in the limit, which is of $\mathcal{O}(N_f/N_c)$ [20].

The question we wish to address here is what is the ground state of high density quark matter in the large $N_c$ and $N_f$ limit. The question would be reasonable because for the CFL phase the same number of flavor as that of color is essential: $N_f = 3$ is as large as $N_c = 3$. The large $N_c$ and $N_f$ limit would be an appropriate competitor to the CFL phase. In the limit, the color superconductivity disappears for the reason as mentioned above [21]. The chiral density wave state can also be absent in the limit. This can be seen as follows. In the three spatial dimension, the chiral density wave is induced by the infrared singularity stemming from the long range gluon interaction [12]. In quark matter, generally speaking, the infinite range interaction can be cut off by screening due to the quark loops. Actually, the electric gluon is screened perturbatively, which acquires the Debye mass of $M_{\rm ele}^2 \sim N_f g^2 \mu^2$ [22]. As for the magnetic sector, it is known that the screening does not appear within perturbation theory [17]. However, there is still a possibility that the magnetic screening is generated by some nonperturbative mechanism. In fact, recent lattice study shows that we can have the magnetic screening at finite density [23]. In the following, let us adopt this possibility. In the large $N_c$ but small $N_f$ limit, the quark loops are suppressed and these screening effects disappear, resulting in the instability of the chiral density wave. In the large $N_c$ and $N_f$ limit, however, the quark loops are not suppressed as we mentioned. This is explicitly seen by the electric mass $M_{\rm ele}^2 \sim N_f g^2 \mu^2$ which remains finite in the limit. Thus the chiral density wave state cannot occur and we are faced with the problem: What is the ground state of quark matter at the high density region in the large $N_c$ and $N_f$ limit?

In this paper, we will not be able to give a decisive answer to the question. We will only explore one possible candidate.

Apart from the possibility that quark matter is in a normal state in the limit, let us suppose the existence of some condensate breaking some symmetry. In order for the condensate to survive when we take $N_c$ to infinity, it needs to be a color singlet object in the form of $\langle \bar{q}\Gamma q\rangle$ involving some gamma matrices $\Gamma$. Since in the large $N_c$ and $N_f$ limit, spin singlet condensate is not available, the candidate state that should be considered in the first place would be the state with the spin 1 condensate $\langle \bar{q}\gamma_\mu\gamma_5 q\rangle$, that is, the ferromagnetism (FM) [24–34].

FM in dense hadronic and quark matter has been studied by several authors. One of their motivations is to explain the observed strong magnetic field in compact stars [35,36]. The theoretical possibility of FM in quark matter was first argued by Tatsumi [26]. For the nonrelativistic itinerant electron gas, it was suggested by Bloch [37] that FM can appear as a consequence of competition between the kinetic energy and the Coulomb potential energy. Tatsumi extended the argument to quark matter to show that the OGE interaction between quarks can induce the instability for FM in both the ultrarelativistic and nonrelativistic regimes with somewhat different mechanisms. It would be worthwhile to examine FM in quark matter to see whether or not it can survive in the large $N_c$ and $N_f$ limit. It is noted that FM should persist in the large $N_c$ limit because the condensate $\langle \bar{q}\gamma_\mu\gamma_5 q\rangle$ is a color singlet. We are concerned with how FM depends on $N_f$. For this purpose, let us recapitulate Tatsumi's argument using the OGE approximation with special attention to the $N_f$ (and $N_c$) dependence. In the end, we will see that FM can become a candidate of the ground state in the limit if quarks are massive.

