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Improved actions and asymptotic scaling in lattice Yang-Mills theory

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Improved actions in SU(2) and SU(3) lattice gauge theories are investigated with an emphasis on asymptotic scaling. A new scheme for tadpole improvement is proposed. The standard but heuristic tadpole improvement emerges from a mean field approximation from the new approach. Scaling is investigated by means of the large distance static quark potential. Both the generic and the new tadpole scheme yield significant improvements on asymptotic scaling when compared with loop improved actions. A study of the rotational symmetry breaking terms, however, reveals that only the new improvement scheme efficiently eliminates the leading irrelevant term from the action.

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I. INTRODUCTION

At the beginning of lattice gauge theories, Wilson pointed out that it is important to maintain exact gauge invariance for finite lattice spacings a thus enforcing gauge invariance in the critical limit of the lattice model. The minimal choice for an action which satisfies this criterion is nowadays known as the Wilson action [1]. Relying on the concept of universality, any lattice action which incorporates the correct symmetries in the continuum limit should work in principle. However, some lattice actions do give better approximations for continuum results for coarser lattices. This issue is central for computer simulations and has led to a continuous development of the so-called improved and perfect actions [2–16]. The basic idea is to add terms to the action which are *irrelevant* in the continuum limit, but which give better approximations at finite lattice spacing [2,3]. Different proposals have been made for such actions on the basis of perturbation theory [4–6] or using renormalization group techniques [11–14,16]. It is widely accepted that the so-called tadpole improvement is important for good properties of these actions [9,10]. To our knowledge, however, a systematic study of different implementations of tadpole improvement has not yet been carried out.

In the context of computer simulations, an extrapolation of data to the limit of vanishing lattice spacing is necessary. Such extrapolations can be made very trustworthy if a relation to an *ab initio* continuum calculation can be established. If we use the string tension σ as the fundamental energy scale of Yang-Mills theory, the perturbative treatment of continuum SU(N_c) Yang-Mills theory predicts the dependence of the lattice spacing on $\beta = 2N_c/g^2$ (with g the bare gauge coupling) to be

$$\ln[\sigma a^2(\beta)] = -\frac{4\pi^2}{\beta_0}\beta + \frac{2\beta_1}{\beta_0^2}\ln\beta + c_\sigma + \mathcal{O}(1/\beta). \quad (1)$$

There, the 1-loop and 2-loop coefficients

$$\beta_0 = \frac{11N_c^2}{6}, \qquad \beta_1 = \frac{17N_c^4}{12} \quad (2)$$

are universal. The dimensionless parameter c_σ depends on the observable and must be determined by nonperturbative methods such as lattice simulations: any physical mass scale (call it m) in units of the reference scale is independent of the lattice spacing for sufficiently large β and is obtained from

$$\ln\left(\frac{m^2}{\sigma}\right) = \ln[m^2a^2(\beta)] - \ln[\sigma a^2(\beta)] = c_m - c_\sigma + \mathcal{O}(1/\beta). \quad (3)$$

Let N denote the number of lattice points in one direction of the lattice. In actual lattice simulations, β cannot be chosen too large if we wish to work with reasonable lattice sizes, $N\ a(\beta)$. Using the Wilson lattice action, it turns out that these values of β are still too small to observe perturbative scaling; for moderate β values, large corrections to the scaling function (1) are observed. Note, however, that the ratio $m^2a^2/\sigma a^2$ is almost independent of the lattice spacing at these β values which let us reliably calculate low energy observables. This property, called "scaling" in the literature, must not be confused with "asymptotic scaling," i.e. perturbative scaling, which is the focal point of the present paper.

In this paper, a new tadpole improved action is proposed. The construction of this action offers a new understanding of the otherwise heuristic "derivation" of the standard tadpole action commonly used in simulations nowadays. We will find that tadpole improvement is highly important for the approach to asymptotic scaling for reasonably sized lattices. Finally, a thorough study of rotational symmetry breaking effects obtained from the static quark potential will reveal that the standard tadpole action does not cancel the leading order irrelevant terms of the action. The numerical data suggest that a complete cancellation might be achieved by means of our new action.

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II. ACTION AND IMPROVEMENT

A. Standard tadpole improvement

Because of the nonlinear relation between the link field $U_\mu(x)$ and the continuum gauge potential, lattice perturbation theory suffers from large tadpole contributions which, however, must cancel for an extrapolation to the continuum limit. That these tadpole contributions are indeed large can be easily anticipated from the expectation value of the link in Landau gauge, $U^\Omega_\mu(x)$. A naive expansion with respect to the lattice spacing, i.e.,

$$\left\langle \frac{1}{N_c} \mathrm{tr} U^\Omega_\mu(x) \right\rangle = \left\langle 1 - \frac{1}{2N_c} \mathrm{tr} A^2_\mu(x) a^2 + \ldots \right\rangle = 1 + \mathcal{O}(a^2), \tag{4}$$

implies that this expectation value should be of order 1. Actual simulations show, however, that the latter expectation value is at most of order 0.8 although σa^2 is as small as 0.05. In order to improve the approach to the continuum limit, one considers the ratio between the link and its expectation value. One assumes that the deviation from unity now provides a better definition of the (continuum) gauge field $A_\mu(x)$:

$$U^\Omega_\mu(x)/\frac{1}{N_c}\langle \mathrm{tr} U^\Omega_\mu(x)\rangle = 1 + iA_\mu(x)a - \ldots \tag{5}$$

Since Landau gauge fixing is problematic because of the Gribov problem, an *ad hoc* description for tadpole improvement has become standard: defining

$$\tilde U_\mu(x) = U_\mu(x)/u_0, \tag{6}$$

where u_0 is the fourth root of the plaquette expectation value, each link field in a lattice operator should be replaced by $\tilde U_\mu(x)$. Note that this procedure is heuristic, and that many other choices for tadpole improvement exist; one could also choose for u_0 the eighth root of the expectation value of the 2×2 planar Wilson loop. Nevertheless, the prescription outlined above has become standard.

