Photon propagator for axion electrodynamics

Yakov Itin*

Institute of Mathematics, Hebrew University of Jerusalem, Givat Ram, Jerusalem, 91904 Israel, and Jerusalem College of Technology, P.O.B. 16031, Jerusalem, 91160 Israel

(Received 22 July 2007; published 25 October 2007)

The axion modified electrodynamics is usually used as a model for description of possible violation of Lorentz invariance in field theory. The low-energy manifestation of Lorentz violation can hopefully be observed in experiments with electromagnetic waves. It justifies the importance of studying how a small axion addition can modify the wave propagation. Although a constant axion does not contribute to the dispersion relation at all, even a slowly varying axion field destroys the light cone structure. In this paper, we study the wave propagation in the axion modified electrodynamics in the framework of the premetric approach. In addition to the modified dispersion relation, we derive the axion generalization of the photon propagator in Feynman and Landau gauge. Our consideration is free of the usual restriction to the constant gradient axion field. It is remarkable that the axion modified propagator is Hermitian. Consequently, the dissipation effects are absent even in the phenomenological model considered here.

DOI: 10.1103/PhysRevD.76.087505 PACS numbers: 04.20.Cv, 03.50.De, 04.50.+h

1. Axion electrodynamics in premetric formalism.—Although Lorentz invariance is a basic assumption of the standard classical and quantum field theory, this invariance is probably violated in quantum gravity and string theory. One believes that the low-energy manifestation of Lorentz violation can be observed in experiments with the ordinary electromagnetic waves. Axion electrodynamics [1], i.e., the standard electrodynamics modified by an additional axion field, provides a theoretical framework for a possible violation of parity and Lorentz invariance—the Carroll-Field-Jackiw (CFJ) model [2–4]. The non-Abelian extensions of the axion modified electrodynamics for the standard model [5], gravity [6,7], and for supersymmetric models [8] were worked out.

The axion itself can be considered as a fundamental field. Recently, some signals on the axion field observations in PVLAS experiments were reported [9]. The new observations, however, do not show the presence of a rotation and ellipticity signals and thus stand a strong upper limit on axion contributions to an optical rotation generated in vacuum by a magnetic field [9].

Alternatively, the axion can be viewed as an effective field constructed, for instance, from torsion [10–13]. Astrophysics consequences of such torsion induced axion models are recently studied intensively, see [14,15]. Moreover, the linear magnetoelectric effects of Cr_2O_3 find a satisfactory explanation in terms of a macroscopic axion field, see [16]. Some mechanisms that actually lead to axion-type modifications of electrodynamics were proposed recently in [17] and in [18].

The axion modified electrodynamics is usually formulated by adding a topological Chern-Simons term to the Maxwell Lagrangian [1–3]. We apply here an alternative premetric approach to classical electrodynamics [19–21]. In this construction, the axion field emerges in a natural way as an irreducible part of a general constitutive tensor. In the premetric formalism, one starts with two independent antisymmetric fields: the electromagnetic strength F_{ij} and the excitation field H^{ij}. Here the Roman indices range from 0 to 3. The Maxwell equations are given by

$$\epsilon^{ijkl}F_{ij,k} = 0, \qquad H^{ij}{}_{,j} = \mathcal{J}^i, \tag{1}$$

where the commas denote the ordinary partial derivatives, ϵ_{ijkl} is the Levi-Civita permutation tensor normalized with $\epsilon^{0123} = -\epsilon_{0123} = 1$. The fields F_{ij} and H^{ij} are not independent one of another. For a wide range of physical effects, they can be assumed to be related by a linear homogeneous constitutive law

$$H^{ij} = \tfrac{1}{2}\chi^{ijkl}F_{kl}. \tag{2}$$

By definition, the constitutive pseudotensor is antisymmetric in two pairs of indices. Hence it has, in general, 36 independent components. Its irreducible decomposition under the group of linear transformations involves three independent pieces. One of these three pieces is the axion field, which is a subject of our interest. The axion electrodynamics is reinstated in the generic premetric framework by a specialization of the constitutive tensor. It is assumed to be of the following form [21]:

$$\chi^{ijkl} = (g^{ik}g^{jl} - g^{il}g^{jk})\sqrt{-g} + \psi\epsilon^{ijkl}. \tag{3}$$

Here g^{ij} is a metric tensor with the signature $\{+,-,-,-\}$ and with the determinant g. The axion ψ is a pseudoscalar field. It is invariant under transformations of coordinates with a positive determinant and changes its sign under transformations with a negative determinant. In this paper, we restrict to a flat Minkowski spacetime with Cartesian coordinates, so the square root factor in (3) can be omitted.

