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Some thermodynamic aspects of pure glue, fuzzy bags, and gauge/string duality

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The thermodynamic properties of an $SU(3)$ gauge theory without quarks are calculated using a string formulation for $1.2T_c \leq T \leq 3T_c$. The results are in good agreement with the lattice data. We also comment on $SU(N)$ gauge theories.

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I. INTRODUCTION

The results at RHIC indicate that the matter created in heavy ion collisions behaves like a strongly coupled liquid [1]. Thus, there is a need for new approaches to strongly coupled gauge theories. Until recently, the lattice formulation was a unique theoretical tool to deal with strongly coupled gauge theories. The subject has taken an interesting turn with Maldacena duality [2]. Although the original proposal was for conformal theories, various modifications have been found that produce gauge/string duals with a mass gap, confinement, and supersymmetry breaking [3].

In this paper we address some issues of thermodynamics of $SU(3)$ pure gauge theory in a dual formulation. Clearly, finding the dual from first principles of string theory is beyond our ability. Instead, we attempt the inverse problem and use some phenomenologically successful five-dimensional models of anti-de Sitter (AdS)/QCD.

II. THE MODEL

Let us first explain the model to be considered. We take the following ansatz for the 10-dimensional background geometry which turns out to be applicable for the temperature range $1.2T_c \leq T \leq 3T_c$ [4]

$$ds^2 = \frac{R^2}{z^2} H(f dt^2 + d\vec{x}^2 + f^{-1} dz^2) + H^{-1} d\Omega_X, \qquad f = 1 - \left(\frac{z}{z_T}\right)^4, \qquad H = e^{(1/2)cz^2}, \tag{1}$$

where $z_T = 1/\pi T$. The value of c will be fixed shortly. The metric is a deformed product of the Euclidean AdS_5 black hole and a 5-dimensional sphere (compact space X). The deformation is due to a z-dependent factor H. Such a deformation is crucial for breaking conformal invariance of the original supergravity solution and introducing Λ_{QCD}. We also take a constant dilaton.

Apart from the language of 10-dimensional string theory, there is a more phenomenological way to attack QCD. This approach called AdS/QCD deals with a 5-dimensional effective description and tries to fit data as much as possible. For our model, its AdS/QCD cousin is obtained by discarding the compact space in (1).

At $T = 0$, then what we get is the slightly deformed AdS_5 metric. In this background linearized Yang-Mills equations are effectively reduced to a Laguerre differential equation. As a result, the spectrum turns out to be like that of the linear Regge models [5,6]. This fact allows one to fix the value of c from the ρ meson trajectory. It is of order $c \approx 0.9\ \mathrm{GeV}^2$ [6]. In addition, this AdS/QCD model provides the phenomenologically acceptable heavy quark potentials as well as the value of the gluon condensate [7,8].

At finite T, the model provides the spatial string tension of pure gauge theory [9]. The agreement with the lattice data is very good for temperatures lower than 2.5–$3T_c$. Because of this reason we set the upper bound on T in (1). Moreover, the model describes in a qualitative way a heavy quark-antiquark pair and the expectation value of the Polyakov loop [10].

Thus, there are reasons to believe that (1) is a good approximation for a string dual to a pure gauge theory.

III. THE ENTROPY DENSITY

One of the bedrocks of the duality is a conjecture that the entropy of gauge theories is equal to the Bekenstein-Hawking entropy of their string (gravity) duals [3]. As is known, the Bekenstein-Hawking entropy is proportional to an (8-dimensional) area of the horizon. The metric (1) has the horizon at $z = z_T$. Therefore, the temperature dependence of the entropy density is

$$s(T) = s_0 T^3 \exp\left\{-\frac{1}{2}\frac{T_c^2}{T^2}\right\}, \tag{2}$$

where s_0 is a factor independent of temperature. In this formula T_c is given by [11]

$$T_c = \frac{1}{\pi}\sqrt{c}. \tag{3}$$

It follows from (2) that the entropy density can be represented as a series in powers of $\frac{1}{T^2}$ with the leading T^3 term

$$s(T) = s_0 T^3 \sum_{n=0}^{\infty} a_n \tau^n, \qquad \tau = \frac{T_c^2}{T^2}, \tag{4}$$

where $a_n = \frac{(-)^n}{2^n n!}$.