B. Motivation of the new action

Let us consider a quadratic (planar) Wilson loop of side length s with an orientation specified by μ, ν located at site x. A naive expansion of this operator yields (see e.g. (2) of [10]):

$$\frac{1}{N_c} \mathrm{Re}\, \mathrm{tr} W_{\mu\nu}(x) = 1 - \frac{1}{N_c} \mathrm{Re}\, \mathrm{tr}\left[\frac{1}{2} O_4 s^4 - \frac{1}{24} O_6 s^6 + \ldots\right], \tag{7}$$

$$O_4(x) = F_{\mu\nu}F_{\mu\nu}(x),\qquad O_6(x) = (D_\mu F_{\mu\nu})^2(x) + (D_\nu F_{\mu\nu})^2(x). \tag{8}$$

The subscripts μ, ν at $O_{4,6}$ have been suppressed. Already terms of order $\mathcal{O}(s^6)$ break rotational symmetry. Choosing the minimal length $s = a$, W coincides with the (minimal) plaquette, which is the only term in the Wilson action.

As outlined in the previous subsection, numerically the expansion (7) converges badly for reasonable lattice sizes. Without any simulation, this fact can be also understood from continuum perturbation theory; although manifestly gauge invariant, in lattice regularization the high energy modes are cut off at a momentum scale $\Lambda_{\mathrm{UV}} \approx \pi/a$. It is well known that in cutoff regularizations expectation values such as O_4 and O_6 in (8) diverge with the cutoff:

$$\langle O_4\rangle \propto \Lambda^4_{\mathrm{UV}} \propto a^{-4}, \qquad \langle O_6\rangle \propto \Lambda^6_{\mathrm{UV}} \propto a^{-6}.$$

The origin of these divergences are quantum fluctuation of the order of the cutoff scale. Obviously, these fluctuations invalidate the expansion (7) (choose $s = a$ for the moment). However, they do not spoil the calculation of physical observables well below the cutoff scale (as will be detailed below). Note, however, that if the desired goal is to match with *asymptotic scaling* provided by continuum Yang-Mills theory, an expansion such as (7) should cover high energy modes too.

One choice for such an action is obtained by replacing *all* operators O_n of the action by

$$\bar O_n = O_n - \langle O_n\rangle. \tag{9}$$

Only the deviation of the gauge invariant operator from its (potentially) divergent expectation value contributes to the action. In practice, this construction can be realized by considering the ratio between the Wilson loop and its expectation value. Using (7), one can show to all orders that in this case the operators O_n only appear in the combination (9).

Without resorting to the naive expansion anymore, we now assume that the above ratio has a sensible expansion with respect to s:

$$\mathrm{Re}\, \mathrm{tr} W_{\mu\nu}(x)/\langle \mathrm{Re}\, \mathrm{tr} W_{\mu\nu}(x)\rangle = 1 - \frac{1}{N_c} \mathrm{Re}\, \mathrm{tr}\left[\frac{1}{2} \bar O_4 s^4 - \frac{1}{24} \bar O_6 s^6 + \ldots\right]. \tag{10}$$

Note that the term $\bar O_4$ gives rise to the continuum action proportional to F^2. The subject of improvement is to remove terms of higher order in s from the action.

A popular choice (Symanzik improvement) is to use a rectangular 1×2 loop. This scheme invokes yet another expansion, i.e., of the rectangular loop with respect to s similar to the one in (10), and relies on a matching of the expansion coefficients to eliminate the irrelevant terms. Here, we are going to use a 2×2 quadratic loop which is a scale transform of the plaquette. The motivation for this choice is that we need to invoke only one type of expansion evaluated at two values for the scale parameter. The hope is that, because of the scale relation between the 1×1 and 2×2 loops, the cancellation of irrelevant terms is more complete at finite values of the lattice spacing where the naive Taylor expansion (such as (10)) appears unjustified.

Assuming that the expansion (10) is valid at least for $s \leq 2a$, we use the expansion for $s = a$ (plaquette) and $s = 2a$ (2×2 Wilson loop) to get rid of the irrelevant terms. Defining

$$\bar{P}_{\mu\nu}(x) = \mathrm{Re\,tr} W^{1\times 1}_{\mu\nu}(x) / \langle \mathrm{Re\,tr} W^{1\times 1}_{\mu\nu}(x)\rangle,$$
$$\bar{P}^{(2)}_{\mu\nu}(x) = \mathrm{Re\,tr} W^{2\times 2}_{\mu\nu}(x) / \langle \mathrm{Re\,tr} W^{2\times 2}_{\mu\nu}(x)\rangle, \tag{11}$$

we choose for the action

$$S = \beta \sum_{\mu > \nu, x} [\kappa_1 \bar{P}_{\mu\nu}(x) + \kappa_2 \bar{P}^{(2)}_{\mu\nu}(x)]. \tag{12}$$

Using the expansion (10) for $s = a$ and $s = 2a$, we are led to

$$\kappa_1 + 16\kappa_2 = 1, \tag{13}$$

$$\kappa_1 + 64\kappa_2 = 0. \tag{14}$$

This first line ensures compatibility with continuum Yang-Mills theory whereas the choice of the second line eliminates the order a^6 terms. The solution of the latter set of equations is given by

$$\kappa_1 = \tfrac{4}{3}, \qquad \kappa_2 = -\tfrac{1}{48}. \tag{15}$$

The present improvement scheme eliminates from the action contributions from tadpole loops. The main purpose for this elimination is that these loops are absent in the *ab initio* continuum formulation of Yang-Mills theory. The impact of these loops is therefore to spoil proper scaling which is familiar from continuum perturbation theory. I point out that, once the tadpole contribution was eliminated, further improvements might be achieved by invoking the standard perturbative improvement scheme. We leave such an investigation to future work. Here, we will justify by numerical calculations that the new action (without further perturbative improvements) already gives rise to much better scaling properties.