In the premetric formalism, the axion field is considered only phenomenologically—as an intrinsic characteristic of

*itin@math.huji.ac.il

a "media." The dynamics model of the axion field is usually constructed by involving an additional kinematic term in the Lagrangian, see for instance [22]. The mathematical methods similar to those used here were shown to be useful also in ray optics applications to GR [23,24] and in quantum plasmadynamics [25,26].

2. Momentum representation.—In the premetric electrodynamics, the wave propagation is usually studied in the framework of the geometric approximation [19]. In this case, the variation of the media characteristics (represented by the constitutive tensor χ^{ijkl}) is neglected relative to the change of the wave parameters. Consequently, one comes to the conclusion that the axion field does not affect the wave propagation at all [19]. It is in a contradiction with the standard CFJ electrodynamics predictions [2]. The discrepancy is certainly originated in the restrictions of the geometric approximation [21].

In order to go beyond the geometric approximation, we start with an ansatz of the form

$$F_{ij} = f_{ij}e^{i\phi}. \tag{4}$$

Here the amplitude f_{ij} and the eikonal ϕ are arbitrary functions of a point, i is the imaginary unit. In order to represent a wave-type solution by (4), we require the amplitude to vary much more slowly than the eikonal function. In other words, we apply a condition

$$\frac{\|f_{ij,k}\|}{\|f_{ij}\|} \ll \|\phi_{,k}\|, \tag{5}$$

where the maximal (matrix) norms are assumed. We substitute (4) into Eqs. (1) and (2) and apply the condition (5) to get

$$\epsilon^{ijkl}f_{kl}q_{,j} = 0, \tag{6}$$

and

$$\tfrac{1}{2}(\chi^{ijkl}f_{kl}q_j - i\chi^{ijkl}{}_{,j}f_{kl}) = j^i. \tag{7}$$

Here the notations $j^i = (-i)\mathcal{J}^i e^{-i\phi}$ and $q_j = \phi_{,j}$ are used. A most general solution of the linear system (6) involves an arbitrary covector a_k,

$$f_{kl} = a_k q_l - a_l q_k. \tag{8}$$

Substituting into (7), we come to an algebraic system of four linear equations for four components of the covector a_k:

$$M^{ik}a_k = j^i. \tag{9}$$

Here the matrix of the system is denoted by

$$M^{ik} = \chi^{ijkl}q_l q_j - i\chi^{ijkl}{}_{,j}q_l. \tag{10}$$

This matrix will play a central role in our analysis [27]. After substitution of (3) we write the M-matrix of the axion modified electrodynamics in the form

$$M^{ij} = (g^{ij}q^2 - q^i q^j) + \Pi^{ij}, \tag{11}$$

where the polarization tensor

$$\Pi^{ij} = i\psi_{,k}q_l\epsilon^{ijkl} \tag{12}$$

is involved. Because of its symmetry properties, the M-matrix satisfies

$$M^{ij}q_i = 0, \qquad M^{ij}q_j = 0. \tag{13}$$

These two relations have a pure physical sense: The former one represents the charge conservation law, while the latter relation represents the gauge invariance of the field equation.

Quite remarkable that the matrix M is Hermitian. Its metric part is a standard real symmetric tensor of vacuum electrodynamics. The nonmetric part represents a polarization tensor Π^{ij} which is antisymmetric and pure imaginary.

3. Dispersion relation.—Let us consider the wave solutions of (9). Four components of the covector a_k satisfy now a homogeneous linear system

$$M^{ij}a_j = 0, \tag{14}$$

which has a nonzero solution if and only if its determinant is equal to zero. For the system (14), this condition holds identically, which can be seen even without an explicit calculation of the determinant. Indeed, the identities (13) express a linear relation between the rows (and between the columns) of the matrix M^{ik}. So this matrix is singular. Moreover, (13_b) also means that the linear system (14) has a nonzero solution of the form $a_j = Cq_j$ with an arbitrary constant C. This solution is evidently unphysical since it can be obtained by a gauge transformation of a trivial zero solution. To describe an observable physically meaningful situation, we must have an additional linear independent solution of (14). A 4×4 linear system (14) has two linear independent solutions (one for gauge and one for physics) if and only if its matrix M^{ij} is of rank 2 or less. An algebraic expression of this requirement is