For future use, we define the truncated model by keeping the two leading terms in (4). We have

$$s_{tr}(T) = s_0 T^3 (1 - \tfrac{1}{2}\tau). \tag{5}$$

IV. THE PRESSURE: FUZZY BAGS

Recently, it has been suggested by Pisarski that for the temperature range $T_{\max} < T < T_{\text{pert}}$ the pressure in QCD with quarks is given by a series in powers of $\frac{1}{T^2}$ times the ideal T^4 term [12]

$$p_{\text{QCD}}(T) \approx f_{\text{pert}} T^4 - B_{\text{fuzzy}} T^2 - B_{\text{MIT}} + \ldots \tag{6}$$

It was called a fuzzy bag model for the pressure. So, B_{MIT} stands for the MIT bag constant. $T_{\max}$ is close to a critical temperature T_c (or some approximate "T_c" for a crossover). A small difference between T_c and $T_{\max}$ may vary with the model. T_{pert} is set by perturbation theory such that it is applicable only for temperatures higher than T_{pert}.

For pure glue, Pisarski argued, based on lattice simulations of [13], that (6) reduces to a sum of two terms

$$p(T) \approx f_{\text{pert}}(T^4 - T_c^2 T^2). \tag{7}$$

This means that $B_{\text{fuzzy}} = f_{\text{pert}} T_c^2$ and B_{MIT} is much smaller than the first two terms. Note that an important consequence of (7) is that $p(T_c) \approx 0$.

V. THE PRESSURE: STRING DUAL

Given the entropy density as a function of T, in the homogeneous case one can find the temperature dependence of the pressure by integrating $\frac{dp}{dT} = s$ [14]. From (4), we get

$$p(T) = \frac{1}{4} s_0 T^4 \left(1 - \tau - \frac{1}{4}\tau^2 \ln\tau - b\tau^2 + \sum_{n=3}^{\infty} b_n \tau^n\right), \tag{8}$$

where b is an integration constant and $b_n = \frac{2a_n}{2-n}$.

Let us now consider whether the proposal of Pisarski is reasonable in the model under consideration. The two leading terms in (8) look similar to those of (7). So, we find that the critical temperature is given by T_c. A simple estimate based on $c \approx 0.9\ \text{GeV}^2$ [6] then gives $T_c \approx 300$ MeV. In $SU(3)$ pure gauge theory T_c is of order 270 MeV. Since the agreement is not bad, we may use this value of c in (1). It is worth noting that it also means that c slowly depends on a number of quarks. Alternatively, the value of c can be fixed from the critical temperature.

We now use $p(T_c) = 0$ to determine the integration constant. As a result, we have

$$b = \sum_{n=3}^{\infty} b_n \approx 0.039. \tag{9}$$

The value of b is indeed small compared to the coefficients in front of the two leading terms. Thus, the agreement is very satisfactory at this point.

To complete the picture, we present the results of numerical calculations. We split the series (8) into two pieces, the first containing the two leading terms, and the second presenting the rest. Then we define

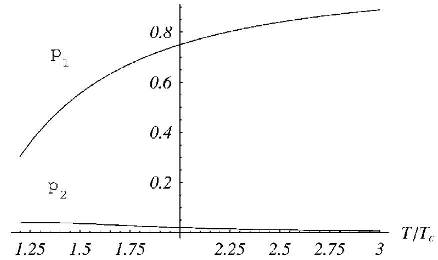

FIG. 1. Values of p_1 and p_2 versus the ratio $\frac{T}{T_c}$.

$$p_1(T) = 1 - \tau,$$

$$p_2(T) = -\frac{1}{4}\tau^2 \ln\tau - b\tau^2 + \sum_{n=3}^{\infty} b_n \tau^n.$$

For simplicity, we have omitted the overall factor $\frac{1}{4} s_0 T^4$. The values of p_1 and p_2 can be read off of Fig. 1. We see that at $T \approx 1.2T_c$ the value of p_2 is 1 order of magnitude smaller than that of p_1. Above $1.2T_c$ the value of p_1 increases, while p_2 decreases and becomes negligible for $T \gtrsim 2T_c$. Thus, $p_1(T)$ provides a reliable approximation whose error is less then 10% for the pressure.

In sum, the truncated model which is equivalent to the proposal of [12] is valid with accuracy better than 10%.

VI. THE SPEED OF SOUND

Having derived the entropy density, we can easily obtain the speed of sound. For the model of interest, we have

$$C_S^2(T) = \frac{s}{Ts'} = \frac{1}{3}\left(1 + \frac{1}{3}\tau\right)^{-1}. \tag{10}$$

For completeness, we also present the result obtained for the truncated model (5). In this case (10) is replaced by

$$C_S^2(T) = \tfrac{1}{3}(1 - \tfrac{1}{2}\tau)(1 - \tfrac{1}{6}\tau)^{-1}. \tag{11}$$

Note that C_S is independent of s_0. Thus, we do not have any free fitting parameter at this point.