C. Comparison with the standard tadpole improved action

Let us assume that we are dealing with an action which features the plaquette and the 2×2 Wilson loop. In the case of standard tadpole improvement, the rule (6) implies that

$$\bar{P}_{\mu\nu}(x) = \frac{1}{N_c} \mathrm{Re\,tr} W^{1\times 1}_{\mu\nu}(x)/u_0^4$$
$$= \mathrm{Re\,tr} W^{1\times 1}_{\mu\nu}(x) / \langle \mathrm{Re\,tr} W^{1\times 1}_{\mu\nu}(x)\rangle,$$
$$\bar{P}^{(2)}_{\mu\nu}(x) = \frac{1}{N_c} \mathrm{Re\,tr} W^{2\times 2}_{\mu\nu}(x)/u_0^8.$$

While for our new action the numerical burden is a self-consistent calculation of the expectation values

$$\langle \mathrm{Re\,tr} W^{1\times 1}_{\mu\nu}(x)\rangle, \qquad \langle \mathrm{Re\,tr} W^{2\times 2}_{\mu\nu}(x)\rangle,$$

standard tadpole improvement appears as an approximation to this numerical problem. There, only

$$\langle \mathrm{Re\,tr} W^{1\times 1}_{\mu\nu}(x)\rangle$$

is calculated self-consistently, and the expectation value of the 2×2 Wilson loop is obtained with the help of the mean field approximation (in Landau gauge):

$$\left\langle \frac{1}{N_c} \mathrm{Re\,tr} W^{2\times 2}_{\mu\nu}(x) \right\rangle \approx u_0^8 = (u_0^4)^2$$
$$\approx \left\langle \frac{1}{N_c} \mathrm{Re\,tr} W^{1\times 1}_{\mu\nu}(x) \right\rangle^2.$$

Having identified the standard approach as an approximation to the present scheme, the crucial question is whether the properties of our action fully justify the higher level of numerical sophistication. The remaining two sections will answer this question.

III. NUMERICAL SIMULATION SETUP

A. Thermalization

The dynamical degrees of freedom are the $\mathrm{SU}(N_c)$ matrices $U_\mu(x)$ which are associated with the links of an N^4 cubic lattice. The partition function is given by

$$Z = \int \mathcal{D}U_\mu \exp\{S[U](w_{11}, w_{22})\}, \tag{16}$$

$$S[U](w_{11}, w_{22}) = \beta \sum_{\mu < \nu, x} \left[\frac{4}{3 w_{11}(\beta)} \mathrm{Re\,tr} W^{1\times 1}_{\mu\nu}(x) - \frac{1}{48 w_{22}(\beta)} \mathrm{Re\,tr} W^{2\times 2}_{\mu\nu}(x) \right], \tag{17}$$

where the action S depends on the expectation values:

$$w_{11}(\beta) = \langle \mathrm{Re\,tr} W^{1\times 1}_{\mu\nu}(x)\rangle, \qquad w_{22}(\beta) = \langle \mathrm{Re\,tr} W^{2\times 2}_{\mu\nu}(x)\rangle. \tag{18}$$

Expanding the expectation values of the latter equations in terms of their functional integrals, we arrive at a set of two nonlinear equations which determine the two unknown parameters $w_{11}(\beta)$ and $w_{22}(\beta)$:

$$w_{11}(\beta) = \frac{1}{Z} \int \mathcal{D}U_\mu \, \mathrm{Re\,tr} W^{1\times 1}_{\mu\nu}(x) \exp\{S[U](w_{11}, w_{22})\}, \tag{19}$$

$$w_{22}(\beta) = \frac{1}{Z} \int \mathcal{D}U_\mu \, \mathrm{Re\,tr} W^{2\times 2}_{\mu\nu}(x) \exp\{S[U](w_{11}, w_{22})\}. \tag{20}$$

Before we can start to accumulate statistically independent lattice configurations $\{U_\mu\}$ for each β, we must solve the latter set of equations for $w_{11}(\beta)$ and $w_{22}(\beta)$, and we must generate a "statistically important" configuration by means of thermalization.

While the reader is invited to develop their own methodology for this task, we here briefly outline our approach which serves the purpose. It appears to be quite natural to solve the set of Eqs. (19) and (20) and to generate the thermalized configuration within one process. We here used a simple iterative procedure: denoting $w_{ii}^{(n)}(\beta)$, $i = 1, 2$, by the approximate solutions to $w_{ii}(\beta)$ of the nth iteration, better approximations are generated by

$$w_{11}^{(n+1)}(\beta) = \frac{1}{Z} \int \mathcal{D}U_\mu \, \mathrm{Re}\,\mathrm{tr} W_{\mu\nu}^{1\times1}(x) \exp\{S[U] \times (w_{11}^{(n)}, w_{22}^{(n)})\}, \tag{21}$$

$$w_{22}^{(n+1)}(\beta) = \frac{1}{Z} \int \mathcal{D}U_\mu \, \mathrm{Re}\,\mathrm{tr} W_{\mu\nu}^{2\times2}(x) \exp\{S[U] \times (w_{11}^{(n)}, w_{22}^{(n)})\}. \tag{22}$$

As starting points for the iteration we chose the naive tree-level values

$$w_{11}^{(0)}(\beta) = 1, \qquad w_{22}^{(0)}(\beta) = 1. \tag{23}$$

In order to monitor the convergence of the above iteration, we introduce the error

$$\epsilon^{(n+1)} = |w_{22}^{(n+1)}(\beta) - w_{22}^{(n)}(\beta)|. \tag{24}$$

It turns out that measuring $w_{22}^{(n)}(\beta)$ is sufficient for monitoring convergence. In practice, the integrals in (21) and (22) are not calculated exactly. Only Monte Carlo estimates $\tilde{w}_{ii}^{(n)}(\beta)$ with statistical errors $\sigma_{ii}^{(n)}(\beta)$ are available. At the beginning of the iteration, it does not make sense to obtain a high precision estimate for an anyhow unconverged value of $\tilde{w}_{ii}^{(n)}(\beta)$. We therefore adopted the following procedure: at the start of the iteration, only 10 iterations are used to obtain the estimates $\tilde{w}_{ii}^{(n)}(\beta)$ and their statistical errors $\sigma_{ii}^{(n)}(\beta)$. As soon as the error of convergence reaches the order of the statistical error, i.e.

$$\epsilon^{(n)} \approx \sigma_{22}^{(n)}(\beta), \tag{25}$$

the number of iterations which are used for the estimators is increased by 10. The iteration stops when $\epsilon^{(n)}$ (and therefore also $\sigma_{22}^{(n)}(\beta)$) drops below a certain number which specifies the precision to be achieved for the parameters. Figure 1 shows the "thermalization history" of the parameters $\tilde{w}_{11}^{(n)}(\beta)/N_c$ and $\tilde{w}_{22}^{(n)}(\beta)/N_c$ as a function of the total number lattice sweeps performed. Data are shown for $\beta = 1.25$ (SU(2)) and $\beta = 3.10$ (SU(3)), which will turn out to correspond to a rather coarse lattice, and for $\beta = 1.55$ (SU(2)) and $\beta = 3.5$ (SU(3)), which are in the scaling regime. After an initial oscillation, the estimators rapidly converge. Note that the spacing between two data points in Fig. 1 shows the number of lattice sweeps which were needed to estimate the integrals (21) and (22).