$$A_{ij} = 0, \tag{15}$$

where A_{ij} is the adjoint matrix. This matrix is obtained by removing the ith row and the jth column from the original matrix M^{ij}. The determinants of the remaining 3×3 matrices are calculated and assembled in a new matrix A_{ij}. The entries of the adjoint matrix are expressed via the entries of the matrix M^{ij} as

$$A_{ij} = \frac{1}{3!}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}M^{i_1j_1}M^{i_2j_2}M^{i_3j_3}. \tag{16}$$

For a matrix satisfying (13), the adjoint matrix A_{ij} is symmetric and proportional to the wave covector [28]

$$A_{ij} = \lambda(q)q_i q_j. \tag{17}$$

Since A_{ij} is symmetric and M^{ij} is Hermitian, the adjoint matrix A_{ij} is real. Thus, also the dispersion function $\lambda(q)$ is real. Consequently, to guarantee the existence of a physically meaningful solution, we have to require only one real condition,

$$\lambda(q) = 0, \tag{18}$$

instead of 16 conditions (15). We calculate now the adjoint matrix for the axion modified electrodynamics model. Substituting (11) into (16) and calculating in turn the powers of the imaginary unit, we derive to the adjoint matrix in the following form:

$$A_{ij} = -[q^4 + (\psi_{,m}\psi^{,m})q^2 - (\psi_{,m}q^m)^2]q_i q_j. \tag{19}$$

Thus we come to the known expression of the dispersion relation for the electromagnetic waves in the axion electrodynamics [3,20,21]:

$$q^4 + (\psi_{,m}\psi^{,m})q^2 - (\psi_{,m}q^m)^2 = 0. \tag{20}$$

Note that this fourth order polynomial does not admit a covariant factorization to a product of two second order polynomials. It is in spite of the fact that in special coordinates such factorization exists for timelike, spacelike and null covectors $\psi_{,m}$, see [29]. The expression (20) is not positive defined so the nonbirefringence condition [30] is violated. It is in correlation with the result of [31] that typically in axion electrodynamics the Lorentz group is broken down to the little group associated with the external 4-vector.

4. Photon propagator.—Let us return now to the full inhomogeneous Maxwell equation with a nonzero current:

$$M^{ij}a_j = j^i. \tag{21}$$

The solution of this equation is useful to represent via the Green function or photon propagator, $D_{ij}(q)$. This matrix is defined in such a way that the vector

$$a_j = -D_{ij}j^i \tag{22}$$

is a formal solution of (21). Because of the gauge invariance and charge conservation, the propagator, $D_{ij}(q)$, is defined only up to terms proportional to the wave covector q_i,

$$D_{ij} \to D_{ij} + \alpha_i q_j + \beta_j q_i. \tag{23}$$

Here the components of the covectors α_i and β_i are arbitrary functions of the wave covector.

Since the matrix M^{ik} is singular, the propagator $D_{ij}(q)$ cannot be taken to be proportional to the inverse of M^{ik}. To overcome this obstacle, we use a construction that involves the second adjoint matrix [26,27]. This tensor is defined as

$$B_{ijkl} = \tfrac{1}{2}\epsilon_{ikm_1n_1}\epsilon_{jlm_2n_2}M^{m_1m_2}M^{n_1n_2}. \tag{24}$$

The photon propagator is expressed via this tensor as

$$D_{ik} = \frac{g^{mn}B_{imnk}}{\lambda q^2}. \tag{25}$$

Calculate the second adjoint matrix for the axion expression (11). It is expressed as a sum of three terms:

$$B_{imnk} = {}^{(1)}B_{imnk} + {}^{(2)}B_{imnk} + {}^{(3)}B_{imnk}. \tag{26}$$

After removing the gauge terms (23), the pure metric piece remains in the form

$${}^{(1)}B_{imnk} = -q^2 g_{ik}q_m q_n. \tag{27}$$

Up to the gauge depending terms, the metric-axion piece is

$${}^{(2)}B_{imnk} = -iq^2\psi^{,j}\epsilon_{injk}q_m. \tag{28}$$

The third pure axion piece (without the gauge depending terms) takes the form

$${}^{(3)}B_{imnk} = -\psi_{,i}\psi_{,k}q_m q_n. \tag{29}$$

Thus we have derived the axion modified photon propagator in the following form:

$${}^{(F)}D_{ik} = \frac{q^2 g_{ik} + i\psi^{,j}q^n\epsilon_{injk} + \psi_{,i}\psi_{,k}}{q^4 + (\psi_{,m}\psi^{,m})q^2 - (\psi_{,m}q^m)^2}. \tag{30}$$