In the Landau theory for the weakly interacting Fermi liquid, the total energy density is given by [38]

$$\begin{aligned}\epsilon_{\rm total} &= \epsilon_{\rm kinetic} + \epsilon_{\rm potential} \\ &= \sum_\sigma \int \frac{d^3p}{(2\pi)^3} E_p n_{\boldsymbol{p}} + \frac{1}{2}\sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \\ &\quad \times \int \frac{d^3p'}{(2\pi)^3} f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'} n_{\boldsymbol{p}} n_{\boldsymbol{p}'}, \end{aligned} \tag{1}$$

where the Landau Fermi-liquid interaction $f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'}$ is related to the two-particle forward scattering amplitude $\mathcal{M}_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'}$ as

$$f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'} = \frac{m}{E_p}\frac{m}{E_{p'}}\mathcal{M}_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'}.$$

We have used obvious notations: $m$ is the quark mass and $E_p = \sqrt{\boldsymbol{p}^2 + m^2}$. $\sigma$ stands for the spin degree of freedom.

For the kinetic energy density, there arises an overall factor of $N_c N_f$ because we have $N_c N_f$ Fermi spheres. The potential energy density would consist of two terms associated with the direct and exchange scattering processes, respectively. However, the former vanishes because it involves $\mathrm{tr}\lambda_a$ in the OGE approximation. In the remaining exchange process, scattering of two quarks with different flavors cannot contribute (i.e., the processes such as $ud \to du$ are forbidden) because the OGE interaction does not mix flavors.[1] Only the processes involving the identical flavor such as $uu \to uu$ are allowed. This means that the Fermi sphere of each flavor makes an independent contribution. Thus the potential energy density receives a factor $N_f$. On the other hand, the quarks with different colors can take part in the exchange process, giving rise to a factor $N_c^2$. Eventually, the potential energy density is proportional to $N_f N_c^2 g^2$, which is the same order as the kinetic energy density. Thus, the factor $N_c N_f$ factorizes out of the total energy density and the competition between the kinetic and potential energies is not influenced by the numbers of color and flavor. If FM appears at some arbitrary numbers of color and flavor, it will persist in the large $N_c$ and $N_f$ limit. The large number of flavor neither encourages nor discourages FM.

Now we consider the higher order diagrams beyond the OGE approximation. Actually, in Refs. [28,33] attempts were made to resum some kind of the infinite number of diagrams. Unfortunately, in $(3 + 1)$ dimension, it is not possible to resum all the leading order diagrams in the large $N_c$ expansion, in contrast to the $(1 + 1)$ dimension where the Hartree-Fock approximation gives the exact solution in the large $N_c$ limit [9]. In fact, in Ref. [28] the analysis was performed in the Hartree-Fock approximation

[1] For finite $N_c$ and $N_f$, we have finite instanton effects [39]. It would be interesting to consider the instanton effect which mixes flavors.

based on the one-gluon-exchange scattering, that is, the so-called ladder QCD, which is not complete in view of the large $N_c$ and $N_f$ expansion. There exist three types of diagrams that give rise to the leading contributions in the large $N_c$ and $N_f$ expansion, which are missed in the ladder QCD analysis. First, in the ladder QCD, interactions between gluons are ignored. However, planar diagrams with gluonic interactions give a leading order contribution. Second, we may consider the multigluon exchange scattering process, which has the same topology as the OGE process and thus can give the leading contribution in the expansion. The Fock terms in the multigluon exchange scattering process should be included. Third, in the multigluon exchange process, not only the Fock but also the Hartree (or direct) terms survive because the trace over color does not vanish generally. The Hartree terms are of the leading order in the large $N_c$ and $N_f$ expansion. This counting can be understood as follows. If we consider the self-energy diagrams for the Hartree and Fock contributions, we find that the Hartree diagrams involve one quark loop. Thus the Hartree terms are next-to-leading order in the large $N_c$ expansion. However, $N_f$ coming from the quark loop makes the Hartree terms of the same order as the Fock term. Thus, in the large $N_c$ and $N_f$ expansion (but not in the large $N_c$ expansion), the Hartree term can become the leading contribution.