In particular for small values of β, several solutions of the nonlinear equations (19) and (20) might exist. If one chooses to perform simulations in this regime of parameter space, a sensible choice would be to pick the solution with least rotational symmetry breaking (see Sec. V).

B. Static quark potential

To investigate scaling, we will express the lattice spacing in units of the string tension σ in order to calculate $a(\beta)$. The static quark potential $V(r)$ is so obtained from planar Wilson loops. These loops are of size $r \times t$, and the spatial links have been smeared to enhance the overlap with the quark antiquark ground state. For the smearing procedure, we consider the spatial hypercube for a given time t: spatial links belonging to this cube are then cooled with respect to the 3-dimensional Wilson action. Cooling is performed by visiting each link of the lattice and replacing it with the (normalized) sum of the adjacent staples.

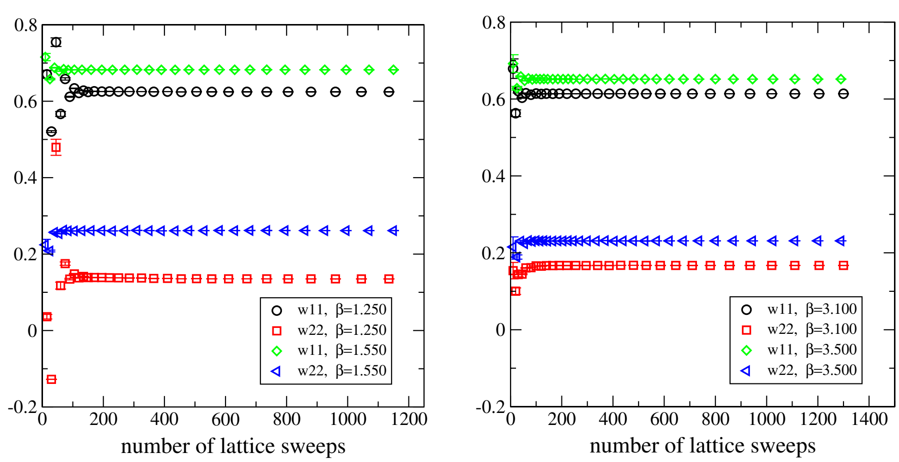

FIG. 1 (color online). Convergence of the action parameters w_{11} and w_{22} as a function of the number of Monte Carlo sweeps: SU(2) (left) and SU(3) (right).

Instead of the Wilson action, one could use the 3-dimensional version of the present action rather than Wilson's action. It turns out, however, that this choice is more time consuming and does not produce better overlaps. Ten cooling sweeps through the lattice are performed to obtain one set of smeared links. Timelike links are unaffected by smearing. The advantages of this smearing procedure are that it is easy to implement, it is fast compared with other smearing techniques, and it is known to yield excellent overlap for SU(2) and SU(3) gauge theories [17] and even for more exotic gauge theories such as for G(2) [18].

In practice, the Wilson loops are fitted to a straight line:

$$- \ln\langle \mathrm{Re\,tr} W_{\mu\nu}^{r\times t}\rangle = V(r)t + \mathrm{const}, \tag{26}$$

where only data with $t \geq t_{\mathrm{low}}$ are included. This suppresses the contribution from excited states. Because of the overlap enhancement, choosing $t_{\mathrm{low}} = 2a$ is sufficient: the linear t-fit represents the data with a $\chi^2/\mathrm{dof} \approx 2$ or better for the β ranges explored in the present paper. Using values as high as $t_{\mathrm{low}} = 4a$, we checked that larger values t_{low} yield the same potential.

In order to explore rotational symmetry breaking effects by the underlying lattice, "off-axis" distances for the quark antiquark pair are considered as well. Potentials corresponding to crystallographic directions

(100) (on-axis), (110) (111)

are taken into account. Our final result for the static potential is shown in Fig. 2. A fit of the on-axis data to

$$V(r) = V_0 - \frac{\alpha}{r} + \sigma r \tag{27}$$

is shown as well. The results of the fit for SU(2) and SU(3) are summarized in the table below:

	β	N_{conf}	$V_0 a$	α	σa^2	χ^2/dof
SU(2)	1.45	400	0.527(2)	0.267(1)	0.0695(6)	1.1
SU(3)	3.30	800	0.631(1)	0.317(1)	0.0666(3)	1.2

Here, N_{conf} denotes the number of independent lattice configurations used to estimate the Wilson loop expectation values. Priority has been given to the SU(3) simulations because of their relevance for QCD. Because we are using an improved action with very good rotational symmetry (see Sec. V), the point $r = a$ can be included in the potential fit without hampering the value for χ^2/dof.

A lattice sweep consists out of a Cabbibo-Marinari update followed by 4 reflections (for SU(2)) and 5 (for SU(3)), respectively. Each reflection replaces the actual configuration by another one which possess the same action. We observe that this process enhances the ergodicity of the algorithm: autocorrelations are reduced and a speedup of thermalization is observed. Twenty "dummy" lattice sweeps are performed until the configuration is used for measurements. Especially for small β values, a smaller number of dummy sweeps might be sufficient. There is room for a further fine-tuning of the algorithm. Note that these sweeps are carried out for fixed $\kappa_{1,2}$ in (12) the values of which were determined during the initial stage of thermalization.

IV. ASYMPTOTIC SCALING WITH IMPROVED ACTIONS

A. The Wilson action—a case study

The Wilson action has been widely studied and is widely used even nowadays. It has been known, however, for a long time that large deviations of the lattice spacing $a(\beta)$ from the perturbative scaling (1) are common with this type of action. The purpose of the present subsection is to quantify this statement.