This is an axion generalization of the standard Feynman photon propagator. Note that in QED one usually multiplies D_{ij} by $-i$.

Observe that, for a constant axion field, the axion modified photon propagator takes the standard vacuum electrodynamics form. Moreover, the propagator expression has poles only on the solutions of the dispersion relation $\lambda = 0$. It is remarkable that the axion modified photon propagator is Hermitian, so the corresponding wave solutions are not damped. The antisymmetric imaginary part appearing in the numerator of (30) is usual in axion modified models.

Let us compare the expression (30) with the standard QED result:

$$(D^{-1})^{ik} = (\Delta^{-1})^{ik} + \Pi^{ik}, \tag{31}$$

where the free photon propagator $\Delta_{ik} = g_{ik}/q^2$ and the polarization tensor $\Pi^{ij} = -i\psi_{,k}q_l\epsilon^{ijkl}$ are involved. Recall that this expression is derived by summation of an infinite sequence of Feynman diagrams. Multiplying the right-hand sides of (30) and (31), we have

$$\begin{aligned} {}^{(F)}D_{ik}(D^{-1})^{im} &= \frac{1}{\lambda}(q^2 g_{ik} + i\psi^{,j}q^n\epsilon_{injk} + \psi_{,i}\psi_{,k}) \\ &\quad \times (q^2 g^{im} - i\psi_{,k}q_l\epsilon^{imkl}) \\ &= \frac{1}{\lambda}(q^4 + (\psi_{,k}\psi^{,k})q^2 - (\psi_{,k}q^k)^2)\delta^i_m = \delta^i_m. \end{aligned} \tag{32}$$

Here we removed the terms proportional to q^i and q_m which are gauge dependent. Consequently, our expression (30) is indeed inverse to (31).

The Landau photon propagator is derived by removing the transversal terms, ${}^{(L)}D_{ik}q^i = {}^{(L)}D_{ki}q^i = 0$. It is given by

$$ {}^{(L)}D_{ik} = \frac{1}{\lambda}\left[g_{ik} - \frac{q_i q_k}{q^2} + i\psi^{,j} q^n \epsilon_{injk} + \psi_{,i}\psi_{,k} + \frac{(\psi_{,m}q^m)^2}{q^4} q_i q_k - \frac{\psi_{,m}q^m}{q^2}(\psi_{,i}q_k + \psi_{,k}q_i)\right]. \tag{33} $$

Consider a special version of axion modified electrodynamics with a timelike covector of axion field derivatives—the CFJ model [2]. In this case, the coordinates can be taken in such a way that the axion field derivatives covector is parametrized as $\psi_i = (\mu, 0, 0, 0)$. Write the wave vector as $q^i = (w, \mathbf{k})$. Substituting into (30) we get

$$ D_{ik} = \frac{1}{\lambda}\begin{pmatrix} w^2 - k^2 + \mu^2 & 0 & 0 & 0 \\ 0 & w^2 - k^2 & -i\mu k_3 & i\mu k_2 \\ 0 & i\mu k_3 & w^2 - k^2 & -i\mu k_1 \\ 0 & -i\mu k_2 & i\mu k_1 & w^2 - k^2 \end{pmatrix}. $$

5. Conclusions.—In this paper, we treat the axion modified electrodynamics as a special case of premetric electrodynamics formalism. In this construction, the axion field emerges as an irreducible part of a general constitutive tensor. In addition to the known covariant dispersion relation of axion electrodynamics, we have derived a covariant expression of the axion modified photon propagator. The axion modified photon propagator in different gauges was constructed recently in [22,32,33]. These expressions share the main properties of (30) and (33). In particular, they are represented by a fraction with a numerator which is a quadratic function of the wave covector and with a denominator proportional to the dispersion relation expression. Note, however, that the previous considerations are restricted to an axion field with a constant gradient. In our consideration, this restriction is removed. It is a remarkable fact that the propagator (30) is Hermitian. Consequently, a properly defined energy-momentum tensor has to be conserved without dissipation. This issue lies, however, beyond the scope of the current note.