It is ideal that all the above diagrams are considered in order to investigate FM in the $N_c$ and $N_f$ limit. In particular, the Hartree terms in the multigluon exchange would be important, which might tend to disfavor FM, contrary to the Fock terms. However, the diagrammatic analysis of the whole leading contributions is a formidable task. It would be necessary to use other nonperturbative methods such as the lattice QCD and the AdS/QCD [40–45], although both the methods involve some difficulty at present; the former suffers from the so-called sign problem and the latter is restricted to the $N_f \ll N_c$ case only. In this situation, it would not be meaningless to discuss the results derived in the analyses in Refs. [28,33], which partly take account of the leading order diagrams in the $N_c$ and $N_f$ expansion. Especially, we believe that the Hartree terms do not overwhelm the Fock terms because the former is a contribution beyond the OGE diagram. Thus, at least the qualitative features in Refs. [28,33] would hold.

We repeat the results of these two works, which will be our conclusion in this paper.

(i) If quarks are massless, it can be analytically proven that FM does not appear [46]. This is because the OGE conserves chirality and thus the helicity of massless quarks, and the Fermi seas for the right-handed (i.e., positive helicity) and left-handed (negative helicity) quarks become the same. Thus, in this case, FM cannot be a candidate of the ground state of quark matter at high density in the large $N_c$ and $N_f$ limit.

(ii) When quarks have finite masses, FM can appear. The large quark mass is favorable for developing the ferromagnetic condensate. The density region where FM shows up depends on the interaction range between quarks: If we assume the range to be infinite [33], FM emerges in the lower density region below some critical density. On the other hand, if we adopt the zero range approximation [28], FM is developed in the higher density region including the asymptotically high density above a critical density. The higher the density, the larger the spin polarization arises. This approximation may be justified if we take account of the finite screening mass, which survives in the large $N_c$ and $N_f$ limit. In reality, the range will be nonzero finite even with the screening effect, and thus a careful examination is necessary to determine which limiting situation is more appropriate. In either case, for massive quarks, FM can become a candidate of the ground state of quark matter in the large $N_c$ and $N_f$ limit.

Before our summary, two comments are in order.

First, in the strong coupling regime which may be relevant to the intermediate density region, the chiral density wave can emerge irrespective of the interaction range [14]. The ground state in the large $N_c$ and $N_f$ limit would be the chiral density wave state. Interplaying with the chiral density wave, the spin density wave associated with FM may appear [30,47,48].

Second, it does not seem that $N_c = N_f = 3$ is so large that FM becomes the ground state to compete with CFL in the real world. However, $N_c = N_f = 3$ might be large enough that FM gives a prominent metastable state relevant to compact stars [26]. If this is the case, the strong magnetic field in compact stars could be regarded as a remnant of FM prevailing in the large $N_c$ and $N_f$ limit.

To summarize, we have addressed the question of what is the ground state of high density quark matter in the large $N_c$ and $N_f$ limit. If we assume the nonperturbative magnetic screening as suggested by lattice calculation, the chiral density wave state as well as the color superconductivity are absent. We have proposed the ferromagnetism as a candidate. It would be interesting to note that even if there is no magnetic screening, FM and the chiral density wave might compete with each other because the latter is weakened, which is now induced only by the magnetic gluon and to which the electric gluon does not contribute.[2] We have seen that FM can remain in the limit within the OGE approximation. We have discussed the leading diagrams in the large $N_c$ and $N_f$ expansion beyond OGE to

[2]As discussed in the summary of Ref. [11], within the ladder approximation, the electric and magnetic gluons contribute to the chiral density wave independently so that only the coefficient of the gap equation is modified when the electric gluon is screened and inactive.

find that there exist contributions that are not taken care of within the ladder QCD. It will be necessary to study further the large $N_c$ and $N_f$ limit of FM. Based on the works of Refs. [28,33], which we believe are reliable at least qualitatively, we conclude that FM can become a candidate if quarks are massive. FM may appear below or above a critical density, depending on the interaction range between quarks. In the massless case, FM is totally absent. We are left with an open question for this idealized world. The normal state would be one of the candidates. We note that even in the massive case there still remains the possibility that another state gives the true ground state with lower free energy. Drawing the phase diagram in the $(N_c, N_f)$ plane will be a theoretical challenge [49,50].

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