The partition function employing Wilson action is given by

$$Z = \int \mathcal{D}U_\mu \exp\{S_{\mathrm{wil}}[U]\}, \tag{28}$$

$$S_{\mathrm{wil}}[U] = \beta \sum_{\mu<\nu,x} \frac{1}{N_c} \mathrm{Re\,tr} W_{\mu\nu}^{1\times 1}(x). \tag{29}$$

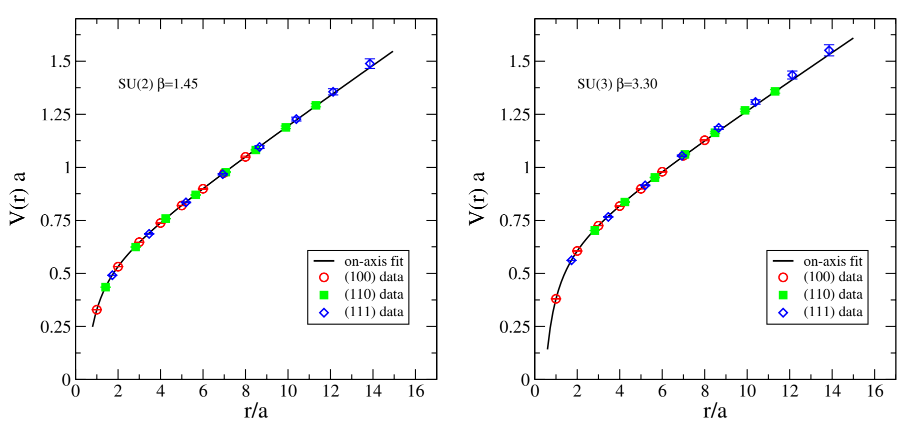

FIG. 2 (color online). Static quark antiquark potential using the present improved action for SU(2) (left) and SU(3) (right).

TABLE I. Measured scaling function $\sigma a^2(\beta)$ for a 16^4 lattice using the Wilson action for SU(2) and SU(3) gauge theories.

SU(2)	β	σa^2	SU(3)	β	σa^2
	2.20	0.28(1)		5.60	0.278(6)
	2.25	0.194(4)		5.65	0.219(4)
	2.30	0.145(2)		5.70	0.171(2)
	2.35	0.1022(9)		5.75	0.133(1)
	2.40	0.0738(5)		5.80	0.1051(7)
	2.45	0.0523(3)		5.85	0.0854(5)
	2.50	0.0390(2)		5.90	0.0731(4)
	2.55	0.0281(2)		5095	0.0601(3)
	2.60	0.0211(2)		6.00	0.0517(2)
				6.05	0.0447(2)
				6.10	0.0387(2)

Using the techniques outlined in the previous section, we calculated the static quark potential and the scaling function $\sigma a^2(\beta)$. The results, obtained from $N_{\rm conf} = 800$ independent configurations on a 16^4 lattice, are summarized in Table I. Finite size effects are expected at the 1% level if the side length of the lattice exceeds 1.5 fm [19]. For a 16^4 lattice, finite size effects therefore play a minor role as long as $\sigma a^2 > 0.04$.

Figure 3 visualizes the data of Table I. In order to bring out any onset of asymptotic scaling, these data are compared with the perturbative scaling function at the 1-loop and 2-loop level (see (1) and (2)):

$$\ln[\sigma a^2(\beta)]^{\text{1-loop}}_{\text{asym}} = -\frac{4\pi^2}{\beta_0}[\beta - \beta_{\rm ref}] + \ln[\sigma a^2(\beta_{\rm ref})], \tag{30}$$

$$\ln[\sigma a^2(\beta)]^{\text{2-loop}}_{\text{asym}} = -\frac{4\pi^2}{\beta_0}[\beta - \beta_{\rm ref}] + \frac{2\beta_1}{\beta_0^2}\ln\frac{\beta}{\beta_{\rm ref}} + \ln[\sigma a^2(\beta_{\rm ref})]. \tag{31}$$

The perturbative scaling functions are normalized to reproduce the measured data for $\beta = \beta_{\rm ref}$. In Fig. 3, the highest considered value β is chosen for $\beta_{\rm ref}$. It is remarkable that both for SU(2) and SU(3), the 2-loop scaling function (31) does not yield an improvement on the agreement of the data with the 1-loop formula (30).

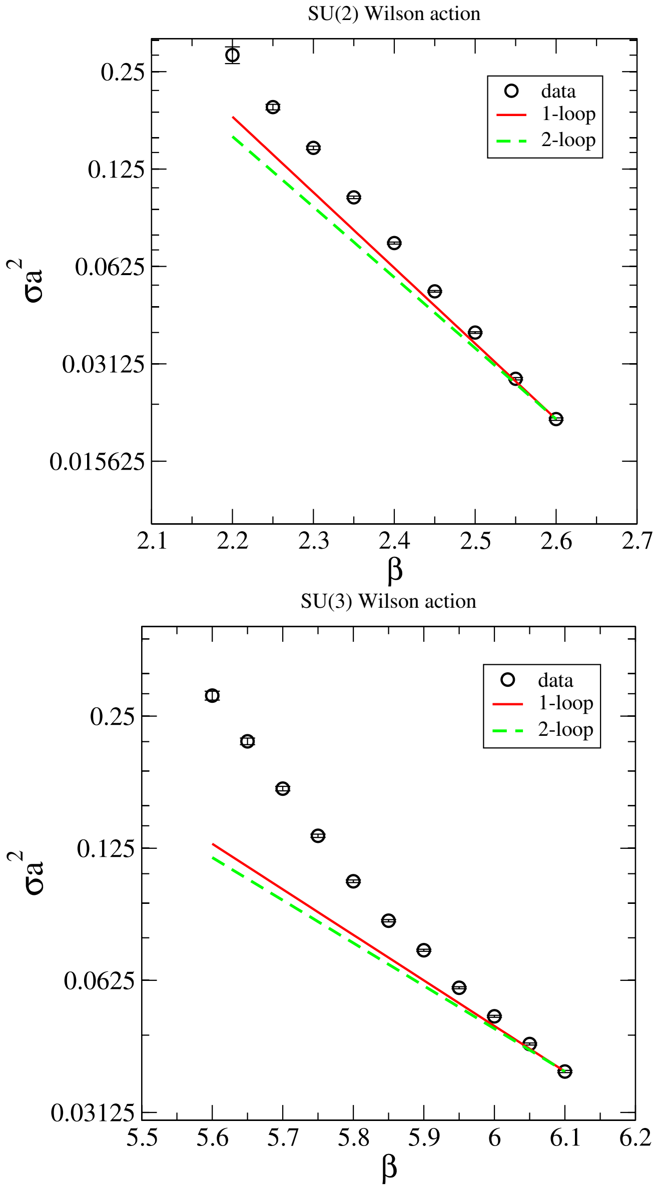

FIG. 3 (color online). Scaling function $\sigma a^2(\beta)$ for a 16^4 lattice using Wilson action for SU(2) (top) and for SU(3) (bottom) gauge theories. Asymptotic scaling according to (30) and (31) is shown as well.