I am grateful to Roman Jackiw and Friedrich Hehl for fruitful discussions. I thank the referee for most useful comments and for very relevant references.

---

[1] F. Wilczek, Phys. Rev. Lett. **58**, 1799 (1987).
[2] S. M. Carroll, G. B. Field, and R. Jackiw, Phys. Rev. D **41**, 1231 (1990).
[3] R. Jackiw, Comments Mod. Phys. A **1**, 1 (1999).
[4] V. A. Kostelecky and M. Mewes, Phys. Rev. D **66**, 056005 (2002).
[5] D. Colladay and V. A. Kostelecky, Phys. Rev. D **58**, 116002 (1998).
[6] V. A. Kostelecky, Phys. Rev. D **69**, 105009 (2004).
[7] R. Jackiw and S. Y. Pi, Phys. Rev. D **68**, 104012 (2003).
[8] H. Belich, J. L. Boldo, L. P. Colatto, J. A. Helayel-Neto, and A. L. M. Nogueira, Phys. Rev. D **68**, 065030 (2003).
[9] E. Zavattini *et al.* (PVLAS Collaboration), Phys. Rev. Lett. **96**, 110406 (2006); arXiv:hep-ex/0706.3419.
[10] V. De Sabbata and M. Gasperini, Phys. Lett. **83A**, 115 (1981).
[11] G. F. Rubilar, Y. N. Obukhov, and F. W. Hehl, Classical Quantum Gravity **20**, L185 (2003).
[12] Y. Itin and F. W. Hehl, Phys. Rev. D **68**, 127701 (2003).
[13] C. Lammerzahl and F. W. Hehl, Phys. Rev. D **70**, 105022 (2004).
[14] O. Preuss, S. K. Solanki, M. P. Haugan, and S. Jordan, Phys. Rev. D **72**, 042001 (2005).
[15] D. Maity, S. SenGupta, and S. Sur, Phys. Rev. D **72**, 066012 (2005).
[16] F. W. Hehl, Y. N. Obukhov, J. P. Rivera, and H. Schmid, arXiv:0707.4407.
[17] F. R. Klinkhamer, Nucl. Phys. **B578**, 277 (2000).
[18] V. A. Kostelecky, R. Lehnert, and M. J. Perry, Phys. Rev. D **68**, 123511 (2003).
[19] F. W. Hehl and Yu. N. Obukhov, *Foundations of Classical Electrodynamics: Charge, Flux, and Metric* (Birkhäuser, Boston, MA, 2003).
[20] F. W. Hehl, Y. Itin, and Yu. N. Obukhov, in *Proceedings of the 3rd Summer School in Modern Mathematical Physics, Zlatibor, Serbia and Montenegro*, edited by B. Dragovich *et al.* (Institute of Physics, Belgrade, 2005), p. 375.
[21] Y. Itin, Phys. Rev. D **70**, 025012 (2004).
[22] A. A. Andrianov, R. Soldati, and L. Sorbo, Phys. Rev. D **59**, 025002 (1998).
[23] V. Perlick, *Ray Optics, Fermat's Principle, and Applications to General Relativity* (Springer, Berlin, 2000).
[24] V. Perlick, Gen. Relativ. Gravit. **38**, 365 (2006).
[25] D. B. Melrose, Plasma Phys. **15**, 99 (1973).
[26] D. B. Melrose, "Quantum Plasmadynamics" (Springer, Berlin, to be published).
[27] Y. Itin, J. Phys. A **40**, F737 (2007).
[28] Y. Itin, "On light propagation in premetric electrodynamics" (unpublished).
[29] Y. Itin, arXiv:0706.2991.
[30] Y. Itin, Phys. Rev. D **72**, 087502 (2005).
[31] A. J. Hariton and R. Lehnert, Phys. Lett. A **367**, 11 (2007).
[32] C. Adam and F. R. Klinkhamer, Nucl. Phys. **B607**, 247 (2001).
[33] R. Lehnert and R. Potting, Phys. Rev. D **70**, 125010 (2004).