We close the discussion of the speed of sound by comparing the results with those of lattice simulations [15]. The curves are shown in Fig. 2. We see that our model is in very good agreement with the lattice for $T \gtrsim 1.7T_c$, while near $1.2T_c$ the discrepancy is of order 15%. The agreement between the truncated model and the lattice is spectacular.

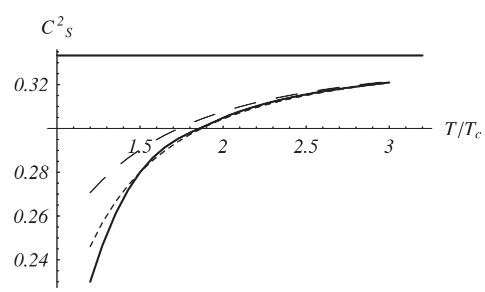

FIG. 2. The square of the speed of sound versus $\frac{T}{T_c}$. The upper and lower dashed curves correspond to (10) and (11), respectively. The solid curve represents the lattice result of [13], while the horizontal line is that of AdS/CFT $C_S^2 = \frac{1}{3}$.

The maximum discrepancy occurred at $T = 1.2T_c$ is of order 6%.

VII. THE GLUON CONDENSATE AT FINITE TEMPERATURE

We will next describe the gluon condensate at finite temperature [16]. It is obtained from the trace anomaly of the energy-momentum tensor [17]. We have

$$G_2(T) = G_2 + 4p - Ts, \tag{12}$$

where G_2 is the condensate at zero temperature.

Unlike the speed of sound, the condensate depends on the parameter s_0. There are two different ways to fix its value which fortunately yield very similar results. The first is to fit the interaction measure $(\epsilon - 3p)/T^4$ as it follows from (4) and (8) to the lattice data of [13] at some normalization point T_n. As a result, we get

$$s_0 = 6.8 \pm 0.3. \tag{13}$$

At first glance it may seem curious that the result is almost independent of the normalization point. As we will see in a moment, this is indeed the case.

The second is to match the coefficient in front of the T^4 term in (8) with that of the bag model [18]. For $SU(3)$ (pure) gauge theory, the latter is $\frac{8}{45}\pi^2$. So, we find

$$s_0 = \tfrac{32}{45}\pi^2 \approx 7.0 \tag{14}$$

that is really the same as (13).

Having determined the value of s_0, we can now write down the expression for the condensate. Combining (4), (8), and (12), we get

$$G_2(T) = -s_0T^4\bigg(\frac{1}{2}\tau + \frac{1}{4}\tau^2\ln\tau + g\tau^2 + \sum_{n=3}^{\infty}(a_n - b_n)\tau^n\bigg), \tag{15}$$

where $g = \frac{1}{8} + b - \frac{k}{s_0}$. Note that the condensate at zero temperature $G_2 = kT_c^4$ has been included in the τ^2 term. For the background geometry (1), the estimate of [8] gives $k \approx 1.20$. Interestingly, the value of g turns out to be small. For $s_0 = 6.8$ it is of order -0.01. The result is shown in Fig. 3.

The expression (15) is cumbersome and difficult of any practical use. We should therefore seek a simpler (nearly equal) expression. To this end, we split the series (15) into two pieces and define

$$g_1(T) = -\frac{1}{2}\tau,$$

$$g_2(T) = -\frac{1}{4}\tau^2\ln\tau - g\tau^2 + \sum_{n=3}^{\infty}(b_n - a_n)\tau^n.$$

For simplicity, we have omitted the overall factor s_0T^4. The values of g_1 and g_2 can be read off of Fig. 4. We see that the value of g_2 is approximately 15% of g_1. Thus, in the temperature range under consideration we may approximate the infinite series (15) by g_1. Finally, the gluon condensate takes the form predicted by the truncated model

$$G_2(T) \approx -\frac{s_0}{2}T_c^2T^2. \tag{16}$$

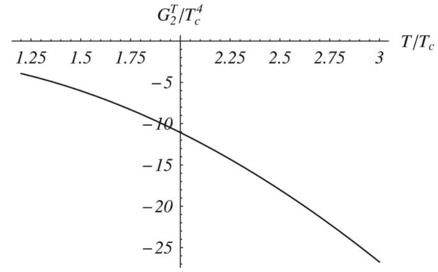

FIG. 3. The gluon condensate in units of T_c^4 versus $\frac{T}{T_c}$. Here $s_0 = 6.8$.

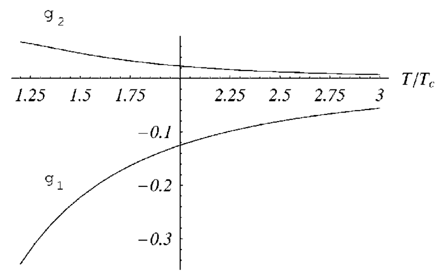

FIG. 4. Values of g_1 and g_2 versus the ratio $\frac{T}{T_c}$.