B. Improved action

Employing the procedure discussed in subsection III A, we generated well "thermalized" configurations (as well as the simulation parameters $w_{11}(\beta)$ and $w_{22}(\beta)$), see (18), for a range of β values which give reasonably sized lattice spacings for the present 16^4 lattice. The simulation parameters as well as the calculated value of the lattice spacing a in units of the string tension σ are summarized in Table II for the SU(2) gauge theory and in Table III for the SU(3) case. The calculated scaling functions $\sigma a^2(\beta)$ are shown in Figs. 4 and 5, respectively. As with the Wilson action, a comparison with the asymptotic scaling functions (30) and (31) is made. A satisfactory agreement with asymptotic scaling on coarse lattices with σa^2 as large as $\sigma a^2 \approx 0.1$ is observed for both gauge groups. In both cases, the agreement with the 2-loop formula seems to be better than with the 1-loop result for $\sigma a^2 \leq 0.06$.

C. Comparison with other actions

In this subsection, the more important case of an SU(3) gauge group is investigated. Two popular actions which do not involve tadpole improvement, but invoke a renormal-

TABLE II. Simulation parameters of the improved action (17) and the calculated scaling function $\sigma a^2(\beta)$; **SU(2)** gauge theory, 16^4 lattice, $N_{\rm conf}$ independent configurations.

β	$w_{11}(\beta)/2$	$w_{22}(\beta)/2$	$N_{\rm conf}$	σa^2
1.250	0.624 55(3)	0.134 99(6)	400	0.279(2)
1.275	0.634 16(3)	0.159 44(6)	400	0.215(5)
1.300	0.641 97(3)	0.177 23(6)	400	0.175(3)
1.325	0.648 85(3)	0.192 11(6)	400	0.1473(7)
1.350	0.655 26(3)	0.205 84(6)	400	0.1244(7)
1.375	0.661 23(3)	0.218 18(6)	400	0.1068(9)
1.400	0.666 90(3)	0.229 86(6)	400	0.0922(7)
1.425	0.672 31(3)	0.241 21(6)	400	0.0787(5)
1.450	0.677 44(3)	0.251 69(6)	400	0.0695(6)
1.475	0.682 29(3)	0.261 43(6)	400	0.0599(3)
1.500	0.686 97(2)	0.270 76(6)	400	0.0528(3)
1.525	0.691 41(2)	0.279 64(6)	800	0.0452(8)
1.550	0.695 72(2)	0.288 09(6)	800	0.0400(3)
1.575	0.699 88(2)	0.296 38(6)	800	0.0351(4)
1.600	0.703 88(2)	0.304 27(6)	800	0.0311(2)

ization group (RG) investigation, are the RG-Iwasaki action [7,8] and the DBW2 [16]. These actions are of the type

$$S[U] = \beta \sum_{\mu<\nu,x}\left[c_0 \frac{1}{N_c} \operatorname{Re}\operatorname{tr} W^{1\times 1}_{\mu\nu}(x) + c_1 \frac{1}{N_c} \operatorname{Re}\operatorname{tr} W^{1\times 2}_{\mu\nu}(x)\right], \tag{32}$$

and differ by the choice of c_1 (note that $c_0 = 1 - 8c_1$ for a proper definition of the bare gauge coupling):

$$c_1 \approx -0.331 \qquad \text{(RG-Iwasaki)},$$
$$c_1 \approx -1.4088 \qquad \text{(DBW2)}.$$

A detailed investigation of the scaling properties of these actions can be found in [20]. We will here focus on their properties concerning *asymptotic scaling*.

TABLE III. Simulation parameters of the improved action (17) and the calculated scaling function $\sigma a^2(\beta)$; **SU(3)** gauge theory, 16^4 lattice, $N_{\rm conf}$ independent configurations.

β	$w_{11}(\beta)/3$	$w_{22}(\beta)/3$	$N_{\rm conf}$	σa^2
2.90	0.585 67(2)	0.114 24(3)	800	0.231(4)
3.00	0.601 35(2)	0.145 53(3)	800	0.151(2)
3.10	0.613 78(2)	0.167 26(3)	800	0.1122(7)
3.15	0.619 23(2)	0.176 67(3)	800	0.0985(5)
3.20	0.624 50(2)	0.185 60(3)	800	0.0851(4)
3.25	0.629 50(2)	0.194 03(3)	800	0.0765(4)
3.30	0.634 29(2)	0.202 15(3)	800	0.0666(3)
3.35	0.638 83(2)	0.209 59(3)	800	0.0589(3)
3.40	0.643 30(2)	0.217 11(3)	800	0.0532(2)
3.45	0.647 64(2)	0.224 32(3)	800	0.0482(2)
3.50	0.651 73(2)	0.231 08(3)	800	0.0424(2)

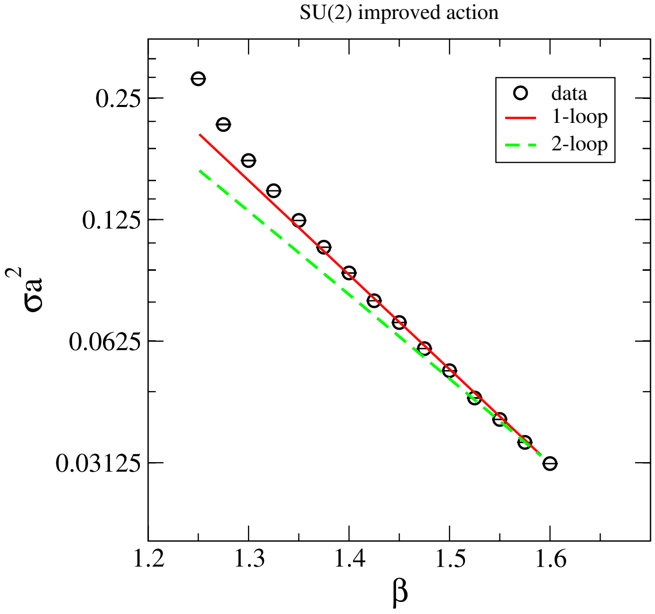

FIG. 4 (color online). Scaling function $\sigma a^2(\beta)$ for a 16^4 lattice using the present improved action for **SU(2)**. Asymptotic scaling according to (30) and (31) is shown as well.

We will need the lattice spacing a in units of the string tension. For the case of the RG-Iwasaki action and the DBW2 action, data for a/r_0 with the Sommer parameter r_0 are taken from the work by Necco [20]. Using $r_0 \approx 0.5$ fm and $\sqrt{\sigma} \approx 440$ MeV, a factor

$$\sigma r_0^2 \approx 1.21$$

is used to convert a^2/r_0^2 to $a^2\sigma$.

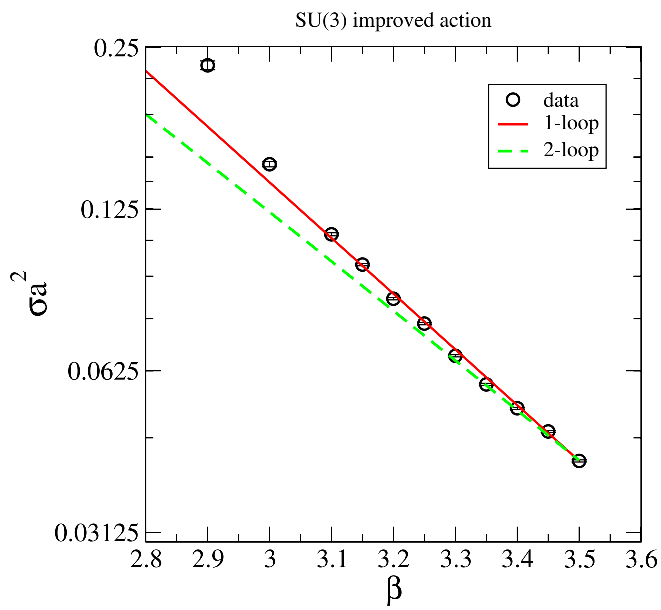

FIG. 5 (color online). Scaling function $\sigma a^2(\beta)$ for a 16^4 lattice using the present improved action for **SU(3)**. Asymptotic scaling according to (30) and (31) is shown as well.

In order to study whether the present improved action (see Eqs. (16)–(18)) is superior to an action with standard tadpole improvement, we here also study the "2×2" action with tree-level coefficients and standard tadpole removal:

$$S[U](w_{11}) = \beta \sum_{\mu<\nu,x} \left[\frac{4}{3w_{11}(\beta)} \mathrm{Re\,tr} W^{1\times1}_{\mu\nu}(x) - \frac{1}{48 w_{11}^2(\beta)} \mathrm{Re\,tr} W^{2\times2}_{\mu\nu}(x) \right], \quad (33)$$

where w_{11} must be self-consistently calculated from

$$w_{11}(\beta) = \frac{1}{Z} \int \mathcal{D}U_\mu \,\mathrm{Re\,tr} W^{1\times1}_{\mu\nu}(x) \exp\{S[U](w_{11})\}. \quad (34)$$

This action was used in [21] to study thermodynamics. There it was observed that tadpole improvement largely reduces the cutoff effects which hamper the calculation of the pressure and the thermal energy density in the SU(3) high temperature phase. Our findings for $w_{11}(\beta)$ and for the scaling function $\sigma a^2(\beta)$ are summarized in Table IV.

For a more quantitative investigation of *asymptotic scaling*, the deviation from asymptotic scaling is measured by the ratio

$$R(\beta) = \frac{a^2(\beta)}{a^2_{\mathrm{asym}}(\beta)}, \quad (35)$$

where the lattice spacing squared a^2 is either provided in units of the string tension or in units of the Sommer parameter r_0 (as e.g. done in [20]). The function $a^2_{\mathrm{asym}}(\beta)$ is provided at 2-loop level by (31). Because the definition of $a^2_{\mathrm{asym}}(\beta)$ involves an arbitrary normalization, the absolute value of R in (35) is meaningless. The data are normalized such that $R = 1$ is attained for the smallest lattice spacing considered. Asymptotic scaling will from (35) be signalled by the function $R(\beta)$ becoming flat for sufficiently large values of β. Since the absolute size of the bare gauge coupling g (and therefore of $\beta = 6/g^2$) depends on the details of the regularization scheme and the action, we will study R as a function of the lattice spacing squared in physical units.

TABLE IV. Simulation parameter of the *standard tadpole improved action* (33) and the calculated scaling function $\sigma a^2(\beta)$; **SU(3)** gauge theory, 16^4 lattice, N_{conf} independent configurations.

β	$3w_{11}(\beta)$	N_{conf}	σa^2
2.60	0.573 13(3)	600	0.115(1)
2.70	0.585 59(2)	600	0.0875(6)
2.80	0.597 15(2)	600	0.0683(4)
2.90	0.607 83(2)	600	0.0520(3)
3.00	0.617 84(2)	600	0.0431(3)

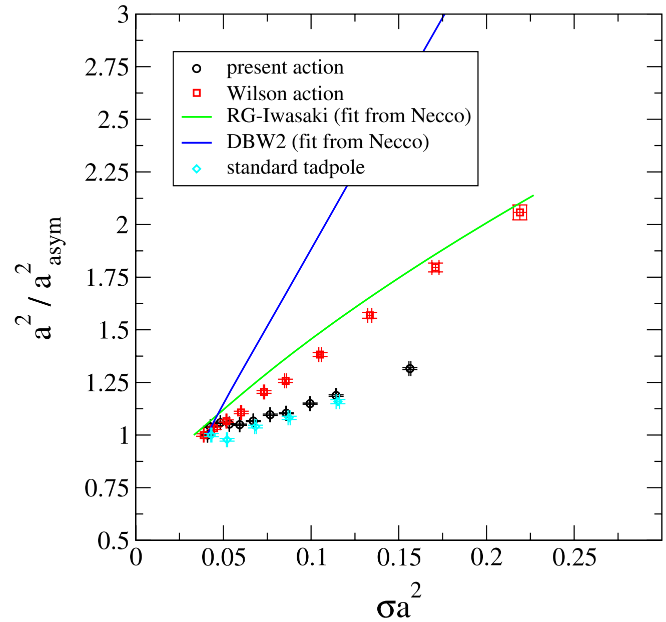

FIG. 6 (color online). Approaching asymptotic scaling using several actions.

The results are shown in Fig. 6 for the actions under investigation. It turns out that any sort of tadpole improvement largely improves scaling along the lines of the asymptotic formula. The Wilson action, but also the DBW2 and Iwasaki actions, show large deviations from asymptotic scaling. In contrast, both the standard and our new tadpole improved action, seem to perform equally well. A possible explanation could be that both actions remove the O_6 irrelevant terms such that asymptotic scaling sets in for rather coarse lattice spacings. The next section will, however, reveal that this interpretation is only justified for the new action proposed in this paper.