VIII. THE INTERACTION MEASURE

Using (4) and (8), one can easily find the expression for the interaction measure. It is

$$\frac{\epsilon - 3p}{T^4} = s_0\bigg(\frac{1}{2}\tau + \frac{1}{4}\tau^2\ln\tau + \Big(b + \frac{1}{8}\Big)\tau^2 + \sum_{n=3}^{\infty}(a_n - b_n)\tau^n\bigg). \tag{17}$$

The truncated model provides a simpler expression

$$\frac{\epsilon - 3p}{T^4} = \frac{s_0}{2}\tau, \tag{18}$$

as expected. In Fig. 5 we have plotted the results. As can be

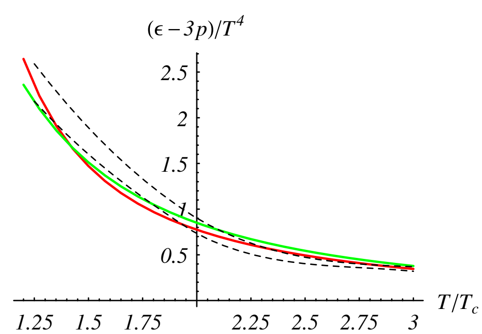

FIG. 5 (color online). The interaction measure $(\epsilon - 3p)/T^4$ versus $\frac{T}{T_c}$. The darker and lighter curves correspond to (17) and (18), respectively. The lattice data of [13] lie between the two dashed lines. Here $s_0 = 6.8$.

seen, the agreement with the lattice data is very satisfactory. An important observation is that varying s_0 over the range (13) has little effect.

IX. $SU(N)$

Can one think of the model (1) as a string dual to an $SU(N)$ pure gauge theory? We will be exploring the consequences of assuming that the pressure vanishes at $T = T_c$ and the parameter c depends on N. This assumption leads to the same expression for the pressure as (8) with b defined by (9). The overall constant s_0 is fixed from the T^4 term. Fitting the bag model, we have

$$s_0 = \frac{4\pi^2}{45}(N^2 - 1). \tag{19}$$

Clearly, the analysis of Sec. II is not sensitive to N. So, the conclusion we draw is that the truncated model is valid with accuracy better than 10%.

Moreover, we can obtain a formula for the pressure normalized by the leading term $p_0 = \frac{1}{4} s_0 T^4$. It is

$$\frac{p}{p_0}(\tau) = \left(1 - \tau - \frac{1}{4}\tau^2 \ln\tau - b\tau^2 + \sum_{n=3}^{\infty} b_n \tau^n\right). \tag{20}$$

Thus our model predicts that the ratio is a function of τ. It does not explicitly depend on N. It is worth noting that in addition to $N = 3$ the prediction is also supported by lattice simulations for $N = 4$ and $N = 8$ [19].

We can gain some understanding of the N dependence of a parameter $\mathfrak{g} = \frac{R^2}{\alpha'}$. Here α' is the usual string parameter coming from the Nambu-Goto action. The lattice data are well fitted by $\frac{T_c}{\sqrt{\sigma}} = 0.596 + \frac{0.453}{N^2}$ [20], where σ is the string tension at zero temperature. For the AdS/QCD cousin of (1) it is given by $\sigma = \mathfrak{g}\frac{e}{4\pi}c$ [7]. Combining with (3), we learn

$$\mathfrak{g} = \frac{4}{\pi e}\left(0.596 + \frac{0.453}{N^2}\right)^{-2}. \tag{21}$$

Simple algebra shows that $\mathfrak{g}$ is a slowly varying function of N. It takes values between 0.93 at $N = 2$ and 1.32 at $N = \infty$. For $N = 3$, $\mathfrak{g}$ is approximately equal to 1.12. It is interesting to compare this value with the estimate of [7]. The latter was made by using the Cornell potential. The result is $\mathfrak{g} \approx 0.94$. The estimates are relatively close. This might be a hint that $\mathfrak{g}$ is also a slowly varying function of a number of quarks.

X. CONCLUDING COMMENTS

First, the model we have proposed predicts the entropy density as a series in $\frac{1}{T^2}$. It differs from the proposal of Pisarski [12] by having a term $\ln T$ in the pressure. In the pure glue case this term turns out to be subdominant. Second, the spatial string tension calculated within the AdS/QCD cousin of (1) can be written as a series in powers of $\frac{1}{T^2}$ times T^2 [9]. However, the first two terms of the series do not provide a reasonable approximation.

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