V. ROTATIONAL SYMMETRY BREAKING

The irrelevant terms of O_6 in (8) are built up from expressions which explicitly violate rotational symmetry. Hence, the absence of the O_6 terms can be checked by calculating the amount [22] of rotational symmetry breaking as a function of the lattice spacing a.

The present improvement scheme belongs to the class of tree-level improvements; it relies on the expansion (10) of the action in powers of the lattice spacing a. A cancellation of the O_6 terms can be hampered by terms which depend logarithmically on a. The cancellation can still be made complete if loop corrections are considered as well as tadpole improvement. At the present stage, there are two crucial questions: Are loop corrections still large for the present range of lattice spacings? Is the new improvement scheme superior to the standard approach so that the additional complexity of the new scheme is justified?

For an answer to these question, we need to quantify the amount of rotational symmetry breaking. For this purpose, we invoke the method introduced by the QCD-Taro collaboration in [16]. Let $V_{\rm on}(r)$ denote the "on-axis" static quark potential obtained from quarks positioned along the main crystallographic direction previously called the (100) direction. Data for which r is not an integer multiple of a are made available by means of the fit (27). Furthermore, call $V(r)$ all data arising from quarks positioned along the (110) and (111) directions. These data are called the "off-axis" data. $\delta V(r)$ denotes their statistical errors. With these definitions, the measure of rotational symmetry breaking is given by

$$\delta_v^2 = \sum_{\rm off} \frac{[V(r) - V_{\rm on}(r)]^2}{V(r)^2 \delta V^2(r)} \bigg/ \bigg(\sum_{\rm off} \frac{1}{\delta V^2(r)} \bigg), \qquad (36)$$

where the sum extends over all "off-axis" data.

Figure 7 shows δ_v as a function of the lattice spacing squared. For the case of the Wilson action, we observe a linear scaling of δ_v with a^2:

$$\delta_v \propto \sigma a^2 + \delta_v(0), \qquad \text{(Wilson action)}.$$

In the case of the action employing standard tadpole improvement, the symmetry breaking effects are significantly reduced, but, still, δ_v rises linearly with a^2. Using the new action proposed in this paper, we find an additional drastic reduction of rotational symmetry breaking effects. Moreover, it seems that this time the functional dependence of δ_v on a^2 seems to change qualitatively. The data now indicate that δ_v is of higher order in a^2:

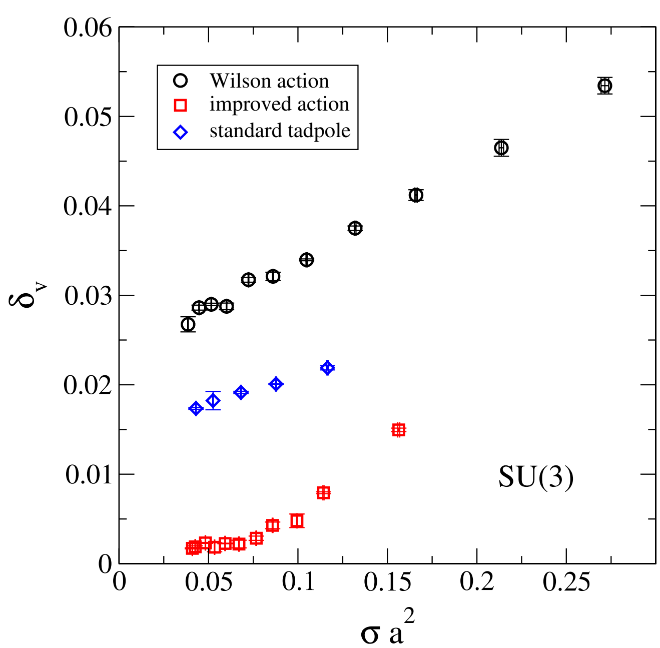

FIG. 7 (color online). The measure δ_v of rotational symmetry breaking as a function of the scaling function σa^2 for SU(3) gauge theory with different actions.

$$\delta_v \propto \sigma^2 a^4, \qquad \text{(SU(3) imroved action)}.$$

However, further numerical investigations are necessary to support this claim. If it is supported by numerical simulations, it would imply that only the present action can completely remove the irrelevant O_6 contributions. It is already clear that the standard tadpole improved action certainly fails this task.

Note also that the data suggest that the Wilson and standard tadpole data approach a small but finite value at $\sigma a^2 = 0$. The size of this value clearly depends on the amount of tadpole contributions. Our preliminary interpretation of this finding is as follows: contributions of tadpole loops solely arise in lattice regularization and, therefore, add substantially to the amount of rotational symmetry breaking present in the static potential. In addition, tadpole loops are generically UV divergent. This might lead to a small, but finite value of δ_v even for very small values of the lattice spacing. An alternative interpretation is that δ_v is quite sensitive to the finite volume leaving the offsets δ_v as artifacts of the analyzing procedure. Further investigations are clearly needed to settle this question.

VI. CONCLUSIONS

The properties of improved actions with respect to asymptotic scaling have been thoroughly investigated in this paper. A focal point of the present study is tadpole improved tree-level actions. A new scheme for tadpole improvement has been proposed and it has been contrasted to the heuristic tadpole approach, which is standard in the literature. It has been shown that the standard tadpole scheme is a mean field approximation to the scheme proposed here.

The numerical results for the scaling function $\sigma a^2(\beta)$ reveal that both types of tadpole improved actions yield results of equal quality as far as asymptotic scaling is concerned. By contrast, loop improved actions (which do not make use of tadpole improvement) produce much bigger deviations from asymptotic scaling.

The amount δ_v of rotational symmetry breaking (see (36)) in the static quark potential was used to compare the quality of both tadpole improvement schemes. Although the function $\delta_v(a)$ is much smaller for the standard tadpole action than for the Wilson action, the functional dependence on the lattice spacing a is the same in both cases. In contrast, we have seen first numerical evidence that δ_v is of higher order in a^2 if the new tadpole improved action is used. We here argue that the generic tadpole scheme fails to eliminate the leading order irrelevant terms of the action. The data indicate that the new action cancels these terms from the action for the range of lattice spacings considered without taking into account loop corrections.

Our approach to tadpole improvement can in principle be extended to kill off the next-to-leading order irrelevant terms as well. The question whether loop corrections must be considered then is left to future work.

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